

**NASA**  
**Technical**  
**Paper**  
**3317**

May 1993

169946  
P.16

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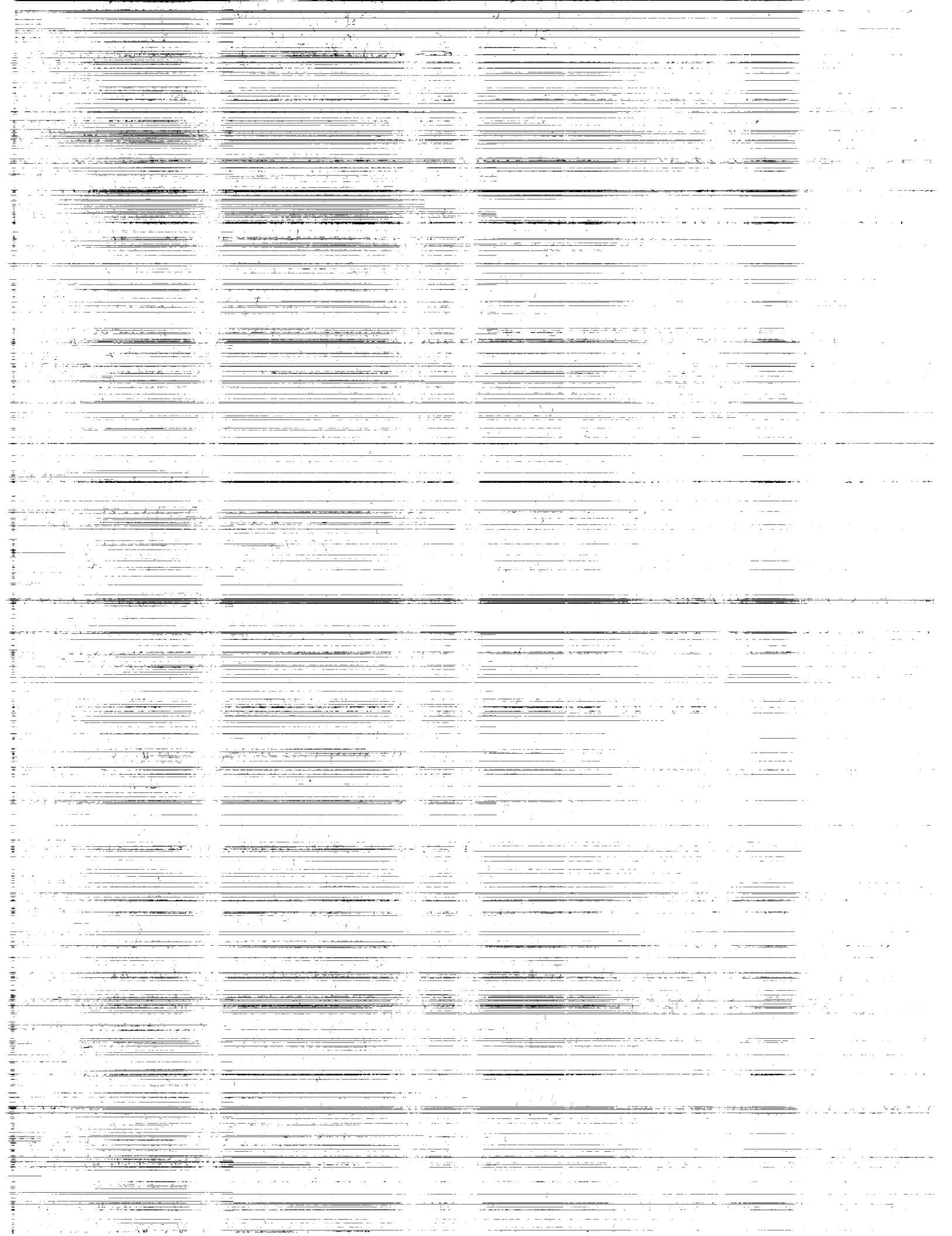
(NASA-TP-3317) EXACT METHODS FOR  
MODAL TRANSIENT RESPONSE ANALYSIS  
INCLUDING FEEDBACK CONTROL (NASA)  
16 p

N93-27585

Unclas

H1/39 0169946

**NASA**



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# Exact Methods for Modal Transient Response Analysis Including Feedback Control

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## Abstract

*This paper presents a modal method for the analysis of controlled structural systems that retains the uncoupled nature of the classical transient response analysis of a structure subjected to a prescribed time-varying load. The control force is expanded as a Taylor series that remains on the right side of the equations, and it does not lead to a computational approach that requires coupling between modes on the left side. Retaining a sufficient number of terms in the series produces a solution to the modal equations that is accurate to machine precision. The approach is particularly attractive for large problems in which standard matrix exponential methods become computationally prohibitive. Numerical results are presented to show the accuracy and efficiency of the proposed approach for dynamic feedback compensation of a truss structure with local member modes in the controller bandwidth.*

## Introduction

Interest has been increasing in analyzing complex structures subjected to time-dependent loadings including loads from a control system designed to permit controls-structures integration (CSI). The control problem is more acute when dealing with structures that may respond both locally and globally in a large number of frequencies due to the applied load. The main challenge in such an analysis is achieving accurate results with a reasonable amount of computational effort. The method presented in this paper is based on a modal approach and can achieve accurate results for the number of modes used.

