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Covariant Harmonic Oscillators - 1973 Revisited

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Abstract

Using the relativistc harmonic oscillator, we give a physical basis to the phenomenological wave function of Yukawa which is covariant and normalizable. We show that this wave function can be interpreted in terms of the unitary irreducible representations of the Poincaré group. The transformation properties of these covariant wave functions are also demonstrated.

1 Introduction

Because wave functions play a central role in nonrelativistic quantum mechanics, one method of combining quantum mechanics and special relativity takes the form of efforts to construct relativistic wave functions with an approrpriate probability interpretation. The harmonic oscillator, which has the useful property of mathematical simplicity, has served as the first concrete solution to many new physical theories. It played a key role in the developing stages of nonrelativistc quantum mechanics, statistical mechanics, theory of specific heat, molecular theory, quantum field theory, theory of superconductivity, theory of coherent light, and many others. It is, therefore, quite natural to expect that the first nontrivial relativistic wave function would be a relativistic harmonic oscillator wave function[1, 2].

In connection with relativistic particles with internal space-time structure, Yukawa attempted to construct relativistic oscillator wave functions in 1953[3]. Yukawa observed that an attempt to solve a relativistic oscillator wave equation in general leads to infinite-component wave functions, and that finite-component wave functions may be chosen if a subsidiary condition involving the four-momentum of the particle is considered. This proposal of Yukawa was further developed by Markov,[4] Takabayasi,[5, 6] Sogami[7] and Ishida.[8]

The effectiveness of Yukawa's oscillator wave function in the relativistic quark model was first demonstrated by Fujimura *et al.*[9] who showed that the Yukawa wave function leads to the correct high-energy asymptotic behavior of the nucleon form factor. The harmonic oscillator wave function was also rediscovered by Feynman *et al.*[10] who advocated the use of relativistic oscillators instead of Feynman diagrams for studying hadronic structures and interactions. The paper of Feynman *et al.* contains all the troubles expected from relativistic wave equations, and the authors of this paper did not make any attempt to hide those troubles.

The basic problem facing any relativistic harmonic oscillator equation is the negative-energy spectrum due to time-like excitations. It had once been widely believed that any attempt to obtain finite- component wave functions by eliminating time-like excitations would lead to a violation of probability conservation. This belief did not turn out to be true. It is now possible to construct harmonic oscillator wave functions without time-like wave functions which form the vector spaces for unitary irreducible representations of the Poincaré group.

In Section 2, we formulate the problem by writing down the relativistically invariant differential equation which leads to the covariant harmonic oscillator formalism. In Section 3, we study solutions of the oscillator differential equation which are normalizable in the four-dimensional x, y, z, t space. In Section 4, representations of the Poincaré group for massive hadrons are constructed from the normalizable harmonic oscillator wave functions. It is shown that they form the basis for unitary irreducible representations of the Poincaré group, as well as that for the O(3)-like little group for massive particles. In Section 5, Lorentz transformation properties of the harmonic oscillator wave functions are studied. The linear unitary representation of Lorentz transformation is provided for the harmonic oscillator wave functions.

2 Covariant Harmonic Oscillator Differential Equations

We first consider the differential equation of Feynman *et al.*[10] for a hadron consisting of two quarks bound together by a harmonic oscillator potential of unit strength:

$$\left\{-2\left[\left(\frac{\partial}{\partial x_a^\mu}\right)^2 + \left(\frac{\partial}{\partial x_b^\mu}\right)^2\right] + \left(\frac{1}{16}\right)\left(x_a^\mu - x_b^\mu\right)^2 + m_0^2\right\}\phi(x_a, x_b) = 0, \tag{1}$$

where x_a and x_b are space-time coordinates for the first and second quarks respectively. This partial differential equation has many different solutions depending on the choice of variables and boundary conditions.

