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# THE UNCERTAINTY PRINCIPLE IN RESONANT GRAVITATIONAL WAVE ANTENNAE AND QUANTUM NON-DEMOLITION MEASUREMENT SCHEMES

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## Abstract

A review on the current efforts to approach and to surpass the fundamental limit in the sensitivity of the Weber type gravitational wave antennae is reported. Applications of quantum non-demolition techniques to the concrete example of an antenna resonant with the transducer are discussed in detail. Analogies and differences from the framework of the squeezed states in quantum optics are finally discussed.

## 1 Introduction

The importance of detecting gravitational waves, as frequently pointed out, consists not only in verifying one of the most direct and astonishing predictions of the simplest metric theory of gravitation, i.e. General Relativity, but also in the possibility to open new windows on phenomena in the Universe in which only violent releases of gravitational energy occur [1]. Gravitational waves have not yet been directly observed because of the extreme smallness of the energy released in actual detectors even if they are emitted by astronomical systems. The hypothetical sources which are strong candidates for emitting gravitational waves, according to our understanding of them due to information collected via the electromagnetic astronomy, are divided into two classes based upon the time evolution. Impulsive sources can be catastrophic events such as supernovae explosions and collapsing binary systems. The frequency spectrum of gravitational waves of this kind is flat up to  $10^3$  Hz, these impulsive phenomena having a characteristic duration of the order of milliseconds. One expects a perturbation of the metric tensor  $h \approx 10^{-21} - 10^{-18}$  for events in our Galaxy and  $h \approx 10^{-23} - 10^{-21}$  for events in the Virgo Cluster. Periodic sources can be pulsars if they deviate substantially from axial symmetry. The expected frequencies range is in this case

between  $10^{-2}$  and  $10^2$  Hz, while  $h \approx 10^{-27} - 10^{-25}$ . The efforts to detect gravitational waves have been concentrated from the very beginning on the impulsive events because of the larger expected perturbation to the metric tensor. It turns out that the modulation of the space-time induced by a gravitational wave on an extended body can also be seen as a production of a force field in it. Detecting the gravitational wave is therefore translated into the problem of detecting this small force of geometrical nature and the displacements produced by it in a test mass. The displacement induced in a body of reasonable sizes,  $\approx 1$  m, has therefore an amplitude of the order of  $10^{-21}$  if the event is due to a supernovae in the Virgo Cluster. The accuracy required to measure such a small displacement is so high that the quantum nature of the detector has to be taken into account because the De Broglie wavelength of a macroscopic test mass is of the same order of magnitude of the expected signal due to the gravitational waves. Here we report on the status of the art of the measurement techniques developed to allow monitoring of a class of gravitational wave detectors in a quantum regime. After a brief introduction for schematizing the detectors of gravitational waves and the sensitivity limit due to the fundamental noise in part 2, we introduce, in part 3, the quantum non-demolition measurement schemes for overcoming these limitations. The applications of stroboscopic and continuous quantum non-demolition schemes for a gravitational bar antenna resonant with the transducer are described respectively in part 4 and part 5. Conclusions deal also with the analogies and the differences from the quantum optics framework and the importance of this topic for understanding quantum mechanics applied to single macroscopic degrees of freedom repeatedly monitored.

## 2 Weber gravitational antennae: fundamental sensitivity limits

The gravitational wave detectors devised so far are based upon monitoring of the distance between two masses localized at different points. The equivalence principle requires a non-local, extended, structure of a gravitational wave detector because it is possible to nullify locally the effects of a gravitational field by means of a suitable choice of the reference frame.

Let us consider two masses in free fall: what is then measured is their variable distance which is supposed to be much smaller than the gravitational wavelength. The effect of a gravitational wave coming along  $z$  axis with proper polarization is to increase of  $h/2$  the distance along  $y$  axis and to decrease by  $h/2$  the distance along  $x$  axis. A classification of the gravitational wave detectors divides these into non resonant and resonant detectors if the two masses are respectively free or elastically coupled.

In non resonant detectors the distance between the two masses is measured by means of interferometric devices. The arms of the interferometer proposed so far are of the order of Km and use of multiple reflections allows an increase in the physical path by several orders of magnitude. In this contribution we will not be concerned with this kind of detector but we shall instead consider the resonant detectors (Weber type gravitational wave antennae), the quantum limit in a interferometric antenna being enforced by the shot noise and the momentum fluctuations imparted by the photon flux to the central mirror of the interferometer [2].

Resonant antennae are typically cylindrical bars of materials having low internal dissipation. The materials used are silicon, sapphire, niobium or a particular aluminum alloy (Al 5056) and

the mass of the antennae is a few tons.

One can show that the motion of the ends of a cylindrical bar of mass  $M$  and length  $L$  oscillating in its fundamental longitudinal mode is equivalent to that of a harmonic oscillator of mass  $M/2$  and equivalent length  $4L/\pi^2$ . If  $x$  is the displacement from equilibrium position the equation of motion of the Weber oscillator is

$$\ddot{x} + \frac{\dot{x}}{\tau_0} + \omega_0^2 x = \frac{2}{\pi^2} L \ddot{h}(t) \quad (1)$$

where  $\tau_0$  is the damping time,  $\omega_0$  is the proper frequency and  $h(t)$  is the amplitude of the incoming gravitational wave. The forcing term due to the gravitational field is proportional to the distance between the two masses. From this formula one can calculate the cross section for the transfer of energy from the wave to the antenna and one finds that this is proportional to the mass of the antenna and to  $L^2$ . The proper frequency  $\omega_0$  is chosen to be tuned with the frequency of the expected wave ( $10^3 Hz$ ) and the corresponding wavelength is very large compared to the size of the antenna. To amplify the extremely small oscillations coupling of the bar with another oscillator of very small mass is used [3],[4]. In this case a system of two coupled harmonic oscillators is obtained in which the energy is continuously transferred back and forth from  $M$  to  $m$  via beating. If the dissipations in the two oscillators are made negligible the amplitude of the oscillations in the second resonator is increased by a factor  $1/\sqrt{\mu}$  with respect to the first resonator, where  $\mu = m/M$ , provided that the frequencies of the two uncoupled oscillators are made coincident. The motion of the transducer is transformed into an electric signal by means of a variable capacitor and an amplifier schematized as an ideal amplifier of gain  $A$  and two noise sources generators with current and voltage spectral densities respectively  $SI_n$  and  $SV_n$ . The sources of noise are the thermal noise, i.e. Brownian motion of antenna, which gives a contribution  $K_B T$  to the energy of the oscillator, being  $K_B$  the Boltzmann's constant and  $T$  the thermodynamical temperature of the antenna and the amplifier noise, which is expressed by means of the parameter  $T_n = (SV_n SI_n)^{1/2}/K_B$ , called noise temperature of the amplifier. This last noise has two effects: it contributes directly as an additive noise source at the output and it acts on the transducer leading to an increase of the temperature. In other words every transducer is at the same time an actuator and the amplifier noise gives rise to a back-action force acting on the mechanical oscillator.

If we define a noise temperature  $T_{eff}$  as the temperature which corresponds to the minimum detectable energy  $E_{eff} = K_B T_{eff}$  transferred to the bar by an impulsive signal with an output signal/noise ratio equal to 1, we find, using a Wiener algorithm in the data analysis [5]

$$T_{eff} = 2 T_n \sqrt{\left(1 + \frac{1}{\lambda_0^2}\right) \left(1 + \frac{2T\lambda_0}{\beta Q T_n}\right)} \quad (2)$$

where  $Q = \omega_0 \tau_0$  is the quality factor of the mechanical system,  $\beta$  is substantially the fraction of energy transferred to the electromagnetic circuit by the bar through the capacitive coupling and  $\lambda_0$  the impedance matching factor defined as

$$\lambda_0 = \frac{SV_n}{SI_n} \frac{1}{Z_0} \quad (3)$$

For the antenna of the Rome group continuously operating since one year at CERN one has a thermodynamical temperature of  $\approx 4.2K$ ; the other parameters are  $Q \simeq 5 \cdot 10^6$  and an amplifier

noise temperature  $T_n \simeq 10^{-7}\text{K}$  [6]. It has been possible to achieve this last result making use of a SQUID amplifier. So one gets for  $T_{eff}$  a value of  $\approx 10\mu\text{K}$ , which is not far from the quantum limit temperature

$$T_{QL} = \frac{\hbar\omega}{K_B} \simeq 10^{-8}\text{K}. \quad (4)$$

One expects that the force with which a gravitational wave acts on the antenna is by many orders of magnitude below the thermal noise even at thermodynamical temperatures as low as 10mK which is the temperature at which the third generation antennae will operate. However, due to the particular features of the data analysis based on the variation of energy in the oscillator in the time, the quantum regime is reached earlier than as expected by (4). By writing the amount of energy which is exchanged during the measurement time  $\Delta t$  between the harmonic oscillator and the thermal reservoir and the quantized energy introduced by the measuring apparatus is easy to show that the quantum regime is obtained when the following condition is satisfied

$$\frac{K_B T \Delta t}{Q} \ll \hbar \quad (5)$$

This can be also shown by reasoning in terms of displacements instead of energy. The variation of the length of the bar due to a gravitational wave with amplitude  $h$  is, according to (1)

$$\frac{\Delta l}{l} \simeq \frac{h}{2}. \quad (6)$$

Because typical values for  $h$  are  $h = 10^{-21}$  (which corresponds to a supernova explosion in the center of the Galaxy) taking  $L = 1$  m, one gets from (5) a variation of the length of the bar  $\Delta L \simeq 10^{-19}$  cm which coincides with the standard quantum limit (i.e. the root square mean of the position of a harmonic oscillator in his fundamental mode)

$$\sqrt{\langle \Delta l^2 \rangle} = \sqrt{\frac{\hbar}{2M\omega}}. \quad (7)$$

It follows therefore that if we do not overcome this limit no information can be obtained on the evolution of the harmonic oscillator.

In these conditions one can find a method to measure the position of the quantum oscillator and to see if an external force has acted on it. However in doing this one must take into account that the position operator  $\hat{x}(t)$  does not commute with itself at different times. Indeed with a measurement of  $\hat{x}(t)$  at time  $t$  one puts the oscillator into an eigenstate of  $\hat{x}(t)$ ; if one repeats this measurement at the instant  $t + \tau$  one puts the oscillator into another eigenstate. It turns out that it is not possible to know if the change in  $\hat{x}(t)$  is caused by a very weak classical external force or by the demolition of the state due to the previous measurement. What is needed is therefore a measurement which does not prevent the execution of the next measurements of the same observable avoiding the demolition of the projection of the state on that observable. This is possible in non-relativistic quantum mechanics as we will discuss in the following considerations, because this theory makes limitations only on a simultaneous, perfect knowledge of two canonical observables.