The computational approach developed herein to predict the dynamic behavior of controlled structural systems is based on a partitioning of the structure and controller equations such that the diagonal modal form of the equations of the structure is preserved. Although this idea is not new to the literature, the treatment of the interaction forces between the structure and controller is described for the first time by a series representation. The number of terms in the series controls the accuracy of the solution which can be a priori prescribed or automatically set to be equal to the machine precision. The high accuracy obtainable with this approach is shown to permit relatively large integration time steps. This feature, in conjunction with the uncoupled modal equations for the structure, leads to highly efficient computations.

The governing equations for the controlled structural system are described. Subsequently, a Taylor-series expansion of the control forces is presented. Through the use of a Taylor-series expansion, a set of recursive relations to compute the modal states and their time derivatives are developed. The proposed approach and a more conventional approach, which fully couples the equations, are applied to the solution of a two-dimensional truss with dynamic feedback compensation. The accuracy and efficiency of the two approaches are compared. The efficiency of the proposed approach is shown to be substantially better than that of the conventional approach. A large number of local member modes in the truss structure are used to demonstrate the importance of developing computational tools for large structural dynamic models.

## Nomenclature

<b>A, B, C, D</b>	control matrices defined in equations (2) and (4)
<b>a</b>	vector of time-dependent modal amplitude coefficients
<b>d( )</b>	derivative operator

<b>E</b>	control-force influence matrix
<b>E<sub>J</sub></b>	matrix defined in equation (A5)
<b>F</b>	applied time-varying load vector
<b>F<sub>J</sub></b>	matrix defined in equation (A2)
<b>f</b>	modal applied force vector
<b>G</b>	damping matrix
<b>g</b>	diagonal modal damping matrix
<b>H<sub>J</sub></b>	matrices defining sensed output measurements used in equation (3)
<b>h</b>	modal forcing vector
<b>I</b>	identity matrix
<b>J</b>	variable index ( $J = 1, 2, 3, 4$ )
<b>K</b>	stiffness matrix
<b>K<sub>e</sub></b>	exact frequency-dependent stiffness matrix
<b>k</b>	diagonal modal stiffness matrix
<b>M</b>	mass matrix
<b>m</b>	diagonal modal mass matrix
<b>N</b>	number of terms used in Taylor series
<b>n</b>	integer identifying term in Taylor series
<b>P<sub>J</sub></b>	matrices defined in equation (A4)
<b>q</b>	displacement vector
<b>R, S</b>	matrices used in matrix exponential solution of equation (20)
<b>S<sub>1</sub>, S<sub>2</sub></b>	coefficients used in solution of differential equation (see eq. (12))
<b>s</b>	variable
<b>T<sub>J</sub></b>	coefficients defined in equations (13) and (14)
<b>t</b>	time
<b>u</b>	control force vector
<b>x</b>	controller state vector
<b>y</b>	output measurement vector
<b>z<sub>rn</sub></b>	particular solution of differential equation defined in equation (12)
<b>Γ, Φ</b>	matrices formed from $\Gamma_{in}$ and $\Phi_{ij}$ , respectively
<b>Γ<sub>in</sub>, Φ<sub>ij</sub></b>	submatrices in full matrix solution given in equation (18), where $i, j = 1-3$
<b>γ, ω</b>	parameters defined in equation (12)
<b>δ</b>	time step
<b>Ψ</b>	matrix formed from vibration modes
<b>0</b>	null matrix

Subscript:

$r$  integer identifying a particular mode

Superscripts:

$(n)$   $n$ th derivative with respect to time

$T$  transpose of matrix or vector

Boldface quantities denote vectors or matrices; dots denote derivatives with respect to time.

## Governing Equations

A standard finite element representation is used to describe the response of a structure to time-varying forces including state-dependent forces typical of feedback control systems. The controlled structural system is governed by the equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F} + \mathbf{E}\mathbf{u} \quad (1)$$

where  $\mathbf{q}$  is the real displacement vector,  $\mathbf{M}$  is the mass matrix,  $\mathbf{G}$  is the damping matrix, and  $\mathbf{K}$  is the stiffness matrix. The time-varying applied load is  $\mathbf{F}$ , and a feedback control system produces control forces  $\mathbf{u}$  where  $\mathbf{E}$  is a control force influence matrix.

The control forces are assumed to be

$$\mathbf{u} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{y} \quad (2)$$

where  $\mathbf{y}$  denotes the sensed output measurements given by

$$\mathbf{y} = \mathbf{H}_1\mathbf{q} + \mathbf{H}_2\dot{\mathbf{q}} + \mathbf{H}_3\ddot{\mathbf{q}} + \mathbf{H}_4\mathbf{u} \quad (3)$$

and  $\mathbf{x}$  denotes the controller states that are governed by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \quad (4)$$