In order to simplify the above differential equation, we introduce new coordinate variables:

$$\begin{aligned} X &= (x_a + x_b)/2, \\ x &= (x_a - x_b)/2. \end{aligned} \tag{2}$$

The four-vector X specifies where the hadron is located in space-time, while the variable x measures the space-time separation between the quarks. In terms of these variables, Eq. (1) can be written as

$$\left(\frac{\partial^2}{\partial X_\mu^2} - m_0^2 + \frac{1}{2}\left[\frac{\partial^2}{\partial x_\mu^2}\right]\right)\phi(X,x) = 0. \tag{3}$$

This equation is separable in the X and x variables. Thus

$$\phi(X,x) = f(X)\psi(x), \tag{4}$$

and $f(X)$ and $\psi(x)$ satisfy the following differential equations respectively:

$$\left(\frac{\partial^2}{\partial X_\mu^2} - m_0^2 - (\lambda+1)\right) f(X) = 0, \tag{5}$$

$$\frac{1}{2}\left(-\frac{\partial^2}{\partial x_\mu^2} + x_\mu^2\right)\psi(x) = (\lambda+1)\psi(x). \tag{6}$$

Eq. (6) is a Klein-Gordon equation, and its solution takes the form

$$f(X) = \exp\left[\pm i p_\mu X^\mu\right], \tag{7}$$

with

$$-P^2 = -P_\mu P^\mu = M^2 = m_0^2 + (\lambda+1).$$

where M and P are the mass and four-momentum of the hadron respectively. The eigenvalue λ is determined from the solution of Eq. (7). We are using the same notation for the operator and eigenvalue for the hadronic four-momentum. This should not cause any confusion since we are dealing only with free hadronic states with a definite four-momentum.

As for the four-momenta of the quarks p_a and p_b, we can combine them into the total four-momentum and momentum-energy separation between the quarks:

$$\begin{aligned} P &= p_a + p_b, \\ q &= (p_a - p_b). \end{aligned} \tag{8}$$

P is the hadronic four-momentum conjugate to X. The internal momentum- energy separation q is conjugate to x provided that there exist wave functions which can be Fourier-transformed. If the momentum-energy wave functions can be obtained from the Fourier transformation of the space-time wave function, the differential equation in the q space is identical to the harmonic oscillator equation for the x space given in Eq. (7)

3 Normalizable Solutions of the Relativistic Oscillator Equation

Since we are quite familiar with the three-dimensional harmonic oscillator equation from nonrelativistic quantum mechanics, we are naturally led to consider the separation of the space and time variables and write the four-dimensional harmonic oscillator equation of Eq.(1.6) as

$$\left(-\nabla^2 + \frac{\partial^2}{\partial t^2} + \left[\mathbf{x}^2 - t^2\right]\right)\psi(x) = (\lambda + 1)\psi(x). \tag{9}$$

However, the $\mathbf{x}t$ system is not the only coordinate system in which the differential equation takes the above form.

If the hadron moves along the Z direction which is also the z direction, then the hadronic factor $f(X)$ of Eq. (8) is Lorentz-transformed in the same manner as the scalar particles are transformed. The Lorentz transformation of the internal coordinates from the laboratory frame to the hadronic rest frame takes the form

$$\begin{aligned} x' &= x, \quad y' = y, \\ z' &= (z - \beta t)/(1 - \beta^2)^{1/2}, \\ t' &= (t - \beta z)/(1 - \beta^2)^{1/2}, \end{aligned} \tag{10}$$

where β is the velocity of the hadron moving along the z direction. The primed quantities are the coordinate variables in the hadronic rest frame. In terms of the primed variables, the oscillator differential equation is

$$\left(-\nabla'^2 + \frac{\partial^2}{\partial t'^2} + \left[\mathbf{x}'^2 - t'^2\right]\right)\psi(x) = (\lambda + 1)\psi(x). \tag{11}$$

This form is identical to that of Eq. (10) , due to the fact that the oscillator differential equation is Lorentz-invariant.[1]

Among many possible solutions of the above differential equation, let us consider the form

$$\begin{aligned} \psi_\beta &= \left(\frac{1}{\pi}\right)\left(\frac{1}{2}\right)^{(a+b+n+k)/2}\left(\frac{1}{a!b!n!k!}\right)^{1/2} H_a(x')H_b(y')H_n(z')H_k(t') \\ &\quad \times \exp\left[-\frac{1}{2}(x'^2 + y'^2 + z'^2 + t'^2)\right], \end{aligned} \tag{12}$$

where a, b, n and k are integers, and $H_a(x')$, $H_b(y')$ ...are the Hermite polynomials. This wave function is normalizable, but the eigenvalues are:

$$\lambda = (a + b + n - k) \tag{13}$$

Thus for a given finite value of λ, there are infinitely many possible combinations of a, b, n and k. The most general solution of the oscillator differential equation is infinitely degenerate.[3]