### 3 Quantum non demolition measurements

The introduction of the quantum non-demolition measurements (QND) dates back to an article by Landau and Peierls [7] in 1931. However only recently, after understanding the role of quantum mechanics in the fundamental limits to the amplifier sensitivity [8],[9] and under the request to surpass the quantum limit in detectors of small displacements [10],[11], the problem has been studied in detail [12],[13]. The idea of a QND strategy is to perform a series of measurements of one observable of a single object in such a way that the act of the measurement itself does not affect the predictability of the result of the next measurements of the same observable. In order to do this the observable, the instants of time in which it is observed and the interaction Hamiltonian should be all carefully chosen for a given dynamical system. For instance, a first high precision measurement of the position of a free particle implies a large dispersion in the possible values of measurements of momentum. If a second measurement of position is made, due to the Heisenberg evolution, the result will have a large dispersion too. Instead, if a measurement of momentum in a free particle is made at a given instant of time, a second possible measurement will give the same result due to the constant value of the momentum between the two consecutive measurements, provided that the interaction due to the first measurement has not demolished the state. This simple example shows the route to define quantum non-demolition measurements. Only particular observables which satisfy a commutation relation at different times  $t_i$  and  $t_j$  are allowed to be monitored in a QND way, i.e. if

$$[\hat{x}(t_i), \hat{x}(t_j)] = 0. \quad (8)$$

Moreover, we must also take into account the perturbation on  $\hat{x}(t)$  induced by the measuring apparatus which is coupled to the observed system by means of the Hamiltonian operator  $\widehat{H}_i$ . To avoid changes in the expected value of the observable during the measurement the following condition must be satisfied:

$$[\hat{x}(t), \widehat{H}_i] = 0. \quad (9)$$

This condition assures that the interaction Hamiltonian is simultaneously diagonalizable with the measured observable, no changes are induced in the measured observable during the measurement time in which only the interaction Hamiltonian will be responsible for the time evolution. A sequence of measurements performed under conditions (8) and (9) will give always the same result. This is a definition of a QND measurement. If the instants of time in which it is satisfied (8) are discrete the QND scheme is named stroboscopic or, in a realistic configuration with a duration of the measurement small with respect to the characteristic timescale of the motion of the observed system, quasi-stroboscopic [14],[15],[16]. Otherwise, having a continuous set of instants of time, the QND scheme is named continuous.

In the case of a single oscillator one introduces the two components of the complex amplitude

$$\begin{cases} \widehat{X}_1 = \text{Re}[(\hat{x} + i\frac{\hat{p}}{m\omega})e^{i\omega t}] \\ \widehat{X}_2 = \text{Im}[(\hat{x} + i\frac{\hat{p}}{m\omega})e^{i\omega t}] \end{cases} \quad (10)$$

such that  $\hat{x}(t) = \widehat{X}_1 \cos \omega t + \widehat{X}_2 \sin \omega t$ . Their properties are

$$(a) \quad \frac{d\widehat{X}_1}{dt} = \frac{d\widehat{X}_2}{dt} = 0 \Rightarrow [\widehat{X}_1(t), \widehat{X}_1(t + \tau)] = [\widehat{X}_2(t), \widehat{X}_2(t + \tau)] = 0 \quad (11)$$

$$(b) \quad [\widehat{X}_1(t), \widehat{X}_2(t)] = \frac{i\hbar}{m\omega}. \quad (12)$$

By using (a) and (b) we get

$$[\hat{x}(t), \hat{x}(t + \tau)] = -[\widehat{X}_1, \widehat{X}_2] \{\cos \omega t \sin \omega(t + \tau) - \sin \omega t \cos \omega(t + \tau)\} = \frac{i\hbar}{m\omega} \sin \omega\tau. \quad (13)$$

This means that to do a QND measurement of the operator  $\hat{x}(t)$  in a single harmonic oscillator one needs the Hamiltonian (here  $\hat{q}$  is the variable of the measuring apparatus which couples with the oscillator)

$$\widehat{H}_i = E_0 \delta\left(t - \frac{n\pi}{\omega}\right) \hat{x} \hat{q} \quad (14)$$

such that the interaction between the system and the measuring apparatus is turned on only when  $\hat{x}(t)$  commutes with itself, that is why this kinds of measurements are called stroboscopic Q.N.D..

For a component of the complex amplitude,  $\widehat{X}_1$ , a QND interaction Hamiltonian should be [12]

$$\widehat{H}_i = E_0 \widehat{X}_1 \hat{q} \quad (15)$$

that is approximately obtained by using the interaction Hamiltonian

$$\widehat{H}_i = 2E_0 \cos \omega_m t \hat{x} \hat{q} \quad (16)$$

provided a low-pass filter at  $\omega_c \ll \omega_m$  is used. For practical reasons a different pumping scheme is used, namely an up-conversion around an electrical frequency  $\omega_e$  such that the interaction Hamiltonian is now

$$\widehat{H}_i = E_0 \cos \omega_e t \cos \omega_m t \hat{x} \hat{q} = \frac{E_0}{2} [\cos(\omega_e + \omega_m)t + \cos(\omega_e - \omega_m)t] \hat{x} \hat{q} \quad (17)$$

which allows an approximate measurement of  $\widehat{X}_1$  if a filtering around  $\omega_e$  is performed with a selectivity such that the terms oscillating at  $\omega_e \pm 2\omega_m$  are made negligible. It has been pointed out that the continuous approximate QND measurement scheme of one component of the complex amplitude is obtained as a first order approximation of the corresponding stroboscopic scheme [17]. If we start from the interaction Hamiltonian of a stroboscopic measurement of  $\widehat{X}_1$  expressed in terms of the physical observable  $\hat{x}$

$$\widehat{H}_i = E_0 \cos \omega_e t \sum_n \delta\left(t - \frac{n\pi}{\omega_1}\right) \widehat{X}_1 \hat{q} = E_0 \cos \omega_e t \sum_n (-1)^n \delta\left(t - \frac{n\pi}{\omega_1}\right) \hat{x} \hat{q} \quad (18)$$

we will see that, by Fourier expanding the Dirac-distribution, it is obtained

$$\widehat{H}_i = E_0 \cos \omega_e t \sum_n \cos(2n + 1)\omega_1 t \hat{x} \hat{q} \quad (19)$$

that, at the first order, is

$$\widehat{H}_i = E_0 \cos \omega_e t \cos \omega_1 t \hat{x} \hat{q} \quad (20)$$

i.e. the usual approximate scheme for monitoring of  $\widehat{X}_1$ . Thus knowing a QND stroboscopic strategy it is simple to write the corresponding QND approximate continuous strategy. This property will be particularly useful in the following considerations, where the more complicated but realistic case of two coupled harmonic oscillators will be treated.

