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CONDITION FOR EQUIVALENCE OF q-DEFORMED AND ANHARMONIC OSCILLATORS

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Abstract

We discuss the equivalence between the q-deformed harmonic oscillator and a specific anharmonic oscillator model, by which some new insight into the problem of the physical meaning of the parameter q can be attained.

1 Introduction

Recently there has been a great deal of interest in the study of quantum groups. Of particular interest here is the development by Macfarlane [1] and independently by Biedenharn [2] of the realization of the quantum group $SU(2)_q$ in terms of the q-analogue of the quantum harnomic oscillator. Although many aspects of the q-deformation of the bose harmonic oscillator algebra have been investigated, still one of the most appealing issues is perhaps the physics behind the parameter q. Here an attempt is made in this direction.

We show that the q-deformed harmonic oscillator model can be used to describe a specific anharmonic oscillator. Thus a q-deformation can be understood as an effective anharmonic deformation, where q is proportional to the strength of the harmonicity. The anharmonic and the q-deformed oscillator models are presented respectively in section 2 and 3 and their equivalence is therein discussed. The latter can in turn be used to examine interesting non-classical features induced by a q-deformation during the time-evolution of a $SU(2)$ coherent state. This is put forward in section 4, and discussed in [3]

2 Anharmonic oscillator

The *anharmonic* oscillator we wish to discuss has the hamiltonian

$$H_\lambda = H_0 + \frac{\mu}{\omega_0} N^3 \equiv N + \frac{1}{2} + \frac{\mu}{\omega_0} N^3 \tag{1}$$

where H_0 is the free hamiltonian of the simple *harmonic* oscillator whose fundermental frenquency is ω_0. $N = b^\dagger b$ is the number operator, whereas $b^\dagger$ and b are respectively the lowering and raising bose operators. H_λ is in units of ω_0 when H_0 is in units of ω_0. The anharmonic term is taken proportional to N^3, and the anharmonicity parameter is positive: specifically we take here $\mu \equiv \omega_0\gamma^2/6$. In the limit of small anharmonic deformations the hamiltonian in Eq.(1) can be discussed in terms of

$$a_\gamma = \sqrt{\Omega_\gamma^{-1}}\,[1 + \gamma^2\frac{(b^\dagger b + 1)^2}{2\cdot 3!}] \qquad \Omega_\gamma = \gamma^{-1}\sinh\gamma \tag{2}$$

It is readily seen that in this representation

$$H_\gamma = \Omega_\gamma(a_\gamma^\dagger a_\gamma + 1/2) \tag{3}$$

is indeed equivalent [4] to H_λ in Eq.(1).

States of our anharmonic oscillator can be constructed as quantum states for H_γ. First note that the vacuum $|0\rangle_\gamma$, defined as $a_\gamma|0\rangle_\gamma = 0$, is the same as the vacuum $|0\rangle_\gamma$ for the harmonic oscillator. However, eigenstates of the number operator $N_\gamma = a_\gamma^\dagger a_\gamma$ substantially differ from those for the harmonic oscillator. The former can be defined as

$$|n\rangle_\gamma = \frac{(a_\gamma^\dagger)^n}{\sqrt{c_{n,\gamma}}}|0\rangle_\gamma \qquad N_\gamma|n_\gamma\rangle_\gamma = \frac{c_{n,\gamma}}{c_{n-1,\gamma}}\,|n\rangle_\gamma \tag{4}$$

while the normalization condition ${}_\gamma\langle m|n\rangle_\gamma = \delta_{m,n}$ determines the $c_{n,\gamma}$'s:

$$c_{n,\gamma} = n!\Omega_\gamma^{-n}\prod_{k=1}^{n}(1 + \frac{\gamma^2 k^2}{2\cdot 3!})^2 = n!\Omega_\gamma^{-n}[(1 + \frac{\gamma^2 n^2}{2\cdot 3!})^2]!, \qquad c_{0,\gamma} = 1 \tag{5}$$

Here we will be concerned, in particular, with coherent states. In the basis $\{|n\rangle_\gamma\}$ ($n = 0, 1, 2, \ldots$) these can be expressed as [5]

$$|\alpha\rangle_\gamma = C_\gamma\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{c_{n,\gamma}}}|n\rangle_\gamma, \qquad C_\gamma^{-2} = \sum_{n=0}^{\infty}\frac{\alpha^{2n}}{c_{n,\gamma}} \tag{6}$$

Where C_γ derives from the normalization condition ${}_\gamma\langle\alpha|\alpha\rangle_\gamma = 1$. The resemblance of the $|\alpha\rangle_\gamma$'s with coherent states of the harmonic oscillator is resdily seen: however, we should stress that only in the limit $\gamma \to \infty$ the anharmonic and harmonic oscillator models are exactly the same.

3 q-deformed harmonic oscillator

Let us recall the $(b, b^\dagger)$ bose operators for the harmonic oscillator introduced earlier. They satisfy the Weyl-Heisenberg algebra

$$[b, b^\dagger] = 1 \qquad [N, b^\dagger] = b^\dagger \qquad N = b^\dagger b \tag{7}$$

Macfarlane [1] and Biedenharm [2] have discussed a deformation of this algebra so that

$$a_q a_q^\dagger - q a_q^\dagger a_q = q^{-N} \qquad [N, a_q^\dagger] = a_q^\dagger \tag{8}$$

and, in particular, its realization in terms of a q-deformed harmonic oscillator. The parameter q [6] characterizes the strength of the deformation.

We explore in this section the connection between q-deformations and anharmonic deformations of the harmonic oscillator. We will first study the effect of a q-deformation on the states of the harmonic oscillator, similarly to what was done in the previous section for the anharmonic oscillator model. By recalling that the q-operators can be realized in terms of the bose operators of the form [1, 2]

$$a_q = \sqrt{\frac{[N+1]_q}{N+1}}\,b; \qquad a_q^\dagger = b^\dagger\sqrt{\frac{[N+1]_q}{N+1}}\,, \tag{9}$$

where $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$, we first construct the quantum states for the q-harmonic oscillator. The q-deformed vacuum is defined as $a_q|0\rangle_q = 0$, and since a_q is a function of b and power of $b^\dagger b$, $|0\rangle_q$ and the vacuum $|0\rangle$ of the harmonic oscillator turn out to be the same. Eigenstates of the number operator $N_q = a_q^\dagger a_q$ can be defined as

$$|n\rangle_q = \frac{(a_q^\dagger)^n}{\sqrt{c_{n,q}}}|0\rangle_q \qquad N_q|n_q\rangle_q = \frac{c_{n,q}}{c_{n-1,q}}\,|n\rangle_q \tag{10}$$

With the choice of $c_{n,q} \equiv \sqrt{[n]_q!}$, where $[n]_q! = [n]_q[n-1]_q\cdots[1]_q$, the set of eigenvectors $\{|n\rangle_q\}$ $(n = 0, 1, 2, \ldots)$ is orthonormal $({}_q\langle m|n\rangle_q = \delta_{m,n})$ and generates the Fock space for the q-deformed oscillator. On the basis $\{|n\rangle_q\}$ $(n = 0, 1, 2, \ldots)$ one can express the coherent states of the q-deformed harmonic oscillator as

$$|\alpha\rangle_q = C_q \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{c_{n,q}}}|n\rangle_q \qquad C_q = [exp_q\alpha^2]^{-1/2} \tag{11}$$

where the factor C_q is again set by the normalization condition ${}_q\langle\alpha|\alpha\rangle_q = 1$. Here exp_q stands for the q-exponential, i.e. $exp_q = \sum_{n=0}^{\infty} x/[n]_q!$. Again note that as $q \to 1$ this q-deformed model exactly reduces to that of a simple harmonic oscillator.

A connection can be established between coherent states of q-deformed harmonic oscillator and coherent states of the anharmonic oscillator in the sense that there exists a condition under which the $|\alpha\rangle_q$'s and the $|\alpha\rangle_\gamma$'s are equivalent. Namely, for oscillator displacements α and γ (or q) such that [3]

$$\alpha(\alpha+8) < \ln^{-1} q^{1/4} \tag{12}$$

we have $|\alpha\rangle_q \to |\alpha\rangle_\gamma$, provided $\gamma = \ln q$. An analytic proof of this equivalence is beyond the aim of this paper and will be reported elsewhere [3]. However, we can compare here the probability number distribution for the $|\alpha\rangle_\gamma$'s to that for the $|\alpha\rangle_q$'s, that is, $P_n^\gamma(\alpha) = |\langle n|\alpha\rangle_\gamma|^2$ and $P_n^q(\alpha) = |\langle n|\alpha\rangle_q|^2$. Owing to the definition of probability as overlap over the same state $|n\rangle$, equal distributions would infer the equivalence of the states $|\alpha\rangle_\gamma$ and $|\alpha\rangle_q$. A numerical evaluation is reported in Fig.1 for values of q and α respectitively conforming and not conforming with the condition (12). In this latter case $P_n^{\gamma_2}(\alpha_2)$ is strongly shifted with respect to $P_n^{q_2}(\alpha_2)$, whereas in the former case the two distribution are nearly the same.

In conclusion, for appropriate displacements (α) and anharmonic couplings (μ) coherent states of an oscillator with anharmonicity $\sim N^3$ (N is the number of particles) are correctly described in terms of coherent states of the q-deformed Lie algebra of $SU(2)$, where $q \simeq exp(\mu/\omega_0)^{1/2}$. This result is particularly important because the parameter q can be given a direct physical meaning: it is proportional to the square root of the anharmonic coupling strength.

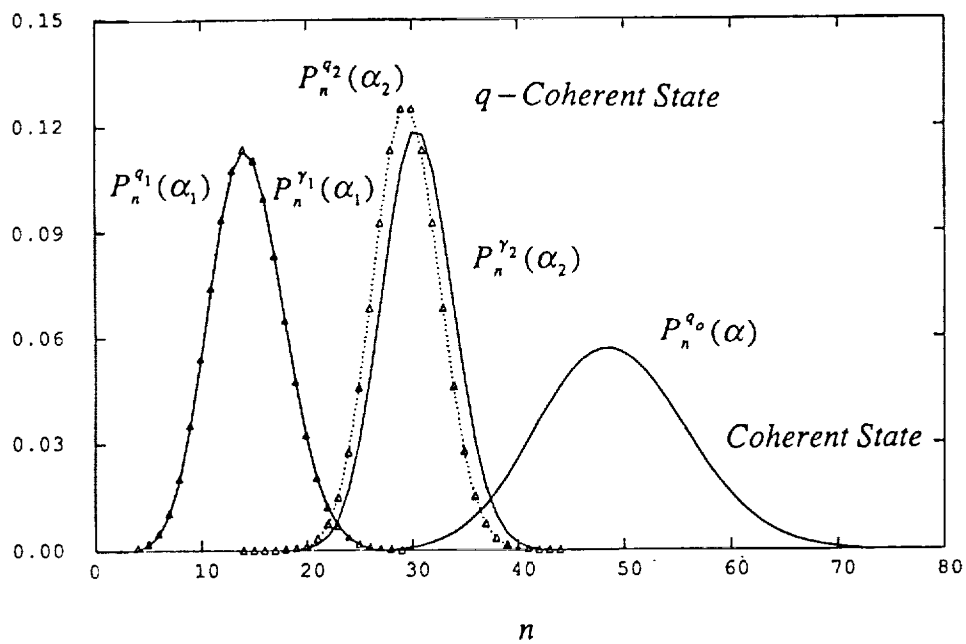

FIG.1. Probability number distributions for coherent states ($|\alpha\rangle_{q_1}, |\alpha\rangle_{q_2}$) of a q-deformed quantum oscillator and for coherent states ($|\alpha\rangle_{\gamma_1}, |\alpha\rangle_{\gamma_2}$) of a quantum oscillator with a third order anharmonicity in the particle number. From their equivalence one can infer the equivalence between the corresponding states, which holds depending on whether the oscillator parameters satisfy ($\alpha_1 = 4,\ \gamma_1 = 0.05$) or do not satisfy ($\alpha_1 = 10,\ \gamma_1 = 0.1$) the condition (12), respectively. Here $q = e^{\gamma}$. $P_n^{q_0}(\alpha)$ is a reference Poïssion ($q_0 = 1$) distribution with $\alpha = 7$.

4 q-deformation and non-classical harmonic oscillator

The equivalence we have established between anharmonicity and q-deformation of a harmonic oscillator is a very helpful one: not only does it provide the q parameter with a definite physical meaning, but also dose it turn out to be useful for investigating and attaining a sound physical interpretation of interesting non-calssical effects induced by a q-deformation during time-evolution of a $SU(2)$ coherent state. The most important of these effects is a q-dependent *self-squeezing*: i.e. a reduction of the uncertainty expactations of the two orthogonal components (quadratures) of the oscillator field below their vacuum values that varies with q. A q-deformation does also alter the *minimality* properties of an initial mimimum uncertainty coherent state, but not its *possionian counting statistics.* The connection between q-deformations of the harmonic oscillator and these rather interesting phenonena is however beyond the purpose of this paper and will be discussed elsewhere [3].

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[4] We here retain terms only of the order γ^3 or lower, as typically done for small anharmonic deformations at ordinary energies;

[5] For simplicity, we take α real;

[6] q is in general complex: however, here $q > 1$ and real;