Because the wave functions are normalizable, all the generators of the Lorentz transformations are Hermitian operators. The Lorentz transformation applicable to this function space is therefore a *unitary* transformation. Indeed, we can write any function of the coordinate variables x, y, z and t as a linear combination of the above solutions. In particular, a solution of the oscillator equation with a given set of quantum numbers in the hadronic rest frame can be written as a linear sum of infinitely many solutions in the hadronic rest frame as we shall see in Section 4.

It is very difficult, if not impossible, to give physical interpretations to infinite-component wave functions. For this reason, it is quite natural to seek a finite set from the infinite number of wave functions at least in one Lorentz frame. The simplest way to obtain such a finite set of wave functions is to invoke the restriction that there be no time-like oscillations in the Lorentz frame in which the hadron is at rest and that the integer k in Eqs. (13) and (14) be zero. In doing so, we are led to the following two fundamental questions:

(a). Is it possible to give physical interpretations to the wave functions belonging to the resulting finite set?

(b). Is it still possible to maintain Lorentz covariance with this condition?

Let us examine question (b) closely.

When the hadron moves along the z axis, the $k = 0$ condition is equivalent to

$$\left(t' + \frac{\partial}{\partial t'}\right) \psi_\beta(x) = 0. \tag{14}$$

The most general form of the above condition is

$$p_\mu \left(x^\mu - \frac{\partial}{\partial x_\mu}\right) \psi_\beta(x) = 0. \tag{15}$$

Thus the $k = 0$ condition is covariant. Once this condition is set, we can write the wave function belonging to this finite set as

$$\psi_\beta(x) = (1/\pi)[1/(2^a 2^b 2^n a! b! n!)]^{1/2} H_a(x') H_b(y') H_n(z') \exp\left[-\frac{1}{2}(x'^2 + y'^2 + z'^2 + t'^2)\right]. \tag{16}$$

Except for the Gaussian factor in the t' variable, the above expression is the wave function for the three-dimensional isotropic harmonic oscillator. This means that we can use the spherical coordinate system for the x', y' and z' variables. We shall see in Section 3 how these ideas form the basis for constructing representations of the Poincaré group.

Since the above oscillator wave functions are separable in the Cartesian coordinate system, and since the transverse coordinate variables are not affected by the boost along the z direction, we can omit the factors depending on the x and y variables when studying their Lorentz transformation properties. The most general form of the wave function given in Eq. (13) becomes

$$\psi_\beta^{n,k}(z', t') = \left[1/(\pi 2^n 2^k n! k!)\right] H_n z' H_k(t') \exp\left[-(\frac{1}{2})(z'^2 + t'^2)\right], \tag{17}$$

with

$$\lambda = (n - k)$$

.

The wave functions satisfying the subsidary condition of Eq. (16) take the simple form

$$\psi_\beta^n(x,t) = [1/(\pi 2^n n!)]^{1/2} H_n(z') \exp[-(1/2)(z'^2 + t'^2)], \tag{18}$$

with

$$\lambda = n$$

. This normalizable wave function without excitations along the t' axis describes the internal space-time structure of the hadron moving along the z direction with the velocity parameter β. If $\beta = 0$, then the wave function becomes

$$\psi_0^n(x,t) = [1/(\pi 2^n n!)]^{1/2} H_n(z) \exp[-(1/2)(z^2 + t^2)], \tag{19}$$

Thus

$$\psi_\beta^n(z,t) = \psi_0^n(z',t')$$

. We have therefore obtained the Lorentz-boosted wave function by making a passive coordinate transformation on the z and t coordinate variables.