It has been pointed out that also in the classical regime, i.e. when the amplifier is not quantum limited, the QND measurement schemes provide a better sensitivity because one phase of the signal is shielded by the back-action force of the amplifier. A quantitative model in the classical limit has been developed in [18]: it turns out that by writing the noise temperature as

$$T_b = \frac{\omega_m}{\omega_e} T_n \frac{1}{r} \quad (21)$$

for a standard 'amplitude and phase' monitoring is  $r < 1$ , and for a QND/BAE scheme  $r$  may be greater than unity. This is due to the squeezing of the electrical noise into one mechanical phase. A generalized uncertainty relation for the two classical conjugate observables due to the back-action of the amplifier noise is introduced as

$$\Delta X_1 \Delta X_2 \simeq \frac{K_B T_n}{2m\omega_m\omega_e} \quad (22)$$

which may be obtained through a replacement on the right hand side in the standard quantum uncertainty relationship

$$\Delta X_1 \Delta X_2 \simeq \frac{\hbar}{2m\omega_m} \quad (23)$$

of  $\hbar$  with  $K_B T_n / \omega_2$ . If a squeezing factor  $\rho$  such that  $\Delta X_1 = \rho \Delta X_2$  is introduced ( $\rho \rightarrow 0$  means a noise-free measurement of  $\widehat{X}_1$ ) the minimum burst noise temperature can be written as

$$T_b = \frac{m\omega_m^2 \Delta X_1^2}{2} \simeq \frac{1}{4} T_n \frac{\omega_1}{\omega_2} \rho \quad (24)$$

showing that the  $r$  figure of merit has a dynamical interpretation in terms of a squeezing factor. Recently, an interpretation of the back-action evasion strategies in which they are seen as an alternative to the usual impedance matching for maximizing the signal to noise ratio has been discussed [19].

The description of the QND measurement suggests how to measure small forces below the standard quantum limit. By means of a simple integration of the Heisenberg equation in presence of an external force  $F(t)$ , one gets for the QND operator  $\widehat{X}_1$

$$\widehat{X}_1(t) = \widehat{X}_1(t_0) - \hat{I} \int_{t_0}^t \frac{F(t')}{m\omega} \sin \omega t' dt'. \quad (25)$$

A sequence of measurement of  $\widehat{X}_1$  will then give as a result a sequence of eigenstates linked to the value of the external force

$$\xi(t, \tau) = \xi(t_0) - \int_{t_0}^t \frac{F(t')}{m\omega} \sin \omega t' dt' \quad (26)$$

By means of successive measurements it is possible to study the form of  $F(t)$  simply inverting (26)

$$F(\tau) = -\frac{m\omega}{\sin \omega \tau} \frac{d}{dt} \xi(t_0, t) \Big|_{t=\tau} \quad (27)$$

The singularities for  $t = n\pi/\omega$  correspond to a null information on the force acting on the harmonic oscillator on some instants of time. This can be compensated by using a second oscillator (i.e. a second antenna) with complex amplitude  $\hat{Y}_1 + i\hat{Y}_2$  which has eigenvalues

$$\zeta(t, \tau) = \zeta(t_0) - \int_{t_0}^t \frac{F(t')}{m\omega} \sin \omega t' dt' \quad (28)$$

here obviously the singularities are in  $t_n = (2n + 1)\pi/2\omega$ .

## 4 QND quasi-stroboscopic scheme for coupled harmonic oscillators

The current generation of gravitational wave antenna of the Weber type operates by means of an antenna coupled to a small mechanical resonator. In such a way the energy deposited in the antenna by a gravitational wave burst is transferred to the transducer. In the case of an ideal transfer of energy, i.e. with both a perfect tuning of the two uncoupled frequencies and negligible dissipations during the beating period, the amplitude of the oscillations in the transducer is larger than that in the antenna by a factor equal to the square root of the ratio of the equivalent masses of the two resonators. All the detectors operating in coincidence as described in [6] were equipped with a resonant transducer and the same is also planned for the third generation of gravitational wave antennas cooled at 50 mK now under development. It is therefore important to generalize the previous considerations on the QND schemes to this situation, as already outlined in [20]. As we have seen, it is possible to schematize the gravitational cryogenic antenna and the resonant transducer with two coupled harmonic oscillator having masses respectively  $m_x$  and  $m_y$  (with  $\mu = \frac{m_y}{m_x} \ll 1$ ). The two coupled mechanical oscillators are described by the Lagrangian

$$L = \frac{1}{2}m_x \dot{x}^2 + \frac{1}{2}m_y \dot{y}^2 - \frac{1}{2}m_x \omega_x^2 x^2 - \frac{1}{2}m_y \omega_y^2 (y - x)^2 = \frac{1}{2}(\dot{\xi} \ \dot{\eta}) T \begin{pmatrix} \xi \\ \eta \end{pmatrix} - \frac{1}{2}(\xi \ \eta) V \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad (29)$$

where the normalized coordinates  $\xi = \sqrt{m_x}x$  and  $\eta = \sqrt{m_y}y$  have been introduced, together with the matrices  $T$  and  $V$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (30)$$

$$V = \begin{pmatrix} \omega_x^2 + \mu\omega_y^2 & -\sqrt{\mu}\omega_y^2 \\ -\sqrt{\mu}\omega_y^2 & \omega_y^2 \end{pmatrix}. \quad (31)$$



As we have already cited to obtain the maximum coupling the two oscillators should have the same frequency  $\omega_x = \omega_y = \omega$ , i.e. they should be tuned. In this case one finds the solutions

$$\omega_{\pm}^2 = \omega_0^2 \left( 1 + \frac{\mu}{2} \pm \sqrt{\mu \left( 1 + \frac{\mu}{4} \right)} \right) \quad (32)$$

which we can write more easily introducing  $\alpha_{\pm} = \frac{\mu}{2} \pm \sqrt{\mu \left( 1 + \frac{\mu}{4} \right)}$  obtaining  $\omega_{\pm}^2 = \omega_0^2 (1 + \alpha_{\pm})$ .

The normal coordinates  $\Xi_{\pm}$  corresponding to the eigenfrequencies  $\omega_{\pm}$  are linked to the physical coordinates by means of an orthogonal matrix

$$\begin{pmatrix} \hat{\Xi}_+ \\ \hat{\Xi}_- \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\alpha_- + 2}} & \frac{\alpha_-}{\sqrt{\mu}\sqrt{\alpha_- + 2}} \\ \frac{1}{\sqrt{\alpha_+ + 2}} & \frac{\alpha_+}{\sqrt{\mu}\sqrt{\alpha_+ + 2}} \end{pmatrix} \begin{pmatrix} \sqrt{m_x} \hat{x} \\ \sqrt{m_y} \hat{y} \end{pmatrix}. \quad (33)$$

Let us introduce the complex amplitudes of the normal modes

$$\begin{cases} \hat{X}_1^{\pm} = \hat{\Xi}_{\pm} \cos \omega_{\pm} t - \frac{\hat{P}_{\Xi_{\pm}}}{\omega_{\pm}} \sin \omega_{\pm} t \\ \hat{X}_2^{\pm} = \hat{\Xi}_{\pm} \sin \omega_{\pm} t + \frac{\hat{P}_{\Xi_{\pm}}}{\omega_{\pm}} \cos \omega_{\pm} t \end{cases} \quad (34)$$

which satisfy the relations

$$[\hat{X}_1^+, \hat{X}_2^+] = \frac{i\hbar}{\omega_+} \quad [\hat{X}_1^-, \hat{X}_2^-] = \frac{i\hbar}{\omega_-} \quad (35)$$

as well as

$$[\hat{X}_{1,2}^+(t), \hat{X}_{1,2}^+(t + \tau)] = [\hat{X}_{1,2}^-(t), \hat{X}_{1,2}^-(t + \tau)] = 0. \quad (36)$$

We can also rewrite the Hamiltonian  $\hat{H}$  of the system as

$$\hat{H} = \frac{\omega_+^2}{2} [(\hat{X}_1^+)^2 + (\hat{X}_2^+)^2] + \frac{\omega_-^2}{2} [(\hat{X}_1^-)^2 + (\hat{X}_2^-)^2]. \quad (37)$$

The commutator  $[\hat{y}(t) - \hat{x}(t), \hat{y}(t + \tau) - \hat{x}(t + \tau)]$  is calculated by writing  $\hat{y}$  and  $\hat{x}$  in terms of the complex amplitudes  $\hat{X}_{1,2}^{\pm}$  of the normal modes which are integral of the motion and by using the same computation procedure which led us to formula (13). Using (35),(36) we obtain, finally, the expression

$$[\hat{y}(t) - \hat{x}(t), \hat{y}(t + \tau) - \hat{x}(t + \tau)] = \frac{i\hbar}{M\omega\mu\sqrt{\mu+4}} \left[ \frac{\omega_+^2}{\omega^2} \sin \omega_+ \tau + \frac{\omega_-^2}{\omega^2} \sin \omega_- \tau \right]. \quad (38)$$

This quantity becomes, in the limit  $\mu \rightarrow 0$

$$[\hat{y}(t) - \hat{x}(t), \hat{y}(t + \tau) - \hat{x}(t + \tau)] = \frac{i\hbar}{2m_y\omega} \frac{\omega_+^2 + \omega_-^2}{\omega^2} \sin \bar{\omega}\tau \cos \omega_B\tau \quad (39)$$

where  $\bar{\omega} = \frac{\omega_+ + \omega_-}{2} = \frac{\omega}{2}\sqrt{\mu + 4} \rightarrow \omega$  and  $\omega_B = \frac{\omega_+ - \omega_-}{2} = \frac{\omega}{2}\sqrt{\mu}$ . Eqns. (38) and (39) show that the commutator of the operator  $\hat{y} - \hat{x}$  with itself at different times is time dependent and it has a characteristic beating behaviour. We have seen that in a quasi-stroboscopic scheme for a single harmonic oscillator the commutator is zero each half a period of the motion and the stroboscopicity is defined whenever measurements with a duration small compared to the period of the motion are performed. This implies a measurement time, a duty cycle, very small and a consequent small value of the effective electromechanical quality factor. In the case of a double harmonic oscillator this drawback is less pronounced because the commutativity is assured every half of a beating period for a time of the order of a period of oscillation. Thus quasi-stroboscopic QND schemes already proposed as a generalization of the conventional BAE scheme based upon a continuous monitoring [17] and already tested on a single oscillator system [21] can be adapted to this situation. In the case of a single harmonic oscillator the duration of the measurement must be small compared to the period of the harmonic oscillator  $T$ , in the case of two coupled harmonic oscillators this duration is of the order of some periods of the uncoupled oscillator, although the interaction must be turned on every quarter of a beating period. The interaction Hamiltonian for a two coupled harmonic oscillator system is therefore

$$\widehat{H}_i = \frac{E_0}{2} \sum_n [\theta(t - \frac{nT_B}{2} + \frac{\Delta T}{2}) + \theta(-t + \frac{nT_B}{2} + \frac{\Delta T}{2})](\hat{y} - \hat{x})\hat{q} \quad (40)$$

where  $T_B$  is the beat period and  $\Delta T$  is of the order of the period of a single harmonic oscillator. Practical values are  $T_B \simeq 40\text{ms}$  and  $\Delta T \simeq 2\text{ms}$ . To calculate the error in a quasi stroboscopic measurement of the operator  $\hat{y} - \hat{x}$  performed for instance in the interval  $\frac{\pi}{2\omega_B} - \frac{2\pi}{\omega}, \frac{\pi}{2\omega_B} + \frac{2\pi}{\omega}$  we identify the conjugate observable of  $\hat{y} - \hat{x}$  as the quantity  $(\hat{p}_y - \hat{p}_x)/2$ . This last can be expressed in terms of the components of the amplitudes of the normal modes and the commutator at different times of the two conjugate observables is obtained as

$$[\hat{y}(t + \tau) - \hat{x}(t + \tau), \frac{1}{2}(\hat{p}_y(t) - \hat{p}_x(t))] = \frac{i\hbar}{2} \left( \frac{\alpha_- - 1}{\alpha_-(\alpha_- + 2)} \cos \omega_+ \tau + \frac{\alpha_+ - 1}{\alpha_+(\alpha_+ + 2)} \cos \omega_- \tau \right). \quad (41)$$

When  $\tau = 0$  the commutator relationship (41) is written as

$$[\hat{y}(t) - \hat{x}(t), \frac{1}{2}(\hat{p}_y(t) - \hat{p}_x(t))] = i\hbar \quad (42)$$

which is exactly the quantity  $[\hat{x}(t), \frac{1}{2}\hat{p}_x(t)] + [\hat{y}(t), \frac{1}{2}\hat{p}_y(t)]$ .

By expressing  $\omega_+$  and  $\omega_-$  in terms of the frequencies  $\bar{\omega}$  and  $\omega_B$  and substituting in  $\alpha_{\pm}$  their expressions in terms of  $\mu$  we get finally

$$[\hat{y}(t + \tau) - \hat{x}(t + \tau), \frac{1}{2}(\hat{p}_y(t) - \hat{p}_x(t))] = i\hbar (\cos \bar{\omega}\tau \cos \omega_B\tau - \frac{1 + \mu}{\sqrt{\mu(\mu + 4)}} \sin \bar{\omega}\tau \sin \omega_B\tau) \quad (43)$$

If the measurement is performed in the interval  $[\frac{\pi}{2\omega_B} - \frac{2\pi}{\omega}, \frac{\pi}{2\omega_B} + \frac{2\pi}{\omega}]$ , we can approximate  $\cos \omega_B\tau \simeq 1$  and  $\sin \omega_B\tau \simeq \omega_B\tau - \frac{\pi}{2}$  and a measurement of infinitesimal duration  $t'$  performed in such interval and with a precision  $\Delta[\hat{y}(t) - \hat{x}(t)]$  allows to evaluate the error introduced in the measurement process on the uncertainty product as

$$\Delta[\hat{y}(t+t') - \hat{x}(t+t')] \cdot \frac{1}{2}\Delta[\hat{p}_y(t) - \hat{p}_x(t)] \approx \frac{\hbar}{2} \left| \cos \bar{\omega}t' - \frac{1+\mu}{\sqrt{\mu(\mu+4)}} \sin \bar{\omega}t'(\omega_B t' - \frac{\pi}{2}) \right| \quad (44)$$

from which, under the approximation for the trigonometric functions, we obtain

$$\frac{1}{2}\Delta[\hat{p}_y(t) - \hat{p}_x(t)] \approx \frac{\hbar}{2\Delta[\hat{y}(t) - \hat{x}(t)]}. \quad (45)$$

The error due to a measurement of duration  $t'$  on the operator  $\hat{y} - \hat{x}$  is calculated starting from  $\Delta[\hat{y}(t) - \hat{x}(t)]$  because

$$\Delta[\hat{y}(t+t') - \hat{x}(t+t')] \approx \Delta[\hat{y}(t) - \hat{x}(t)] \left| \cos \bar{\omega}t' - \frac{1+\mu}{\sqrt{\mu(\mu+4)}} \sin \bar{\omega}t'(\omega_B t' - \frac{\pi}{2}) \right| \quad (46)$$

If the notation now is changed defining  $\Delta_t = \Delta[\hat{y}(t) - \hat{x}(t)]$  we have

$$\frac{\Delta_t}{t'} \left[ \cos \bar{\omega}t' - \frac{1+\mu}{\sqrt{\mu(\mu+4)}} \sin \bar{\omega}t'(\omega_B t' - \frac{\pi}{2}) \right] \quad (47)$$

and in the limit of  $t' \rightarrow 0$  we get

$$\frac{d\Delta_t}{dt} = \frac{1+\mu}{\sqrt{\mu(\mu+4)}} \frac{\pi\bar{\omega}}{2} \cdot \Delta_t \quad (48)$$

from which, by integrating, we obtain the error on a measurement performed around  $t = \frac{\pi}{2\omega_B}$  as

$$\Delta[\hat{y}(t+\tau) - \hat{x}(t+\tau)] \approx \Delta[\hat{y}(t) - \hat{x}(t)] \exp\left[\frac{\pi(\mu+1)}{2\sqrt{\mu(\mu+4)}}\bar{\omega}\tau\right]. \quad (49)$$

For instance, for a choice  $t = [\frac{\pi}{2\omega_B} - \frac{2\pi}{\bar{\omega}}]$  and  $\tau = \frac{4\pi}{\bar{\omega}}$  we obtain

$$\Delta\left[\hat{y}\left(\frac{\pi}{2\omega_B} + \frac{2\pi}{\bar{\omega}}\right) - \hat{x}\left(\frac{\pi}{2\omega_B} + \frac{2\pi}{\bar{\omega}}\right)\right] \approx \Delta\left[\hat{y}\left(\frac{\pi}{2\omega_B} - \frac{2\pi}{\bar{\omega}}\right) - \hat{x}\left(\frac{\pi}{2\omega_B} - \frac{2\pi}{\bar{\omega}}\right)\right] \exp\left[\frac{2\pi^2(\mu+1)}{\sqrt{\mu(\mu+4)}}\right] \quad (50)$$

A drawback of these measurement scheme appears when  $\mu$  is very small and the frequency of the measurement is consequently very small too. To overcome this problem a multimode configuration can be used. In this case the commutator at different times approaches zero more frequently when compared to a two-mode configuration of the same final mass ratio. A more detailed description of this point can be found in [22].

## 5 QND continuous schemes for coupled harmonic oscillators

Also QND continuous schemes can be used for coupled harmonic oscillator. A first example is given by a monitoring of the complex amplitude of the physical modes  $\hat{x}$  and  $\hat{y}$  [23]. Introducing the complex amplitudes such that

$$\begin{cases} \hat{y} = \text{Re}[(\hat{Y}_1 + i\hat{Y}_2)e^{-i\omega_y t}] \\ \hat{p}_y/m_y\omega_y = \text{Im}[(\hat{Y}_1 + i\hat{Y}_2)e^{-i\omega_y t}] \end{cases} \quad (51)$$

we can rewrite the Hamiltonian in terms of  $\hat{Y}_1$  and  $\hat{Y}_2$  and, by writing the Heisenberg equations for the time evolution of  $\hat{Y}_1$ , we obtain

$$\frac{d\hat{Y}_1}{dt} = -\omega_y \hat{x} \sin \omega_y t. \quad (52)$$

The complex amplitude is not a constant of the motion. However it is easily proved that it is a QND observable. A relationship valid for an infinitesimal time  $\tau$  is derived for the time evolution

$$\hat{Y}_1(t + \tau) = \hat{Y}_1(t) - \omega_y \hat{x} \sin \omega_y t \tau \quad (53)$$

and this implies the commutation rule for  $\hat{Y}_1$  at different times

$$[\hat{Y}_1(t + \tau), \hat{Y}_1(t)] = [\hat{Y}_1(t) - \omega_y \hat{x} \sin \omega_y t, \hat{Y}_1] = 0 \quad (54)$$

because of the commutativity between  $\hat{Y}_1$  and  $\hat{x}$ . Thus  $\hat{Y}_1$  (or  $\hat{Y}_2$ , for which similar relationships hold) is a QND observable, although it is not conserved during the motion. From (52) the coordinate  $\hat{x}$  is inferred as

$$\hat{x}(t) = -\frac{1}{\omega_y \sin \omega_y t} \frac{d\hat{Y}_1}{dt} \quad (55)$$

apart from the singularities already discussed appearing when  $\sin \omega_y t = 0$ . When a classical force  $F(t)$  acts on the system the Hamiltonian operator is modified and the added term is

$$\hat{H}_f = -(\hat{x} + \hat{y}) F(t) \quad (56)$$

obtaining, in this case, the following expression for the time evolution of  $\hat{Y}_1$

$$\frac{d\hat{Y}_1}{dt} = -\omega_y \hat{x} \sin \omega_y t - \frac{\sin \omega_y t}{m_y \omega_y} F(t). \quad (57)$$

However the effect of the external force to be detected, in our case of geometrical nature, on the transducer is negligible compared to the effect on the antenna, due to the smaller size of the transducer. Thus  $\hat{H}_f \simeq -\hat{x}F(t)$  and the second term in (57) can be omitted. In this reasonable approximation, i.e.  $F(t)$  acting only on the antenna,  $\hat{Y}_1$  is also QNDF, i.e. QND also in presence of an external force. To obtain a continuous monitoring of  $\hat{Y}_1$  we need a QND interaction Hamiltonian of the type

$$\hat{H}_i = E_0 \cos \omega_e t \cos \omega_y t (\hat{y} - \hat{x}) \hat{q} \quad (58)$$

that is a coherent superposition of pumpings at frequencies  $\omega_e \pm \omega_y$ . Analogous considerations can be made for the monitoring of the real or the imaginary part of the complex amplitude of one normal mode expressed in terms of the physical modes through (33). The advantage in this case is that the quantity  $\hat{X}_1^+$  is a constant of the motion and its monitoring is the standard one

already discussed for a single harmonic oscillator. This is obtained by means of the interaction Hamiltonian

$$\widehat{H}_i = E_0 \cos \omega_e t \cos \omega_+ t (\hat{y} - \hat{x}) \hat{q} \quad (59)$$

and the analogous for monitoring a component of the complex amplitude  $\widehat{X}_1^-$  by substituting  $\omega_+$  with  $\omega_-$ . One drawback of monitoring one component of the complex amplitude of the normal modes is that the information on the other mode is lost, and it is crucial to have information on both the modes to take full advantage of the resonant schemes.

An alternative scheme suggested by the time dependence of the commutator consists in a monitoring corresponding to the following Hamiltonian:

$$\widehat{H}_i = E_0 \cos \omega_e t \cos \bar{\omega} t \cos \omega_B t (\hat{y} - \hat{x}) \hat{q}. \quad (60)$$

This coupling allows one to infer information on both the modes because, upon filtering around  $\omega_e$  in such a way to neglect terms oscillating at  $\omega_e \pm 2\omega_B$ ,  $\omega_e \pm 2\bar{\omega}$ ,  $\omega_e \pm 2(\bar{\omega} \pm \omega_B)$ , it can be rewritten as

$$\widehat{H}_i = \frac{E_0}{4} \cos \omega_e t (\beta_+ \hat{X}_1^+ + \beta_- \hat{X}_1^-) \hat{q} \quad (61)$$

where  $\beta_{\pm}$  are coefficients related to the coefficients of the matrix (33) and are expressed as

$$\beta_{\pm} = [m_x(2 + \frac{\mu}{2} \mp \sqrt{\mu(1 + \frac{\mu}{4})})]^{-1/2} (\frac{1}{2\sqrt{\mu}} \mp \sqrt{\frac{(1 + \frac{\mu}{4})}{\mu} - \frac{1}{\mu^{3/2}}}) \quad (62)$$

which, in the limit of  $\mu \rightarrow 0$ , goes to  $\beta_{\pm} = \mp 1/\sqrt{2m_x\mu}$ . In this limit the interaction Hamiltonian assumes a simple form

$$\widehat{H}_i = \frac{E_0}{\sqrt{32m_x\mu}} \cos \omega_e t (\hat{X}_1^+ - \hat{X}_1^-) \hat{q} \quad (63)$$

which contains information on both the normal modes and in such a way that QND measurements can be performed on both the modes. In all the three cases here discussed the selectivity requirements on the electrical circuit are more stringent than in the case of a single harmonic oscillator, because now the electrical oscillator must have a quality factor  $Q_e \gg \omega_e/\omega_B$  in order to avoid detection of sidebands contributions. The interaction Hamiltonian (60) can also be written as

$$\widehat{H}_i = \frac{E_0}{2} \cos \omega_e t (\cos \omega_+ t + \cos \omega_- t) (\hat{y} - \hat{x}) \hat{q}. \quad (64)$$

With the analogy to the multipump scheme discussed for a single oscillator we can imagine a interaction Hamiltonian of which (64) is only the first order approximation

$$\widehat{H}_i \simeq \frac{E_0}{2} \cos \omega_e t \left[ \sum_{n=0}^{+\infty} \cos(2n+1)\omega_+ t + \sum_{m=0}^{+\infty} \cos(2m+1)\omega_- t \right] (\hat{y} - \hat{x}) \hat{q} \quad (65)$$

which corresponds, in the limit of a stroboscopic pumping of the kind

$$\widehat{H}_i = E_0 \left[ \sum_{n=0}^{+\infty} (-1)^n \delta(t - \frac{n\pi}{\omega}) + \sum_{m=0}^{+\infty} (-1)^m \delta(t - \frac{n\pi}{\omega}) \right] (\hat{y} - \hat{x}) \hat{q} \quad (66)$$

It is interesting to observe that after a time equal to  $T_B/2$  both the trains of Dirac distributions will coincide, i.e.  $T_B/2 = n\pi/\omega_+ = m\pi/\omega_-$  where  $n = m + 2$  (the fact that  $n$  and  $m$  have the same parity assures the same sign of the corresponding Dirac pulses at those times). So each half a period the two trains are summed and the quasi-stroboscopic scheme discussed in the previous section can be considered as the first order approximation of the stroboscopic scheme resulting from (66). This completes the connection between the multipump continuous schemes and the quasi-stroboscopic scheme introduced in the previous section.

## 6 Conclusions

We have shown the scenario under which quantum non-demolition measurement schemes should be demanded for detecting gravitational waves in the generation of resonant gravitational wave antennae currently under development, particularly ultra-low temperature resonant bar antennae such as the Rome, Legnaro and Stanford ones which will work at a thermodynamical temperature of  $\simeq 50$  mK. Both QND stroboscopic and continuous schemes have been discussed as well as their link and practical schemes to implement them. However the interest of quantum non-demolition measurement schemes goes beyond the detectability of the gravitational radiation, involving also the quantum measurement theory and the predictions of it for repeated measurements on a single macroscopic oscillator. Feasibility of the generation of macroscopically distinguishable states using a QND scheme has been recently discussed in quantum optics [24], [25]. It has been pointed out that the generation of Schroedinger cats using micromechanical oscillators with quantum limited sensitivity is also feasible [26]. Unlike the optical case, in which the QND measurement is obtained with a frequency mixing due to non-linear susceptibility, the QND measurement for the mechanical case is obtained using an electric field which can be large as one wants. Dissipations in a mechanical oscillator also are quite low compared to electrical or optical oscillators. Moreover, analogies to the production and the detection of squeezed states in optics [27] have been shown. We want to point out a fundamental difference between the two topics: in the case of the optical squeezed states we deal with a quantized field in which its quantum nature is responsible for the limitation to the sensitivity, in the case of quantum non-demolition measurements on a harmonic oscillator the eventual force field which has to be monitored is considered classical and the fundamental limitations comes from the process of the measurement and the interaction of the meter with the external environment. What is squeezed in a QND measure is the back-action noise generated by the amplifier and the squeezing is made in a phase orthogonal to the one which is detected [21]. Despite this conceptual difference the formalisms to deal with QND strategies are similar to the one used to deal with squeezed states. This analogy is so narrow that also multipump [28], [29] and quasi-stroboscopic [30], [31] schemes have been independently and successfully implemented for squeezing the light. Further thoughts on the analogies and the differences between quantum non-demolition measurements on a harmonic oscillator and the squeezing of the quantum noise can give rise to a better understanding on the same interpretation of Quantum Electrodynamics and the operative origin of the vacuum fluctuations of the field in terms of a measurement process [32], an aspect of this fascinating and successful theory which has been very little investigated until now.

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