In equations (1)-(4),  $\mathbf{u}$ ,  $\mathbf{x}$ ,  $\mathbf{y}$ , and  $\mathbf{q}$  are vectors, whereas the remaining boldfaced quantities are matrices. The form of the control system equations is of a very general nature that can be applied to a variety of specific treatments that have been used in the literature. The time history of response is expressed in terms of the natural vibration modes as

$$\mathbf{q} = \mathbf{\Psi}\mathbf{a} \quad (5)$$

where  $\mathbf{\Psi}$  is a matrix whose  $i$ th column is the natural vibration mode (eigenvector) corresponding to the  $i$ th eigenvalue of the undamped system, and  $\mathbf{a}$  is a vector of time-dependent amplitude coefficients. This response leads to the differential equation

$$\mathbf{m}\ddot{\mathbf{a}} + \mathbf{g}\dot{\mathbf{a}} + \mathbf{k}\mathbf{a} = \mathbf{h} \quad (6a)$$

where  $\mathbf{m}$ ,  $\mathbf{g}$ , and  $\mathbf{k}$  are modal values of mass, damping, and stiffness, and  $\mathbf{h}$  is the vector of modal loads that includes the applied load and the control force. These quantities are

$$\mathbf{m} = \mathbf{\Psi}^T\mathbf{M}\mathbf{\Psi} \quad \mathbf{g} = \mathbf{\Psi}^T\mathbf{G}\mathbf{\Psi} \quad \mathbf{k} = \mathbf{\Psi}^T\mathbf{K}\mathbf{\Psi} \quad (6b)$$

and

$$\mathbf{h} = \Psi^T \mathbf{F} + \Psi^T \mathbf{E} \mathbf{u} = \mathbf{f} + \Psi^T \mathbf{E} \mathbf{u} \quad (6c)$$

Because of the orthogonality of the mode shapes  $\Psi$  with respect to  $\mathbf{K}$  and  $\mathbf{M}$ , the first three matrices in equation (6a) are diagonal if proportional damping is assumed; that is,  $\mathbf{G}$  is a linear combination of  $\mathbf{K}$  and  $\mathbf{M}$ . Nonproportional damping can be treated by putting the damping terms on the right side of the equations and treating them as part of the control system. In reference 1, results from the solution of equations (6) were presented for the case of no control forces. When  $\mathbf{h}$  includes feedback control forces, all the elements of  $\mathbf{a}$  will be coupled. If the specific dependence on  $\mathbf{a}$  and  $\dot{\mathbf{a}}$  of  $\mathbf{h}$  is brought to the left side of the equations, we no longer have a simple diagonal set of equations to solve but a set of fully populated matrix equations. The full matrix solution of these equations leads to large computer storage and run times. The next section presents a partitioned solution approach that retains the control-force loading terms on the right side of the equations.

### Partitioned Integration Formulas

In reference 2, a partitioning approach that retains the symmetry and sparsity of the structural equations was developed in conjunction with an approximate implicit method for time integration. The approach of reference 2 is applicable to both the coupled physical coordinate equations and the uncoupled modal coordinate equations. However, the implicit integration procedure of reference 2 requires small integration time steps to maintain accuracy. An alternate approach for the modal coordinate equations that retains the diagonal nature, but uses an exact integration approach, is to express the control force as a Taylor series.

#### Taylor-Series Solution for Control Forces

The control force  $\mathbf{h}$  as a Taylor series about  $t = 0$  is given by

$$\mathbf{h} = \sum_{n=0}^N \frac{{}^{(n)}\mathbf{h} t^n}{n!} \quad (7)$$

where  ${}^{(n)}\mathbf{h}$  is the  $n$ th derivative of  $\mathbf{h}$  evaluated at  $t = 0$ , and  $N$  is the number of terms in the Taylor-series expansion. To facilitate the calculation of  ${}^{(n)}\mathbf{h}$ , one can modify the control equations as shown in the appendix to eliminate  $\mathbf{y}$  and  $\ddot{\mathbf{a}}$ . The following sequence of equations is used to calculate  ${}^{(n)}\mathbf{h}$ :

$$\ddot{\mathbf{a}} = \mathbf{m}^{-1}(\mathbf{f} + \Psi^T \mathbf{E} \mathbf{u} - \mathbf{g}\dot{\mathbf{a}} - \mathbf{k}\mathbf{a}) \quad (8)$$

$$\dot{\mathbf{x}} = \mathbf{E}_1 \mathbf{a} + \mathbf{E}_2 \dot{\mathbf{a}} + \mathbf{E}_3 \mathbf{f} + \mathbf{E}_4 \mathbf{x} \quad (9)$$

$$\mathbf{u} = \mathbf{P}_1 \mathbf{a} + \mathbf{P}_2 \dot{\mathbf{a}} + \mathbf{P}_3 \mathbf{f} + \mathbf{P}_4 \mathbf{x} \quad (10)$$

with the equation for  $\mathbf{h}$  given by

$$\mathbf{h} = \mathbf{f} + \Psi^T \mathbf{E}(\mathbf{P}_1 \mathbf{a} + \mathbf{P}_2 \dot{\mathbf{a}} + \mathbf{P}_3 \mathbf{f} + \mathbf{P}_4 \mathbf{x}) \quad (11)$$

The determination of the  $n$ th derivative of  $\mathbf{h}$  requires the  $n$  and  $n + 1$  derivatives of  $\mathbf{a}$  and the  $n$ th derivative of  $\mathbf{x}$ . For example,  ${}^{(1)}\mathbf{h}$  involves  $\ddot{\mathbf{a}}$  and  $\dot{\mathbf{x}}$  as determined from equations (8) and (9). To calculate  ${}^{(2)}\mathbf{h}$ ,  ${}^{(3)}\mathbf{a}$  and  ${}^{(3)}\mathbf{x}$  are obtained by taking the derivative of equations (8) and (9), which can be evaluated using the just-determined  $\dot{\mathbf{x}}$ ,  $\ddot{\mathbf{a}}$ , and  ${}^{(1)}\mathbf{h}$ . This process can be repeated recursively by



taking time derivatives of equations (8)–(11) to obtain a sufficient number of terms in the Taylor series to give accurate results.