Let us next study the orthogonality relations of the wave functions. Since the volume element is Lorentz-invariant:

$$dzdt = dz'dt', \tag{20}$$

there is no difficulty in understanding the orthogonality relation:

$$\int \psi_\beta^{n'}(z,t)\psi_\beta^n(z,t)dzdt = \int \psi_0^{n'}(z,t)\psi_0^n(z,t) = \delta_{n',n}. \tag{21}$$

However, a more interesting problem is the inner product of two wave functions belonging to different Lorentz frames. This inner product becomes

$$\int \psi_0^{n'}(z,t)\psi_\beta^n(z,t)dzdt = \delta_{n',n}\left[1 - \beta^2\right]^{(n+1)/2}. \tag{22}$$

The remarkable fact is that the orthogonality in the quantum number n is still preserved because of the Lorentz invariance of the harmonic oscillator differential equation. The oscillator equation does not depend on the velocity parameter β. As for the factor $[1 - \beta^2]^{(n+1)/2}$ in Eq. (23), we note first that, when the oscillator is in the ground state, it becomes like a Lorentz contraction of a rigid rod by $[1 - \beta^2]^{1/2}$. Excited-state wave functions are obtained from the ground state wave function through repeated applications of the step operator:

$$\mid n, \beta >= \sqrt{1/n!}\left(z' - \frac{\partial}{\partial z'}\right)^n \mid 0, \beta > . \tag{23}$$

The transformation property of each step-up operator is like that of z. Therefore, if the ground-state wave function is like a rigid rod along the z direction, the $n^{\underline{\text{th}}}$ excited state should behave like a multiplication of $(n+1)$ rigid rods.[11, 2]

4 Irreducible Unitary Representations of the Poincaré Group

The Poincaré group consists of space-time translations and Lorentz transformations. Let us go back to the quark coordinates x_a and x_b in Eq. (1) and consider performing Poincaré transformations on the quarks. The same Lorentz transformation matrix is applicable to x_a, x_b, x as well as X. However, under the space-time translation which changes x_a and x_b to $x_a + a$ and $x_b + b$ respectively,

$$\begin{aligned} X &\rightarrow X + a, \\ x &\rightarrow x. \end{aligned} \tag{24}$$

The quark separation coordinate x is not affected by translations. For this reason, the generators of translations for this system are

$$P_\mu = -i\frac{\partial}{\partial X^\mu}, \tag{25}$$

while the generators of Lorentz transformations are

$$M_{\mu\nu} = L^*_{\mu\nu} + L_{\mu\nu}, \tag{26}$$

where

$$\begin{aligned} L^*_{\mu\nu} &= i\left(X_\mu\frac{\partial}{\partial X^\nu} - X_\nu\frac{\partial}{\partial X^\mu}\right), \\ L_{\mu\nu} &= i\left(x_\mu\frac{\partial}{\partial x^\nu} - x_\nu\frac{\partial}{\partial x^\mu}\right). \end{aligned}$$

It is straight-forward to check that the ten generators defined in Eqs. (26) and (27) satisfy the commutation relations of the Poincaré group. We are interested in constructing normalizable wave functions which are diagonal in the Casimir operators P^2 and W^2:

$$\begin{aligned} P^2 &= -\left(\frac{\partial}{\partial X^\mu}\right)^2, \\ &= \frac{1}{2}\left(-\frac{\partial^2}{\partial x_\mu^2} + x_\mu^2\right) + m_0^2, \end{aligned} \tag{27}$$

$$W^2 = M^2(\mathbf{L}')^2, \tag{28}$$

where

$$L'_i = -i\epsilon_{ijk}x'_j\frac{\partial}{\partial x'_k}.$$

The eigenvalue of P^2 is $M^2 = m_0^2 + (\lambda + 1)$, and that for W^2 is $M^2\ell(\ell + 1)$. M is the hadronic mass, and ℓ is the total intrinsic angular momentum of the hadron due to internal motion of the spinless quarks.[12] In addition, we can choose the solutions to be diagonal in the component of the intrinsic angular momentum along the direction of the motion. This component is often called the helicity. If the hadron moves along the Z direction, the helicity operator is L_3.

