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ON HARMONIC OSCILLATORS AND THEIR KEMMER RELATIVISTIC FORMS

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Abstract

We show that Dirac (Kemmer) equations are intimately connected with (para)supercharges coming from (para)supersymmetric quantum mechanics, a nonrelativistic theory. The dimensions of the irreducible representations of Clifford (Kemmer) algebras play a fundamental role in such an analysis. These considerations are illustrated through oscillatorlike interactions, leading to (para)relativistic oscillators.

1 Introduction

Supersymmetric quantum mechanics (SSQM) as initiated by Witten [1] is characterized by a superposition of bosonic and fermionic operators, leading from an algebraic point of view to the so-called Lie superalgebras [2]. It is now well known [3] that some of their generators i.e. the supercharges, can be related to Dirac hamiltonians if the spin-orbit coupling procedure [4] is under study. In particular, when oscillatorlike interactions are considered, this connection gives rise to the Dirac oscillator [5] whose nonrelativistic limit corresponds to an ordinary harmonic oscillator with a strong spin-orbit coupling term.

Moreover, extensive studies have recently combined bosons and parafermions [6] leading to parasupersymmetric quantum mechanics (PSSQM) [6,7,8]. The Lie structures subtended by this generalized context are now referred to as parasuperalgebras [9]. We plan to show [10] here that the corresponding parasupercharges can be connected with Kemmer hamiltonians [11] when a specific procedure of parasupersymmetrization, compatible with Kemmer algebras [12], is considered. We also prove [13] that the nonrelativistic limit of the associated Kemmer oscillator is an ordinary harmonic oscillator still coupled with a spin-orbit term but where the values of the spin involved in this term are now zero or one.

The contents of this communication are then distributed as follows. In Section 2, we point out the connection between SSQM and Dirac hamiltonians through the free case and the harmonic oscillator one. Then, in Section 3, we visit the extension to the parasupersymmetric context by considering again the free and the harmonic oscillator contexts.

2 Supersymmetric Quantum Mechanics and Dirac Formalism

Let us recall that the N=2-supersymmetric quantum mechanics [1] is characterized by the existence of two supercharges Q_1 and Q_2 satisfying the relations

$$\{Q_\alpha, Q_\beta\} = 2\,\delta_{\alpha\beta} H_{SS}\,, \tag{2.1}$$

$$[H_{SS}, Q_\alpha] = 0\,, \quad \alpha, \beta = 1, 2\,, \tag{2.2}$$

where H_{SS} is the supersymmetric hamiltonian. In the so-called spin-orbit coupling procedure [4], these supercharges are realized through 4 by 4 matrices associated with the irreducible unitary representations of the unitary Lie superalgebra su (2|2) [14], when the 3-dimensional spatial context is under study. More precisely the free case corresponds to

$$H_{SS} = \frac{\vec{p}^{\,2}}{2m}\,, \tag{2.3}$$

$$Q_\alpha = \frac{1}{\sqrt{2m}}\,\varphi^j_\alpha\, p^j\,, \quad \alpha = 1, 2\,, \tag{2.4}$$

where the sum over repeated indices is understood. As the hermitian matrices φ^j_α generate su (2|2), we can propose the following choice

$$\varphi^j_1 = \alpha^j\,, \quad \varphi^j_2 = i\,\beta^j_\alpha\,, \tag{2.5}$$

leading to identify the free Dirac hamiltonian H_D with

$$H_D \equiv \vec{\alpha}\cdot\vec{p} + m\beta = \sqrt{2m}\; Q_1 + m\beta . \tag{2.6}$$

