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NOVEL PROPERTIES OF THE q-ANALOGUE QUANTIZED RADIATION FIELD

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Abstract

The "classical limit" of the q-analogue quantized radiation field is studied paralleling conventional quantum optics analyses. The q-generalizations of the phase operator of Susskind and Glogower (circa 1964) and that of Pegg and Barnett (circa 1988) are constructed. Both generalizations and their associated number-phase uncertainty relations are manifestly q-independent in the $|n>_q$ number basis. However, in the q-coherent state $|z>_q$ basis, the variance of the generic electric field, $(\Delta E)^2$, is found to be increased by a factor $\lambda(z)$ where $\lambda(z) > 1$ if $q \neq 1$. At large amplitudes, the amplitude itself would be quantized if the available resolution of unity for the q-analogue coherent states is accepted in the formulation. These consequences are remarkable versus the conventional $q = 1$ limit.

1 Introduction

On several occasions during the last fifty years, new mathematical symmetries have been constructed in theoretical physics but only found to be relevant to nature five or more years later. If this is occurring now in the case of quantum algebras, we need to know the physical implications of these new and distinctly novel symmetry structures. If there are q-oscillators in nature which realize these new algebras, surely there must be a quantum field which has such q-oscillators as its normal modes. Until we know the physical properties of such a field , say in its "classical limit", we may not be able to glean its distinct relevance to problems and phenomena in quantum optics, many body physics, particle physics ....

2 A Completeness Relation for the q-Analogue Coherent States by q-Integration

The q-analogue coherent states $|z>_q$ satisfy $a|z>_q = z|z>_q$ where the q-oscillator algebra is [1] ($q \rightarrow 1$, usual bosons)

$$aa^\dagger - q^{1/2}a^\dagger a = q^{-N/2} \tag{1}$$

$$[N, a^\dagger] = a^\dagger \qquad [N, a] = -a \tag{2}$$

It is physically very important that there remains the mathematically trivial bosonic $[a, a] = 0$.

In the $|n>_q$ basis, $< m|n >= \delta_{mn}$ and[1]

$$a^\dagger|n >= \sqrt{[n+1]}|n+1> \qquad a|n >= \sqrt{[n]}|n-1> \qquad a|0>= 0 \tag{3}$$

where $[x]_q = [x] \equiv (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2})$ is the "q-deformation" of x. More simply $[x] = \sinh(sx/2)/\sinh(s/2)$ where $q = \exp s$, $0 \le q \le 1$.

The q-analogue coherent states $|z>_q$ are good candidates for studying the classical limit of the q-analogue quantized radiation field because (i) there exists a resolution of unity [2]

$$I = \int |z >< z|\, d\mu(z) \tag{4}$$

(ii) they indeed are "minimum uncertainty states" for they do minimize the fundamental commutation relation

$$U_{Q,P} \equiv \frac{2\Delta Q\Delta P - |<[Q,P]>|}{|<[Q,P]>|} \ge 0 \tag{5}$$

with $U|_{|z>} = 0$ but $U|_{|n>\neq|0>} = \frac{(3[n]+[n+1])}{([n+1]-[n])}$, and (iii) the n^{th} order correlation function factorizes, i.e.

$$Tr(\rho E^-(x)E^+(y)) = \mathcal{E}^-(x)\mathcal{E}^+(y), \ldots \tag{6}$$

But, simultaneously, there are intriguing differences in the $|z>_q$ basis for other coherence and uncertainty properties of the q-analogue quantized field. Some of these will be discussed as we go along.

In the $|z>_q$ basis, from $a|z>= z|z>$ it follows that for $< z|z >= 1$

$$|z>_q = N(z)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]!}}|n>, \qquad N(z) = e_q(|z|^2)^{-1/2} \tag{7}$$

in terms of the "q-exponential function"

$$e_q(z) \equiv \sum_{n=0}^{\infty}\frac{z^n}{[n]!}, \qquad [n]! \equiv [n][n-1]\cdots[1], \qquad [0]! = 1 \tag{8}$$

which is an entire function $|e_q(z)| \le e_q(|z|) \le exp(|z|)$. For $x > 0$, it's positive, but for $x < 0$ it wildly oscillates within these bounds!