Because the spatial part of the harmonic oscillator equation in Eq. (12) is separable also in the spherical coordinate system, we can write its solution using spherical variables in the hadronic rest frame space spanned by x', y' and z'. The most general form of the solution is

$$\psi^{k,m}_{\beta\lambda\ell}(x) = R^{\ell}_{\mu}(r')Y^{\mu}\ell(\theta',\phi')[1/([\sqrt{\pi}2^k k!)]^{1/2}H_k(t')e^{-t'^2/2}, \tag{29}$$

where

$$\begin{aligned} r' &= [x'^2 + y'^2 + z'^2]^{1/2}, \\ \cos\theta' &= z'/r', \\ \tan\phi &= y'/x', \end{aligned}$$

and

$$\lambda = 2\mu + \ell - k. \tag{30}$$

$R^{\ell}_{\mu}(r')$ is the normalized radial wave function for the three-dimensional harmonic oscillator:

$$R^{\ell}_{\mu}(r) = (2(\mu!)/[\Gamma(\mu + \ell + 3/2)]^3)^{1/2} r^{\ell} L^{\ell+1/2}_{\mu}(r^2)e^{-r^2/2}, \tag{31}$$

where $L^{\ell+1/2}_{\mu}(r^2$ is the associated Laguerre function.[13] The above radial wave function satisfies the orthonormality condition:[14]

$$\int_0^{\infty} nr^2 R^{\ell}_{\mu}(r)R^{\ell}_{\nu}(r)dr = \delta_{\mu\nu}. \tag{32}$$

The spherical form given in Eq. (30) can of course be expressed as a linear combination of the wave functions in the Cartesian coordinate system given in Eq. (17).

The wave function of Eq. (30) is diagonal in the Casimir operators of Eqs. (28) and (29), as well as in L^3. It indeed forms a vector space for the $O(3)$-like little group.[15, 16] However, the system is infinitely degenerate due to excitations along the t' axis. As we did in Section 2, we can suppress the time-like oscillation by imposing the subsidiary condition of Eq. (16), or by restricting k to be zero in Eq. (31). The solution then takes the form

$$\psi^{m}_{\beta\lambda\ell}(x) = R^{\ell}_{\mu}(r')Y^{m}_{\ell}(\theta',\phi')[(1/\pi)^{1/4}\exp(-t'^2/2)], \tag{33}$$

with

$$\lambda = 2\mu + \ell.$$

Thus for a given λ, there are only a finite number of solutions. The above spherical form can be expressed as a linear combination of the solutions without time-like excitations in the Cartesian coordinate system given in Eq. (17).

We can now write the solution of the differential equation of Eq. (1) as

$$\phi(X, x) = e^{\pm iP\cdot X}\phi^{m}_{\beta\lambda\ell}(x). \tag{34}$$

This wave function describes a free hadron with a definite four-momentum having an internal space-time structure which can be described by an irreducible unitary representation of the Poincaré group. The representation is unitary because the portion of the wave function depending on the internal variable x is square-integrable, and all the generators of Lorentz transformations are Hermitian operators. We shall study in the next section how these wave functions are Lorentz transformed.

5 Transformation Properties of the Harmonic Oscillator Wave Functions

If the hadronic velocity is zero, then its rest frame coincides with the laboratory frame. The wave function then is

$$\psi_0(x) = R^{\ell}_{\mu}(r)Y^{m}_{\ell}(\theta, \phi)[(1/\pi)^{1/4}\exp(-t^2/2)]. \tag{35}$$

The simplest way to obtain the wave function for the moving hadron is to replace the r, θ and ϕ variables in the above expression by their primed counterparts. This produces Eq. (30). However, we are interested in obtaining the wave function for a moving hadron as a linear combination of the wave functions for the rest frame. If we apply the boost operator to the wave function for the hadron at rest,

$$\psi^{\ell m}_{\beta\lambda}(x) = [e^{-i\eta K_3}]\psi^{\ell m}_{0}\lambda(x), \tag{36}$$

where K_3 is the boost generator along the z axis, its form is

$$K_3 = -i\left(z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}\right), \tag{37}$$

and η is related to velocity parameter β by

$$\sinh\eta = \beta/(1-\beta^2)^{1/2}.$$

Both the rest-frame and moving-frame wave functions have the same set of eigenvalues for the Casimir operators P^2 and W^2 of the Poincaré group.

These eigenstate wave functions are linear combinations of the Cartesian forms in their respective coordinate systems. If the hadron moves along the z direction, the x and y variables remain invariant. Therefore, we use the wave function of Eq. (19) with $\beta = 0$:

$$\psi^{n,0}_{0} = [1/(\pi 2^n n!)]^{1/2}H_n(z)\exp[-(1/2)(z^2 + t^2)]. \tag{38}$$

The superscipt 0 indicates that there are no time-like excitations: $k = 0$. We are now led to consider the transformation

$$\begin{aligned}\psi_\beta^{n,0}(z,t) &= [\exp(-i\eta K_3)]\psi_0^{n,0}(z,t) \\ &= \psi_0^{n,0}(z',t'),\end{aligned} \tag{39}$$

and ask what the boost operator $\exp(-i\eta K_3)$ does to $\psi_0^{n,0}(z,t)$.

This boost operator of course changes z and t to z' and t' respectively as is indicated above. However, we are interested in whether the transformation can take the linear form

$$\psi_\beta^{n,0}(z,t) = \sum_{n',k'} A_{n'k'}^{n,0}(\beta)\psi_0^{n'k'}(z,t). \tag{40}$$

Because the oscillator differential equation is Lorentz invariant, the eigenvalue λ of Eq. (18) remains invariant, and only the terms which satisfy the condition

$$n = (n' - k') \tag{41}$$

make non-zero contributions in the sum. Thus the above expression can be simplified to

$$\psi_\beta^{n,0}(z,t) = \sum_{k=0}^{\infty} A_k^n(\beta)\psi_0^{n+k,k}(z,t). \tag{42}$$

This is indeed a *linear unitary representation of the Lorentz group.* The representation is infinte-dimensional because the sum over k is extended from zero to infinity.[17]

The remaining problem is to determine the coefficient $A_k^n(\beta)$. Using the orthogonality relation, we can write

$$\begin{aligned}A_k^n(\beta) &= \int dzdt\psi_0^{n+k,k}(z,t)\psi_\beta^{n,0}(z,t) \\ &= \frac{1}{\pi}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{1/2}\left(\frac{1}{n!(n+k)!}\right)^{1/2} \\ &\quad \times\int dzdtH_{n+k}(z)H_k(t)Hn(z') \\ &\quad \times\exp\left(-\frac{1}{2}(z^2+z'^2+t^2+t'^2)\right).\end{aligned} \tag{43}$$

In this integral, the Hermite polynomials and the Gaussian form are mixed with the kinematics of Lorentz transformation. However, if we use the generating function for the Hermite polynomial, the evaluation of the integral is straightforward, and the result is

$$A_k^n(\beta) = (1-\beta^2)^{(1+n)/2}\beta^k\left(\frac{(n+k)!}{n!k!}\right)^{1/2}. \tag{44}$$

Thus the linear expansion given in Eq. (41) can be written as

$$\begin{aligned}\psi_\beta^{n,0}(z,t) &= [1/(2^n\pi)]^{1/2}(1-\beta)^{(n+1)/2}(\exp[-(z^2+t^2)/2]) \\ &\quad \times\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k\left(\frac{\beta}{2}\right)^k H_{n+k}(z)H_k(t).\end{aligned} \tag{45}$$

As was indicated with respect to Eq. (20), this linear transformation has to be unitary. Let us check this by calculating the sum

$$S = \sum_{k=0}^{\infty} \mid A_k^n(\beta) \mid^2 . \tag{46}$$

According to Eq. (45), this sum is

$$S = [1 - \beta^2]^{(n+1)} \sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!} (\beta^2)^k. \tag{47}$$

On the other hand, the binomial expansion of $[1 - \beta^2]^{-(n+1)}$ takes the form

$$[1 - \beta^2]^{-(n+1)} = \sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!} \beta^{2k}. \tag{48}$$

Therefore the sum S is equal to one. The linear transformation of Eq. (43) is indeed a unitary transformation.

It is also of interest to see how this transformation can be achieved directly in terms of solutions which are eigenstates of the Casimir operators. For this purpose we construct the solutions in terms of the spherical coordinate variables for the three-dimensional (x, y, z) space and treat t separately. If the hadron is at rest,

$$\psi_{0\lambda\ell}^{k,m}(x) = R_\mu^\ell(r) Y_\ell^m(\theta, \phi)[1/(\sqrt{\pi} 2^k k!)]^{1/2} H_k(t) e^{-t^2/2}. \tag{49}$$

Thus we have to write the generators of Lorentz transformations in terms of these variables. The three rotation generators can be written as[13]

$$\begin{aligned} L_3 &= -i\frac{\partial}{\partial\phi}, \\ L_\pm &= L_1 \pm L_2 \\ &= \pm e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i cot\theta \frac{\partial}{\partial\phi}\right). \end{aligned} \tag{50}$$