Here the velocity of light is taken to be one. The second supercharge leads to a new Dirac hamiltonian unitarily equivalent to the usual one given by Eq. (2.6), as it is easily verified through

$$U = \frac{1}{\sqrt{2}}\left(I + i\beta\right). \tag{2.7}$$

The harmonic oscillator case can be studied in a completely parallel way. Its supersymmetrized version is characterized by

$$H_{SS} = \frac{\vec{p}^{\,2}}{2m} + \frac{1}{2} m\omega^2 \vec{x}^{\,2} + \frac{\omega}{2}\left(3 + 2\vec{L}\cdot\vec{\sigma}\right) \otimes \sigma_3 , \tag{2.8}$$

$$Q_1 = \frac{1}{\sqrt{2m}}\left(\vec{\alpha}\cdot\vec{p} - i m\omega\beta\vec{\alpha}\cdot\vec{x}\right), \tag{2.9a}$$

$$Q_2 = \frac{1}{\sqrt{2m}}\left(i\beta\vec{\alpha}\cdot\vec{p} + m\omega\vec{\alpha}\cdot\vec{x}\right). \tag{2.9b}$$

These two supercharges (2.9) give rise to two (unitarily equivalent) Dirac hamiltonians by using analogous identifications to (2.6), the first one coinciding with the Dirac oscillator proposed by Moshinsky and Szczepaniak [5]. These two operators lead to the same nonrelativistic limit [15] i.e. an ordinary harmonic oscillator with a strong spin-orbit coupling term.

3 Parasupersymmetric Quantum Mechanics and Kemmer Formalism

The relations characterizing the N=2-parasupersymmetric quantum mechanics in terms of the two parasupercharges Q_1 and Q_2 write [10]

$$Q_\alpha^3 - 3\,Q_\beta Q_\alpha Q_\beta = Q_\alpha H_{PSS} , \tag{3.1}$$

$$2\left\{Q_\alpha, Q_\beta^2\right\} - Q_\beta Q_\alpha Q_\beta - Q_\alpha^3 = Q_\alpha H_{PSS} , \tag{3.2}$$

$$\left[H_{PSS}, Q_\alpha\right] = 0 , \quad \alpha, \beta = 1, 2 , \quad \alpha \neq \beta , \tag{3.3}$$

where H_{PSS} is the parasupersymmetric hamiltonian and where there is no summation on repeated indices.

The 3-dimensional free case is still associated with (2.3) and (2.4) but now the matrices φ_α^j correspond to the Kemmer algebra K (4) [12]

$$\varphi_1^j = \left[\beta^0, \beta^j\right] , \quad \varphi_2^j = i\left\{\beta^0, \beta^j\right\} . \tag{3.4}$$

Here the matrices β_μ satisfy

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu \beta_\mu = 2 g_{\mu\nu} \beta_\lambda + 2 g_{\nu\lambda} \beta_\mu , \quad g_{00} = -g_{ii} = 1 . \tag{3.5}$$

This realization enables the following identification for the free Kemmer hamiltonian [11]

$$H_\kappa \equiv \left[\beta^0, \beta^j\right] p^j + m \beta^0 = 2\sqrt{2m}\, Q_1 + m \beta^0 , \tag{3.6}$$

and it also ensures a new proposal with respect to the second parasupercharge, unitarily equivalent [10] to the one proposed in (3.6).

The harmonic oscillator context is subtended by parallel identifications leading in particular to

$$H_\kappa = \left[\beta^0, \beta^j\right] p^j + i m \omega \left\{\beta^0, \beta^j\right\} x^j . \tag{3.7}$$

The nonrelativistic hamiltonians corresponding to (3.7) in the spin 0 and spin 1 cases are respectively given by [13]

$$H_0 = \frac{\vec{p}^{\,2}}{2m} + \frac{1}{2} m \omega^2 \vec{x}^{\,2} - \frac{3}{2}\omega , \tag{3.8}$$

$$H_1 = \frac{\vec{p}^{\,2}}{2m} + \frac{1}{2} m \omega^2 \vec{x}^{\,2} - \frac{\omega}{2}\left(3 + 2 \vec{L}\cdot\vec{S}\right) , \tag{3.9}$$

where S_j (j = 1, 2, 3) refer to the 3 by 3 spin 1 matrices. Through these results, the system described by the hamiltonian (3.7) is called the Kemmer or pararelativistic oscillator [10].

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