### Differential Equation Solution for Modal Amplitudes

The solution to equations (6) with  $\mathbf{h}$  given by equation (7) is

$$\mathbf{a}_r = \exp(-\gamma_r t) [S_{1r} \cos(\omega_r t) + S_{2r} \sin(\omega_r t)] + \sum_{n=0}^N z_{rn} t^n \quad (12)$$

where

$$\gamma_r = \frac{\mathbf{g}_{rr}}{2\mathbf{m}_{rr}}$$

$$\omega_r = \sqrt{\frac{\mathbf{k}_{rr}}{\mathbf{m}_{rr}} - \gamma_r^2}$$

$$z_{rn} = \left[ \frac{\binom{n}{n}}{(n+1)!} - (n+2)\mathbf{m}_{rr}(z_r)_{n+2} - \mathbf{g}_{rr}(z_r)_{n+1} \right] \frac{n+1}{\mathbf{k}_{rr}}$$

$$(z_r)_{N+2} = (z_r)_{N+1} = 0$$

The two constants  $S_{1r}$  and  $S_{2r}$  are determined by knowing  $\mathbf{a}_r$  and  $\dot{\mathbf{a}}_r$  at  $t = 0$ . The solution at  $t = \delta$  is

$$\mathbf{a}_r(\delta) = T_{1r} [\mathbf{a}_r(0) - z_{r0}] + T_{2r} [\dot{\mathbf{a}}_r(0) - z_{r1}] + \sum_{n=0}^N z_{rn} \delta^n \quad (13)$$

$$\dot{\mathbf{a}}_r(\delta) = T_{3r} [\mathbf{a}_r(0) - z_{r0}] + T_{4r} [\dot{\mathbf{a}}_r(0) - z_{r1}] + \sum_{n=1}^N n z_{rn} \delta^{n-1} \quad (14)$$

where

$$T_{1r} = \exp(-\gamma_r \delta) \left[ \cos(\omega_r \delta) + \frac{\gamma_r}{\omega_r} \sin(\omega_r \delta) \right]$$

$$T_{2r} = \exp(-\gamma_r \delta) + \frac{\sin(\omega_r \delta)}{\omega_r}$$

$$T_{3r} = -\exp(-\gamma_r \delta) \frac{\omega_r^2 + \gamma_r^2}{\omega_r} \sin(\omega_r \delta)$$

$$T_{4r} = \exp(-\gamma_r \delta) \left[ \cos(\omega_r \delta) - \frac{\gamma_r}{\omega_r} \sin(\omega_r \delta) \right]$$

Because the equations for the control state are coupled, an analytic solution is not attempted. However, with all the derivatives up to  $N$  available, the controller state at time  $\delta$  can be obtained as

$$\mathbf{x} = \sum_{n=0}^N \frac{\binom{n}{n} \delta^n}{n!} \mathbf{x} \quad (15)$$

When the problem has no control forces, this solution is exact for any time step  $\delta$  as long as all the derivatives of  $\mathbf{f}$  are included. When the control forces are present, a variable number of terms are taken so that the control force is known to a certain accuracy at a given time step. Alternatively, a fixed number of terms may be used for all time steps.

If the number of terms is too large, the series in equation (12) involving  $z_{rn}$  has been found to contain large terms of opposite sign such that accurate results cannot be obtained. When this event occurs, a more accurate solution for the modal amplitudes is obtained by simply using the result

$$\mathbf{a} = \sum_{n=0}^{N+1} \frac{{}^{(n)}\delta^n}{n!} \mathbf{a} \quad (16)$$

which is easily calculated because all the derivatives of  $\mathbf{a}$  are available.

For discrete time control systems, a special case can be obtained by taking only one term in the series of equation (7). This procedure has the effect of a zero-order hold which applies a control force having a value corresponding to the conditions at the beginning of the step and remaining constant throughout the time step.

### Nonpartitioned Integration Formulas

To assess the accuracy and efficiency of the partitioned solution approach presented in this paper, the nonpartitioned integration formulas are presented here and used for comparisons in the "Results and Discussion" section.