It is not difficult to calculate the three boost generators. They take the form

$$\begin{aligned} iK_3 &= \cos\theta\left(r\frac{\partial}{\partial t} + t\frac{\partial}{\partial r}\right) - \frac{t}{r}\sin\theta\frac{\partial}{\partial\theta}, \\ iK_\pm &= K_1 \pm iK_2 \\ &= e^{\pm i\phi}\left(r\frac{\partial}{\partial t} + t\sin\theta\frac{\partial}{\partial r} - \frac{t}{r}\cos\theta\frac{\partial}{\partial\theta} \pm \frac{t}{r\sin\theta}\frac{\partial}{\partial\theta}\right). \end{aligned} \tag{51}$$

The rotation generators affect only the spherical harmonics in the wave function of Eq. (50). Thus

$$\begin{aligned} L_3\psi_{0\lambda\ell}^{k,m} &= m\psi_{0\lambda\ell}^{k,m}, \\ L_\pm\psi_{0\lambda\ell}^{k,m} &= \sqrt{(\ell \mp m)(\ell \pm m + 1)}\psi_{0\lambda\ell}^{k,m\pm1}. \end{aligned} \tag{52}$$

The above relations mean that rotations do not change the quantum numbers λ, ℓ and k. They only change m. Eq. (53) indeed corresponds to the fact that the little group for massive hadrons is like $SO(3)$.

On the other hand, if we apply the boost generators, we end up with somewhat complicated formulas:

$$\begin{aligned} iK_3\psi_{\lambda\ell}^{km} &= \left[\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right]^{1/2} Y_{\ell+1}^{m}(\theta,\phi)Q_{-\ell}F_{\lambda\ell}^{k}(r,t) \\ &+ \left[\frac{(\ell+1)(\ell-m)}{(2\ell+1)(2\ell-1)}\right]^{1/2} Y_{\ell-1}^{m}(\theta,\phi)Q_{\ell+1}F_{\lambda\ell}^{k}, \\ iK_\pm &= \left[\frac{(\ell\pm m+1)(\ell\pm m+2)}{(2\ell+1)(2\ell+3)}\right]^{1/2} Y_{\ell+1}^{m\pm1}(\theta,\phi)Q_{\pm(\ell}F_{\lambda\ell}^{k} \\ &+ \left[\frac{(\ell\mp m)(\ell\mp m-1)}{(2\ell+1)(2\ell-1)}\right]^{1/2} Y_{\ell+1}^{m\pm1}(\theta,\phi)Q\pm(\ell+1)F_{\lambda\ell}^{k}(r,t). \end{aligned} \tag{53}$$

where

$$Q_\ell = \left(t\frac{\partial}{\partial r} + r\frac{\partial}{\partial t} + \ell\frac{t}{r}\right),$$

and

$$F_{\lambda\ell}^{k}(r,t) = R_{\mu}^{\ell}(r)[1/(\sqrt{\pi}2^k k!)]^{1/2}H_k(t)\exp(-t^2/2).$$

K_3 does not change the value of m, while K_+ and K_- change m by $+1$ and -1 respectively. In addition, unlike the rotation operators, the boost generators change λ, ℓ and k. This is a manifestation of the fact that the unitary representation is infinite-dimensional as is indicated in Eq. (43).

It is possible to finish the calculation by explicitly carrying out the differentiations contained in the Q_ℓ operators. However, this does not appear necessary, because we already know what the answer should be from our experience with the Cartesian coordinate sytem.

6 Conclusion

The harmonic oscillator applied to the symmetric quark model has withstood the test of time. The work of Karr[18, 19] has fully integrated the field theorectic aspects of this work. Below we present the experimental present status of the non-strange baryon in relation to the harmonic oscillator.

TABLE I. Mass spectrum of nonstrange baryons. The calculated masses based on Eqs. (9.1) and (9.2) in Kim and Noz,[2] **Theory and Applications of the Poincaré Group**. The experimental masses are from "Physical Review D" **45**, No. 11, (June, 1992). The last column contains the identification code of the pion-nucleon resonance used in Particle Data Group. For N = 0 and N = 1, the quark model multiplet scheme is in excellent agreement with the experimental world. For N = 2, the model seems to work well, but more work is needed on both the theoretical and experimental fronts. There are still very few particles in N = 3. Baryonic masses are measured in MeV.

