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DEFORMATION OF SUPERSYMMETRIC AND CONFORMAL QUANTUM MECHANICS THROUGH AFFINE TRANSFORMATIONS

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Abstract

Affine transformations (dilatations and translations) are used to define a deformation of one-dimensional $N = 2$ supersymmetric quantum mechanics. Resulting physical systems do not have conserved charges and degeneracies in the spectra. Instead, superpartner Hamiltonians are q-isospectral, i.e. the spectrum of one can be obtained from another (with possible exception of the lowest level) by q^2-factor scaling. This construction allows easily to rederive a special self-similar potential found by Shabat and to show that for the latter a q-deformed harmonic oscillator algebra of Biedenharn and Macfarlane serves as the spectrum generating algebra. A general class of potentials related to the quantum conformal algebra $su_q(1,1)$ is described. Further possibilities for q-deformation of known solvable potentials are outlined.

1. Introduction

Standard Lie theory is known to provide very useful tools for description of physical systems. Elegant applications were found in quantum mechanics within the concept of spectrum generating, or, dynamical (super)symmetry algebras [1]. The most famous example is given by the harmonic oscillator problem (so the name of this workshop) where spectrum is generated by the Heisenberg-Weyl algebra. Some time ago a wide attention was drawn to the deformations of Lie algebras which nowdays are loosely called "quantum algebras", or, "quantum groups" [2] (below we do not use the second term because Hopf algebra structure is not relevant in the present context). Spin-chain models were found [3] where Hamiltonian commutes with generators of the quantum algebra $su_q(2)$, deformation parameter q being related to a coupling constant. Thus, an equivalence of a particular perturbation of the interaction between "particles" to the deformation of symmetry algebra governing the dynamics was demonstrated.

Biedenharn and Macfarlane introduced q-deformed harmonic oscillator as a building block of the quantum algebras [4, 5]. Various applications of q-oscillators appeared since that time [6-13] (an overview of the algebraic aspects of q-analysis is given in Ref.[7]). Physical models refering to q-oscillators can be conditionally divided into three classes. The first one is related to systems on lattices [8]. In the second class dynamical quantities are defined on "quantum planes" – the spaces

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with non-commutative coordinates [9]. Although Schrödinger equation in this approach looks similar to the standard one, all suggested explicit realizations of it in terms of the normal calculus result in purely finite-difference equations. Parameter q responsible for the non-commutativity of quantum space coordinates serves as some non-local scale on the continuous manifolds and, therefore, the basic physical principles are drastically changed in this type of deformation. We shall not pursue here the routes of these two groups of models.

The third – dynamical symmetry realization class – is purely phenomenological: one deforms already known spectra by postulating the form of a Hamiltonian as some combination of formal quantum algebra generators [10], or, as an anticommutator of q-oscillator creation and annihilation operators [4, 8]. This application, in fact, does not have straightforward physical meaning because of the non-uniqueness of deformation procedure. Even exact knowledge of a spectrum is not enough for precise reconstruction of an interaction. For a given potential with some number of bound states one can associate another potential containing new parameters and exhibiting the same spectrum [14]. Therefore the physics behind such deformations is not completely fixed. Moreover, for a rich class of spectral problems there are powerful restrictions on the asymptotic growth of discrete eigenvalues [15] so that not any ordered set of numbers can represent a spectrum. All this means that one should more rigorously define physical interaction responsible for a prescribed deformation of a given simple spectrum. q-Analogs of the harmonic oscillators were also used for the description of small violation of statistics of identical particles [13] (general idea on the treatment of this problem on the basis of a parametric deformation of commutation relations was suggested in Ref.[16]). The papers listed above represent only a small fraction of works devoted to quantum algebras and q-analysis. For an account of unmentioned here applications we refer to reviews [17, 18].

Recently Shabat have found one-dimensional reflectionless potential showing peculiar self-similar behavior and describing an infinite number soliton system [19]. Following this development the author proposed [20] to take known exactly solvable Schrödinger potentials and try to deform their shape in such a way that the problem remains to be exactly solvable but the spectrum acquires complicated functional character. So, the Shabat's potential was identified in Ref.[20] as a q-deformation of conformally invariant harmonic and particular forms of Rosen-Morse and Pöschl-Teller potentials. The hidden q-deformed Heisenberg-Weyl algebra was found to be responsible for purely exponential character of the spectrum. In comparison with the discussed above third group of models present approach to "quantum" symmetries is the direct one – physical interaction is fixed first and the question on quantum algebra behind prescribed rule of q-deformation is secondary.

In accordance with this guiding principle a deformation of supersymmetric (SUSY) quantum mechanics [21, 22] was proposed in Ref.[23]. This talk is devoted to description of the results of Refs.[19, 20, 23] and subsequent developments. We start by giving in Sect.2 a brief account of the properties of simplest $(0+1)$-dimensional SUSY models. In Sect.3 we describe a deformation of these models on the basis of pure scaling transformation of a superpartner potential, namely, we find q-SUSY algebra following from this rule and analyze its properties. Sect.4 outlines possible extensions of the simplest potential deformation. In Sect.5 we show that mentioned above self-similar potential naturally appears within q-SUSY as that characterized by the simplest structure of Hamiltonian. In this case factorization operators entering the supercharges are well defined on the Hilbert space of square integrable functions and generate q-oscillator algebra. As a result, a representation of q-deformed conformal algebra $su_q(1,1)$ is obtained. In Sect.6 we give short description of further generalizations of the Shabat's potential which correspond to general q-deformed conformal quantum mechanics and q-deformation of (hyper)elliptic potentials. Sect.7 contains some conclusions. We would like to stress once more that suggested realizations of q-algebras are continuous (i.e. they are not purely finite-difference ones) and they are used within the standard physical concepts.

2. SUSY quantum mechanics

The simplest $N = 2$ SUSY quantum mechanics is fixed by the following algebraic relations between the Hamiltonian of a system H and supercharges $Q^\dagger$, Q [21]

$$\{Q^\dagger, Q\} = H, \quad Q^2 = (Q^\dagger)^2 = 0, \quad [H, Q] = [H, Q^\dagger] = 0. \tag{1}$$

All operators are supposed to be well defined on the relevant Hilbert space. Then, independently on explicit realizations the spectrum is two-fold degenerate and the ground state energy is semipositive, $E_{vac} \geq 0$.

Let us consider a particle moving in one-dimensional space. Below, the coordinate x is tacitly assumed to cover the whole line, $x \in R$, if it is not explicitly stated that it belongs to some cut. Standard representation of the algebra (1) contains one free superpotential $W(x)$ [22]:

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \quad A = (p - iW(x))/\sqrt{2}, \quad [x, p] = i, \tag{2}$$

$$H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} = \tfrac{1}{2}(p^2 + W^2(x) - W'(x)\sigma_3), \tag{3}$$

$$W'(x) \equiv \frac{d}{dx} W(x), \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It describes a particle with two-dimensional internal space the basis vectors of which can be identified with the spin "up" and "down" states.

The subhamiltonians $H_\pm$ are isospectral as a result of the intertwining relations

$$H_- A^\dagger = A^\dagger H_+, \qquad AH_- = H_+ A. \tag{4}$$

The only possible difference concerns the lowest level. Note that the choice $W(x) = x$ corresponds to the harmonic oscillator problem and then $A^\dagger$, A coincide with the bosonic creation and annihilation operators $a^\dagger$, a which satisfy the algebra

$$[a, a^\dagger] = 1, \qquad [N, a^\dagger] = a^\dagger, \qquad [N, a] = -a, \tag{5}$$

where N is the number operator, $N = a^\dagger a$. This, and another particular choice, $W(x) = \lambda/x$, correspond to the conformally invariant dynamics [24].

3. q-Deformed SUSY quantum mechanics

Now we shall introduce the tools needed for the quantum algebraic deformation of the above construction. Let T_q be smooth q-scaling operator defined on the continuous functions

$$T_q f(x) = f(qx), \tag{6}$$

where q is a real non-negative parameter. Evident properties of this operator are listed below

$$T_q f(x)g(x) = [T_q f(x)][T_q g(x)], \qquad T_q \frac{d}{dx} = q^{-1}\frac{d}{dx}T_q,$$
$$T_q T_p = T_{qp}, \qquad T_q^{-1} = T_{q^{-1}}, \qquad T_1 = 1. \tag{7}$$

On the Hilbert space of square integrable functions $\mathcal{L}_2$ one has

$$\int_{-\infty}^{\infty} \phi^*(x)\psi(qx)dx = q^{-1}\int_{-\infty}^{\infty} \phi^*(q^{-1}x)\psi(x)dx, \tag{8}$$

where from the hermitian conjugate of T_q can be found

$$T_q^\dagger = q^{-1}T_q^{-1}, \qquad (T_q^\dagger)^\dagger = T_q. \tag{9}$$

As a result, $\sqrt{q}\,T_q$ is a unitary operator. Because we take wave functions to be infinitely differentiable, an explicit realization of T_q is provided by the operator

$$T_q = e^{\ln q\, x\, d/dx} = q^{x\, d/dx}. \tag{10}$$

Expanding (10) into the formal series and using integration by parts one can prove relations (9) on the infinite line and semiline $[0,\infty]$. A special care should be taken for finite cut considerations since T_q moves boundary point(s).

Let us define the q-deformed factorization operators

$$A^\dagger = \frac{1}{\sqrt{2}}(p + iW(x))\,T_q, \qquad A = \frac{q^{-1}}{\sqrt{2}}\,T_q^{-1}(p - iW(x)), \tag{11}$$

where $W(x)$ is arbitrary function and for convinience we use the same notations as in the undeformed case (3). A and $A^\dagger$ are hermitian conjugates of each other on $\mathcal{L}_2$. Now one has

$$A^\dagger A \;=\; \tfrac{1}{2}q^{-1}(p^2 + W^2(x) - W'(x)) \equiv q^{-1}H_-, \tag{12}$$
$$\begin{aligned} A\,A^\dagger &= \tfrac{1}{2}q^{-1}T_q^{-1}(p^2 + W^2(x) + W'(x))T_q \\ &= \tfrac{1}{2}q\,(p^2 + q^{-2}W^2(q^{-1}x) + q^{-1}W'(q^{-1}x)) \equiv qH_+. \end{aligned} \tag{13}$$

We define q-deformed SUSY Hamiltonian and supercharges to be

$$H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = \begin{pmatrix} qA^\dagger A & 0 \\ 0 & q^{-1}AA^\dagger \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}. \tag{14}$$

These operators satisfy the following q-deformed version of the $N = 2$ SUSY algebra

$$\{Q^{\dagger}, Q\}_q = H, \quad \{Q, Q\}_q = \{Q^{\dagger}, Q^{\dagger}\}_q = 0, \quad [H, Q]_q = [Q^{\dagger}, H]_q = 0, \tag{15}$$

where we introduced q-brackets

$$[X, Y]_q \equiv qXY - q^{-1}YX, \qquad [Y, X]_q = -[X, Y]_{q^{-1}}, \tag{16}$$

$$\{X, Y\}_q \equiv qXY + q^{-1}YX, \qquad \{Y, X\}_q = \{X, Y\}_{q^{-1}}. \tag{17}$$

Note that the supercharges are not conserved because they do not commute with the Hamiltonian (in this respect our algebra principally differs from the construction of Ref.[11]). An interesting property of the algebra (15) is that it shares with (1) the semipositiveness of the ground state energy which follows from the observation that $Q^{\dagger}$, Q and the operator $q^{-\sigma_3}H$ satisfy ordinary SUSY algebra (1). Evidently, in the limit $q \to 1$ one recovers conventional SUSY quantum mechanics.

For the subhamiltonians $H_{\pm}$ the intertwining relations look as follows

$$H_- A^{\dagger} = q^2 A^{\dagger} H_+, \qquad A H_- = q^2 H_+ A. \tag{18}$$

Hence, $H_{\pm}$ are not isospectral but rather q-isospectral, i.e. the spectrum of H_- can be obtained from the spectrum of H_+ just by the q^2-factor scaling:

$$H_+ \psi^{(+)} = E^{(+)}\psi^{(+)}, \qquad H_- \psi^{(-)} = E^{(-)}\psi^{(-)},$$

$$E^{(-)} = q^2 E^{(+)}, \qquad \psi^{(-)} \propto A^{\dagger}\psi^{(+)}, \quad \psi^{(+)} \propto A\psi^{(-)}. \tag{19}$$

Possible exception concerns only the lowest level in the same spirit as it was in the undeformed SUSY quantum mechanics. If $A^{\dagger}, A$ do not have zero modes then there is one-to-one correspondence between the spectra. We name this situation as a spontaneously broken q-SUSY because for it $E_{vac} > 0$. If A (or, $A^{\dagger}$) has zero mode then q-SUSY is exact, $E_{vac} = 0$, and H_+ (or, H_-) has one level less than its superpartner H_- (or, H_+).

As a simplest physical example let us consider the case $W(x) = qx$. The Hamiltonian takes the form

$$\begin{aligned} H &= \tfrac{1}{2}p^2 + \tfrac{1}{4}(q^2 + q^{-2})x^2 + \tfrac{1}{4}(q^{-1} - q) + \tfrac{1}{4}((q^2 - q^{-2})x^2 - q - q^{-1})\sigma_3 \\ &= \tfrac{1}{2}p^2 + \tfrac{1}{2}q^{2\sigma_3}x^2 - \tfrac{1}{2}q^{\sigma_3}\sigma_3, \end{aligned} \tag{20}$$

and describes a spin-1/2 particle in the harmonic potential and related magnetic field along the third axis. The physical meaning of the deformation parameter q is analogous to that in the XXZ-model [3] – it is a specific interaction constant in the standard physical sense. This model has exact q-SUSY and if q^2 is a rational numbeı then the spectrum exhibits accidental degeneracies.

4. General deformation of superpartner Hamiltonians

Described above q-deformation of the SUSY quantum mechanics is by no means unique. If one chooses in the formulas (11) T_q to be not q-scaling operator but, instead, the shift operator

$$T_q f(x) = f(x+q), \qquad T_q = e^{q\, d/dx}, \tag{21}$$

then SUSY algebra will not be deformed at all. The superpartner Hamiltonians will be isospectral and the presence of T_q-operator results in the very simple deformation of old superpartner potential $U_+(x) \to U_+(x-q)$ (kinetic term is invariant). Evidently such deformation does not change the spectrum of $U_+(x)$ and that is why SUSY algebra remains intact. Nevertheless it creates new physically relevant SUSY quantum mechanical models. The crucial point in generating of them was the implication of essentially infinite order differential operators as the intertwining operators.

A more general T_q is given by the shift operator in arbitrary coordinate system

$$T_q f(z(x)) = f(z(x)+q), \qquad T_q = e^{q\, d/dz(x)}, \quad \frac{d}{dz} = \frac{1}{z'(x)}\frac{d}{dx}. \tag{22}$$

The effects of choices $z = \ln x$ and $z = x$ were already discussed above. In general, operator T_q will not preserve the form of kinetic term in H_+-Hamiltonian. Physically, such change would correspond to the transition from motion of a particle on flat space to the curved space dynamics. Below we shall assume the definition (6) but full affine transformation on the line

$$T_q f(x) = f(qx + a)$$

may be used in all formulas without changes.

An interesting question is whether inversion transformation can be joined to the affine part so that a complete $SL(2)$ group element $z \to (az+b)/(cz+d)$ will enter the formalism in a meaningful way? Application of the described construction to the higher dimensional problems is not so straightforward. If variables separate (spherically symmetric or other special potentials) then it may work in a parallel with the non-deformed models. In the many-body case one can perform independent affine transformations for each of the superselected by fermionic number subhamiltonians and thus to "deform" these SUSY models as well.

5. q-Deformed conformal quantum mechanics

Particular form of the $su(1,1)$ algebra generators can be realized via the harmonic oscillator creation and annihilation operators (5)

$$K_+ = \tfrac{1}{2}(a^\dagger)^2, \qquad K_- = \tfrac{1}{2}a^2, \qquad K_0 = \tfrac{1}{2}(N+\tfrac{1}{2}), \tag{23}$$

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = -2K_0. \tag{24}$$

This means that harmonic potential has $su(1,1)$ as the dynamical symmetry algebra, physical states being split into two irreducible representaions according to their parity. Let us show that the potential introduced in Ref.[19] obeys the quantum conformal symmetry algebra $su_q(1,1)$ in complete parallel with (23),(24).

First, we shall rederive this potential within q-SUSY physical situation. Let us consider the Hamiltonian of a spin-1/2 particle in an external potential $\frac{1}{2}U(x)$ and a magnetic field $\frac{1}{2}B(x)$ along the third axis

$$H = \tfrac{1}{2}(p^2 + U(x) + B(x)\sigma_3) \tag{25}$$

and impose two conditions: we take magnetic field to be homogeneous

$$B = -\beta^2 q^{-2} = constant \tag{26}$$

and require the presence of q-SUSY (15). Equating (25) and (14) we arrive at the potential

$$U(x) = W^2(x) - W'(x) + \beta^2 q^{-2}, \tag{27}$$

where $W(x)$ satisfies the following mixed finite-difference and differential equation

$$W'(x) + qW'(qx) + W^2(x) - q^2 W^2(qx) = 2\beta^2. \tag{28}$$

This is the condition of a self-similarity [19] which bootstraps the potential in different points (in Ref.[20] $\beta^2 = \gamma^2(1+q^2)/2$ parametrization was used). Smooth solution of (28) for symmetric potentials $U(-x) = U(x)$ is given by the following power series

$$W(x) = \sum_{i=1}^{\infty} c_i \, x^{2i-1}, \qquad c_i = \frac{q^{2i}-1}{q^{2i}+1}\frac{1}{2i-1}\sum_{m=1}^{i-1} c_{i-m} c_m, \quad c_1 = \frac{2\beta^2}{1+q^2}. \tag{29}$$

In different limits of the parameters several well known exactly solvable problems arise: 1) Rosen-Morse - at $q \to 0$; 2) Pöschl-Teller - at $\beta \propto q \to \infty$; 3) harmonic potential - at $q \to 1$; 4) $1/x^2$-potential - at $q \to 0$ and $\beta \to 0$. However, strictly speaking for all these limits to be valid one has to prove their smoothness, e.g., for 4) there may be solutions for which two limiting procedures do not commute, etc. Note also that for the case 2) the coordinate range should be restricted to finite cut because of the presence of singularities. Infinite soliton solution of Shabat corresponds to the range $0 < q < 1$ at fixed β. If $q \neq 0, 1, \infty$, there is no analytical expression for $W(x)$ but some general properties of this function may be found along the analysis of Ref.[19].

The spectrum can be derived by pure algebraic means. We already know that the spectra of $H_\pm$ subhamiltonians are related via the q^2-scaling

$$E_{n+1}^{(-)} = q^2 E_n^{(+)}, \tag{30}$$

where the number n numerates levels from below for both spectra. Because q-SUSY is exact in this model the lowest level of H_- corresponds to the first excited state of H_+. But due to the restriction (26) the spectra differ only by a constant,

$$E_n^{(-)} = E_n^{(+)} - \beta^2 q^{-2}, \tag{31}$$

Conditions (30) and (31) give us the spectrum of H

$$E_{n,m} = \beta^2 \frac{q^{-2m} - q^{2n}}{1-q^2}, \qquad m = 0, 1; \; n = 0, 1, \ldots, \infty. \tag{32}$$

At $q < 1$ there are two finite accumulation points, i.e. (32) looks similar to two-band spectrum. At $q > 1$ energy eigenvalues seem to grow exponentially to the infinity but there is a catch which does not allow to identify (32) in this case with real physical spectrum. In Ref.[19] it was proven that for $0 < q < 1$ the superpotential is smooth and positive at $x = +\infty$. But then $\psi_0^{(-)}(x) = \exp(-\int^x W(y)dy)$ is a normalizable wave function defining the ground state of H_--subhamiltonian and all other states are generated from it without violation of the normalizability condition. Therefore relation (32) at $0 < q < 1$ defines real physical spectrum.

At $q > 1$ the series defining $W(x)$ converges only on a finite interval $|x| < r < \infty$. From inequalities

$$\rho^2 \equiv \frac{q^2-1}{q^2+1} < \frac{q^{2i}-1}{q^{2i}+1} < 1, \qquad i > 1$$

we have $0 < c_i^{(1)} < c_i < c_i^{(2)}$, where $c_i^{(1,2)}$ are defined by the rule (29) when q-factor on the right hand side is replaced by ρ^2 and 1 respectively ($c_1^{(1,2)} = c_1$). As a result, $1 < 2\sqrt{c_1}r/\pi < \rho^{-1}$, which means that $W(x)$ is smooth only on a cut at the ends of which it has some singularities. From the basic relation (28) it follows that these are simple poles with negative unit residues. In fact there should be an infinite number of simple "primary" and "secondary" poles. The former ones are characterized by negative unit residues and location points x_m tending to $\pi(m+1/2)/\sqrt{c_1}$, $m \in Z$, at $q \to \infty$ (c_1 is fixed). "Secondary" poles are situated at $x = q^n x_m$, $n \in Z^+$, with corresponding residues defined by some algebraic equations. Unfortunately, general analytical structure of the function $W(x)$ is not known yet, presented above hypothesis needs rigorous proof with exact identification of all singularities and this is quite challenging problem.

On the other hand, existence of singularities in superpotential does not allow to take formal consequences of SUSY as granted. Namely, isospectrality (or, q-isospectrality) of H_+ and H_- for the whole line problem is broken at this point. Hence one is forced to consider Shrödinger operator (25) on a cut $[-r, r]$ with boundary conditions $\psi_n(\pm r) = 0$. Pole character of $W(x)$ singularities leads to $\psi_0^{(-)}(\pm r) = 0$, i.e. $\psi_0^{(-)}$ is true ground state of H_-. It also garantees that $U_-(x)$ is finite on the physical boundaries, $U_-(\pm r) < \infty$. Note, however, that the spectrum E_n for such type of problems can not grow faster than n^2 at $n \to \infty$ [15] in apparent contradiction with (32). This discrepancy is resolved by observation that action of T_q-operator creates singularities inside the interval $[-r, r]$ so that $U_+(x)$ and $q^2 U_+(qx)$ are not isospectral potentials (in ordinary sense) as it was at $q < 1$. Hence, the $q > 1$ case of (32) does not correspond to real physical spectrum of the model.

The number of deformations of a given function is not limited. The crucial property preserved by the presented above q-curling is the property of exact solvability of "undeformed" Rosen-Morse, harmonic oscillator, and Pöschl-Teller potentials. It is well known that potentials at infinitely small and exact zero values of a parameter may obey completely different spectra. In our case, deformation with $q < 1$ converts one-level Rosen-Morse problem into the infinite-level one with exponentially small energy eigenvalues. Whether one gets exactly solvable potential at $q > 1$ is an open question but this is quite plausible because at $q = \infty$ a problem with known spectrum arises.

Derivation of the dynamical symmetry algebra is not difficult. To find that we rewrite relations

(12), (13) for the superpotential (28)

$$A^{\dagger}A = q^{-1}H + \frac{\beta^2 q^{-1}}{1-q^2}, \qquad A\,A^{\dagger} = q\,H + \frac{\beta^2 q^{-1}}{1-q^2}, \tag{33}$$

where H is the Hamiltonian with purely exponential spectrum

$$H = \tfrac{1}{2}(p^2 + W^2(x) - W'(x)) - \frac{\beta^2}{1-q^2}, \qquad E_n = -\frac{\beta^2}{1-q^2}\,q^{2n}. \tag{34}$$

Evidently,

$$AA^{\dagger} - q^2 A^{\dagger}A = \beta^2 q^{-1}. \tag{35}$$

Normalization of the right hand side of (35) to unity results in the first relation entering the definition of q-deformed Heisenberg-Weyl algebra.

The shifted Hamiltonian (34) and $A^{\dagger}$, A operators satisfy braid-type commutation relations

$$[A^{\dagger}, H]_q = [H, A]_q = 0,$$

or,

$$H\,A^{\dagger} = q^2 A^{\dagger} H, \qquad A\,H = q^2 H\,A. \tag{36}$$

Energy eigenfunctions $|n\rangle$ can be uniquely determined from the ladder operators action

$$A^{\dagger}|n\rangle = \beta q^{-1/2}\sqrt{\frac{1-q^{2(n+1)}}{1-q^2}}\,|n+1\rangle, \qquad A\,|n\rangle = \beta q^{-1/2}\sqrt{\frac{1-q^{2n}}{1-q^2}}\,|n-1\rangle. \tag{37}$$

It is convinient to introduce the formal number operator

$$N = \frac{\ln[(q^2-1)H/\beta^2]}{\ln q^2}, \qquad N\,|n\rangle = n|n\rangle, \tag{38}$$

which is defined only on the eigenstates of the Hamiltonian. Now one can check that operators

$$a_q = \frac{q}{\beta}\,A\,q^{-N/2}, \qquad a_q^{\dagger} = \frac{q}{\beta}\,q^{-N/2}A^{\dagger} \tag{39}$$

satisfy original q-deformed harmonic oscillator algebra of Biedenharn and Macfarlane [4, 5]

$$a_q a_q^{\dagger} - q a_q^{\dagger} a_q = q^{-N}, \quad [N, a_q^{\dagger}] = a_q^{\dagger}, \quad [N, a_q] = -a_q. \tag{40}$$

The quantum conformal algebra $su_q(1,1)$ is realized as follows,

$$K^+ = \frac{1}{q+q^{-1}}(a_q^{\dagger})^2, \qquad K^- = (K^+)^{\dagger}, \qquad K_0 = \tfrac{1}{2}(N + \tfrac{1}{2}),$$

$$[K_0, K^{\pm}] = \pm K^{\pm}, \qquad [K^+, K^-] = -\frac{q^{4K_0} - q^{-4K_0}}{q^2 - q^{-2}}. \tag{41}$$

Since $H \propto q^{4K_0}$, the dynamical symmetry algebra of the model is $su_q(1,1)$. Generators $K^{\pm}$ are parity invariant and therefore even and odd wave functions belong to different irreducible representations of this algebra. We conclude that quantum algebras have useful applications even within the continuous dynamics described by ordinary differential equations. A different approach to q-deformation of conformal quantum mechanics on the basis of pure finite difference realizations was suggested in Ref.[25].

Let us compare presented model with the construction of Ref.[26]. Kalnins, Levine, and Miller called as the conformal symmetry generator any differential operator $L(t)$ which maps solutions of the time-dependent Schrödinger equation to the solutions, i.e. which satisfies the relation

$$i\frac{\partial}{\partial t}L - [H, L] = R\,(i\frac{\partial}{\partial t} - H), \tag{42}$$

where R is some operator. On the shell of Schrödinger equation solutions $L(t)$ is conserved and all higher powers of space derivative, entering the definition of $L(t)$, can be replaced by the powers of $\partial/\partial t$ and linear in $\partial/\partial x$ term. But any analytical function of $\partial/\partial t$ is replaced by the function of energy when applied to stationary states. This trick allows to simulate any infinite order differential operator by the one linear in space derivative and to prove that a solution with energy E can always be mapped to the not-necessarily normalizable solution with the energy $E + f(E)$ where $f(E)$ is arbitrary analytical function. "On-shell" raising and lowering operators always can be found if one knows the basis solutions of the Schrödinger equation but sometimes it is easier to find symmetry generators and use them in search of the spectrum. In our construction we have "off-shell" symmetry generators, which map physical states onto each other and satisfy quantum algebraic relations in the rigorous operator sense. In this respect our results are complimentary to those of the Ref.[26].

It is clear that affine transformations provide a particular example of possible potential deformations leading just to scaling of spectra. In general one can try to find a map of a given potential with spectrum E_n to a particular related potential with the spectrum $f(E_n)$ for any analytical function $f(E)$. A problem of arbitrary non-linear deformation of Lie algebras was treated in Ref.[12] using the symbols of operators which were not well defined on proper Hilbert space. Certainly, the method of Ref.[26] should be helpful in the analysis of this interesting problem in a more rigorous fashion and the model presented above shows that sometimes an "off-shell" realization of symmetry generators can be found.

6. Factorization method and new potentials

SUSY quantum mechanics is related to the factorization method of solving of Schrödinger equation [27-29]. Within the latter approach one has to find solutions of the following nonlinear chain of coupled differential equations for superpotentials $W_j(x)$

$$W_j' + W_{j+1}' + W_j^2 - W_{j+1}^2 = k_{j+1} \equiv \lambda_{j+1} - \lambda_j, \quad j = 0, 1, 2\ldots \tag{43}$$

where k_j, λ_j are some constants. The Hamiltonians associated to (43) are

$$2H_j = p^2 + U_j(x) = p^2 + W_j^2(x) - W_j'(x) + \lambda_j, \tag{44}$$

$$U_0(x) = W_0^2 - W_0' + \lambda_0, \qquad U_{j+1}(x) = U_j(x) + 2W_j'(x),$$

where λ_0 is an arbitrary energy shift parameter.

SUSY Hamiltonians are obtained by unification of any two successive pairs H_j, H_{j+1} in a diagonal 2×2 matrix. Analogous construction for a piece of the chain (44) with more entries was called an order N parasupersymmetric quantum mechanics [30, 31]. In the latter case relations (43) naturally arise as the diagonality conditions of a general $(N+1) \times (N+1)$-dimensional parasupersymmetric Hamiltonian.