The explicit dependence of the control force on  $\mathbf{a}$  and  $\mathbf{x}$  can be accounted for directly by substituting equation (10) into equations (6). Thus,

$$m\ddot{\mathbf{a}} + (\mathbf{g} - \Psi^T \mathbf{E} \mathbf{P}_2) \dot{\mathbf{a}} + (\mathbf{k} - \Psi^T \mathbf{E} \mathbf{P}_1) \mathbf{a} - \Psi^T \mathbf{E} \mathbf{P}_4 \mathbf{x} = (\mathbf{I} + \Psi^T \mathbf{E} \mathbf{P}_3) \mathbf{f} \quad (17)$$

By applying the integration method of reference 3 to equations (9) and (17), the following general form is obtained:

$$\begin{bmatrix} \mathbf{a}(t + \delta) \\ \dot{\mathbf{a}}(t + \delta) \\ \mathbf{x}(t + \delta) \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{bmatrix} \begin{bmatrix} \mathbf{a}(t) \\ \dot{\mathbf{a}}(t) \\ \mathbf{x}(t) \end{bmatrix} + \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \dots & \Gamma_{1n} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \dots & \Gamma_{2n} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \dots & \Gamma_{3n} \end{bmatrix} \begin{bmatrix} \mathbf{f}(t) \\ \dot{\mathbf{f}}(t) \\ \ddot{\mathbf{f}}(t) \\ \vdots \\ {}^{(n-1)}\mathbf{f}(t) \end{bmatrix} \quad (18)$$

The matrices in equation (18) are generated from a Taylor series so that sufficient terms can be taken to achieve machine accuracy.

When the applied forces are represented with a zero-order hold, then  ${}^{(n)}\mathbf{f} = 0$  for  $n > 0$ . Thus, equation (18) reduces to

$$\begin{bmatrix} \mathbf{a}(t + \delta) \\ \dot{\mathbf{a}}(t + \delta) \\ \mathbf{x}(t + \delta) \end{bmatrix} = [\Phi] \begin{bmatrix} \mathbf{a}(t) \\ \dot{\mathbf{a}}(t) \\ \mathbf{x}(t) \end{bmatrix} + [\Gamma] [\mathbf{f}(t)] \quad (19)$$

For periodic sampling with period  $\delta$ , the matrices  $\Phi$  and  $\Gamma$  are computed by the matrix exponential (ref. 4)

$$\begin{aligned} \Phi &= e^{\mathbf{R}\delta} \\ \Gamma &= \int_0^\delta e^{\mathbf{R}s} ds \mathbf{S} \end{aligned} \quad (20)$$

where

$$\mathbf{R} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ -\mathbf{m}^{-1}(\mathbf{k} - \Psi^T \mathbf{E} \mathbf{P}_1) & -\mathbf{m}^{-1}(\mathbf{g} - \Psi^T \mathbf{E} \mathbf{P}_2) & \mathbf{m}^{-1} \Psi^T \mathbf{E} \mathbf{P}_4 \\ \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_4 \end{bmatrix}$$

and

$$\mathbf{S} = \begin{bmatrix} \mathbf{0} \\ \mathbf{m}^{-1}(\mathbf{I} + \Psi^T \mathbf{E} \mathbf{P}_3) \\ \mathbf{E}_3 \end{bmatrix}$$

## Results and Discussion

A vibration and transient response analysis has been carried out for the planar-truss beam shown in figure 1. In the results herein, no damping has been assumed. Two truss structures with the properties shown in table I are considered. Case 1 has mass only at the nodes, whereas case 2 has the same total mass, but with distributed mass in the diagonal members such that the first overall beam mode and the lowest diagonal member mode have similar frequencies. These cases allowed the method to be evaluated in a problem in which only a few modes are required for accuracy and also in a problem in which a large number of modes in a narrow frequency range would affect the response. A comparison of the two results gives an indication of the importance of the interaction of local and overall modes in dynamic response. Results have been obtained from the program BUNVIS-RG (ref. 5) which is based on an exact stiffness formulation that yields accurate results for all modes and eigenvalues without the introduction of nodes beyond those at member intersections. In the BUNVIS-RG analysis, the exact global stiffness matrix  $\mathbf{K}_e$  is a transcendental function of frequency with the result that no separate mass and stiffness matrices occur. However, as shown in references 6 and 7, a modal mass matrix may be obtained as

$$\mathbf{M} = -\frac{d\mathbf{K}_e}{d\omega^2}$$

With this  $\mathbf{M}$ , the modal quantities in equations (6) are exact for all eigenvalues, and additional nodes are not required to achieve accuracy.

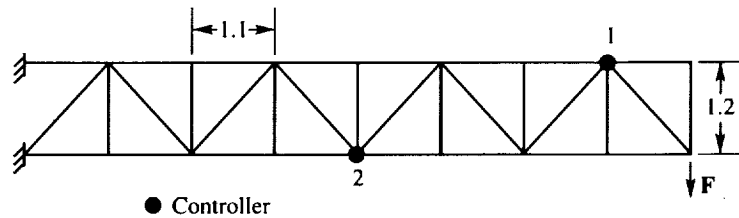


Figure 1. Planar-truss beam. Dimensions are given in meters.

Table I. Beam Member Properties

Case	Axial stiffness, N, for—			Bending stiffness, N-m <sup>2</sup> , for—			Mass of joint, kg	Mass of diagonal, kg/m
	Chord	Batten	Diagonal	Chord	Batten	Diagonal		
1	$2.8 \times 10^7$	$0.84 \times 10^7$	$1.0 \times 10^7$	720	195	120	1.46	0
2	2.8	.84	1.0	720	195	120	.80	.81

## Vibration

The lowest vibration mode for the configuration with only lumped mass is shown in figure 2 and is seen to have the character of a first-beam bending mode. For the configuration with distributed mass in the diagonals, two modes are shown that exhibit both beam bending and local member vibrations. In the same frequency range, seven more modes occur that are almost entirely local in nature. An appreciable local response is expected to occur for any excitation containing a frequency content near these vibration frequencies. Reference 1 shows this local behavior where the harmonic response of these configurations is shown over a large frequency range.