N	L	SU(6)	SU(3)	Spin	J	Calculated Mass	Experimental Mass	PDG-ID
0	0	56	8	1/2	1/2	940	939	P_{11}****
			10	3/2	3/2	1240	1232	P_{33}****
1	1	70	8	1/2	1/2	1520	1535	S_{11}****
					3/2	1520	1520	D_{13} ****
			8	3/2	1/2	1688	1650	S_{11} ****
					3/2	1688	1700	D_{13} ***
					5/2	1688	1675	D_{15} ****
			10	1/2	1/2	1652	1620	S_{31} ****
					3/2	1652	1700	D_{33} ****
2	0	56	8	1/2	1/2	1480	1440	P_{11} ****
			10	3/2	3/2	1780	1600	P_{33} **
		70	8	1/2	1/2	1730	1710	P_{11} ***
			8	3/2	3/2	1898	1900	P_{13} *
			10	1/2	1/2	1862	1750	P_{31} *
2	2	56	8	1/2	3/2	1660	1720	P_{13} ****
					5/2	1660	1680	F_{15} ****
			10	3/2	1/2	1960	1910	P_{31} ****
					3/2	1960	1920	P_{33} ***
					5/2	1960	1905	F_{35} ****
					7/2	1960	1950	F_{37} ****
	2	70	8	1/2	3/2	1900		
					5/2	1900	2000	F_{15} **
			8	3/2	1/2	2078	2100	P_{11} *
					3/2	2078		
					5/2	2078		
					7/2	2078	1990	F_{17} **
			10	1/2	3/2	2030		
					5/2	2030	2000	F_{35} *

Table I. Mass spectrum of nonstrange baryons continued.

N	L	SU(6)	SU(3)	Spin	J	Calculated Mass	Experimental Mass	PDG-ID
3	1	70	8	1/2	1/2	2060		
					3/2	2060	2080	D_{13} **
			8	3/2	1/2	2228	2090	S_{11} *
					3/2	2228		
					5/2	2228		
			10	1/2	1/2	2192	1900	S_{31} ***
					3/2	2192		
		70	8	1/2	1/2	2060		
					3/2	2060		
			8	3/2	1/2	2228		
					3/2	2228		
					5/2	2228		
			10	1/2	1/2	2192		
			10	1/2	1/2	2192		
					3/2	2192		
		56	8	1/2	1/2	1810		
					3/2	1810		
			10	3/2	1/2	2110	2150	S_{31} *
					3/2	2110	1940	D_{33} *
					5/2	2110	1930	D_{35} ***
	2	70	8	1/2	3/2	2180		
					5/2	2180		
			8	3/2	1/2	2348		
					3/2	2348		
					5/2	2348		
					7/2	2348	2190	G_{17} ****
			10	1/2	3/2	2312		
					5/2	2360		
	3	70	8	1/2	5/2	2528		
					7/2	2528		
			8	3/2	3/2	2528		
					5/2	2528	2200	D_{15} **
					7/2	2528		
					9/2	2528	2250	G_{19} ****
			10	1/2	5/2	2492		
					7/2	2492	2200	G_{37} *
		56	8	1/2	5/2	2110		
					7/2	2110		
			10	3/2	3/3	2410		
					5/2	2410	2350	D_{35} *
					7/2	2410	2390	F_{37} *
					9/2	2410	2400	G_{39} **

TABLE II. In addition, there are resonances which do not fit in this table. Since most of these resonaces correspond to even- parity baryons, they presumably belong to $N = 4$ multiplet.

SU(3)	J	Mass	PDG-ID
8	3/2	1540	P_{13} *
8	9/2	2220	H_{19} ****
8	11/2	2600	$I_{1,11}$ ***
8	13/2	2700	$K_{1,13}$ **
10	1/2	1550	P_{31} *
10	9/2	2300	H_{39} **
10	7/2	2390	F_{37} *
10	11/2	2420	$H_{3,11}$ ****
10	13/2	2750	$I_{3,13}$ **
10	15/2	2950	$K_{3,15}$ **

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