If $W_j(x)$'s do not have severe singularities then the spectra of two operators from (44) may differ only by a finite number of lowest levels. Under the additional condition that the functions

$$\psi_0^{(j)}(x) = e^{-\int^x W_j(y)dy} \tag{45}$$

are square normalizable one finds the spectrum

$$H_j\, \psi_n^{(j)}(x) = E_n^{(j)}\, \psi_n^{(0)}(x), \qquad E_n^{(j)} = \tfrac{1}{2}\lambda_{j+n}, \tag{46}$$

where subscript n numerates levels from below. In this case (45) represents ground state wave function of H_j from which one can determine lowest excited states of $H_{j'}$, $j' < j$,

$$\psi_n^{(j)}(x) \propto (p + iW_j)(p + iW_{j+1}) \ldots (p + iW_{j+n-1})\, \psi_0^{(j+n)}. \tag{47}$$

Any exactly solvable discrete spectrum problem can be represented in the form (43)-(47). Sometimes it is easier to solve Schrödinger equation by direct construction of the chain of associated Hamiltonians (44). If $U_0(x)$ has only N bound states then there does not exist $W_N(x)$ making $\psi_0^{(N)}$ normalizable. If $W_N(x) = 0$, then $H_j\,(j < N)$ has exactly $N - j$ levels, the potential $U_j(x)$ is reflectionless and corresponds to $(N-j)$-soliton solution of the KdV-equation.

In order to solve evidently underdetermined system (43) one has to impose some closure conditions. At this stage it is an art of a researcher to find such an Ansatz which allows to generate infinite number of W_j and k_j from fewer entries. Most of old known examples are generated by the choice $W_j(x) = a(x)j + b(x) + c(x)/j$ where a, b, c are some functions determined from the recurrence relations [27, 28] (see also [19]). New look on the equations (43) was expressed in Ref.[32]. It was suggested to consider that chain as some infinite dimensional dynamical system and to analyze general constraints reducing it to the finite-dimensional integrable cases. In particular, it was shown that very simple periodic closure conditions

$$W_{j+N}(x) = W_j(x), \qquad \lambda_{j+N} = \lambda_j, \tag{48}$$

for N odd lead to all known hyperelliptic potentials describing finite-gap spectra (i.e. those with finite number of permitted bands). In this case parameters λ_j do not, of cause, coincide with the spectrum. The first non-trivial example appears at $N = 3$ and corresponds to Lame equation with one finite gap in the spectrum. Equivalently one can consider arising Schrodinger equation in the Weierstrass form (then periodic potential has singular points where wave functions are required to be equal to zero) and again parameters λ_j do not coincide with (purely discrete) spectrum. Note that in the analysis of parasupersymmetric models [30, 31] constants k_j were naturally treated as arbitrary parameters only occasionally giving the energy levels.

The self-similar potential of Sect.5 was found in Ref.[19] by the following Ansatz in the chain (43)

$$W_j(x) = q^j W(q^j x), \tag{49}$$

which gives a solution provided $W(x)$ satisfies the equation (28) and constants k_j are related to each other as follows

$$k_j \propto q^{2j}, \quad j \geq 0. \tag{50}$$

As it was already discussed, the parameters $\lambda_j \propto q^{2j}$ give the spectrum of problem at $0 < q < 1$ and therefore closure (49) seems to be completely different from (48). However, described above q-SUSY quantum mechanics and subsequent derivation of (49),(50) shows that in fact (49) is a q-deformation of the following closure condition

$$W_{j+1}(x) = W_j(x), \qquad k_{j+1} = k_j, \tag{51}$$

which leads to harmonic oscillator potential. Indeed, one may write

$$W_{j+1}(x) = qW_j(qx), \qquad k_{j+1} = q^2 k_j \tag{52}$$

and check that (49), (50) follow from these conditions.

As it was announced in Ref.[23] one can easily generalize deformation of SUSY quantum mechanical models to the parasupersymmetric ones. In the particular case defined by $(N+1)$-member piece of the chain (44) one simply has to act on the successive Hamiltonians by different affine transformation group elements. This would lead to multiparameter deformation of the parasupersymmetric algebraic relations. Following the consideration of Ref.[30] one may impose analogous physical restrictions on the Hamiltonians and look for the explicit form of potentials accepting these constraints. Analyzing such possibilities the author have found the following general q-periodic closure of the chain (43)

$$W_{j+N}(x) = qW_j(qx), \qquad k_{j+N} = q^2 k_j. \tag{53}$$

These conditions describe q-deformation of the finite-gap and related potentials appearing at $q = 1$. Let us find a symmetry algebra behind (53) at $N = 2$.