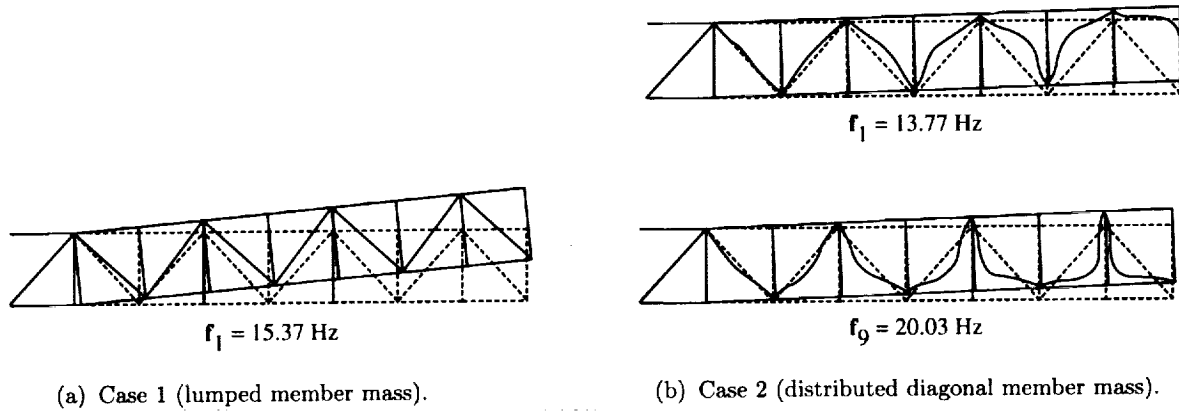


Figure 2. Effect of local member interaction on vibration modes.

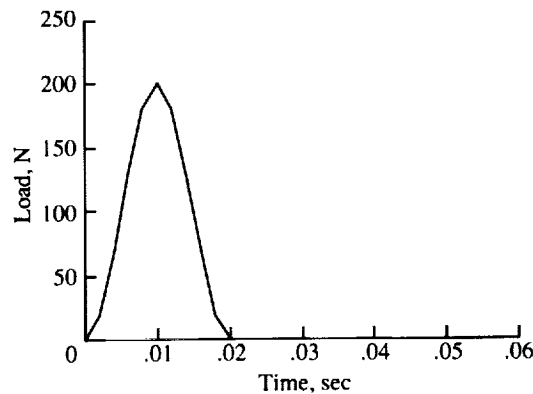


Figure 3. Loading history.

## Closed-Loop Response

The problem considered is the beam of figure 1 with tip load having the time history shown in figure 3. The nonzero portion of the loading is represented by a piecewise linear approximation of the  $(1 - \cos(100\pi t))$  distribution. Ten steps are taken during the time that the load rises to a peak and returns to 0. The two controllers shown in figure 1 were designed to damp the first two beam vibration modes based on the lumped mass model. The controller matrices are shown in table II and are based on the active vibration absorber (AVA) concept of reference 8. The acceleration response at controller location 1 for the disturbance described in figure 3 was determined using 12 modes (frequencies up to 445 Hz) and is shown in figure 4; a rapid decay is evident. The same controller and disturbance applied to the beam with distributed mass in the diagonal members results in the response shown in figure 5. The response was determined using 36 modes in order to capture the effect of the many local modes present. In this case, frequencies up to 150 Hz were present which

correspond to about the fifth mode of the truss having no diagonal mass. The response of the truss having a distributed diagonal mass shows an additional frequency content, higher peak accelerations, and a longer decay time than the response for the truss having no diagonal mass.

Table II. Control Matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ -5905.237 & 0 & -49.26434 & 0 \\ 0 & -94500.35 & 0 & -197.0654 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 14208 & 0 & 118.53 & 0 \\ 0 & 268570 & 0 & 560.06 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

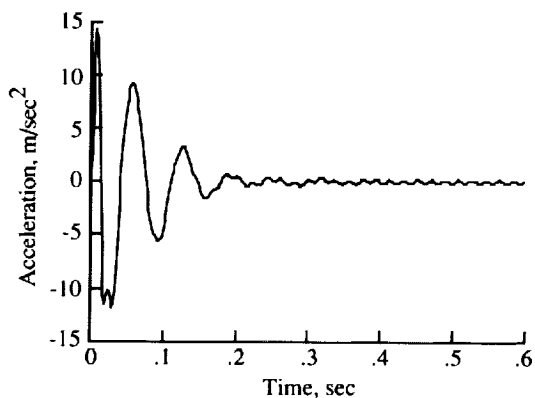


Figure 4. Acceleration response at control location 1 for model having only lumped mass.