To derive the resolution of unity, we need a lemma which is a q-analoque of Euler's formula: We define the q-derivative

$$\frac{d}{d_qx}f(x) \equiv \frac{f(q^{1/2}x) - f(q^{-1/2}x)}{q^{1/2}x - q^{-1/2}x} \tag{9}$$

[1] From now on the sub-q's are usually implicit!

and for $f(x)$ on the interval $[0,a]$, the inverse operation

$$\int_0^a f(x)\, d_q x \equiv a(q^{-1/2} - q^{1/2}) \sum_{n=0}^{\infty} q^{\frac{2n+1}{2}} f(q^{(2n+1)/2} a). \tag{10}$$

So, for instance $\frac{d}{d_q x} a x^n = a[n] x^{n-1}$, $\frac{d}{d_q x} e_q(ax) = a\, e_q(ax)$ and inversely $\int a x^{n-1}\, d_q x = a x^n/[n]$, $\int e_q(ax)\, d_q x = e_q(ax)/a$ up to the constants. It follows that there are two integration by parts formulas

$$\int_0^a f(q^{1/2}x)\,(\frac{d}{d_q x} g(x))\, d_q x = f(x)g(x)|_{x=0}^{x=a} - \int_0^a \frac{d}{d_q x} f(x) g(q^{-1/2}x)\, d_q x \tag{11}$$

and the $q \rightarrow 1/q$ expression.

We define $-\zeta$ =largest zero of $e_q(x)$ and restrict $e_q(z) \equiv [\ \sum_{n=0}^{\infty} \frac{z^n}{[n]!}$ for$-\zeta < x$; 0,otherwise] . Then by the first integration by parts formula

$$\int_0^{\zeta} e_q(-x) x^n\, d_q x = [n]! \tag{12}$$

From this the resolution of unity simply follows for the measure

$$d\mu(z) = \frac{1}{2\pi} e_q(|z|^2) e_q(-|z|^2) d_q |z|^2\, d\theta \tag{13}$$

since

$$\begin{aligned}
\int |z><z|\, d\mu(z) &= \frac{1}{2\pi} \int \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|z|^n |z^*|^m}{\sqrt{[n]![m]!}} e_q(-|z|^2)\, d_q |z|^2 \\
&\quad \int \exp\left(i(n-m)\theta\right) d\theta |n><m| &(14)\\
&= \sum_{n=0}^{\infty} \frac{1}{[n]!} \int_0^{\zeta_i} x^n\, e_q(-x)\, d_q x\, |n><n|, \qquad x = |z|^2 &(15)\\
&= \sum_{n=0}^{\infty} |n><n| \ = I &(16)
\end{aligned}$$

Several remarks are appropriate:(i) states with $|z|^2 \geq \zeta_i$ do not contribute,(ii) arbitrary $|z>_q$ coherent states are not orthogonal since $<\alpha|\beta> = N(\alpha)N(\beta)\, e_q(\alpha^* \beta) \neq 0$,(iii) the $|z>_q$are actually overcomplete, since

$$|\alpha>_q = \int |z><z|\alpha>\ d\mu(z), \qquad\qquad <z|\alpha> \neq 0, \tag{17}$$

(iv) with $f(z) \equiv <z|f>$, the $a^\dagger, a$ act $<z|a^\dagger|f> = z^* f(z)$, and $<z|a|f> = N(z) \frac{d}{d_q z^*} N(z)^{-1} f(z)$, (v) any zero of $e_q(-\zeta_i) = 0$ can be the upper endpoint of integration provided something restricts $e_q(x)$ beyond $-\zeta_i$. If not, on the rhs of (12) there is also $r_n = -[n]! \sum_{k=0}^{n} \frac{1}{[n-k]!} (q^{1/2} x_k)^{n-k} e_q(-x_k)$ where $x_k = q^{k/2}|\zeta_i|$. This restriction occurs if there are q-discrete auxillary states ($|\tilde{z}_k|^2 = x_k$)

$$|\tilde{z}_k>_q = M_k \sum_{j=0}^{\infty} \frac{(q^{1/4}\tilde{z}_k)^j}{\sqrt{[j]!}} |j+k>, \qquad\qquad \tilde{a}_k |\tilde{z}_k>_q = (q^{1/4}\tilde{z}_k)|\tilde{z}_k>_q \tag{18}$$

with $k = 0, 1, \ldots$; $M_k = e_q(q^{1/2}|\tilde{z}_k|^2)^{-1/2}$; with a discrete measure $d\tilde{\mu}_k = \frac{1}{2\pi M_k^2} e_q(-|\tilde{z}_k|^2)\, d\theta$.

3 The q-Analogue Quantized Radiation Field and Its Uncertainty Relations

In analyzing the field in the $|z>_q$ classical limit, we suppress the $\overrightarrow{k}$ mode and $\hat{\epsilon}$ polarization indices for the generic electric and magnetic fields, etc. . There are diagonal representations of operators, e.g. the single-mode density operator

$$\hat{\rho} = \int d\mu(z)\, \phi_N(z, z^*)|z><z| \tag{19}$$

where $\int d\mu(z)\, \phi_N(z,z^*) = 1$ as $Tr(\rho) = 1$; so $<(a^\dagger)^r a^s> = Tr[\rho(a^\dagger)^r a^s] = \int d\mu(z)\,(z^*)^r z^s \phi_N(z,z^*)$. Similarly, $< a^r(a^\dagger)^s > = \int d\mu(z)\, z^r(z^*)^s \psi_N(z,z^*)$ for $\psi_N(z,z^*) \equiv <z|\hat{\rho}|z>$, $\int d\mu(z)\, \psi_N(z,z^*) = 1$, and so

$$\psi_N(z,z^*) = \int d\mu(y)\, \phi_N(y,y^*) N(y)^2 N(z)^2 e_q(yz^*) e_q(zy^*) \tag{20}$$

Note that due to the use of q-integration to obtain (16), a new " q-quantization" in the z complex plane has occurred, e.g. ϕ_N contributes to (19) only when

$$|z|^2 = q^{(2n+1)/2}\zeta_i, \qquad n = 0,1,2,\dots. \tag{21}$$

Consequently, for the generic electric and magnetic fields

$$\hat{E} = i(\hbar\omega/2\epsilon_0 V)^{1/2}[a e^{i(\overrightarrow{k}\cdot\overrightarrow{r}-\omega t)} - a^\dagger e^{-i(\overrightarrow{k}\cdot\overrightarrow{r}-\omega t)}] \tag{22}$$

with $z = |z|\exp(i\theta)$,

$$<z|\hat{E}|z> = -2(\hbar\omega/2\epsilon_0 V)^{1/2}\,|z|\,\sin(\overrightarrow{k}\cdot\overrightarrow{r} - \omega t + \theta) \tag{23}$$

which indeed "*looks* " like a classical field *but* the possible amplitudes are q-quantized; the modulus squared assumes a geometric series of discrete values.

With the usual definitions $\hat{P} = -i(\hbar\omega/2)^{1/2}(a - a^\dagger)$, $\hat{Q} = (\hbar/2\omega)^{1/2}(a + a^\dagger)$, the fractional uncertainties $\frac{\Delta\hat{Q}}{|<\hat{Q}>|}$ and $\frac{\Delta\hat{P}}{|<\hat{P}>|}$ are of $O(1)$ for $|z| \to \infty$ and

$$<z|[Q,P]|z> = <z|[a,a^\dagger]|z> = i\hbar\lambda(z) \geq i\hbar \tag{24}$$

where the important function ($q = \exp s$)

$$\lambda(z) \equiv N(z)^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}\cosh(s(2n+1)/4)}{[n]!\cosh(s/4)} \tag{25}$$

goes as $(q^{-1/2} - 1)|z|^2 + 1$ as $|z| \to \infty$. However, $\Delta Q \Delta P = 1/2|<[Q,P]>|$ for $|z>_q$ expectation values, per (5).

For the generic electric field, in the $|n>_q$ basis

$$(\Delta\hat{E})^2_{|n>} = (\hbar\omega/2\epsilon_0 V)([n+1] + [n]) \tag{26}$$

Instead, in the $|z>_q$ basis

$$(\Delta\hat{E})^2_{|z>} = (\hbar\omega/2\epsilon_0 V) < z|[a,a^\dagger]|z >= (\hbar\omega/2\epsilon_0 V)\,\lambda(z) \tag{27}$$

and so the fractional uncertainty in amp $\hat{E}$ (or $\hat{B}$) is also of $O(1)$. Note that from (25) $\lambda(z) = N(z)^2 e_q(|z|^2/q^{1/2}) - |z|^2(1-q^{1/2})$. There is a curious operator identity for $q \neq 1$

$$(-(i/\hbar)[Q,P]\cosh(s/4))^2 - ((2/\hbar\omega)H\sinh(s/4))^2 = 1 \tag{28}$$

which fundamentally relates the basic commutation relation and the single-mode hamiltonian[2] , (quadratic in $\hat{P}$,$\hat{Q}$)

$$H = (1/2)\hbar\omega(a^\dagger a + aa^\dagger) = (1/2)(\hat{P}^2 + \omega^2\hat{Q}^2). \tag{29}$$

We get for $(1-q)$ small, that $\lambda(z) \simeq \sqrt{1 + ((2\bar{E}/\hbar\omega)^2 - 4\bar{E}/\hbar\omega)\tanh^2(s/4)}$ where $\bar{E} = E_{\bar{n}} - \hbar\omega/2$ for $\bar{n} =< z|N|z >$ or $< z|[N]|z >$, so λ depends on the deviation from the vacuum energy.