First we write out explicitly the system of arising equations

$$\begin{gathered} W_1'(x) + W_2'(x) + W_1^2(x) - W_2^2(x) = 2\alpha, \\ W_2'(x) + qW_1'(qx) + W_2^2(x) - q^2 W_1^2(qx) = 2\beta. \end{gathered} \tag{54}$$

One can check that the operators

$$K^+ = \tfrac{1}{2}(p + iW_1)(p + iW_2)\sqrt{q}T_q, \qquad K^- = (K^+)^\dagger \tag{55}$$

satisfy the relations

$$\begin{gathered} K^+K^- = H(H - \alpha), \qquad K^-K^+ = (q^2H + \beta)(q^2H + \alpha + \beta), \\ H = \tfrac{1}{2}(p^2 + W_1^2(x) - W_1'(x)). \end{gathered} \tag{56}$$

The operator H obeys the following commutation relations with $K^{\pm}$

$$HK^{+}-q^{2}K^{+}H=(\alpha+\beta)K^{+},\qquad K^{-}H-q^{2}HK^{-}=(\alpha+\beta)K^{-}. \tag{57}$$

Note that by adding to H of some constant equations (57) may be rewritten in the form (36).

On the basis of (56) one may define various q-commutation relations between K^{+} and K^{-}. The simplest one would be the following

$$K^{-}K^{+}-q^{4}K^{+}K^{-}=q^{2}(\alpha(1+q^{2})+2\beta)H+\beta(\alpha+\beta). \tag{58}$$

The formal map onto the relations (41) is also available. Therefore relations (57),(58) give a particular form of the "quantization" of the algebra $su(1,1)$ which is explicitly recovered at $q=1$.

Described q-deformation of the conformal quantum mechanics is more general than that presented in Sect.5. Indeed, various limits of q give the following solvable cases: 1) a two-level potential corresponding to two-soliton system appears at $q=0$; 2) a finite cut analog of two-soliton potential arises at $q\to\infty$; 3) the general conformal potential comprising both oscillator and $1/x^{2}$ parts is recovered in the limit $q\to 1$ when $W(x)\propto a/x+bx$. In order to find the spectrum of H at arbitrary q it is neccessary to know general properties of the superpotential W_1. Let us suppose that there exists a solution for positive α and β such that $\exp(-\int^{x}W_{1,2})$ are normalizable wave functions. Then the spectrum consists of two geometric series and by shifting can be represented in the form

$$E_n=\begin{cases} E_0 q^{2m}, & \text{for n=2m}\\ E_1 q^{2m}, & \text{for n=2m+1}\end{cases} \tag{59}$$

with the $E_n<E_{n+1}$ ordering fulfilled. Even and odd wave functions fall into independent irreducible representations of $su_q(1,1)$. A more detailed consideration of potentials and algebraic structures arising from the q-periodic closure of the chain (43) will be given elsewhere.

7. Conclusions

To conclude, we described a deformation of the SUSY quantum mechanics on the basis of affine transformations. The main feature of the construction is that superpartner Hamiltonians satisfy non-trivial braid-type intertwining relations which remove degeneracies of the original SUSY spectra. Obtained formalism naturally leads to the Shabat's self-similar potential describing slowly decreasing solutions of the KdV equation. The latter is shown to have straightforward meaning as a q-deformation of the harmonic oscillator potential. Equivalently, one may consider it as a deformation of a one-soliton system. Corresponding raising and lowering operators satisfy q-deformed Heisenberg-Weyl algebra atop of which a quantum conformal algebra $su_q(1,1)$ can be built. We also outlined a generalization of the Shabat's potential on the basis of q-deformation of periodic closure condition and presented q-deformation of general conformal quantum mechanics potentials.

In this paper the parameter q was taken to be real but nothing prevents from consideration of complex values as well (this changes only hermicity properties). The most interesting cases appear when q is a root of unity [33]. For example, at $q^{3}=1$ eq. (28) generates a potential proportional to the so-called equianharmonic Weierstrass functions. More complicated hyperelliptic potentials are generated at higher roots of unity. The nontrivial Hopf algebra structure of the quantum groups was not considered because it is not relevant in the context of quantum mechanics of one particle in one dimension. Perhaps higher dimensional and many body problems shall elucidate this point. In fact, there seems to be no principle obstacles for higher dimensional generalizations although resulting systems may not have direct physical meaning. Another possibility is that described self-similar systems may arise from higher dimensional ones after the similarity reductions.

In order to illustrate various possibilities we rewrite the simplest self-similarity equation without scaling (i.e. at $q=1$) but with non-trivial translationary part

$$W'(x)+W'(x+a)+W^2(x)-W^2(x+a)=constant. \tag{60}$$

Solutions of this equation provide a realization of the ordinary undeformed Heisenberg-Weyl algebra. The full effect of the presence of the parameter a in (60) is not known to the author but solutions whose absolute values monotonically increase at $x\to\pm\infty$ seem to be forbidden. Note also that in all formulas of SUSY and q-SUSY quantum mechanics superpotential $W(x)$ may be replaced by a hermitian $n\times n$ matrix function. The equations (28), (35), (60) may be equally thought as being the matrix ones with the right hand sides proportional to unit matrices. We end by a speculative conjecture that described machinery may be useful in seeking for q-deformations of the non-linear integrable evolution equations, like KdV, *sin*-Gordon, etc.

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