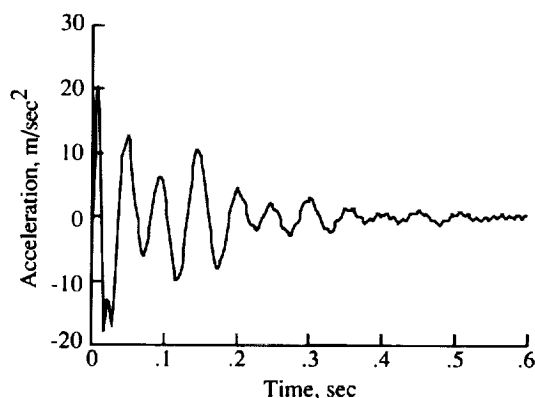


Figure 5. Acceleration response at control location 1 for model having diagonal member mass.

### Accuracy and Efficiency

Accurate results for any time step can be obtained from the nonpartitioned solution (eq. (18)) at the expense of computer time. Identical results can also be obtained by using the present method (eqs. (13)–(16)). The trade-off between accuracy and computer efficiency for these two approaches is shown in figures 6 and 7. In figure 6 the percentage error of the maximum displacement is plotted as a function of the number of terms in the Taylor series. Results are given for two time steps: (1)  $\delta = 0.002$  (solid curve) which was the size of the linear segments of the applied load variation, and (2)  $\delta = 0.001$  (dashed curve) which was half that value. In figure 7 the relative computer time (Central Processing Unit time) for these cases is shown. A greater improvement in accuracy can be obtained with less of a time penalty by increasing the number of terms in the Taylor series rather than by reducing the time step. The relative computer time for the solution done to machine accuracy from equation (18) is shown as a horizontal line at the top of figure 7.

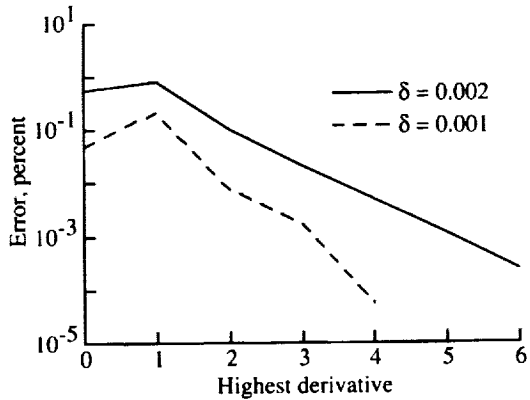


Figure 6. Error as a function of solution parameters.

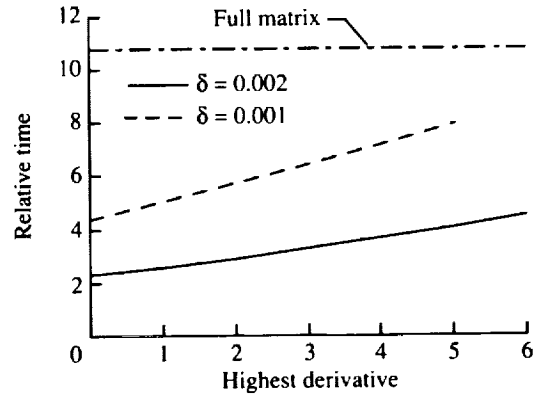


Figure 7. Computer time as a function of solution parameters.

The relative computer time for the two methods is a function of the number of modes used in the solution. Results obtained for the problem described in this paper are shown in figure 8 in which relative computer time is plotted against the number of modes used. The curves are based on results for 12, 36, and 50 modes with best-fit equations developed to extrapolate to the 100 modes shown in the figure. The setup time for equation (18) is a cubic function of the number of modes, and at each time step the computational time is a quadratic function of the number of modes. In contrast, results obtained from equations (13)–(16) which preserve the diagonal nature of the problem are basically a linear function of the number of modes resulting in significant computational savings for large structural models.

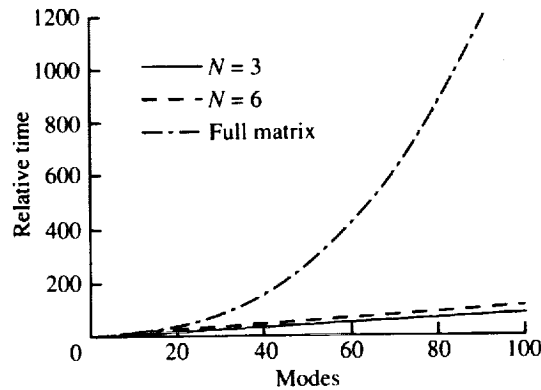


Figure 8. Comparison of computer time for various computational approaches. Number of derivatives used in partitioned approach as denoted by  $N$ .

## Concluding Remarks

A method has been presented for calculating the response of a structure subject to time-varying loads and a general control law that may include the dynamics of the controller. The method uses the natural vibration modes of the structure. By expanding the control forces as a Taylor series, the method achieved accurate results independent of time step size while preserving the uncoupled nature of the classical modal transient response solution. A study of the computer efficiency of the approach showed that the computer time was nearly a linear function of the number of modes used, whereas other commonly used methods involving a full matrix had as much as a cubic variation of computer time with number of modes.

The method was applied to two variations of a planar truss model. One model had only lumped mass, and the other had mass in the diagonal members such that first-beam bending and local member vibration were near the same frequency. Closed-loop simulation results showed that the controller produced good decay of vibrational response for the lumped mass. However, a more complicated response, increased accelerations, and longer decay times were observed for the model with member mass. These results indicate the importance of local modeling and the need to include large numbers of modes in the simulation. The method developed herein is particularly well suited for the simulation of closed-loop structural systems when large numbers of modes must be retained.