4 q-Generalizations of the Phase Operators

Since z's magnitude may be q-quantized as in basic analysis, we next consider possible phase operators. Recall $z = |z|\exp(i\theta)$ and that mathematically a hermitian phase operator conjugate to N, to $[N] \equiv a^\dagger a$, or to H does not exist [3].

An $\widehat{exp}(i\phi)_q$ generalization of the phase operator of Susskind-Glogower [3] is defined by [4]

$$a \equiv ([N+1])^{1/2}\widehat{exp}(i\phi) \qquad a^\dagger \equiv \widehat{exp}(-i\phi)([N+1])^{1/2} \tag{30}$$

and there are hermitian operators

$$\widehat{cos}(\phi) \equiv (1/2)[\widehat{exp}(i\phi) + \widehat{exp}(-i\phi)] \qquad \widehat{sin}(\phi) \equiv (1/2i)[\widehat{exp}(i\phi) - \widehat{exp}(-i\phi)]. \tag{31}$$

These generalizations give many q-independent operator commutation relations , see [4]. So, from $[N, \widehat{cos}(\phi)] = -i\widehat{sin}(\phi), \ldots$ the usual number-phase uncertainty relations follow for arbitrary q:

$$\Delta N\,\Delta\widehat{cos}(\phi) \geq (1/2)| < \widehat{sin}(\phi) > | \qquad \Delta N\,\Delta\widehat{sin}(\phi) \geq (1/2)| < \widehat{cos}(\phi) > | \tag{32}$$

In the $|n>_q$ basis, these definitions (30-31) correspond to

$$\widehat{exp}(i\phi)_q \equiv \sum_{n=0}^{\infty} |n >< n+1| \tag{33}$$

which is manifestly q-independent in $|n>_q$, non-unitary, and a q-analogue of the SG operator.

[2]For H, the energy is not additive for two widely separated systems, violating the usual cluster decomposition "axiom" in quantum field theory. But, for q-quanta this is not unreasonable since the fractional uncertainty in the energy based on H is also $O(1)$ in the $|z>$ basis and the quanta by (1) are compelled to be always interacting,i.e. by exclusion-principle-like q-forces! An alternative hamiltonian is $H_N = \hbar\omega(N+1/2)$ where N is the number operator and it has the usual free-quanta additivity, etc. .

Analogously, a q-generalization of the Pegg and Barnett operator [5] is obtained [4] by introducing a complete, orthonormal basis of $(s+1)$ phase states $|\theta_m>_q = (s+1)^{-1/2}\sum_{m=0}^{s} \exp(in\theta_m)|n>_q$, $\theta_m = \theta_0 + 2m\pi/(s+1)$, with $m = 0, 1, \ldots, s,$. These are eigenstates of the respectively hermitian and unitary

$$\hat{\phi}_q \equiv \sum_{m=0}^{s} \theta_m |\theta_m >< \theta_m| \tag{34}$$

$$\exp(i\hat{\phi})_q \equiv |0 >< 1| + \cdots + |s-1 >< s| + \exp(i(s+1)\theta_0)|s >< 0| \tag{35}$$

which is manifestly q-independent, unitary, and only differs from (33) by the last term. Chaichian and Ellinas' polar operator is the same as $\exp(i\hat{\phi})_q$ when the reference phase in [6] is chosen to be $\phi_R = (s+1)\theta_0$.

Finally, although the $|z>_q$ coherent states do not minimize the $N, \widehat{cos}(\phi), \widehat{sin}(\phi)$ uncertainty relations (32), they do in the PB-case [7] both *give* and *minimize* Dirac's commutation relation, i.e. in $|z>_q$ basis for $|z|$ large

$$[N, \hat{\phi}_q] = i \tag{36}$$

Also $\widehat{cos}(\phi)_q$ and $\widehat{sin}(\phi)_q$ show some "correspondence principle" type behavior:

$$\frac{< z|\widehat{sin}(\phi)|z >}{< z|\widehat{cos}(\phi)|z >} = \frac{\sin(\theta)}{\cos(\theta)}, \qquad < z|\widehat{cos}(\phi)^2 + \widehat{sin}(\phi)^2|z > = 1 - (1/2)e_q(|z|^2)^{-1} \tag{37}$$

and proportionality for $< z|\widehat{cos}(\phi)^2 - \widehat{sin}(\phi)^2|z >$.

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III. QUANTUM OPTICS