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March 24, 1993

## Appendix

### Modified Control Equations

The output measurement  $\mathbf{y}$  can be expressed in terms of  $\mathbf{a}$  by combining equations (3) and (5). Thus,

$$\mathbf{y} = \mathbf{H}_1 \Psi \mathbf{a} + \mathbf{H}_2 \Psi \dot{\mathbf{a}} + \mathbf{H}_3 \Psi \ddot{\mathbf{a}} + \mathbf{H}_4 \mathbf{u} \quad (\text{A1})$$

Eliminating  $\ddot{\mathbf{a}}$  from equation (A1) by using equation (8) gives

$$\mathbf{y} = \mathbf{F}_1 \mathbf{a} + \mathbf{F}_2 \dot{\mathbf{a}} + \mathbf{F}_3 \mathbf{f} + \mathbf{F}_4 \mathbf{u} \quad (\text{A2})$$

where

$$\mathbf{F}_1 = \mathbf{H}_1 \Psi - \mathbf{H}_3 \Psi \mathbf{m}^{-1} \mathbf{k}$$

$$\mathbf{F}_2 = \mathbf{H}_2 \Psi - \mathbf{H}_3 \Psi \mathbf{m}^{-1} \mathbf{g}$$

$$\mathbf{F}_3 = \mathbf{H}_3 \Psi \mathbf{m}^{-1}$$

$$\mathbf{F}_4 = \mathbf{H}_4 + \mathbf{H}_3 \Psi \mathbf{m}^{-1} \Psi^T \mathbf{E}$$

The expression for  $\mathbf{u}$  is

$$\mathbf{u} = \mathbf{C}\mathbf{x} + \mathbf{D}(\mathbf{F}_1 \mathbf{a} + \mathbf{F}_2 \dot{\mathbf{a}} + \mathbf{F}_3 \mathbf{f} + \mathbf{F}_4 \mathbf{u}) \quad (\text{A3})$$

Solving equation (A3) for  $\mathbf{u}$  gives

$$\mathbf{u} = \mathbf{P}_1 \mathbf{a} + \mathbf{P}_2 \dot{\mathbf{a}} + \mathbf{P}_3 \mathbf{f} + \mathbf{P}_4 \mathbf{x} \quad (\text{A4})$$

where

$$\mathbf{P}_J = \mathbf{ZDF}_J \quad (J = 1, 2, 3)$$

$$\mathbf{P}_4 = \mathbf{ZC}$$

$$\mathbf{Z} = (\mathbf{I} - \mathbf{DF}_4)^{-1}$$

By using equations (A2) and (A4), the equation for the controller state can be written as

$$\dot{\mathbf{x}} = \mathbf{E}_1 \mathbf{a} + \mathbf{E}_2 \dot{\mathbf{a}} + \mathbf{E}_3 \mathbf{f} + \mathbf{E}_4 \mathbf{x} \quad (\text{A5})$$

where

$$\mathbf{E}_J = \mathbf{B}(\mathbf{F}_J + \mathbf{F}_4 \mathbf{P}_J) \quad (J = 1, 2, 3)$$

$$\mathbf{E}_4 = \mathbf{A} + \mathbf{BF}_4 \mathbf{P}_4$$



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REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188	
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503				
1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE May 1993	3. REPORT TYPE AND DATES COVERED Technical Paper		
4. TITLE AND SUBTITLE Exact Methods for Modal Transient Response Analysis Including Feedback Control		5. FUNDING NUMBERS WU 590-14-61-01		
6. AUTHOR(S) Melvin S. Anderson and W. Keith Belvin				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) NASA Langley Research Center Hampton, VA 23681-0001		8. PERFORMING ORGANIZATION REPORT NUMBER L-17155		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Washington, DC 20546-0001		10. SPONSORING/MONITORING AGENCY REPORT NUMBER NASA TP-3317		
11. SUPPLEMENTARY NOTES Anderson: Old Dominion University, Norfolk, VA; Belvin: Langley Research Center, Hampton, VA.				
12a. DISTRIBUTION/AVAILABILITY STATEMENT  Unclassified Unlimited  Subject Category 39		12b. DISTRIBUTION CODE		
13. ABSTRACT (Maximum 200 words) This paper presents a modal method for the analysis of controlled structural systems that retains the uncoupled nature of the classical transient response analysis of a structure subjected to a prescribed time-varying load. The control force is expanded as a Taylor series that remains on the right side of the equations, and it does not lead to a computational approach that requires coupling between modes on the left side. Retaining a sufficient number of terms in the series produces a solution to the modal equations that is accurate to machine precision. The approach is particularly attractive for large problems in which standard matrix exponential methods become computationally prohibitive. Numerical results are presented to show the accuracy and efficiency of the proposed approach for dynamic feedback compensation of a truss structure with local member modes in the controller bandwidth.				
14. SUBJECT TERMS Closed-loop simulation; Controlled structures; Nonconservative forces; Series expansion; Controls-structures integration (CSI)			15. NUMBER OF PAGES 14	16. PRICE CODE A03
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT	