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Abstract

A new numerical method is developed for the solution of the two-dimensional, steady Navier-Stokes equations. The method that is presented differs in significant ways from the established numerical methods for solving the Navier-Stokes equations. The major differences are the following: First, the focus of the present method is on satisfying flux conservation in an integral formulation, rather than on simulating conservation laws in their differential form. Second, the present approach provides a unified treatment of the discrete dependent variables and their derivatives. All are treated as unknowns together to be solved for through simulating local and global flux conservation. Third, fluxes are balanced at cell interfaces without the use of interpolation or flux limiters. Fourth, flux conservation is achieved through the use of discrete regions known as conservation elements and solution elements. These elements are not the same as the standard control volumes used in the finite-volume method. Fifth, the discrete approximation obtained on each solution element is a functional solution of both the integral and differential form of the Navier-Stokes equations. Finally, the method that is presented is a highly localized approach in which the coupling to nearby cells is only in one direction for each spatial coordinate, and involves only the immediately adjacent cells.

A general third-order formulation for the steady, compressible Navier-Stokes equations is presented, and then a Newton's method scheme is developed for the solution of incompressible, laminar channel flow. It is shown that the Jacobian matrix is nearly block diagonal if the nonlinear system of discrete equations is arranged appropriately and a proper pivoting strategy is used. Numerical results are presented for Reynolds numbers of 100, 1000, and 2000. Finally, it is shown that the present scheme can resolve the developing channel flow boundary layer using as few as six to ten cells per channel width, depending on the Reynolds number.

A New Flux Conserving Newton's Method Scheme for the Two-Dimensional, Steady Navier-Stokes Equations

I. Introduction

This paper is concerned with the development of a new numerical approach, called the space-time solution element method,* for solving the Navier-Stokes equations. The present work builds on the ideas recently presented by Chang and To[1,2] for the numerical solution of conservation laws, and is part of a larger effort to build from the ground up a new family of flux conserving numerical schemes for solving the unsteady Navier-Stokes equations.

The numerical framework that we are proposing differs in significant ways from the well established traditional methods for solving the Navier-Stokes equations. The major differences are the following: First, our approach is based on satisfying flux conservation in a space-time integral formulation. The focus is not on simulating partial differential equations in their differential form, but rather in satisfying space-time flux conservation of physical quantities. Second, the method we propose is a unified approach, in that it provides a unified treatment of space and time, and a unified treatment of the discrete dependent variables and their derivatives. All are treated as unknowns together to be solved for through simulating local and global flux conservation. Third, fluxes are balanced at cell interfaces without interpolation or the use of flux limiters. This differs fundamentally from a typical finite-volume method.[3] Fourth, local and global flux conservation is achieved through the use of discrete regions in space-time known as conservation elements and solution elements. These elements are not the same as the standard control volumes used in the finite-volume method. Fifth, the numerical approximation obtained on each solution element is a functional solution which satisfies the Navier-Stokes equations in both integral and differential form to some specified order. Finally, our method is a highly localized approach. The coupling to nearby cells is only in one direction for each space-time coordinate (as in a first-order method) and involves only the immediately adjacent cells. This feature holds irrespective of the order of accuracy of a particular numerical scheme based on our general approach.

To better illustrate these distinguishing features, let

$$\vec{\nabla} \cdot \vec{h} = 0 \tag{1.1}$$

be one of the governing equations of fluid mechanics, where the time derivative is part of the divergence operator, and $\vec{h}$ is a space-time flux current density vector. Using Gauss' divergence theorem, equation (1.1) may be recast in an equivalent space-time integral form

$$\oint_{S(V)} \vec{h} \cdot \overrightarrow{ds} = 0. \tag{1.2}$$

*An abbreviation for The Method of Space-Time Conservation Element and Solution Element.

$S(V)$ is the surface inclosing an arbitrary volume V in space-time, and $\overrightarrow{ds}$ is equal to $d\sigma\,\vec{n}$ where $\vec{n}$ is the outward unit normal to $S(V)$ and $d\sigma$ is the area of a surface element of $S(V)$.

Equation (1.2) expresses a physical conservation law, namely, that the total flux out of $S(V)$ of $\vec{h}$ is equal to zero. In general, equation (1.2) implies that there is a physical quantity which is conserved in the space-time volume V. The authors would like to emphasize that, unlike equation (1.1), equation (1.2) remains valid even when the components of $\vec{h}$ become discontinuous (Indeed, equation (1.1) is derived from equation (1.2) assuming that $\vec{h}$ is smooth). This feature, coupled with the fact that equation (1.2) expresses a physical conservation law which is valid for any space-time volume V, suggests that, in general, equation (1.2) is a better starting point for numerical calculations than is equation (1.1). The question then becomes how to best numerically simulate equation (1.2).

The most general and natural way to represent the exact solution $\vec{h}$ by an approximate solution $\vec{\underset{\sim}{h}}$ in some discrete region $\underset{\sim}{V}$ in space-time is through a Taylor series expansion. The discrete analogue to equation (1.2) would then be

$$\oint_{S(\underset{\sim}{V})} \vec{\underset{\sim}{h}} \cdot \overrightarrow{ds} = 0. \tag{1.3}$$

Note that to faithfully represent equation (1.2) in a discrete way, equation (1.3) must be satisfied on every computational cell and every union of computational cells in a mesh which has been used to discretize a flow field. Equation (1.3) is thus the starting point for any numerical algorithm based on the space-time solution element method.

In this paper we are concerned with applying the ideas outlined above to the steady Navier-Stokes equations. Our ultimate aim, as expressed above, is to develop a family of new flux conserving numerical schemes for the unsteady Navier-Stokes equations. To lay the foundation for this effort, the present work will concentrate on the steady equations.

The specific application that we address in this paper is incompressible, laminar channel flow. A flux conserving Newton's method scheme is developed based on the ideas expressed above and applied to the channel flow problem. The numerical method that we present differs in fundamental ways from any other Newton's method scheme that the authors are aware of. Previous work for the Navier-Stokes equations has typically involved a finite-difference formulation. References 4 - 10 are a partial listing of work by other authors.

We begin the development of our scheme in the next section by formulating the steady, compressible Navier-Stokes equations in the form of conservation laws. Following that, we derive the discrete flux conservation equations for a mesh with coinciding rectangular conservation elements and solution elements. We then use the discrete equations to develop an implicit scheme for solving incompressible, laminar channel flow. Finally, we present and discuss numerical results.

II. Conservation Laws for the Navier-Stokes Equations

We consider the two-dimensional, steady, compressible Navier-Stokes equations in dimensionless form. We assume that the ratio of specific heats γ, the viscocity μ, and the coefficient of thermal conductivity κ are all constant, and that the fluid is a perfect gas. Let $Re_L = \frac{\rho_\infty U_\infty L}{\mu}$ denote the Reynolds number, let $Pr = \frac{C_p \mu}{\kappa}$ denote the Prandtl number (where C_p is the specific heat at constant pressure), and let $M_\infty = \frac{U_\infty}{\sqrt{\gamma R T_\infty}}$, denote the free-stream Mach number (where R is a gas constant). The parameters L, U_∞, ρ_∞, and T_∞ refer to some reference length, velocity, density, and temperature, respectively.

Let x and y denote the horizontal and vertical coordinates, respectively, of a two-dimensional Euclidean space E_2. Denoting the fluid density by ρ, the horizontal velocity component by u, the vertical velocity component by v, the static pressure by p, and the temperature by T, the governing continuity, momentum, and energy equations may then be written in Cartesian coordinates as[11]

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{2.1}$$

$$\frac{\partial}{\partial x}(\rho u^2 + p - \tau_{xx}) + \frac{\partial}{\partial y}(\rho u v - \tau_{xy}) = 0 \tag{2.2}$$

$$\frac{\partial}{\partial x}(\rho u v - \tau_{xy}) + \frac{\partial}{\partial y}(\rho v^2 + p - \tau_{yy}) = 0 \tag{2.3}$$

$$\frac{\partial}{\partial x}\left\{\left[\rho\left(e + \frac{u^2+v^2}{2}\right) + p\right] u - u\tau_{xx} - v\tau_{xy} + q_x\right\}$$

$$+ \frac{\partial}{\partial y}\left\{\left[\rho\left(e + \frac{u^2+v^2}{2}\right) + p\right] v - u\tau_{xy} - v\tau_{yy} + q_y\right\} = 0 \tag{2.4}$$

where

$$\tau_{xx} = \frac{2}{3Re_L}\left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \tag{2.5}$$

$$\tau_{xy} = \frac{1}{Re_L}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{2.6}$$

$$\tau_{yy} = \frac{2}{3Re_L}\left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) \tag{2.7}$$

$$e = \frac{1}{M_\infty^2 \gamma(\gamma - 1)} T \tag{2.8}$$

$$q_x = -\frac{1}{(\gamma-1)M_\infty^2 Re_L Pr}\frac{\partial T}{\partial x} \tag{2.9}$$

$$q_y = -\frac{1}{(\gamma-1)M_\infty^2 Re_L Pr}\frac{\partial T}{\partial y} \tag{2.10}$$

By applying Gauss' divergence theorem to equations (2.1) - (2.4), they may be written in integral form as

$$\oint_{S(V)} \vec{h}_M \cdot \vec{ds} = 0 \tag{2.11}$$

$$\oint_{S(V)} \vec{h}_{XM} \cdot \vec{ds} = 0 \tag{2.12}$$

$$\oint_{S(V)} \vec{h}_{YM} \cdot \vec{ds} = 0 \tag{2.13}$$

$$\oint_{S(V)} \vec{h}_E \cdot \vec{ds} = 0 \tag{2.14}$$

where $S(V)$ is the boundary of an arbitrary region V in E_2. The flux current density vectors, $\vec{h}_M$, $\vec{h}_{XM}$, $\vec{h}_{YM}$, and $\vec{h}_E$, corresponding to the continuity, x-momentum, y-momentum, and and energy equations, respectively, are defined by

$$\vec{h}_M \stackrel{def}{=} (\rho u, \rho v) \tag{2.15}$$

$$\vec{h}_{XM} \stackrel{def}{=} (\rho u^2 + p - \tau_{xx}\,,\ \rho u v - \tau_{xy}) \tag{2.16}$$

$$\vec{h}_{YM} \stackrel{def}{=} (\rho u v - \tau_{xy},\ \rho v^2 + p - \tau_{yy}) \tag{2.17}$$

$$\vec{h}_E \stackrel{def}{=} \Big\{\Big[\rho(e + \frac{u^2+v^2}{2}) + p\Big] u - u\tau_{xx} - v\tau_{xy} + q_x\,,$$
$$\Big[\rho(e + \frac{u^2+v^2}{2}) + p\Big] v - u\tau_{xy} - v\tau_{yy} + q_y\Big\} \tag{2.18}$$

Equations (2.11) - (2.14) thus represent physical conservation laws for the conservation of mass, momentum, and energy in an arbitrary region V of E_2.

III. Derivation of the Discrete Flux Conservation Equations

Let E_2 be discretized by a mesh with nonoverlapping rectangular regions. We assume constant spacing Δx in the x direction and constant spacing Δy in the y direction. (See Figure 1.) Each of the rectangular regions in the mesh will be referred to as both a conservation element and a solution element of E_2. A conservation element is a discrete region in E_2 over which the flux conservation constraints (2.11) - (2.14) will be imposed. A solution element is a discrete region in E_2 in which a local Taylor series expansion will be employed to represent the physical solution. In general, they need not refer to the same discrete region. (See [2] for an example where the conservation elements and solution elements do not refer to the same discrete region.) A conservation element will be denoted by $CE(i,j)$ and a solution element by $SE(i,j)$. The boundary of a conservation element will be denoted by $S(CE(i,j))$. We will denote the cell centers of the conservation and solution elements by (x_i, y_j), where the subscript i is an index for the x coordinates, and the subscript j is an index for the y coordinates.

We then assume that the fluid density, u and v velocity, pressure, and temperature can each be represented locally on a solution element by a two-dimensional Taylor series expansion about the cell center (x_i, y_j) as follows:

$$\begin{aligned} \underset{\sim}{\rho}(x,y;i,j) \stackrel{def}{=} & \rho_{2,0}(x-x_i)^2 + \rho_{1,1}(x-x_i)(y-y_j) + \rho_{0,2}(y-y_j)^2 \\ + & \rho_{1,0}(x-x_i) + \rho_{0,1}(y-y_j) + \rho_{0,0} \end{aligned} \tag{3.1}$$

$$\begin{aligned} \underset{\sim}{u}(x,y;i,j) \stackrel{def}{=} & u_{2,0}(x-x_i)^2 + u_{1,1}(x-x_i)(y-y_j) + u_{0,2}(y-y_j)^2 \\ + & u_{1,0}(x-x_i) + u_{0,1}(y-y_j) + u_{0,0} \end{aligned} \tag{3.2}$$

$$\begin{aligned} \underset{\sim}{v}(x,y;i,j) \stackrel{def}{=} & v_{2,0}(x-x_i)^2 + v_{1,1}(x-x_i)(y-y_j) + v_{0,2}(y-y_j)^2 \\ + & v_{1,0}(x-x_i) + v_{0,1}(y-y_j) + v_{0,0} \end{aligned} \tag{3.3}$$

$$\begin{aligned} \underset{\sim}{p}(x,y;i,j) \stackrel{def}{=} & p_{2,0}(x-x_i)^2 + p_{1,1}(x-x_i)(y-y_j) + p_{0,2}(y-y_j)^2 \\ + & p_{1,0}(x-x_i) + p_{0,1}(y-y_j) + p_{0,0} \end{aligned} \tag{3.4}$$

$$\begin{aligned} \underset{\sim}{T}(x,y;i,j) \stackrel{def}{=} & T_{2,0}(x-x_i)^2 + T_{1,1}(x-x_i)(y-y_j) + T_{0,2}(y-y_j)^2 \\ + & T_{1,0}(x-x_i) + T_{0,1}(y-y_j) + T_{0,0} \end{aligned} \tag{3.5}$$

For clarity the i,j subscripts have been omitted from the coefficients in the Taylor series expansions. These coefficients are the unknowns to be solved for.

With this notation, the Taylor series coefficients are related to the derivatives of the discrete dependent variables at the cell center as follows:

$$u_{0,0} = \underset{\sim}{u}$$

$$u_{1,0} = \partial \underset{\sim}{u}/\partial x$$

$$u_{0,1} = \partial \underset{\sim}{u}/\partial y$$

$$u_{2,0} = \frac{1}{2}\,\partial^2 \underset{\sim}{u}/\partial x^2$$

$$u_{0,2} = \frac{1}{2}\,\partial^2 \underset{\sim}{u}/\partial y^2$$

$$u_{1,1} = \partial^2 \underset{\sim}{u}/\partial x \partial y$$

and similarly for $\underset{\sim}{\rho}, \underset{\sim}{v}, \underset{\sim}{p}$, and $\underset{\sim}{T}$.

The discrete analogue to equations (2.11) - (2.14) over an arbitrary conservation element $CE(i,j)$ in E_2 is then given by

$$\oint_{S(CE(i,j))} \vec{\underset{\sim}{h}}_M \cdot \overrightarrow{ds} = 0 \qquad (3.6)$$

$$\oint_{S(CE(i,j))} \vec{\underset{\sim}{h}}_{XM} \cdot \overrightarrow{ds} = 0 \qquad (3.7)$$

$$\oint_{S(CE(i,j))} \vec{\underset{\sim}{h}}_{YM} \cdot \overrightarrow{ds} = 0 \qquad (3.8)$$

$$\oint_{S(CE(i,j))} \vec{\underset{\sim}{h}}_E \cdot \overrightarrow{ds} = 0 \qquad (3.9)$$

where

$$\vec{\underset{\sim}{h}}_M \stackrel{def}{=} (\underset{\sim}{\rho}\,\underset{\sim}{u}, \underset{\sim}{\rho}\,\underset{\sim}{v}) \qquad (3.10)$$

$$\vec{\underset{\sim}{h}}_{XM} \stackrel{def}{=} (\underset{\sim}{\rho}\,\underset{\sim}{u}^2 + \underset{\sim}{p} - \underset{\sim}{\tau}_{xx}\,,\; \underset{\sim}{\rho}\,\underset{\sim}{u}\,\underset{\sim}{v} - \underset{\sim}{\tau}_{xy}) \qquad (3.11)$$

$$\vec{\underset{\sim}{h}}_{YM} \stackrel{def}{=} (\underset{\sim}{\rho}\,\underset{\sim}{u}\,\underset{\sim}{v} \;-\; \underset{\sim}{\tau}_{xy},\; \underset{\sim}{\rho}\,\underset{\sim}{v}^2 + \underset{\sim}{p} \;-\; \underset{\sim}{\tau}_{yy}) \tag{3.12}$$

$$\begin{aligned}\vec{\underset{\sim}{h}}_{E} \stackrel{def}{=} \Big\{ &\Big[\underset{\sim}{\rho}\left(\underset{\sim}{e} + (\underset{\sim}{u}^2 + \underset{\sim}{v}^2)/2\right) \;+\; \underset{\sim}{p}\Big]\,\underset{\sim}{u} \;-\; \underset{\sim}{u}\,\underset{\sim}{\tau}_{xx} \;-\; \underset{\sim}{v}\,\underset{\sim}{\tau}_{xy} \;+\; \underset{\sim}{q}_x\,,\\ &\Big[\underset{\sim}{\rho}\left(\underset{\sim}{e} + (\underset{\sim}{u}^2 + \underset{\sim}{v}^2)/2\right) \;+\; \underset{\sim}{p}\Big]\,\underset{\sim}{v} \;-\; \underset{\sim}{u}\,\underset{\sim}{\tau}_{xy} \;-\; \underset{\sim}{v}\,\underset{\sim}{\tau}_{yy} \;+\; \underset{\sim}{q}_y \Big\}\end{aligned} \tag{3.13}$$

and

$$\underset{\sim}{\tau}_{xx} \;=\; \frac{2}{3Re_L}(2\,\partial\underset{\sim}{u}/\partial x \;-\; \partial\underset{\sim}{v}/\partial y) \tag{3.14}$$

$$\underset{\sim}{\tau}_{xy} \;=\; \frac{1}{Re_L}(\partial\underset{\sim}{u}/\partial y \;+\; \partial\underset{\sim}{v}/\partial x) \tag{3.15}$$

$$\underset{\sim}{\tau}_{yy} \;=\; \frac{2}{3Re_L}(2\,\partial\underset{\sim}{v}/\partial y \;-\; \partial\underset{\sim}{u}/\partial x) \tag{3.16}$$

$$\underset{\sim}{e} \;=\; \frac{1}{M_\infty^2\gamma(\gamma-1)}\,\underset{\sim}{T} \tag{3.17}$$

$$\underset{\sim}{q}_x \;=\; -\,\frac{1}{(\gamma-1)M_\infty^2 Re_L Pr}\,\partial\underset{\sim}{T}/\partial x \tag{3.18}$$

$$\underset{\sim}{q}_y \;=\; -\,\frac{1}{(\gamma-1)M_\infty^2 Re_L Pr}\,\partial\underset{\sim}{T}/\partial y \tag{3.19}$$

Since the discrete flux current density vectors $\vec{\underset{\sim}{h}}_M$, $\vec{\underset{\sim}{h}}_{XM}$, $\vec{\underset{\sim}{h}}_{YM}$, and $\vec{\underset{\sim}{h}}_E$ are in general discontinuous across the boundary of a solution element, the integrations in equations (3.6) - (3.9) are understood to be carried out on the interior of the conservation element but immediately adjacent to the boundary S.

We are now ready to proceed with the evaluation of each of the integrals in equations (3.6) - (3.9). To conserve space, we will only go through the details for equation (3.6). For equations (3.7) - (3.9) we simply give the final results.

Since each $S(CE(i,j))$ is a simple closed curve in E_2, the surface integration required in equation (3.6) can be converted into a line integration form [1, p.14]. With $\overrightarrow{ds} \;=\; d\sigma\,\vec{n}$, where $\vec{n}$ is the outward unit normal to $S(CE(i,j))$ and $d\sigma$ is the length of a surface element in E_2, we have

$$\overrightarrow{ds} \stackrel{def}{=} dy\,\vec{i} - dx\,\vec{j}, \tag{3.20}$$

and

$$\vec{h}_M \cdot \overrightarrow{ds} = -\,\rho\,\underset{\sim}{v}\,dx + \rho\,\underset{\sim}{u}\,dy = \vec{g}_M \cdot \vec{dr} \tag{3.21}$$

where

$$\vec{g}_M \overset{def}{=} (-\,\rho\,\underset{\sim}{v}, \rho\,\underset{\sim}{u}) \tag{3.22}$$

and

$$\vec{dr} \overset{def}{=} dx\,\vec{i} + dy\,\vec{j}. \tag{3.23}$$

The line integration is taken to be positive in the counterclockwise sense. If we denote the vertices of an arbitrary conservation element $CE(i,j)$ by P, Q, R, and S as shown in Figure 2, we have

$$\oint_{S(CE(i,j))} \vec{h}_M \cdot \overrightarrow{ds} = \oint_{PQRS_{i,j}} \vec{g}_M \cdot \vec{dr}$$
$$\overset{def}{=} \left[J(\overline{PQ})_M + J(\overline{QR})_M + J(\overline{RS})_M + J(\overline{SP})_M \right]_{i,j} \tag{3.24}$$

where $[J(\overline{PQ})_M]_{i,j}$ denotes the flux of $\vec{h}_M$ through the line segment $\overline{PQ}_{i,j}$, and similarly for $J(\overline{QR})_M$, $J(\overline{RS})_M$, and $J(\overline{SP})_M$. We then have (omitting i,j subscripts)

$$\begin{aligned} J(\overline{PQ})_M &\overset{def}{=} \int_P^Q -\,\rho\,\underset{\sim}{v}\,dx + \rho\,\underset{\sim}{u}\,dy \\ &= \int_{x_i+\frac{\Delta x}{2}}^{x_i-\frac{\Delta x}{2}} -\,\rho\,\underset{\sim}{v}\,dx + \rho\,\underset{\sim}{u}\,dy \\ &= \int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} \rho\,\underset{\sim}{v}\,dx \qquad \text{with} \quad y = y_j + \frac{\Delta y}{2} \end{aligned} \tag{3.25}$$

$$\begin{aligned} J(\overline{QR})_M &\overset{def}{=} \int_Q^R -\,\rho\,\underset{\sim}{v}\,dx + \rho\,\underset{\sim}{u}\,dy \\ &= \int_{y_j+\frac{\Delta y}{2}}^{y_j-\frac{\Delta y}{2}} -\,\rho\,\underset{\sim}{v}\,dx + \rho\,\underset{\sim}{u}\,dy \\ &= -\int_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}} \rho\,\underset{\sim}{u}\,dy \qquad \text{with} \quad x = x_i - \frac{\Delta x}{2} \end{aligned} \tag{3.26}$$

$$
\begin{aligned}
J(\overline{RS})_M &\stackrel{def}{=} \int_R^S -\underset{\sim}{\rho}\,\underset{\sim}{v}\,dx + \underset{\sim}{\rho}\,\underset{\sim}{u}\,dy \\
&= \int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} -\underset{\sim}{\rho}\,\underset{\sim}{v}\,dx + \underset{\sim}{\rho}\,\underset{\sim}{u}\,dy \qquad (3.27)\\
&= -\int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} \underset{\sim}{\rho}\,\underset{\sim}{v}\,dx \qquad \text{with} \quad y = y_j - \frac{\Delta y}{2}
\end{aligned}
$$

$$
\begin{aligned}
J(\overline{SP})_M &\stackrel{def}{=} \int_S^P -\underset{\sim}{\rho}\,\underset{\sim}{v}\,dx + \underset{\sim}{\rho}\,\underset{\sim}{u}\,dy \\
&= \int_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}} -\underset{\sim}{\rho}\,\underset{\sim}{v}\,dx + \underset{\sim}{\rho}\,\underset{\sim}{u}\,dy \qquad (3.28)\\
&= \int_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}} \underset{\sim}{\rho}\,\underset{\sim}{u}\,dy \qquad \text{with} \quad x = x_i + \frac{\Delta x}{2}
\end{aligned}
$$

where

$$
\begin{aligned}
\underset{\sim}{\rho}\,\underset{\sim}{u} = {} & \rho_{0,0}\,u_{0,0} + (\rho_{1,0}\,u_{0,0} + \rho_{0,0}\,u_{1,0})\,(x - x_i) + (\rho_{0,1}\,u_{0,0} + \rho_{0,0}\,u_{0,1})\,(y - y_j) \\
& + (\rho_{0,0}\,u_{2,0} + \rho_{2,0}\,u_{0,0} + \rho_{1,0}\,u_{1,0})\,(x - x_i)^2 \qquad (3.29)\\
& + (\rho_{0,0}\,u_{1,1} + \rho_{1,1}\,u_{0,0} + \rho_{0,1}\,u_{1,0} + \rho_{1,0}\,u_{0,1})\,(x - x_i)(y - y_j) \\
& + (\rho_{0,2}\,u_{0,0} + \rho_{0,0}\,u_{0,2} + \rho_{0,1}\,u_{0,1})\,(y - y_j)^2 + \text{H. O. T.}
\end{aligned}
$$

$$
\begin{aligned}
\underset{\sim}{\rho}\,\underset{\sim}{v} = {} & \rho_{0,0}\,v_{0,0} + (\rho_{1,0}\,v_{0,0} + \rho_{0,0}\,v_{1,0})\,(x - x_i) + (\rho_{0,1}\,v_{0,0} + \rho_{0,0}\,v_{0,1})\,(y - y_j) \\
& + (\rho_{0,0}\,v_{2,0} + \rho_{2,0}\,v_{0,0} + \rho_{1,0}\,v_{1,0})\,(x - x_i)^2 \qquad (3.30)\\
& + (\rho_{0,0}\,v_{1,1} + \rho_{1,1}\,v_{0,0} + \rho_{0,1}\,v_{1,0} + \rho_{1,0}\,v_{0,1})\,(x - x_i)(y - y_j) \\
& + (\rho_{0,2}\,v_{0,0} + \rho_{0,0}\,v_{0,2} + \rho_{0,1}\,v_{0,1})\,(y - y_j)^2 + \text{H. O. T.}
\end{aligned}
$$

All polynomial terms in equations (3.29) and (3.30) that are higher than second order have been represented through the abbreviation "H. O. T.". These terms will be neglected throughout the remainder of the paper to be consistent with the second-order expansions (3.1) - (3.5) (i.e., $\vec{\underset{\sim}{h}}_M$ is understood to be a second-order Taylor series expansion, and similarly for $\vec{\underset{\sim}{h}}_{XM}, \vec{\underset{\sim}{h}}_{YM}$, and $\vec{\underset{\sim}{h}}_E$).

Carrying out the line integrations in equations (3.25) - (3.28), we obtain

$$J(\overline{PQ})_M = \frac{\Delta x^3}{12}(\rho_{0,0}\,v_{2,0} + \rho_{2,0}\,v_{0,0} + \rho_{1,0}\,v_{1,0}) \tag{3.31}$$

$$+ \Delta x \left[\frac{\Delta y^2}{4}(\rho_{0,0}\,v_{0,2} + \rho_{0,2}\,v_{0,0} + \rho_{0,1}\,v_{0,1}) + \frac{\Delta y}{2}(\rho_{0,1}\,v_{0,0} + \rho_{0,0}\,v_{0,1}) + \rho_{0,0}\,v_{0,0}\right]$$

$$J(\overline{QR})_M = -\frac{\Delta y^3}{12}(\rho_{0,0}\,u_{0,2} + \rho_{0,2}\,u_{0,0} + \rho_{0,1}\,u_{0,1}) \tag{3.32}$$

$$- \Delta y \left[\frac{\Delta x^2}{4}(\rho_{0,0}\,u_{2,0} + \rho_{2,0}\,u_{0,0} + \rho_{1,0}\,u_{1,0}) - \frac{\Delta x}{2}(\rho_{1,0}\,u_{0,0} + \rho_{0,0}\,u_{1,0}) + \rho_{0,0}\,u_{0,0}\right]$$

$$J(\overline{RS})_M = -\frac{\Delta x^3}{12}(\rho_{0,0}\,v_{2,0} + \rho_{2,0}\,v_{0,0} + \rho_{1,0}\,v_{1,0}) \tag{3.33}$$

$$- \Delta x \left[\frac{\Delta y^2}{4}(\rho_{0,0}\,v_{0,2} + \rho_{0,2}\,v_{0,0} + \rho_{0,1}\,v_{0,1}) - \frac{\Delta y}{2}(\rho_{0,1}\,v_{0,0} + \rho_{0,0}\,v_{0,1}) + \rho_{0,0}\,v_{0,0}\right]$$

$$J(\overline{SP})_M = \frac{\Delta y^3}{12}(\rho_{0,0}\,u_{0,2} + \rho_{0,2}\,u_{0,0} + \rho_{0,1}\,u_{0,1}) \tag{3.34}$$

$$+ \Delta y \left[\frac{\Delta x^2}{4}(\rho_{0,0}\,u_{2,0} + \rho_{2,0}\,u_{0,0} + \rho_{1,0}\,u_{1,0}) + \frac{\Delta x}{2}(\rho_{1,0}\,u_{0,0} + \rho_{0,0}\,u_{1,0}) + \rho_{0,0}\,u_{0,0}\right]$$

For clarity, all i, j subscripts have been omitted in equations (3.31) - (3.34). By virtue of equations (3.6) and (3.24) we require that

$$J(\overline{PQ})_M + J(\overline{QR})_M + J(\overline{RS})_M + J(\overline{SP})_M \equiv 0 \tag{3.35}$$

which simply says that the total flux of $\vec{\underset{\sim}{h}}_M$ out of $S(CE(i,j))$ is equal to zero. Applying this condition to equations (3.31) - (3.34), we obtain the *mass flux conservation constraint*

$$\rho_{1,0}\,u_{0,0} + \rho_{0,0}\,u_{1,0} + \rho_{0,1}\,v_{0,0} + \rho_{0,0}\,v_{0,1} \equiv 0. \tag{3.36}$$

Imposing this condition, we obtain from equations (3.31) - (3.34) the following expressions for the normalized mass flux across the boundaries of the solution element $SE(i,j)$:

$$\frac{J(\overline{PQ})_M}{\Delta x} = \frac{\Delta x^2}{12}(\rho_{0,0}\, v_{2,0} + \rho_{2,0}\, v_{0,0} + \rho_{1,0}\, v_{1,0}) \tag{3.37}$$

$$+ \frac{\Delta y^2}{4}(\rho_{0,0}\, v_{0,2} + \rho_{0,2}\, v_{0,0} + \rho_{0,1}\, v_{0,1}) - \frac{\Delta y}{2}(\rho_{1,0}\, u_{0,0} + \rho_{0,0}\, u_{1,0}) + \rho_{0,0}\, v_{0,0}$$

$$-\frac{J(\overline{QR})_M}{\Delta y} = \frac{\Delta y^2}{12}(\rho_{0,0}\, u_{0,2} + \rho_{0,2}\, u_{0,0} + \rho_{0,1}\, u_{0,1}) \tag{3.38}$$

$$+ \frac{\Delta x^2}{4}(\rho_{0,0}\, u_{2,0} + \rho_{2,0}\, u_{0,0} + \rho_{1,0}\, u_{1,0}) - \frac{\Delta x}{2}(\rho_{1,0}\, u_{0,0} + \rho_{0,0}\, u_{1,0}) + \rho_{0,0}\, u_{0,0}$$

$$-\frac{J(\overline{RS})_M}{\Delta x} = \frac{\Delta x^2}{12}(\rho_{0,0}\, v_{2,0} + \rho_{2,0}\, v_{0,0} + \rho_{1,0}\, v_{1,0}) \tag{3.39}$$

$$+ \frac{\Delta y^2}{4}(\rho_{0,0}\, v_{0,2} + \rho_{0,2}\, v_{0,0} + \rho_{0,1}\, v_{0,1}) + \frac{\Delta y}{2}(\rho_{1,0}\, u_{0,0} + \rho_{0,0}\, u_{1,0}) + \rho_{0,0}\, v_{0,0}$$

$$\frac{J(\overline{SP})_M}{\Delta y} = \frac{\Delta y^2}{12}(\rho_{0,0}\, u_{0,2} + \rho_{0,2}\, u_{0,0} + \rho_{0,1}\, u_{0,1}) \tag{3.40}$$

$$+ \frac{\Delta x^2}{4}(\rho_{0,0}\, u_{2,0} + \rho_{2,0}\, u_{0,0} + \rho_{1,0}\, u_{1,0}) + \frac{\Delta x}{2}(\rho_{1,0}\, u_{0,0} + \rho_{0,0}\, u_{1,0}) + \rho_{0,0}\, u_{0,0}$$

Note that with these expressions for $J(\overline{PQ})_M$, $J(\overline{QR})_M$, $J(\overline{RS})_M$, and $J(\overline{SP})_M$, condition (3.35) is satisfied exactly.

The mathematical procedure required to obtain the discrete flux conservation equations corresponding to equations (3.7) - (3.9) is exactly analogous to that given above for equation (3.6).

Corresponding to equation (3.7), we obtain the *x-momentum flux conservation constraint*,

$$\rho_{0,0}\,(u_{1,0}\, u_{0,0} + u_{0,1}\, v_{0,0}) + p_{1,0} - \frac{1}{Re_L}[(2\, u_{0,2} + v_{1,1}) + \frac{2}{3}(4\, u_{2,0} - v_{1,1})] \equiv 0 \tag{3.41}$$

and the following expressions for the normalized flux of $\vec{\underset{\sim}{h}}_{XM}$ across the boundaries of $SE(i,j)$:

$$\frac{J(\overline{PQ})_{XM}}{\Delta x} = \tag{3.42}$$

$$\frac{\Delta x^2}{12}\Big[u_{2,0}\,\rho_{0,0}\,v_{0,0} + u_{1,0}\,(\rho_{1,0}\,v_{0,0}+\rho_{0,0}\,v_{1,0})\Big] + \frac{\Delta y^2}{4}\Big[u_{0,2}\,\rho_{0,0}\,v_{0,0} + u_{0,1}\,(\rho_{0,1}\,v_{0,0}+\rho_{0,0}\,v_{0,1})\Big]$$
$$+\frac{\Delta y}{2}\Big[\rho_{0,0}\,v_{0,0}\,u_{0,1} - \frac{1}{Re_L}(2\,u_{0,2}+v_{1,1})\Big] - \frac{1}{Re_L}(u_{0,1}+v_{1,0}) + u_{0,0}\,\frac{J(\overline{PQ})_M}{\Delta x}$$

$$-\frac{J(\overline{QR})_{XM}}{\Delta y} = \tag{3.43}$$

$$\frac{\Delta y^2}{12}\Big[u_{0,2}\,\rho_{0,0}\,u_{0,0} + u_{0,1}\,(\rho_{0,1}\,u_{0,0}+\rho_{0,0}\,u_{0,1}) + p_{0,2}\Big]$$
$$+\frac{\Delta x^2}{4}\Big[u_{2,0}\,\rho_{0,0}\,u_{0,0} + u_{1,0}\,(\rho_{1,0}\,u_{0,0}+\rho_{0,0}\,u_{1,0}) + p_{2,0}\Big]$$
$$+\frac{\Delta x}{2}\Big[\rho_{0,0}\,v_{0,0}\,u_{0,1} - \frac{1}{Re_L}(2\,u_{0,2}+v_{1,1})\Big] + p_{0,0} - \frac{2}{3Re_L}(2\,u_{1,0}-v_{0,1}) - u_{0,0}\,\frac{J(\overline{QR})_M}{\Delta y}$$

$$-\frac{J(\overline{RS})_{XM}}{\Delta x} = \tag{3.44}$$

$$\frac{\Delta x^2}{12}\Big[u_{2,0}\,\rho_{0,0}\,v_{0,0} + u_{1,0}\,(\rho_{1,0}\,v_{0,0}+\rho_{0,0}\,v_{1,0})\Big] + \frac{\Delta y^2}{4}\Big[u_{0,2}\,\rho_{0,0}\,v_{0,0} + u_{0,1}\,(\rho_{0,1}\,v_{0,0}+\rho_{0,0}\,v_{0,1})\Big]$$
$$-\frac{\Delta y}{2}\Big[\rho_{0,0}\,v_{0,0}\,u_{0,1} - \frac{1}{Re_L}(2\,u_{0,2}+v_{1,1})\Big] - \frac{1}{Re_L}(u_{0,1}+v_{1,0}) - u_{0,0}\,\frac{J(\overline{RS})_M}{\Delta x}$$

$$\frac{J(\overline{SP})_{XM}}{\Delta y} = \tag{3.45}$$

$$\frac{\Delta y^2}{12}\Big[u_{0,2}\,\rho_{0,0}\,u_{0,0} \;+\; u_{0,1}\,(\rho_{0,1}\,u_{0,0} + \rho_{0,0}\,u_{0,1}) \;+\; p_{0,2}\Big]$$

$$+\;\frac{\Delta x^2}{4}\Big[u_{2,0}\,\rho_{0,0}\,u_{0,0} \;+\; u_{1,0}\,(\rho_{1,0}\,u_{0,0} + \rho_{0,0}\,u_{1,0}) \;+\; p_{2,0}\Big]$$

$$-\;\frac{\Delta x}{2}\Big[\rho_{0,0}\,v_{0,0}\,u_{0,1} \;-\; \frac{1}{Re_L}(2\,u_{0,2} + v_{1,1})\Big] \;+\; p_{0,0} \;-\; \frac{2}{3Re_L}(2\,u_{1,0} - v_{0,1}) \;+\; u_{0,0}\,\frac{J(\overline{SP})_M}{\Delta y}$$

The normalized mass flux expressions (3.37) - (3.40) have been used to simplify equations (3.42) - (3.45).

Corresponding to equation (3.8), we obtain the *y-momentum flux conservation constraint*,

$$\rho_{0,0}\,(v_{1,0}\,u_{0,0} + v_{0,1}\,v_{0,0}) \;+\; p_{0,1} \;-\; \frac{1}{Re_L}[(2\,v_{2,0} + u_{1,1}) \;+\; \frac{2}{3}\,(4\,v_{0,2} - u_{1,1})] \;\equiv\; 0 \tag{3.46}$$

and the following expressions for the normalized flux of $\vec{h}_{YM}$ across the boundaries of $SE(i,j)$:

$$\frac{J(\overline{PQ})_{YM}}{\Delta x} = \tag{3.47}$$

$$\frac{\Delta x^2}{12}\Big[v_{2,0}\,\rho_{0,0}\,v_{0,0} \;+\; v_{1,0}\,(\rho_{1,0}\,v_{0,0} + \rho_{0,0}\,v_{1,0}) + \;p_{2,0}\Big]$$

$$+\;\frac{\Delta y^2}{4}\Big[v_{0,2}\,\rho_{0,0}\,v_{0,0} \;+\; v_{0,1}\,(\rho_{0,1}\,v_{0,0} + \rho_{0,0}\,v_{0,1}) + \;p_{0,2}\Big]$$

$$-\;\frac{\Delta y}{2}\Big[\rho_{0,0}\,u_{0,0}\,v_{1,0} - \;\frac{1}{Re_L}(2\,v_{2,0} + u_{1,1})\Big] \;+\; p_{0,0} \;-\; \frac{2}{3Re_L}(2\,v_{0,1} - u_{1,0}) \;+\; v_{0,0}\,\frac{J(\overline{PQ})_M}{\Delta x}$$

$$-\;\frac{J(\overline{QR})_{YM}}{\Delta y} = \tag{3.48}$$

$$\frac{\Delta y^2}{12}\Big[v_{0,2}\,\rho_{0,0}\,u_{0,0} \;+\; v_{0,1}\,(\rho_{0,0}\,u_{0,1} + \rho_{0,1}\,u_{0,0})\Big] \;+\; \frac{\Delta x^2}{4}\Big[v_{2,0}\,\rho_{0,0}\,u_{0,0} \;+\; v_{1,0}\,(\rho_{0,0}\,u_{1,0} + \rho_{1,0}\,u_{0,0})\Big]$$

$$-\;\frac{\Delta x}{2}\Big[\rho_{0,0}\,u_{0,0}\,v_{1,0} \;-\; \frac{1}{Re_L}(2\,v_{2,0} + u_{1,1})\Big] \;-\; \frac{1}{Re_L}(u_{0,1} + v_{1,0}) \;-\; v_{0,0}\,\frac{J(\overline{QR})_M}{\Delta y}$$

$$-\frac{J(\overline{RS})_{YM}}{\Delta x} = \tag{3.49}$$

$$\frac{\Delta x^2}{12}\Big[v_{2,0}\,\rho_{0,0}\,v_{0,0} \;+\; v_{1,0}\,(\rho_{1,0}\,v_{0,0}+\rho_{0,0}\,v_{1,0}) \;+\; p_{2,0}\Big]$$

$$+\;\frac{\Delta y^2}{4}\Big[v_{0,2}\,\rho_{0,0}\,v_{0,0} \;+\; v_{0,1}\,(\rho_{0,1}\,v_{0,0}+\rho_{0,0}\,v_{0,1}) + \;p_{0,2}\Big]$$

$$+\;\frac{\Delta y}{2}\Big[\rho_{0,0}\,u_{0,0}\,v_{1,0} - \frac{1}{Re_L}(2\,v_{2,0}+u_{1,1})\Big] \;+\; p_{0,0} \;-\; \frac{2}{3Re_L}(2\,v_{0,1}-u_{1,0}) \;-\; v_{0,0}\,\frac{J(\overline{RS})_M}{\Delta x}$$

$$\frac{J(\overline{SP})_{YM}}{\Delta y} = \tag{3.50}$$

$$\frac{\Delta y^2}{12}\Big[v_{0,2}\,\rho_{0,0}\,u_{0,0} \;+\; v_{0,1}\,(\rho_{0,0}\,u_{0,1}+\rho_{0,1}\,u_{0,0})\Big] \;+\; \frac{\Delta x^2}{4}\Big[v_{2,0}\,\rho_{0,0}\,u_{0,0} \;+\; v_{1,0}\,(\rho_{0,0}\,u_{1,0}+\rho_{1,0}\,u_{0,0})\Big]$$

$$+\;\frac{\Delta x}{2}\Big[\rho_{0,0}\,u_{0,0}\,v_{1,0} \;-\; \frac{1}{Re_L}(2\,v_{2,0}+u_{1,1})\Big] \;-\; \frac{1}{Re_L}(u_{0,1}+v_{1,0}) \;\;+\; v_{0,0}\,\frac{J(\overline{SP})_M}{\Delta y}$$

Corresponding to equation (3.9), we obtain the *energy flux conservation constraint,*

$$p_{0,0}\,(u_{1,0}+v_{0,1}) \;+\; \frac{1}{M_\infty^2\gamma(\gamma-1)}\,\rho_{0,0}\,(T_{1,0}\,u_{0,0}+T_{0,1}\,v_{0,0})$$

$$-\;\frac{1}{Re_L}\Big[v_{1,0}\,(v_{1,0}+u_{0,1}) \;+\; u_{0,1}\,(v_{1,0}+u_{0,1}) \;+\; \tfrac{2}{3}[u_{1,0}\,(2\,u_{1,0}-v_{0,1}) \;+\; v_{0,1}\,(2\,v_{0,1}-u_{1,0})]\Big]$$

$$-\;\frac{2}{(\gamma-1)M_\infty^2 Re_L Pr}\,(T_{2,0}+T_{0,2}) \;\equiv\; 0 \tag{3.51}$$

and the following expressions for the normalized flux of $\vec{\underset{\sim}{h}}_E$ across the boundaries of $SE(i,j)$:

$$\frac{J(\overline{PQ})_E}{\Delta x} = \tag{3.52}$$

$$\frac{\Delta x^2}{12}\Big\{\rho_{0,0}\,v_{0,0}\,(\frac{u_{1,0}^2+v_{1,0}^2}{2}+u_{0,0}\,u_{2,0}+v_{0,0}\,v_{2,0}) \;+\; (\rho_{1,0}\,v_{0,0}+\rho_{0,0}\,v_{1,0})\,(u_{1,0}\,u_{0,0}+v_{1,0}\,v_{0,0})$$

$$+\;\frac{1}{M_\infty^2\gamma(\gamma-1)}\,[T_{1,0}\,\rho_{0,0}\,v_{1,0} \;+\; v_{0,0}(T_{2,0}\,\rho_{0,0}+T_{1,0}\,\rho_{1,0})] \;+\; (v_{2,0}\,p_{0,0}+v_{1,0}\,p_{1,0}+v_{0,0}\,p_{2,0})$$

$$-\;\frac{2}{3Re_L}[v_{2,0}\,(2\,v_{0,1}-u_{1,0}) \;+\; 2\,v_{1,0}\,(v_{1,1}-u_{2,0})] \;-\; \frac{1}{Re_L}[u_{2,0}\,(v_{1,0}+u_{0,1}) \;+\; u_{1,0}\,(2\,v_{2,0}+u_{1,1})]\Big\}$$

$$+\;\frac{\Delta y^2}{4}\Big\{\rho_{0,0}\,v_{0,0}\,(\frac{u_{0,1}^2+v_{0,1}^2}{2}+u_{0,0}\,u_{0,2}+v_{0,0}\,v_{0,2}) \;+\; (\rho_{0,1}\,v_{0,0}+\rho_{0,0}\,v_{0,1})\,(u_{0,0}\,u_{0,1}+v_{0,0}\,v_{0,1})$$

$$+\;\frac{1}{M_\infty^2\gamma(\gamma-1)}\,[T_{0,2}\,\rho_{0,0}\,v_{0,0} \;+\; T_{0,1}\,(\rho_{0,1}\,v_{0,0}+\rho_{0,0}\,v_{0,1})] \;+\; (v_{0,2}\,p_{0,0}+v_{0,1}\,p_{0,1}+v_{0,0}\,p_{0,2})$$

$$-\;\frac{1}{Re_L}[u_{0,2}\,(v_{1,0}+u_{0,1}) \;+\; u_{0,1}\,(2\,u_{0,2}+v_{1,1})] \;-\; \frac{2}{3Re_L}[v_{0,2}\,(2\,v_{0,1}-u_{1,0}) \;+\; v_{0,1}\,(4\,v_{0,2}-u_{1,1})]\Big\}$$

$$-\;\frac{\Delta y}{2}\,\Big\{\rho_{0,0}\,u_{0,0}\,(u_{1,0}\,u_{0,0}+v_{1,0}\,v_{0,0}) \;+\; (u_{1,0}\,p_{0,0}+p_{1,0}\,u_{0,0}) \;+\; \frac{1}{M_\infty^2\gamma(\gamma-1)}\,T_{1,0}\,\rho_{0,0}\,u_{0,0}$$

$$-\;\frac{2}{3Re_L}\Big[u_{1,0}\,(2\,u_{1,0}-v_{0,1}) \;+\; u_{0,0}\,(4\,u_{2,0}-v_{1,1})\Big]$$

$$-\;\frac{1}{Re_L}\Big[v_{1,0}\,(v_{1,0}+u_{0,1}) \;+\; v_{0,0}\,(2\,v_{2,0}+u_{1,1})\Big] \;-\; \frac{2}{(\gamma-1)M_\infty^2 Re_L Pr}T_{2,0}\Big\}$$

$$-\;\frac{1}{Re_L}\Big[u_{0,0}\,(v_{1,0}+u_{0,1}) \;+\; \tfrac{2}{3}v_{0,0}\,(2\,v_{0,1}-u_{1,0})\Big] \;+\; v_{0,0}\,p_{0,0} \;-\; \frac{1}{(\gamma-1)M_\infty^2 Re_L Pr}T_{0,1}$$

$$+\;\Big(\frac{u_{0,0}^2+v_{0,0}^2}{2} \;+\; \frac{1}{M_\infty^2\gamma(\gamma-1)}\,T_{0,0}\Big)\,\frac{J(\overline{PQ})_M}{\Delta x}$$

$$-\frac{J(\overline{QR})_E}{\Delta y} \quad = \tag{3.53}$$

$$\frac{\Delta y^2}{12}\Big\{\rho_{0,0}\,u_{0,0}\,(\frac{u_{0,1}^2+v_{0,1}^2}{2}+u_{0,0}\,u_{0,2}+v_{0,0}\,v_{0,2}) \;+\; (\rho_{0,1}\,u_{0,0}+\rho_{0,0}\,u_{0,1})(u_{0,1}\,u_{0,0}+v_{0,1}\,v_{0,0})$$

$$+\;\frac{1}{M_\infty^2\gamma(\gamma-1)}\,[T_{0,1}\,\rho_{0,0}\,u_{0,1} \;+\; u_{0,0}(T_{0,2}\,\rho_{0,0}+T_{0,1}\,\rho_{0,1})] \;+\; (u_{0,2}\,p_{0,0}+u_{0,1}\,p_{0,1}+u_{0,0}\,p_{0,2})$$

$$-\;\frac{2}{3Re_L}[u_{0,2}\,(2\,u_{1,0}-v_{0,1}) \;+\; 2\,u_{0,1}\,(u_{1,1}-v_{0,2})] \;-\; \frac{1}{Re_L}[v_{0,2}\,(v_{1,0}+u_{0,1}) \;+\; v_{0,1}\,(2\,u_{0,2}+v_{1,1})]\Big\}$$

$$+\;\frac{\Delta x^2}{4}\Big\{\rho_{0,0}\,u_{0,0}\,(\frac{u_{1,0}^2+v_{1,0}^2}{2}+u_{0,0}\,u_{2,0}+v_{0,0}\,v_{2,0}) \;+\; (\rho_{1,0}\,u_{0,0}+\rho_{0,0}\,u_{1,0})(u_{0,0}\,u_{1,0}+v_{0,0}\,v_{1,0})$$

$$+\;\frac{1}{M_\infty^2\gamma(\gamma-1)}\,[T_{2,0}\,\rho_{0,0}\,u_{0,0} \;+\; T_{1,0}\,(\rho_{1,0}\,u_{0,0}+\rho_{0,0}\,u_{1,0})] \;+\; (u_{2,0}\,p_{0,0}+u_{1,0}\,p_{1,0}+u_{0,0}\,p_{2,0})$$

$$-\;\frac{2}{3Re_L}[u_{2,0}\,(2\,u_{1,0}-v_{0,1}) \;+\; u_{1,0}\,(4\,u_{2,0}-v_{1,1})] \;-\; \frac{1}{Re_L}[v_{2,0}\,(v_{1,0}+u_{0,1}) \;+\; v_{1,0}\,(2\,v_{2,0}+u_{1,1})]\Big\}$$

$$-\;\frac{\Delta x}{2}\,\Big\{\rho_{0,0}\,u_{0,0}\,(u_{1,0}\,u_{0,0}+v_{1,0}\,v_{0,0}) \;+\; (u_{1,0}\,p_{0,0}+p_{1,0}\,u_{0,0}) \;+\; \frac{1}{M_\infty^2\gamma(\gamma-1)}\,T_{1,0}\,\rho_{0,0}\,u_{0,0}$$

$$-\;\frac{2}{3Re_L}\Big[u_{1,0}\,(2\,u_{1,0}-v_{0,1}) \;+\; u_{0,0}\,(4\,u_{2,0}-v_{1,1})\Big]$$

$$-\;\frac{1}{Re_L}\Big[v_{1,0}\,(v_{1,0}+u_{0,1}) \;+\; v_{0,0}\,(2\,v_{2,0}+u_{1,1})\Big] \;-\; \frac{2}{(\gamma-1)M_\infty^2 Re_L Pr}T_{2,0}\Big\}$$

$$-\;\frac{1}{Re_L}\Big[v_{0,0}\,(v_{1,0}+u_{0,1}) \;+\; \tfrac{2}{3}u_{0,0}\,(2\,u_{1,0}-v_{0,1})\Big] \;+\; u_{0,0}\,p_{0,0} \;-\; \frac{1}{(\gamma-1)M_\infty^2 Re_L Pr}T_{1,0}$$

$$-\;\Big(\frac{u_{0,0}^2+v_{0,0}^2}{2} \;+\; \frac{1}{M_\infty^2\gamma(\gamma-1)}\,T_{0,0}\Big)\,\frac{J(\overline{QR})_M}{\Delta y}$$

$$-\frac{J(\overline{RS})_E}{\Delta x} = \tag{3.54}$$

$$\frac{\Delta x^2}{12}\Big\{\rho_{0,0}\,v_{0,0}\,(\frac{u_{1,0}^2+v_{1,0}^2}{2}+u_{0,0}\,u_{2,0}+v_{0,0}\,v_{2,0})\;+\;(\rho_{1,0}\,v_{0,0}+\rho_{0,0}\,v_{1,0})\,(u_{1,0}\,u_{0,0}+v_{1,0}\,v_{0,0})$$

$$+\;\frac{1}{M_\infty^2\gamma(\gamma-1)}\,[T_{1,0}\,\rho_{0,0}\,v_{1,0}\;+\;v_{0,0}(T_{2,0}\,\rho_{0,0}+T_{1,0}\,\rho_{1,0})]\;+\;(v_{2,0}\,p_{0,0}+v_{1,0}\,p_{1,0}+v_{0,0}\,p_{2,0})$$

$$-\;\frac{2}{3Re_L}[v_{2,0}\,(2\,v_{0,1}-u_{1,0})\;+\;2\,v_{1,0}\,(v_{1,1}-u_{2,0})]\;-\;\frac{1}{Re_L}[u_{2,0}\,(v_{1,0}+u_{0,1})\;+\;u_{1,0}\,(2\,v_{2,0}+u_{1,1})]\Big\}$$

$$+\;\frac{\Delta y^2}{4}\Big\{\rho_{0,0}\,v_{0,0}\,(\frac{u_{0,1}^2+v_{0,1}^2}{2}+u_{0,0}\,u_{0,2}+v_{0,0}\,v_{0,2})\;+\;(\rho_{0,1}\,v_{0,0}+\rho_{0,0}\,v_{0,1})\,(u_{0,0}\,u_{0,1}+v_{0,0}\,v_{0,1})$$

$$+\;\frac{1}{M_\infty^2\gamma(\gamma-1)}\,[T_{0,2}\,\rho_{0,0}\,v_{0,0}\;+\;T_{0,1}\,(\rho_{0,1}\,v_{0,0}+\rho_{0,0}\,v_{0,1})]\;+\;(v_{0,2}\,p_{0,0}+v_{0,1}\,p_{0,1}+v_{0,0}\,p_{0,2})$$

$$-\;\frac{1}{Re_L}[u_{0,2}\,(v_{1,0}+u_{0,1})\;+\;u_{0,1}\,(2\,u_{0,2}+v_{1,1})]\;-\;\frac{2}{3Re_L}[v_{0,2}\,(2\,v_{0,1}-u_{1,0})\;+\;v_{0,1}\,(4\,v_{0,2}-u_{1,1})]\Big\}$$

$$+\;\frac{\Delta y}{2}\Big\{\rho_{0,0}\,u_{0,0}\,(u_{1,0}\,u_{0,0}+v_{1,0}\,v_{0,0})\;+\;(u_{1,0}\,p_{0,0}+p_{1,0}\,u_{0,0})\;+\;\frac{1}{M_\infty^2\gamma(\gamma-1)}\,T_{1,0}\,\rho_{0,0}\,u_{0,0}$$

$$-\;\frac{2}{3Re_L}\Big[u_{1,0}\,(2\,u_{1,0}-v_{0,1})\;+\;u_{0,0}\,(4\,u_{2,0}-v_{1,1})\Big]$$

$$-\;\frac{1}{Re_L}\Big[v_{1,0}\,(v_{1,0}+u_{0,1})\;+\;v_{0,0}\,(2\,v_{2,0}+u_{1,1})\Big]\;-\;\frac{2}{(\gamma-1)M_\infty^2 Re_L Pr}T_{2,0}\Big\}$$

$$-\;\frac{1}{Re_L}\Big[u_{0,0}\,(v_{1,0}+u_{0,1})\;+\;\tfrac{2}{3}v_{0,0}\,(2\,v_{0,1}-u_{1,0})\Big]\;+\;v_{0,0}\,p_{0,0}\;-\;\frac{1}{(\gamma-1)M_\infty^2 Re_L Pr}T_{0,1}$$

$$-\;\Big(\frac{u_{0,0}^2+v_{0,0}^2}{2}\;+\;\frac{1}{M_\infty^2\gamma(\gamma-1)}\,T_{0,0}\Big)\,\frac{J(\overline{RS})_M}{\Delta x}$$

$$\frac{J(\overline{SP})_E}{\Delta y} \quad = \tag{3.55}$$

$$\frac{\Delta y^2}{12}\Big\{\rho_{0,0}\,u_{0,0}\,(\frac{u_{0,1}^2+v_{0,1}^2}{2}+u_{0,0}\,u_{0,2}+v_{0,0}\,v_{0,2}) \;+\; (\rho_{0,1}\,u_{0,0}+\rho_{0,0}\,u_{0,1})(u_{0,1}\,u_{0,0}+v_{0,1}\,v_{0,0})$$

$$+\frac{1}{M_\infty^2\gamma(\gamma-1)}[T_{0,1}\,\rho_{0,0}\,u_{0,1} \;+\; u_{0,0}(T_{0,2}\,\rho_{0,0}+T_{0,1}\,\rho_{0,1})] \;+\; (u_{0,2}\,p_{0,0}+u_{0,1}\,p_{0,1}+u_{0,0}\,p_{0,2})$$

$$-\frac{2}{3Re_L}[u_{0,2}\,(2\,u_{1,0}-v_{0,1}) \;+\; 2\,u_{0,1}\,(u_{1,1}-v_{0,2})] \;-\; \frac{1}{Re_L}[v_{0,2}\,(v_{1,0}+u_{0,1}) \;+\; v_{0,1}\,(2\,u_{0,2}+v_{1,1})]\Big\}$$

$$+\frac{\Delta x^2}{4}\Big\{\rho_{0,0}\,u_{0,0}\,(\frac{u_{1,0}^2+v_{1,0}^2}{2}+u_{0,0}\,u_{2,0}+v_{0,0}\,v_{2,0}) \;+\; (\rho_{1,0}\,u_{0,0}+\rho_{0,0}\,u_{1,0})(u_{0,0}\,u_{1,0}+v_{0,0}\,v_{1,0})$$

$$+\frac{1}{M_\infty^2\gamma(\gamma-1)}[T_{2,0}\,\rho_{0,0}\,u_{0,0} \;+\; T_{1,0}\,(\rho_{1,0}\,u_{0,0}+\rho_{0,0}\,u_{1,0})] \;+\; (u_{2,0}\,p_{0,0}+u_{1,0}\,p_{1,0}+u_{0,0}\,p_{2,0})$$

$$-\frac{2}{3Re_L}[u_{2,0}\,(2\,u_{1,0}-v_{0,1}) \;+\; u_{1,0}\,(4\,u_{2,0}-v_{1,1})] \;-\; \frac{1}{Re_L}[v_{2,0}\,(v_{1,0}+u_{0,1}) \;+\; v_{1,0}\,(2\,v_{2,0}+u_{1,1})]\Big\}$$

$$+\frac{\Delta x}{2}\Big\{\rho_{0,0}\,u_{0,0}\,(u_{1,0}\,u_{0,0}+v_{1,0}\,v_{0,0}) \;+\; (u_{1,0}\,p_{0,0}+p_{1,0}\,u_{0,0}) \;+\; \frac{1}{M_\infty^2\gamma(\gamma-1)}T_{1,0}\,\rho_{0,0}\,u_{0,0}$$

$$-\frac{2}{3Re_L}\Big[u_{1,0}\,(2\,u_{1,0}-v_{0,1}) \;+\; u_{0,0}\,(4\,u_{2,0}-v_{1,1})\Big]$$

$$-\frac{1}{Re_L}\Big[v_{1,0}\,(v_{1,0}+u_{0,1}) \;+\; v_{0,0}\,(2\,v_{2,0}+u_{1,1})\Big] \;-\; \frac{2}{(\gamma-1)M_\infty^2 Re_L Pr}T_{2,0}\Big\}$$

$$-\frac{1}{Re_L}\Big[v_{0,0}\,(v_{1,0}+u_{0,1}) \;+\; \tfrac{2}{3}u_{0,0}\,(2\,u_{1,0}-v_{0,1})\Big] \;+\; u_{0,0}\,p_{0,0} \;-\; \frac{1}{(\gamma-1)M_\infty^2 Re_L Pr}T_{1,0}$$

$$+\Big(\frac{u_{0,0}^2+v_{0,0}^2}{2} \;+\; \frac{1}{M_\infty^2\gamma(\gamma-1)}T_{0,0}\Big)\,\frac{J(\overline{SP})_M}{\Delta y}$$

With these expressions for the flux of $\vec{h}_{XM}$, $\vec{h}_{YM}$, and $\vec{h}_E$ across a cell boundary, the following conditions are satisfied on every solution element.

$$J(\overline{PQ})_{XM} \;+\; J(\overline{QR})_{XM} \;+\; J(\overline{RS})_{XM} \;+\; J(\overline{SP})_{XM} \;\equiv\; 0 \tag{3.56}$$

$$J(\overline{PQ})_{YM} \;+\; J(\overline{QR})_{YM} \;+\; J(\overline{RS})_{YM} \;+\; J(\overline{SP})_{YM} \;\equiv\; 0 \tag{3.57}$$

$$J(\overline{PQ})_E \;+\; J(\overline{QR})_E \;+\; J(\overline{RS})_E \;+\; J(\overline{SP})_E \;\equiv\; 0 \tag{3.58}$$

Before concluding this section, one final comment is required. For compressible flows, it will be necessary to assume an equation of state in order to close the system of equations (2.11) - (2.14) and their discrete counterpart (3.6) - (3.9). Since we have assumed that the fluid is a perfect gas, we may take (using dimensional quantities)

$$p^* = \rho^* R T^* \tag{3.59}$$

and its discrete analogue

$$\underset{\sim}{p}^* = \underset{\sim}{\rho}^* R \underset{\sim}{T}^*. \tag{3.60}$$

In dimensionless form, we then have

$$\underset{\sim}{p} = \frac{1}{\gamma M_\infty^2} \underset{\sim}{\rho} \underset{\sim}{T} =$$

$$\begin{aligned}
\frac{1}{\gamma M_\infty^2} \Big[& \rho_{0,0} T_{0,0} + (\rho_{1,0} T_{0,0} + \rho_{0,0} T_{1,0})(x - x_i) + (\rho_{0,1} T_{0,0} + \rho_{0,0} T_{0,1})(y - y_j) \\
& + (\rho_{0,0} T_{2,0} + \rho_{2,0} T_{0,0} + \rho_{1,0} T_{1,0})(x - x_i)^2 \\
& + (\rho_{0,0} T_{1,1} + \rho_{1,1} T_{0,0} + \rho_{0,1} T_{1,0} + \rho_{1,0} T_{0,1})(x - x_i)(y - y_j) \\
& + (\rho_{0,2} T_{0,0} + \rho_{0,0} T_{0,2} + \rho_{0,1} T_{0,1})(y - y_j)^2 \Big]
\end{aligned} \tag{3.61}$$

so that

$$p_{0,0} \stackrel{def}{=} \frac{1}{\gamma M_\infty^2} \rho_{0,0} T_{0,0} \tag{3.62}$$

$$p_{0,1} \stackrel{def}{=} \frac{1}{\gamma M_\infty^2} (\rho_{0,1} T_{0,0} + \rho_{0,0} T_{0,1}) \tag{3.63}$$

$$p_{1,0} \stackrel{def}{=} \frac{1}{\gamma M_\infty^2} (\rho_{1,0} T_{0,0} + \rho_{0,0} T_{1,0}) \tag{3.64}$$

$$p_{0,2} \stackrel{def}{=} \frac{1}{\gamma M_\infty^2} (\rho_{0,2} T_{0,0} + \rho_{0,0} T_{0,2} + \rho_{0,1} T_{0,1}) \tag{3.65}$$

$$p_{1,1} \stackrel{def}{=} \frac{1}{\gamma M_\infty^2} (\rho_{0,0} T_{1,1} + \rho_{1,1} T_{0,0} + \rho_{0,1} T_{1,0} + \rho_{1,0} T_{0,1}) \tag{3.66}$$

$$p_{2,0} \stackrel{def}{=} \frac{1}{\gamma M_\infty^2} (\rho_{0,0} T_{2,0} + \rho_{2,0} T_{0,0} + \rho_{1,0} T_{1,0}). \tag{3.67}$$

IV. Application to Incompressible Channel Flows

Our formulation thus far has been for the compressible Navier-Stokes equations. Application to the incompressible equations can be done very simply as a special case.

When the density is constant, the gradient terms $\rho_{0,1}, \rho_{1,0}, \rho_{0,2}, \rho_{1,1}$, and $\rho_{2,0}$ in the Taylor series expansion (3.1) are all zero, and the constant term $\rho_{0,0}$ may be set equal to one. Equations (3.31) - (3.58) then apply to incompressible flow, and equations (3.59) - (3.67) are no longer applicable.

Consider the channel geometry and mesh shown in Figure 3. On each solution element $SE(i,j)$, there are 4 unknown discrete variables, $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{p}$, and $\underset{\sim}{T}$, each with 6 unknown coefficients, for a total of 24 unknowns per cell. However, the mixed coefficient terms $p_{1,1}$ and $T_{1,1}$ do not appear in any of the discrete equations (3.37) - (3.55). We will assume that $p_{1,1} = T_{1,1} = 0$ as a result. In addition, the energy equation decouples from the continuity and momentum equations, so that the 17 unknowns for the two components of velocity and pressure may be solved for independently of the five remaining unknowns for the temperature.

Let N_j denote the number of solution elements from the lower wall to the upper wall, and let N_i denote the number of solution elements in the downstream direction. Then the total number of unknowns for the velocity and pressure is $17N_iN_j$. We thus require $17N_iN_j$ conditions to close the system.

The flux conservation constraints (3.36), (3.41), and (3.46), which ensure local flux conservation on each cell, provide the starting point of the present formulation. For incompressible flow, these become

$$\left[u_{1,0} + v_{0,1}\right]_{i,j} = 0 \tag{4.1}$$

$$\left[u_{1,0}\,u_{0,0} + u_{0,1}\,v_{0,0} + p_{1,0} - \frac{1}{Re_L}[(2\,u_{0,2} + v_{1,1}) + \frac{2}{3}(4\,u_{2,0} - v_{1,1})]\right]_{i,j} = 0 \tag{4.2}$$

$$\left[v_{1,0}\,u_{0,0} + v_{0,1}\,v_{0,0} + p_{0,1} - \frac{1}{Re_L}[(2\,v_{2,0} + u_{1,1}) + \frac{2}{3}(4\,v_{0,2} - u_{1,1})]\right]_{i,j} = 0 \tag{4.3}$$

for $j = 1, ..., N_j$ and $i = 1, ..., N_i$, for a total of $3N_iN_j$ conditions.

To ensure global flux conservation, mass and momentum fluxes must be balanced at cell interfaces. Consider the vertical interface between $SE(i,j)$ and $SE(i+1,j)$, as shown in Figure 3. Using the cell orientation $PQRS$ previously introduced, the mass flux leaving $SE(i,j)$ through the vertical interface is given by $[J(\overline{SP})_M]_{i,j}$. On the other hand, the mass flux leaving $SE(i+1,j)$ through this same interface is given by $[J(\overline{QR})_M]_{i+1,j}$ (and similarly for the momentum fluxes). Conservation of mass (and momentum) requires that the sum of the two be zero. Balancing fluxes across each vertical interface in the mesh, we thus require that

$$\left[\frac{J(\overline{SP})_M}{\Delta y}\right]_{i,j} + \left[\frac{J(\overline{QR})_M}{\Delta y}\right]_{i+1,j} = 0 \tag{4.4}$$

$$\left[\frac{J(\overline{SP})_{XM}}{\Delta y}\right]_{i,j} + \left[\frac{J(\overline{QR})_{XM}}{\Delta y}\right]_{i+1,j} = 0 \tag{4.5}$$

$$\left[\frac{J(\overline{SP})_{YM}}{\Delta y}\right]_{i,j} + \left[\frac{J(\overline{QR})_{YM}}{\Delta y}\right]_{i+1,j} = 0 \tag{4.6}$$

for $j = 1, ..., N_j$ and $i = 1, ..., N_i - 1$, for a total of $3N_j(N_i - 1) = 3N_iN_j - 3N_j$ conditions. Similarly, the fluxes through the horizontal interface between $SE(i,j)$ and $SE(i,j+1)$ must also balance. We thus require that

$$\left[\frac{J(\overline{PQ})_M}{\Delta x}\right]_{i,j} + \left[\frac{J(\overline{RS})_M}{\Delta x}\right]_{i,j+1} = 0 \tag{4.7}$$

$$\left[\frac{J(\overline{PQ})_{XM}}{\Delta x}\right]_{i,j} + \left[\frac{J(\overline{RS})_{XM}}{\Delta x}\right]_{i,j+1} = 0 \tag{4.8}$$

$$\left[\frac{J(\overline{PQ})_{YM}}{\Delta x}\right]_{i,j} + \left[\frac{J(\overline{RS})_{YM}}{\Delta x}\right]_{i,j+1} = 0 \tag{4.9}$$

for $j = 1, ..., N_j - 1$ and $i = 1, ..., N_i$, for a total of $3N_i(N_j - 1) = 3N_iN_j - 3N_i$ conditions. Equations (4.1) - (4.9) thus represent a total of $9N_iN_j - 3N_i - 3N_j$ conditions.

In addition to the above requirements, there are discrete boundary conditions that must be satisfied. For each solution element along a wall boundary, we must require as a minimum that there be no mass flux through the wall. This can be accomplished by setting the integrated mass flux at the wall to zero, or by setting the v velocity component identically to zero on the wall boundary. The second approach requires three conditions, whereas the first approach requires only one. In this paper, we use the first approach. Note that the zero mass flux condition takes the place of a wall boundary condition for the v velocity component. For the u velocity, we simply set $\underset{\sim}{u} = 0$ at the midpoint of the wall face of each cell. This leads to the following $4N_i$ conditions:

$$\left[\frac{\Delta y^2}{4} u_{0,2} - \frac{\Delta y}{2} u_{0,1} + u_{0,0}\right]_{i,1} = 0 \tag{4.10}$$

$$\left[\frac{\Delta x^2}{12} v_{2,0} + \frac{\Delta y^2}{4} v_{0,2} - \frac{\Delta y}{2} v_{0,1} + v_{0,0}\right]_{i,1} = 0 \tag{4.11}$$

$$\left[\frac{\Delta y^2}{4} u_{0,2} + \frac{\Delta y}{2} u_{0,1} + u_{0,0}\right]_{i,N_j} = 0 \tag{4.12}$$

$$\left[\frac{\Delta x^2}{12} v_{2,0} + \frac{\Delta y^2}{4} v_{0,2} + \frac{\Delta y}{2} v_{0,1} + v_{0,0}\right]_{i,N_j} = 0 \tag{4.13}$$

Boundary conditions must also be specified at the inlet and exit. At the upstream boundary, we specify plug flow inlet conditions:

$$\left[\frac{\Delta x^2}{4} u_{2,0} - \frac{\Delta x}{2} u_{1,0} + u_{0,0}\right]_{1,j} = [u_i]_j \tag{4.14}$$

$$\left[\frac{\Delta x^2}{4} v_{2,0} - \frac{\Delta x}{2} v_{1,0} + v_{0,0}\right]_{1,j} = 0 \tag{4.15}$$

Downstream, we specify the pressure:

$$\left[p_{0,0}\right]_{N_i,j} = p_e. \tag{4.16}$$

The total number of conditions is then increased to $9\,N_iN_j + N_i$.

Equations (4.1) - (4.16) are the minimal conditions that must be satisfied and are the starting point of our numerical formulation. Note that conditions (4.1) - (4.9) are integral relationships, and that these conditions ensure that fluxes of mass and momentum are conserved on every cell and every union of cells in the mesh. Having ensured that these fundamental requirements are satisfied, we may now turn to the differential form of the discrete equations to introduce additional conditions to close the system. We may thus require that

$$\vec{\nabla} \cdot (\underset{\sim}{u}, \underset{\sim}{v}) = \vec{\nabla} \cdot \vec{\underset{\sim}{h}}_M \equiv 0 \tag{4.17}$$

$$\vec{\nabla} \cdot (\underset{\sim}{u}^2 + \underset{\sim}{p} - \underset{\sim}{\tau}_{xx}, \underset{\sim}{u}\,\underset{\sim}{v} - \underset{\sim}{\tau}_{xy}) = \vec{\nabla} \cdot \vec{\underset{\sim}{h}}_{XM} \equiv 0 \tag{4.18}$$

$$\vec{\nabla} \cdot (\underset{\sim}{u}\,\underset{\sim}{v} - \underset{\sim}{\tau}_{xy}, \underset{\sim}{v}^2 + \underset{\sim}{p} - \underset{\sim}{\tau}_{yy}) = \vec{\nabla} \cdot \vec{\underset{\sim}{h}}_{YM} \equiv 0 \tag{4.19}$$

on each solution element. (These relationships are already satisfied at the cell center by virtue of the flux conservation constraints (4.1) - (4.3), but only to first order throughout the rest of the solution element.) This leads to the following $6N_iN_j$ conditions:

$$\Big[2\,u_{2,0} \;+\; v_{1,1}\Big]_{i,j} \;=\; 0 \tag{4.20}$$

$$\Big[2\,v_{0,2} \;+\; u_{1,1}\Big]_{i,j} \;=\; 0 \tag{4.21}$$

$$\Big[2\,(2\,u_{0,0}\,u_{2,0} \;+\; u_{1,0}^2 \;+\; p_{2,0}) \;+\; u_{1,1}\,v_{0,0} \;+\; v_{1,1}\,u_{0,0} \;+\; u_{1,0}\,v_{0,1} \;+\; u_{0,1}\,v_{1,0}\Big]_{i,j} \;=\; 0 \tag{4.22}$$

$$\Big[u_{0,0}\,v_{0,2} \;+\; v_{0,0}\,u_{0,2} \;+\; v_{0,1}\,u_{0,1} \;+\; u_{1,1}\,u_{0,0} \;+\; u_{1,0}\,u_{0,1}\Big]_{i,j} \;=\; 0 \tag{4.23}$$

$$\Big[2\,(2\,v_{0,0}\,v_{0,2} \;+\; v_{0,1}^2 \;+\; p_{0,2}) \;+\; u_{1,1}\,v_{0,0} \;+\; v_{1,1}\,u_{0,0} \;+\; u_{1,0}\,v_{0,1} \;+\; u_{0,1}\,v_{1,0}\Big]_{i,j} \;=\; 0 \tag{4.24}$$

$$\Big[u_{0,0}\,v_{2,0} \;+\; v_{0,0}\,u_{2,0} \;+\; v_{1,0}\,u_{1,0} \;+\; v_{1,1}\,v_{0,0} \;+\; v_{1,0}\,v_{0,1}\Big]_{i,j} \;=\; 0 \tag{4.25}$$

The total number of conditions is then increased to $15\,N_iN_j \;+\; N_i$. If we assume that $p_{2,0} \;=\; 0$ in view of the fact that channel flow is dominated by gradients in the cross-flow direction, then the total number of conditions required is reduced to $16\,N_iN_j$. We thus need $N_iN_j - N_i \;=\; N_i\,(N_j - 1)$ conditions to close the system. Since $N_i\,(N_j - 1)$ is the number of horizontal cell interfaces in the mesh, this suggests imposing a condition at each horizontal cell interface. An obvious choice would be to require that the u velocity component be continuous in the cross-stream direction. We then have as our final condition

$$\left[\frac{\Delta y^2}{4}\,u_{0,2} \;-\; \frac{\Delta y}{2}\,u_{0,1} \;+\; u_{0,0}\right]_{i,j+1} \;=\; \left[\frac{\Delta y^2}{4}\,u_{0,2} \;+\; \frac{\Delta y}{2}\,u_{0,1} \;+\; u_{0,0}\right]_{i,j} \tag{4.26}$$

for $j = 1, ..., N_j - 1$ and $i = 1, ..., N_i$.

We now summarize the present implicit scheme. Using (4.1), (4.20), and (4.21) to eliminate $v_{0,1}$, $v_{1,1}$, and $u_{1,1}$, and dropping the $p_{2,0}$ term, there are 13 unknowns per cell, and the Taylor series expansions become

$$\begin{aligned} \underset{\sim}{u}(x,y;i,j) \;\stackrel{def}{=}\;& u_{2,0}(x-x_i)^2 - 2\,v_{0,2}(x-x_i)(y-y_j) + u_{0,2}(y-y_j)^2 \\ &+\; u_{1,0}(x-x_i) + u_{0,1}(y-y_j) + u_{0,0} \end{aligned} \tag{4.27}$$

$$\begin{aligned} \underset{\sim}{v}(x,y;i,j) \;\stackrel{def}{=}\;& v_{2,0}(x-x_i)^2 - 2\,u_{2,0}(x-x_i)(y-y_j) + v_{0,2}(y-y_j)^2 \\ &+\; v_{1,0}(x-x_i) - u_{1,0}(y-y_j) + v_{0,0} \end{aligned} \tag{4.28}$$

$$\underset{\sim}{p}(x,y;i,j) \;\stackrel{def}{=}\; p_{0,2}(y-y_j)^2 \;+\; p_{1,0}(x-x_i) + p_{0,1}(y-y_j) + p_{0,0} \tag{4.29}$$

Equations (4.1) - (4.3), which are the flux conservation constraints corresponding to equations (3.6) - (3.8), become

$$\left[u_{1,0} + v_{0,1}\right]_{i,j} = 0 \tag{4.30}$$

$$\left[u_{1,0}\,u_{0,0} + u_{0,1}\,v_{0,0} + p_{1,0} - \frac{2}{Re_L}(u_{2,0} + u_{0,2})\right]_{i,j} = 0 \tag{4.31}$$

$$\left[v_{1,0}\,u_{0,0} - u_{1,0}\,v_{0,0} + p_{0,1} - \frac{2}{Re_L}(v_{2,0} + v_{0,2})\right]_{i,j} = 0 \tag{4.32}$$

Equations (4.4) - (4.6), which are streamwise interface conditions, are given by

$$\begin{aligned} \left[\frac{J(\overline{SP})_M}{\Delta y}\right]_{i,j} &+ \left[\frac{J(\overline{QR})_M}{\Delta y}\right]_{i+1,j} = \\ \left[\frac{\Delta y^2}{12}u_{0,2} + \frac{\Delta x^2}{4}u_{2,0} + \frac{\Delta x}{2}u_{1,0} + u_{0,0}\right]_{i,j} & \\ - \left[\frac{\Delta y^2}{12}u_{0,2} + \frac{\Delta x^2}{4}u_{2,0} - \frac{\Delta x}{2}u_{1,0} + u_{0,0}\right]_{i+1,j} &= 0 \end{aligned} \tag{4.33}$$

$$\begin{aligned} &\left[\frac{J(\overline{SP})_{XM}}{\Delta y}\right]_{i,j} + \left[\frac{J(\overline{QR})_{XM}}{\Delta y}\right]_{i+1,j} = \\ &\Bigg[\frac{\Delta y^2}{12}(2\,u_{0,2}\,u_{0,0} + u_{0,1}^2 + p_{0,2}) + \frac{\Delta x^2}{4}(2\,u_{2,0}\,u_{0,0} + u_{1,0}^2) \\ &- \frac{\Delta x}{2}[v_{0,0}\,u_{0,1} - u_{0,0}\,u_{1,0} - \frac{2}{Re_L}(u_{0,2} - u_{2,0})] + p_{0,0} - \frac{2}{Re_L}u_{1,0} + u_{0,0}^2\Bigg]_{i,j} \\ &- \Bigg[\frac{\Delta y^2}{12}(2\,u_{0,2}\,u_{0,0} + u_{0,1}^2 + p_{0,2}) + \frac{\Delta x^2}{4}(2\,u_{2,0}\,u_{0,0} + u_{1,0}^2) \\ &+ \frac{\Delta x}{2}[v_{0,0}\,u_{0,1} - u_{0,0}\,u_{1,0} - \frac{2}{Re_L}(u_{0,2} - u_{2,0})] + p_{0,0} - \frac{2}{Re_L}u_{1,0} + u_{0,0}^2\Bigg]_{i+1,j} \\ &= 0 \end{aligned} \tag{4.34}$$

$$\left[\frac{J(\overline{SP})_{YM}}{\Delta y}\right]_{i,j} + \left[\frac{J(\overline{QR})_{YM}}{\Delta y}\right]_{i+1,j} = \tag{4.35}$$

$$\left[\frac{\Delta y^2}{12}(v_{0,2}\,u_{0,0} + u_{0,2}\,v_{0,0} - u_{1,0}\,u_{0,1}) + \frac{\Delta x^2}{4}(v_{2,0}\,u_{0,0} + u_{2,0}\,v_{0,0} + v_{1,0}\,u_{1,0})\right.$$

$$\left. + \frac{\Delta x}{2}[u_{0,0}\,v_{1,0} + v_{0,0}\,u_{1,0} - \frac{2}{Re_L}(v_{2,0} - v_{0,2})] - \frac{1}{Re_L}(u_{0,1} + v_{1,0}) + v_{0,0}\,u_{0,0}\right]_{i,j}$$

$$- \left[\frac{\Delta y^2}{12}(v_{0,2}\,u_{0,0} + u_{0,2}\,v_{0,0} - u_{1,0}\,u_{0,1}) + \frac{\Delta x^2}{4}(v_{2,0}\,u_{0,0} + u_{2,0}\,v_{0,0} + v_{1,0}\,u_{1,0})\right.$$

$$\left. - \frac{\Delta x}{2}[u_{0,0}\,v_{1,0} + v_{0,0}\,u_{1,0} - \frac{2}{Re_L}(v_{2,0} - v_{0,2})] - \frac{1}{Re_L}(u_{0,1} + v_{1,0}) + v_{0,0}\,u_{0,0}\right]_{i+1,j} = 0$$

Equations (4.7) - (4.9) and (4.26), which are cross-stream interface conditions, become

$$\left[\frac{J(\overline{PQ})_{M}}{\Delta x}\right]_{i,j} + \left[\frac{J(\overline{RS})_{M}}{\Delta x}\right]_{i,j+1} = \tag{4.36}$$

$$\left[\frac{\Delta x^2}{12}v_{2,0} + \frac{\Delta y^2}{4}v_{0,2} - \frac{\Delta y}{2}u_{1,0} + v_{0,0}\right]_{i,j}$$

$$- \left[\frac{\Delta x^2}{12}v_{2,0} + \frac{\Delta y^2}{4}v_{0,2} + \frac{\Delta y}{2}u_{1,0} + v_{0,0}\right]_{i,j+1} = 0$$

$$\left[\frac{J(\overline{PQ})_{XM}}{\Delta x}\right]_{i,j} + \left[\frac{J(\overline{RS})_{XM}}{\Delta x}\right]_{i,j+1} = \tag{4.37}$$

$$\left[\frac{\Delta x^2}{12}(u_{2,0}\,v_{0,0} + v_{2,0}\,u_{0,0} + u_{1,0}\,v_{1,0}) + \frac{\Delta y^2}{4}(u_{0,2}\,v_{0,0} + v_{0,2}\,u_{0,0} - u_{0,1}\,u_{1,0})\right.$$

$$\left. + \frac{\Delta y}{2}\left[v_{0,0}\,u_{0,1} - u_{0,0}\,u_{1,0} - \frac{2}{Re_L}(u_{0,2} - u_{2,0})\right] - \frac{1}{Re_L}(u_{0,1} + v_{1,0}) + u_{0,0}\,v_{0,0}\right]_{i,j}$$

$$- \left[\frac{\Delta x^2}{12}(u_{2,0}\,v_{0,0} + v_{2,0}\,u_{0,0} + u_{1,0}\,v_{1,0}) + \frac{\Delta y^2}{4}(u_{0,2}\,v_{0,0} + v_{0,2}\,u_{0,0} - u_{0,1}\,u_{1,0})\right.$$

$$\left. - \frac{\Delta y}{2}\left[v_{0,0}\,u_{0,1} - u_{0,0}\,u_{1,0} - \frac{2}{Re_L}(u_{0,2} - u_{2,0})\right] - \frac{1}{Re_L}(u_{0,1} + v_{1,0}) + u_{0,0}\,v_{0,0}\right]_{i,j+1} = 0$$

$$\left[\frac{J(\overline{PQ})_{YM}}{\Delta x}\right]_{i,j} + \left[\frac{J(\overline{RS})_{YM}}{\Delta x}\right]_{i,j+1} = \tag{4.38}$$

$$\left[\frac{\Delta x^2}{12}(2\,v_{2,0}\,v_{0,0} + v_{1,0}^2) + \frac{\Delta y^2}{4}(2\,v_{0,2}\,v_{0,0} + u_{1,0}^2 + p_{0,2})\right.$$

$$\left. - \frac{\Delta y}{2}[u_{0,0}\,v_{1,0} + v_{0,0}\,u_{1,0} - \frac{2}{Re_L}(v_{2,0} - v_{0,2})] + p_{0,0} + \frac{2}{Re_L}u_{1,0} + v_{0,0}^2\right]_{i,j}$$

$$- \left[\frac{\Delta x^2}{12}(2\,v_{2,0}\,v_{0,0} + v_{1,0}^2) + \frac{\Delta y^2}{4}(2\,v_{0,2}\,v_{0,0} + u_{1,0}^2 + p_{0,2})\right.$$

$$\left. + \frac{\Delta y}{2}[u_{0,0}\,v_{1,0} + v_{0,0}\,u_{1,0} - \frac{2}{Re_L}(v_{2,0} - v_{0,2})] + p_{0,0} + \frac{2}{Re_L}u_{1,0} + v_{0,0}^2\right]_{i,j+1}$$

$$= 0$$

$$\underset{\sim}{u}(x_i, y_{j+\frac{\Delta y}{2};i,j}) - \underset{\sim}{u}(x_i, y_{j+1-\frac{\Delta y}{2};i,j+1}) = \tag{4.39}$$

$$\left[\frac{\Delta y^2}{4}\,u_{0,2} + \frac{\Delta y}{2}\,u_{0,1} + u_{0,0}\right]_{i,j} - \left[\frac{\Delta y^2}{4}\,u_{0,2} - \frac{\Delta y}{2}\,u_{0,1} + u_{0,0}\right]_{i,j+1} = 0$$

The boundary conditions at the lower and upper walls are

$$\left[\frac{\Delta y^2}{4}\,u_{0,2} - \frac{\Delta y}{2}\,u_{0,1} + u_{0,0}\right]_{i,1} = 0 \tag{4.40}$$

$$\left[\frac{\Delta x^2}{12}\,v_{2,0} + \frac{\Delta y^2}{4}\,v_{0,2} + \frac{\Delta y}{2}\,u_{1,0} + v_{0,0}\right]_{i,1} = 0 \tag{4.41}$$

$$\left[\frac{\Delta y^2}{4}\,u_{0,2} + \frac{\Delta y}{2}\,u_{0,1} + u_{0,0}\right]_{i,N_j} = 0 \tag{4.42}$$

$$\left[\frac{\Delta x^2}{12}\,v_{2,0} + \frac{\Delta y^2}{4}\,v_{0,2} - \frac{\Delta y}{2}\,u_{1,0} + v_{0,0}\right]_{i,N_j} = 0. \tag{4.43}$$

Conditions (4.20) - (4.25), which are derived from the differential form of the governing equations, are given by

$$\left[v_{1,1} + 2\,u_{2,0}\right]_{i,j} = 0 \tag{4.44}$$

$$\left[u_{1,1} + 2\,v_{0,2}\right]_{i,j} = 0 \tag{4.45}$$

$$\left[2\,(u_{0,0}\,u_{2,0} - v_{0,0}\,v_{0,2}) + u_{1,0}^2 + u_{0,1}\,v_{1,0}\right]_{i,j} = 0 \tag{4.46}$$

$$\left[v_{0,0}\,u_{0,2} - u_{0,0}\,v_{0,2}\right]_{i,j} = 0 \tag{4.47}$$

$$\left[2\,(v_{0,0}\,v_{0,2} - u_{0,0}\,u_{2,0} + p_{0,2}) + u_{1,0}^2 + u_{0,1}\,v_{1,0}\right]_{i,j} = 0 \tag{4.48}$$

$$\left[u_{0,0}\,v_{2,0} - v_{0,0}\,u_{2,0}\right]_{i,j} = 0 \tag{4.49}$$

and the upstream and downstream boundary conditions are

$$\left[\frac{\Delta x^2}{4}\,u_{2,0} - \frac{\Delta x}{2}\,u_{1,0} + u_{0,0}\right]_{1,j} - [u_i]_j = 0 \tag{4.50}$$

$$\left[\frac{\Delta x^2}{4}\,v_{2,0} - \frac{\Delta x}{2}\,v_{1,0} + v_{0,0}\right]_{1,j} = 0 \tag{4.51}$$

$$\left[p_{0,0}\right]_{N_i,j} = p_e. \tag{4.52}$$

Equations (4.30) - (4.52) are a coupled system of second-order polynomial equations that take the form

$$F(\bar{X}) = 0. \tag{4.53}$$

The unknowns $p_{1,0}$, $p_{0,1}$ and $p_{0,2}$ may be eliminated using (4.31), (4.32) and (4.48), respectively, so that there are a net 10 unknowns per cell that must be explicity solved for. The total number of equations in the nonlinear system is $10\,N_i\,N_j$.

The Jacobian matrix associated with equation (4.53) is block tridiagonal, with block sizes equal to $10\,N_j$. If the equations in (4.53) are arranged appropriately, the Jacobian matrix has the structure shown in Figure 4. Assuming an initial solution iterate $\bar{X}^0$, Newton's method for this system takes the following form:

$$\left[\frac{\partial F}{\partial \bar{X}}\right]^n \Delta^n \bar{X} = - F(\bar{X}^n) \tag{4.54}$$

where

$$\bar{X}^{n+1} = \bar{X}^n + \Delta^n \bar{X} \tag{4.55}$$

The Jacobian matrix shown in Figure 4 has a highly diagonalized block structure. Ninety per cent of the lower diagonal blocks, and 80% of the upper diagonal blocks, are zero. In addition, there are no equations that span all three blocks in a block row of the matrix. If pivoting is employed during the elimination process, the upper diagonal blocks will fill in, and the storage required is $\frac{21}{10}N_iN^2 - \frac{11}{10}N^2$ where $N = 10N_j$ is the block size. However, if the equations that are located entirely within the diagonal block are pivoted separately from the equations that span two blocks, there will be no fill-in, and the storage required is only $\frac{13}{10}N_iN^2 - \frac{3}{10}N^2$. In the former case, the iteration matrix is nearly block bi-diagonal. In the latter case, the iteration matrix is nearly block diagonal. For both cases, the Jacobian matrix retains a highly diagonalized block structure during the elimination process by virtue of the fact that none of the discrete equations span more than two blocks.

The computational work required is significantly reduced in either case over the situation in which the elimination must be carried out completely in all of the blocks. Calculations performed on a Cray YMP have shown that, for the nearly block bi-diagonal iteration matrix, the computational work required per iteration is reduced by about 47%. For the nearly block diagonal iteration matrix, calculations have shown that the computational work required is reduced by about 60%.

V. Numerical Results

In this section numerical results are presented from calculations of developing channel flows with Reynolds numbers of 100, 1000, and 2000. Our concern here is not with quantitative validation, but rather with demonstrating the overall features of the present scheme. Validation will be considered in a future paper.

For plug flow inlet conditions, the incoming velocity profile may be arbitrarily specified. We consider for convenience velocity profiles of the form

$$u_i(y) = C(\frac{1}{4} - y^2)^{\frac{1}{n}}. \tag{5.1}$$

As n increases, the inlet velocity profile approaches a top-hat shape. In this paper we take n equal to 6. (See Figure 5.) Calculations with n values ranging from 4 to 10 have also been performed but will not be presented in this paper. The constant C is chosen so that the integrated mass flux at the inlet is equal to one.

One of the major goals of the present work is to demonstrate the feasibility of solving developing viscous flows on coarsely spaced uniform grids. To that end, numerous calculations have been performed of developing channel flows on grids using from 4 to 12 cells per channel width. Our results indicate that as few as 6 cells across the channel may be sufficient to resolve the developing boundary layer, depending on the Reynolds number. For each Reynolds number considered, our approach was to repeatedly solve for the developing flow on successively finer grids until the predicted boundary layer thickness remained unchanged from the previous calculation. In the results that follow, we present results from the two finest grids that were used for each Reynolds number. We should point out that, due to symmetry, the number of independent cells to resolve the flow field is half the number of cells per channel width.

The computational grids used in the present study are shown in Figures 6 - 9. In each case the exit boundary is 11 channel heights downstream. Figures 10 - 12, 13 - 15, and 16 - 18 present results for Reynolds numbers of 100, 1000, and 2000, respectively. The predicted streamwise velocity profile is shown at 1, 3, 5, and 11 channel heights downstream. Each figure also shows the inlet velocity profile and the fully developed analytical solution.

Figures 10 - 11 clearly indicate that at a Reynolds number of 100, the flow becomes fully developed in only a few channel heights downstream. At $x = 5$, the predicted streamwise velocity along the centerline differs from the fully developed solution by .35 %. At $x = 11$, the predicted profile and the fully developed solution agree to a minimum of 3 decimal places everywhere. Figure 12 combines the results from Figures 10 and 11. The results show clearly that 6 cells per channel width are as adequate as 8 cells to resolve the developing boundary layer. The CPU times required for these cases were 2.4 and 5.0 seconds, respectively, on a Cray YMP. The solutions were converged to a maximum residual error of 10^{-4}, starting from an initial guess of uniform flow.

At the higher Reynolds numbers of 1000 and 2000, the boundary layer is thinner and develops much more slowly. Consequently, more than 6 cells per channel width are required to resolve the developing flow. The numerical results in Figures 13 - 18 indicate that 8 cells across the channel are sufficient to resolve the boundary layer at a Reynolds number of 1000, and 10 cells are sufficient at a Reynolds number of 2000. The CPU times ranged from 6.1 to 9.9 seconds for the Reynolds number 1000 results, and from 11.1 to 16.6 seconds for the Reynolds number 2000 results.

The numerical results in Figures 13 - 18 also show the occurence of non-physical oscillations in the velocity profile. These occur as a lingering effect of the singularity at the channel inlet. At the lower Reynolds number of 100, the naturally occuring viscocity is sufficient to quickly damp out the effects of the singularity. However, at the higher Reynolds numbers, this is no longer the case, and the oscillations persist further downstream. We should point out that a more physically realistic inlet velocity profile would reduce the strength of the singularity and the resulting non-physical oscillations.

Conclusion

In this paper we have presented a new flux conserving numerical scheme for solving the two-dimensional, steady Navier-Stokes equations. There are numerous advantages to the scheme we have developed. First, fluxes of mass, momentum, and energy are conserved on every cell and every union of cells in a computational mesh which has been used to discretize a flow field. Second, fluxes are balanced at cell interfaces without the use of interpolation, extrapolation, or flux limiters. Third, the discrete solution obtained on each cell is a functional solution of both the integral and differential form of the Navier-Stokes equations. Fourth, the present scheme is highly localized, and concentrates most of the information on a local cell. This results in a nearly block diagonal Jacobian matrix with minimal solution times and storage requirements. Fifth, as shown above, the present approach offers the potential to solve developing viscous flows on coarsely spaced, uniform grids. Finally, the approach we have developed provides a unified treatment of the discrete dependent variables and their derivatives. All are treated as unknowns together to be solved for through simulating local and global flux conservation – i.e., physics.

Dedication:

This paper is dedicated to the memory of my good friend, Bruce L. Scott. (JRS)

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Appendix

Analytical Solution for Fully Developed Flow

The analytical solution for the fully developed velocity is obtained by solving equation (2.2). For flow in an infinitely long straight channel, this equation reduces to

$$\frac{d^2u}{dy^2} = Re_L \frac{dp}{dx} \tag{A.1}$$

where $\frac{dp}{dx}$ is the nondimensional constant pressure gradient, and u must satisfy the boundary conditions

$$u(-\tfrac{1}{2}) = 0 \tag{A.2a}$$

$$u(\tfrac{1}{2}) = 0. \tag{A.2b}$$

The solution to equation (A.1) and boundary conditions (A.2) is

$$u(y) = \frac{Re_L}{2}\frac{dp}{dx}(y^2 - \tfrac{1}{4}). \tag{A.3}$$

For unit nondimensional velocity at the channel inlet, conservation of mass requires that

$$\frac{dp}{dx} = -\frac{12}{Re_L}. \tag{A.4}$$

The analytical solution for the nondimensional fully developed velocity is thus given by

$$u(y) = -6(y^2 - \tfrac{1}{4}). \tag{A.5}$$

When the flow is fully developed, equations (4.30) - (4.52) reduce to a linear set of equations which can be solved analytically. For the fully developed case, the discrete equations reduce to

$$\left[p_{1,0} - \frac{2}{Re_L}u_{0,2}\right]_{i,j} = 0 \qquad \text{for } j = 1, ..., N_j \tag{A.6}$$

$$\left[\frac{\Delta y^2}{4}u_{0,2} + \frac{\Delta y}{2}u_{0,1} + u_{0,0}\right]_{i,j} - \left[\frac{\Delta y^2}{4}u_{0,2} - \frac{\Delta y}{2}u_{0,1} + u_{0,0}\right]_{i,j+1} = 0 \tag{A.7}$$

$$\text{for } j = 1, ..., N_j - 1$$

$$\left[-\Delta y\, u_{0,2} - u_{0,1}\right]_{i,j} - \left[\Delta y\, u_{0,2} - u_{0,1}\right]_{i,j+1} = 0 \tag{A.8}$$

for $j = 1, ..., N_j - 1$

$$\left[\frac{\Delta y^2}{4} u_{0,2} - \frac{\Delta y}{2} u_{0,1} + u_{0,0}\right]_{i,1} = 0 \tag{A.9}$$

$$\left[\frac{\Delta y^2}{4} u_{0,2} + \frac{\Delta y}{2} u_{0,1} + u_{0,0}\right]_{i,N_j} = 0 \tag{A.10}$$

Equation (A.6), which is the x-momentum flux conservation constraint, immediately determines the value of $u_{0,2}$ on each solution element. We have

$$\left[u_{0,2}\right]_{i,j} = \frac{Re_L}{2} p_{1,0} = \frac{Re_L}{2} \frac{dp}{dx} = \frac{Re_L}{2} \left(- \frac{12}{Re_L}\right) = -6 \tag{A.11}$$

since $p_{1,0}$ is the known pressure gradient $\frac{dp}{dx}$. There are thus only two unknowns per solution element, $u_{0,1}$ and $u_{0,0}$. If we consider the special case of only one solution element for the entire channel width so that $N_j = 1$, equations (A.9) and (A.10) form a system of two equations in the two unknowns $u_{0,1}$ and $u_{0,0}$. The solution of this system is given by

$$u_{0,1} = 0, \qquad u_{0,0} = \tfrac{3}{2}. \tag{A.12}$$

Thus,

$$\underset{\sim}{u}(y) = -6\,(y^2 - \tfrac{1}{4}) \equiv u(y) \tag{A.13}$$

so that the analytical solution is recovered.

In general, it can be shown that the solution to the system of discrete equations (A.6) - (A.10) is given by

$$\left[u_{0,2}\right]_{i,j} = -6 \tag{A.14}$$

$$\left[u_{0,1}\right]_{i,j} = -12\, y_j \tag{A.15}$$

$$\left[u_{0,0}\right]_{i,j} = -6\,(y_j^2 - \tfrac{1}{4}) \tag{A.16}$$

so that

$$\underset{\sim}{u}(y; i, j) = -6\,(y - y_j)^2 - 12\, y_j\,(y - y_j) - 6\,(y_j^2 - \tfrac{1}{4}) \tag{A.17}$$

where

$$y_j = -\tfrac{1}{2} + (j - \tfrac{1}{2})\Delta y. \tag{A.18}$$

The analytical solution is thus recovered on each solution element.

A re-examination of equation (A.6) indicates that

$$2\left[u_{0,2}\right]_{i,j} = Re_L \frac{dp}{dx}. \tag{A.19}$$

Since

$$\partial^2 \underset{\sim}{u}/\partial y^2 = 2\left[u_{0,2}\right]_{i,j} \tag{A.20}$$

we have

$$\partial^2 \underset{\sim}{u}/\partial y^2 = Re_L \frac{dp}{dx} \tag{A.21}$$

which corresponds exactly to equation (A.1). The x-momentum flux conservation constraint for the discrete solution thus reproduces the governing differential equation for fully developed channel flow.

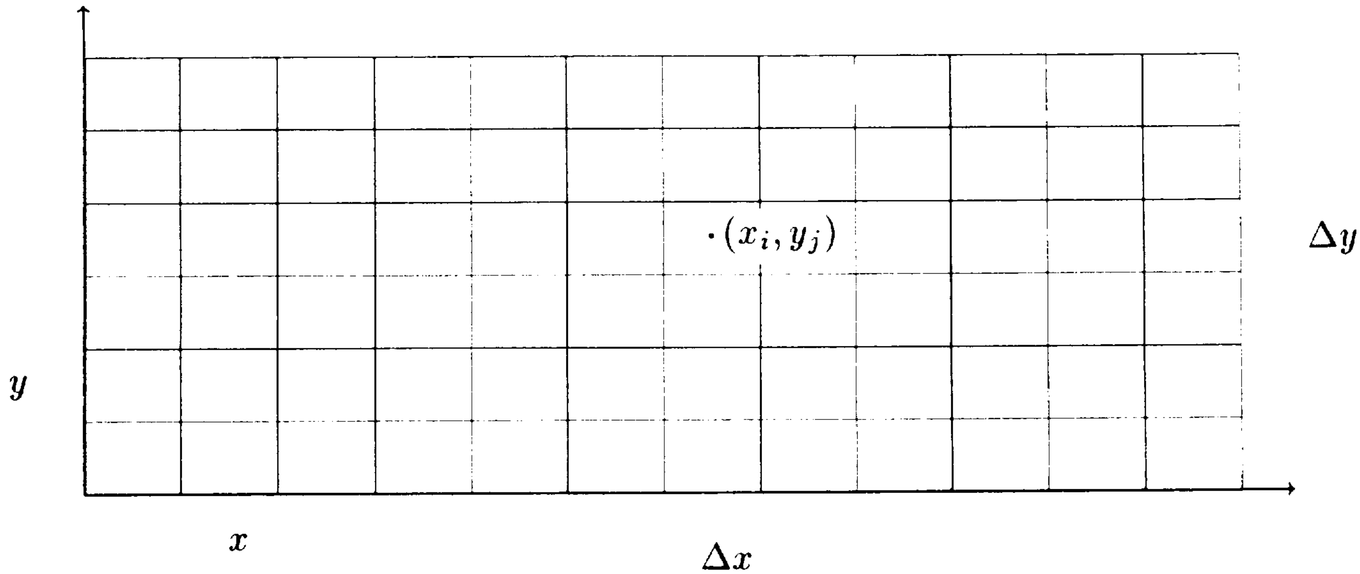

Figure 1. Discretization of E_2.

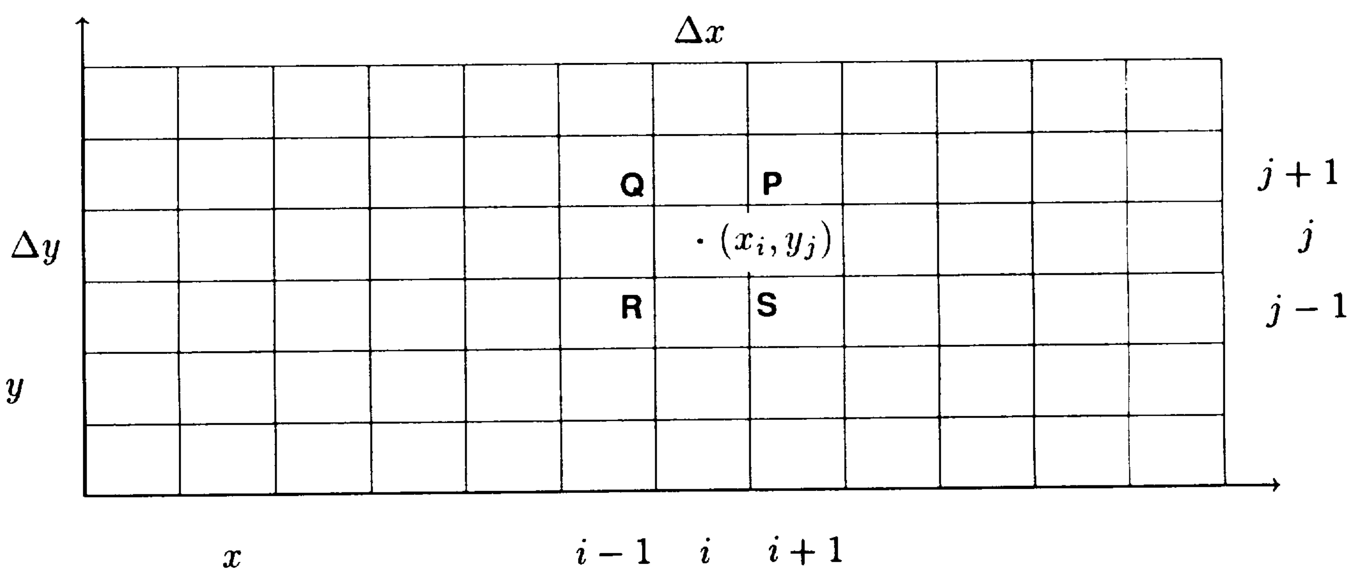

Figure 2. Orientation for Line Integration Around S(CE(i,j)).

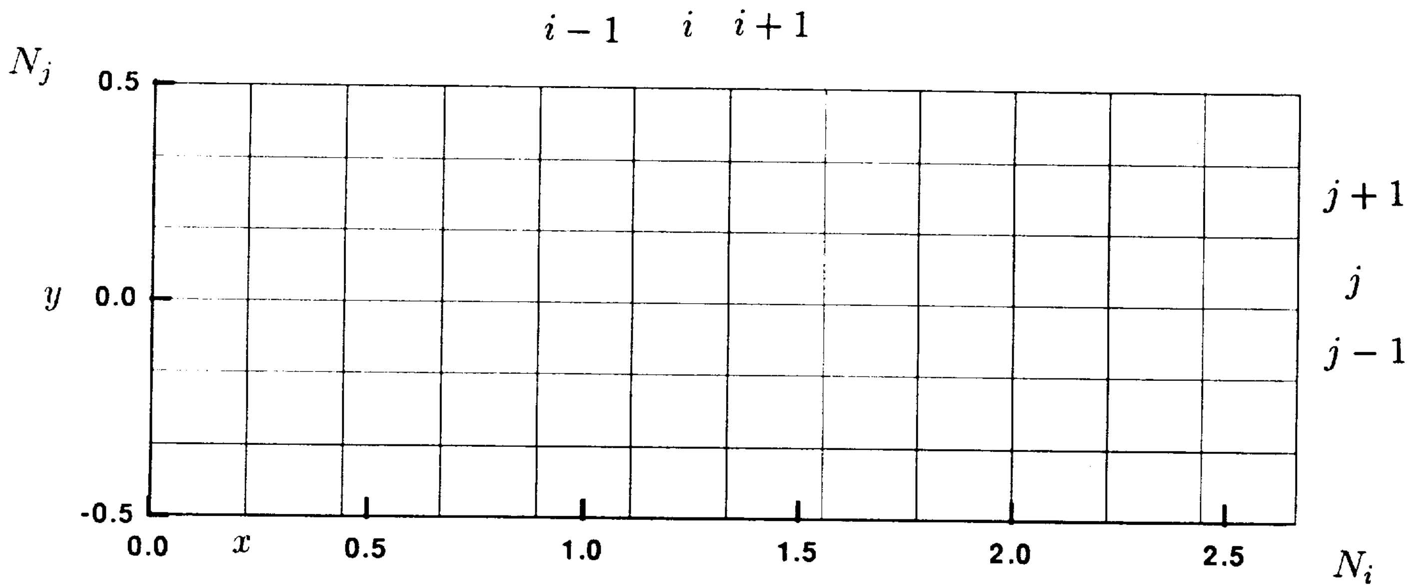

Figure 3. Channel Geometry and Discretization of Flow Field.

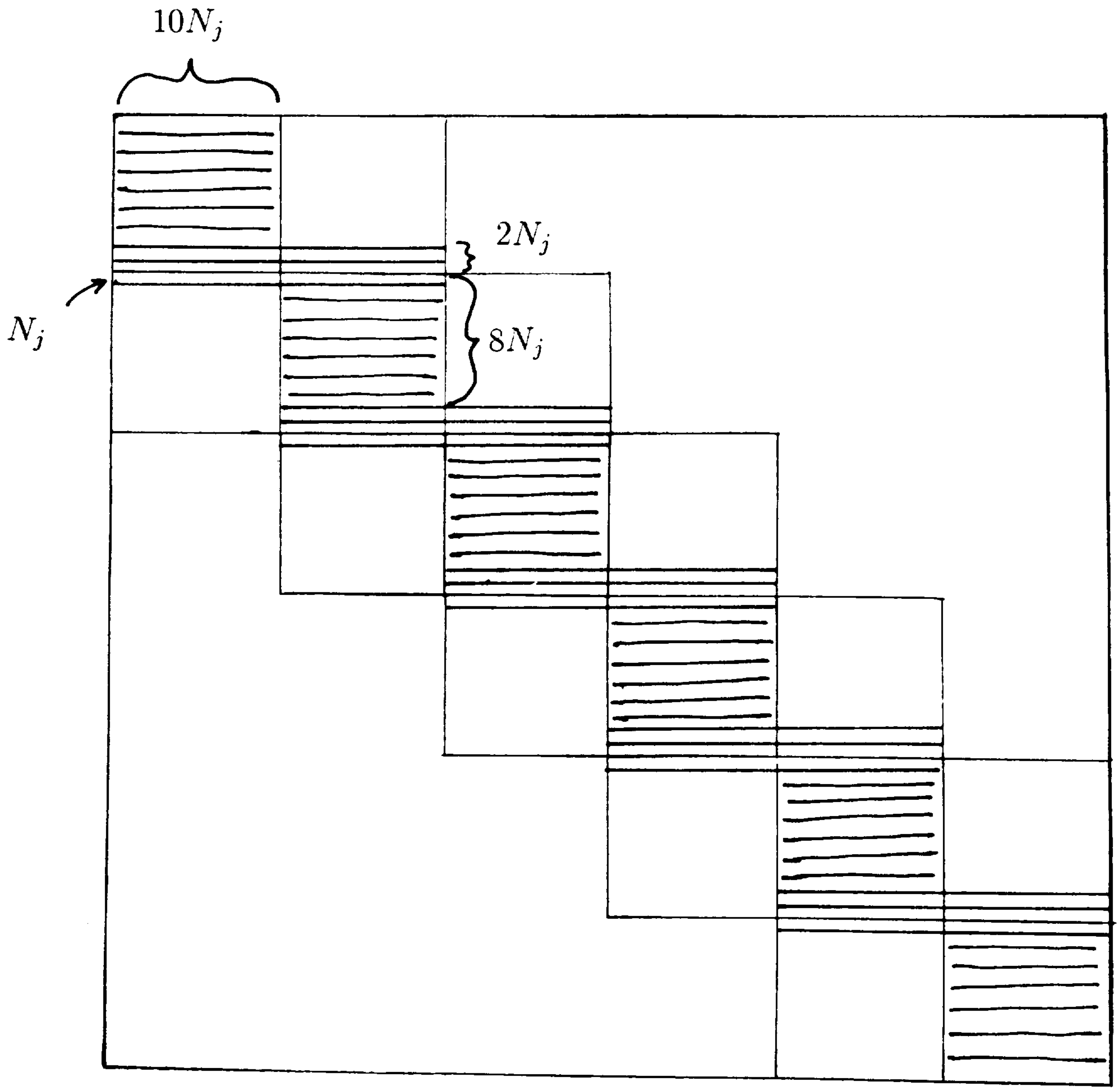

Figure 4. Jacobian Matrix Structure.

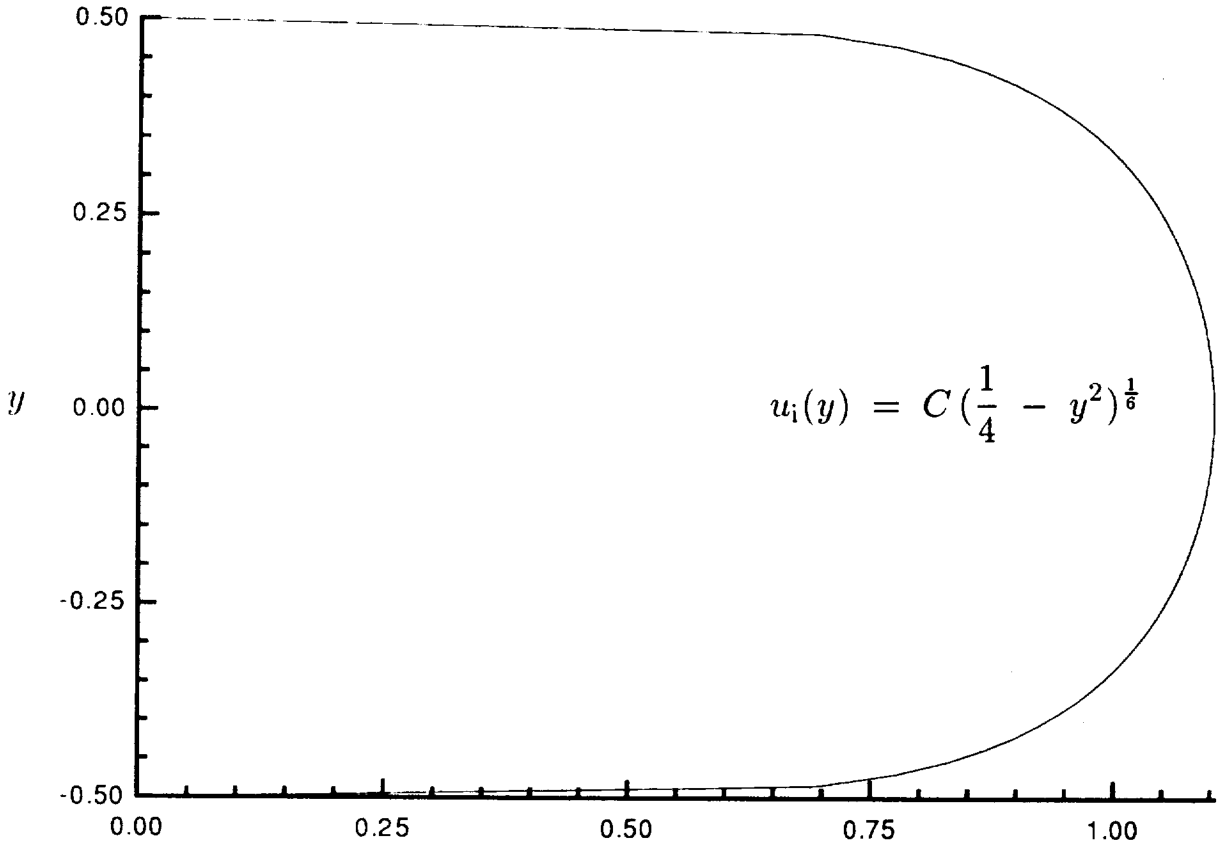

Figure 5. Inlet Velocity Profile.

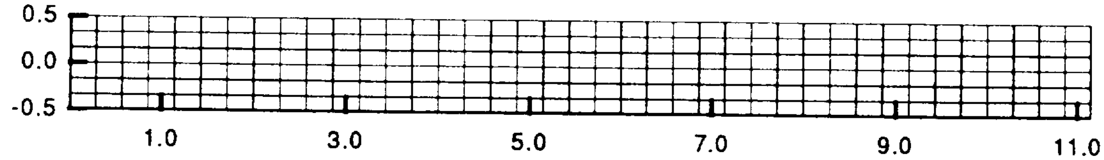

Figure 6. 39 X 6 Computational Grid.

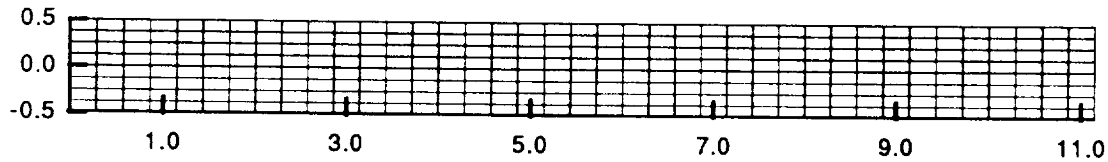

Figure 7. 39 X 8 Computational Grid.

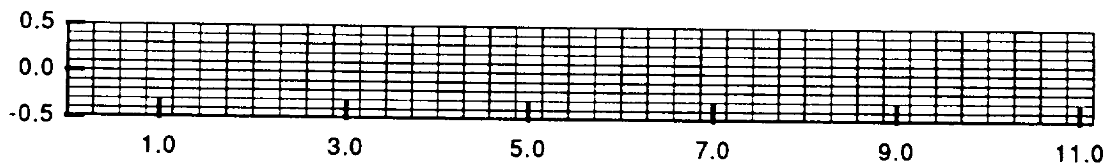

Figure 8. 39 X 10 Computational Grid.

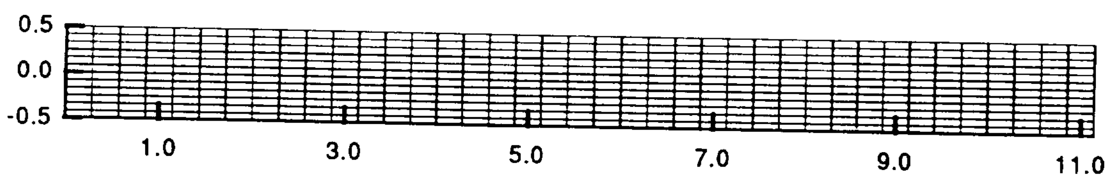

Figure 9. 39 X 12 Computational Grid.

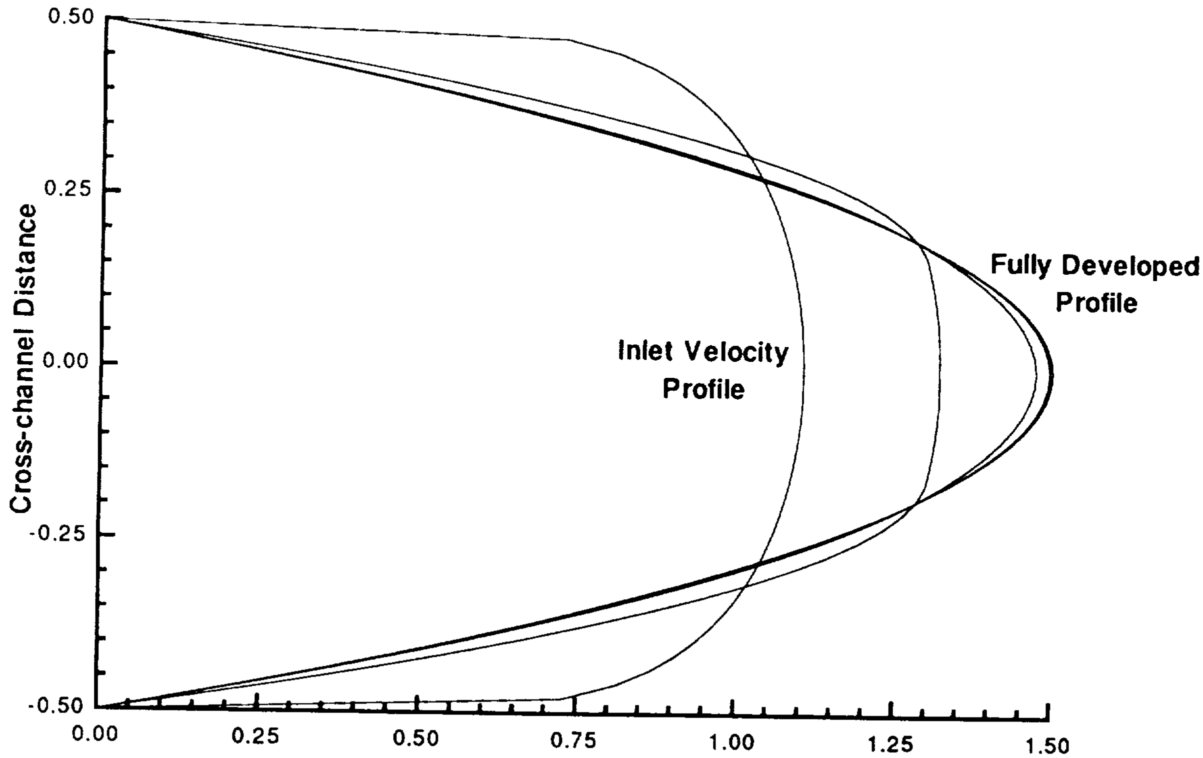

Figure 10. Predicted streamwise velocity profile at 1, 3, 5, and 11 channel heights downstream. Re = 100. 39 X 6 grid.

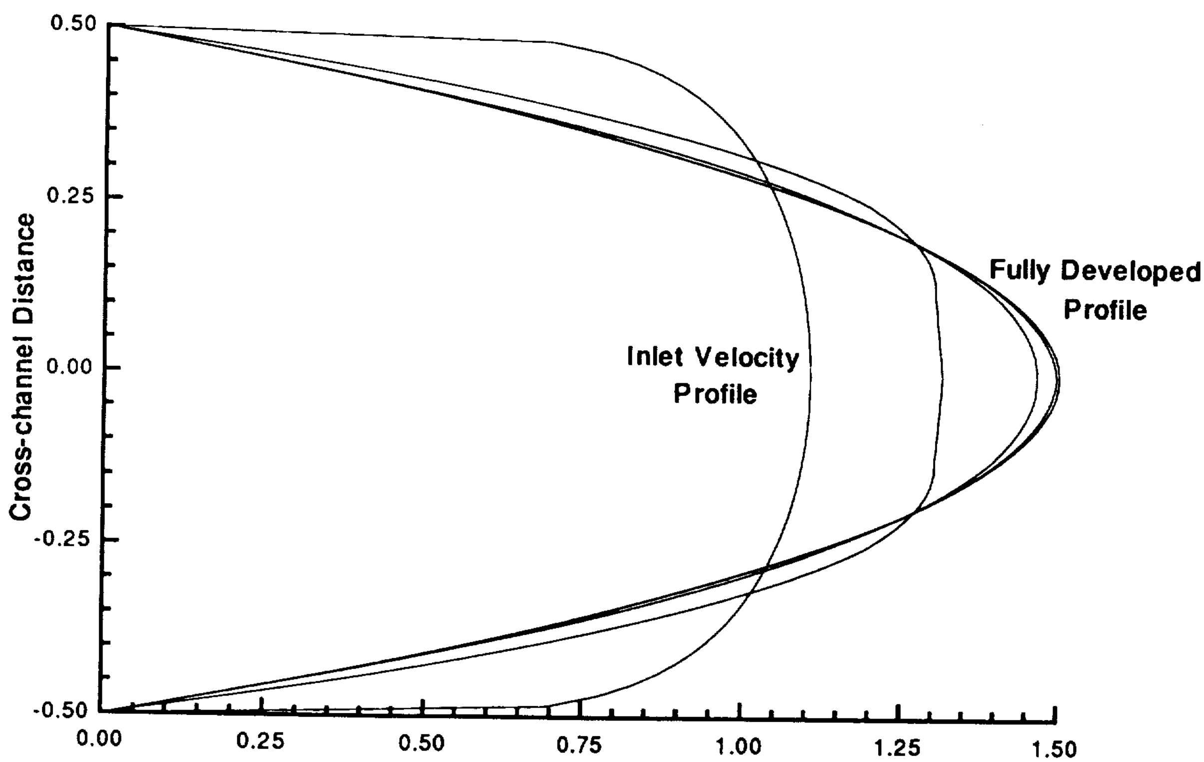

Figure 11. Predicted streamwise velocity profile at 1, 3, 5, and 11 channel heights downstream. Re = 100. 39 X 8 grid.

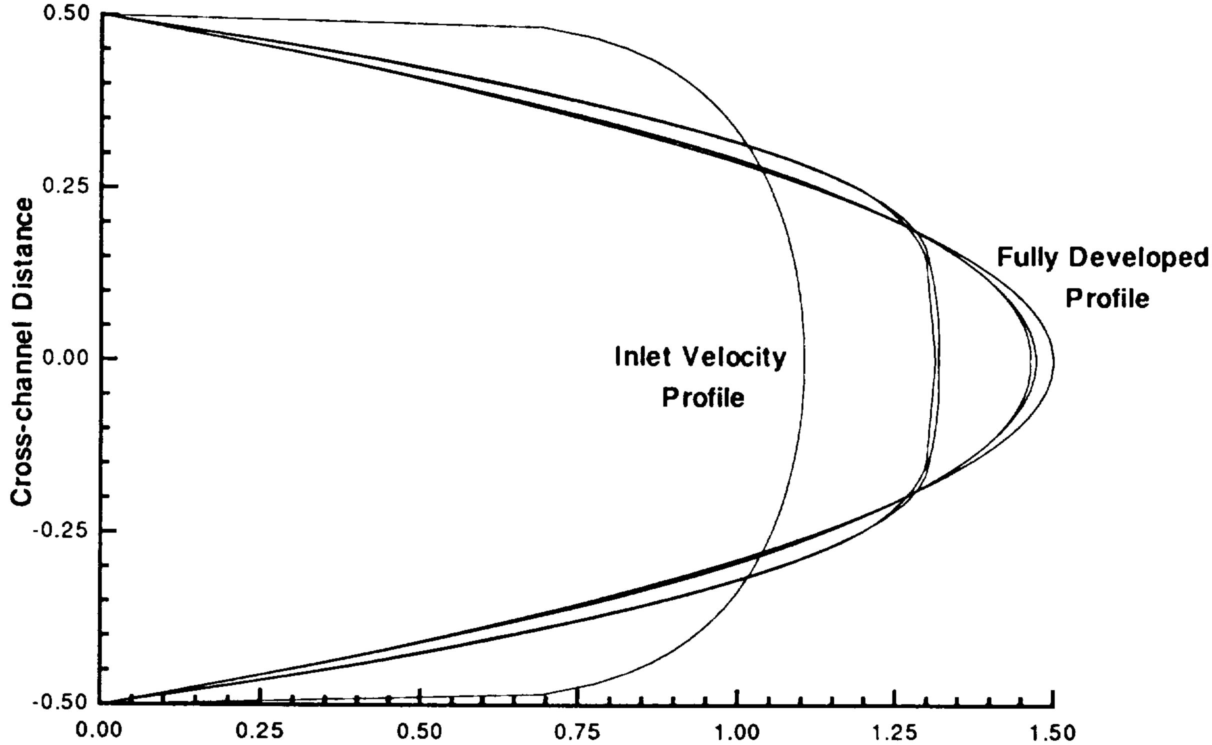

Figure 12. Comparison of predicted streamwise velocity at 1, 3, and 11 channel heights downstream from calculations on 39 X 6 and 39 X 8 grids. Re = 100.

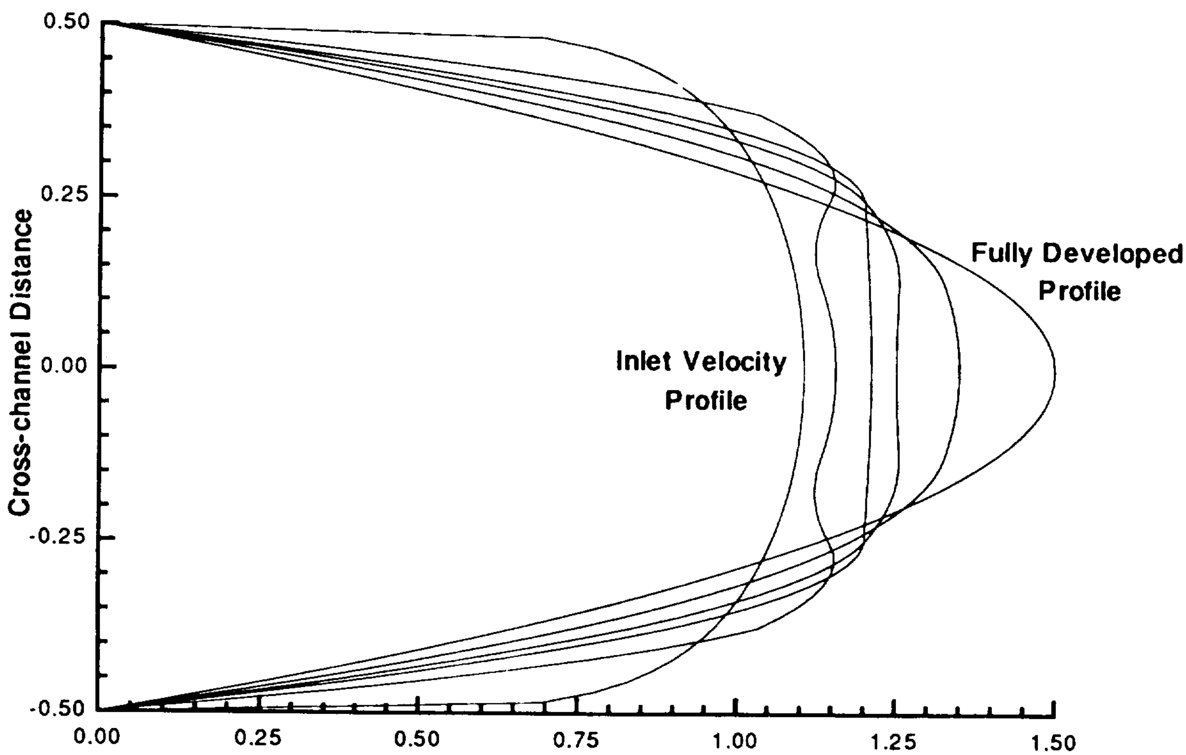

Figure 13. Predicted streamwise velocity profile at 1, 3, 5, and 11 channel heights downstream. Re = 1000. 39 X 8 grid.

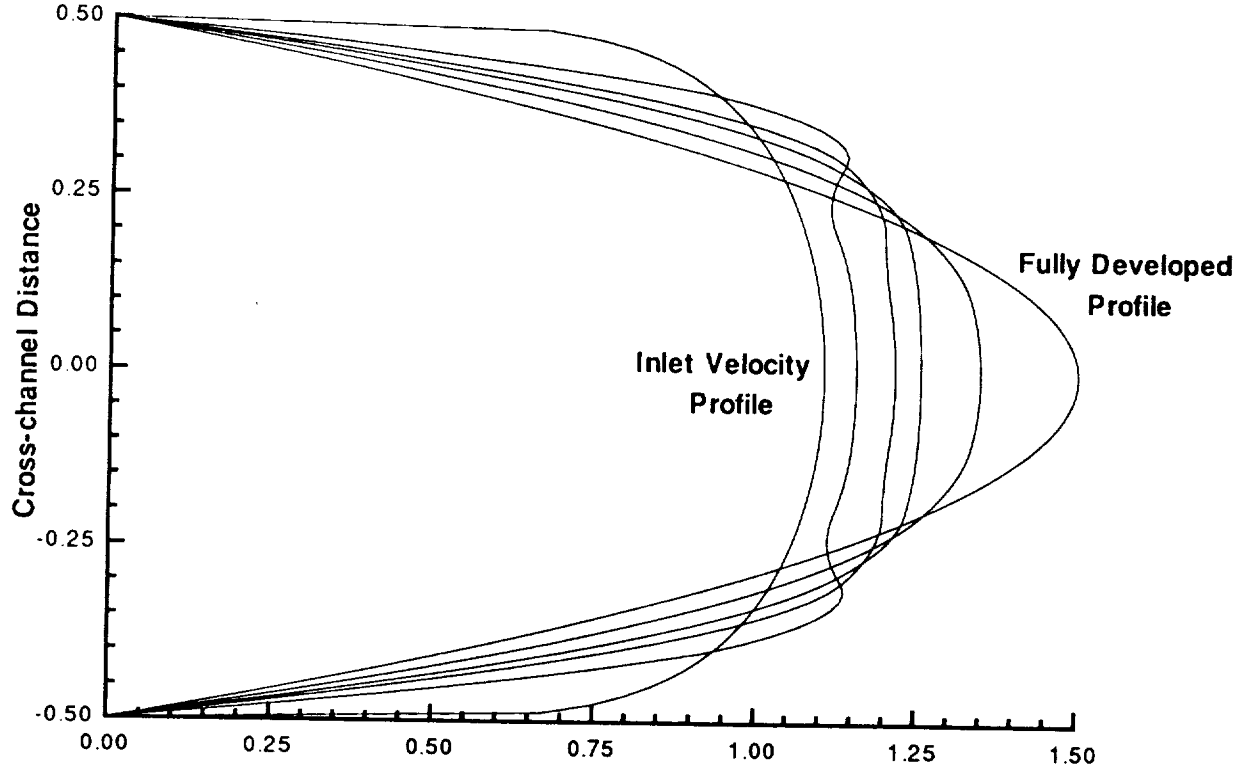

Figure 14. Predicted streamwise velocity profile at 1, 3, 5, and 11 channel heights downstream. Re = 1000. 39 X 10 grid.

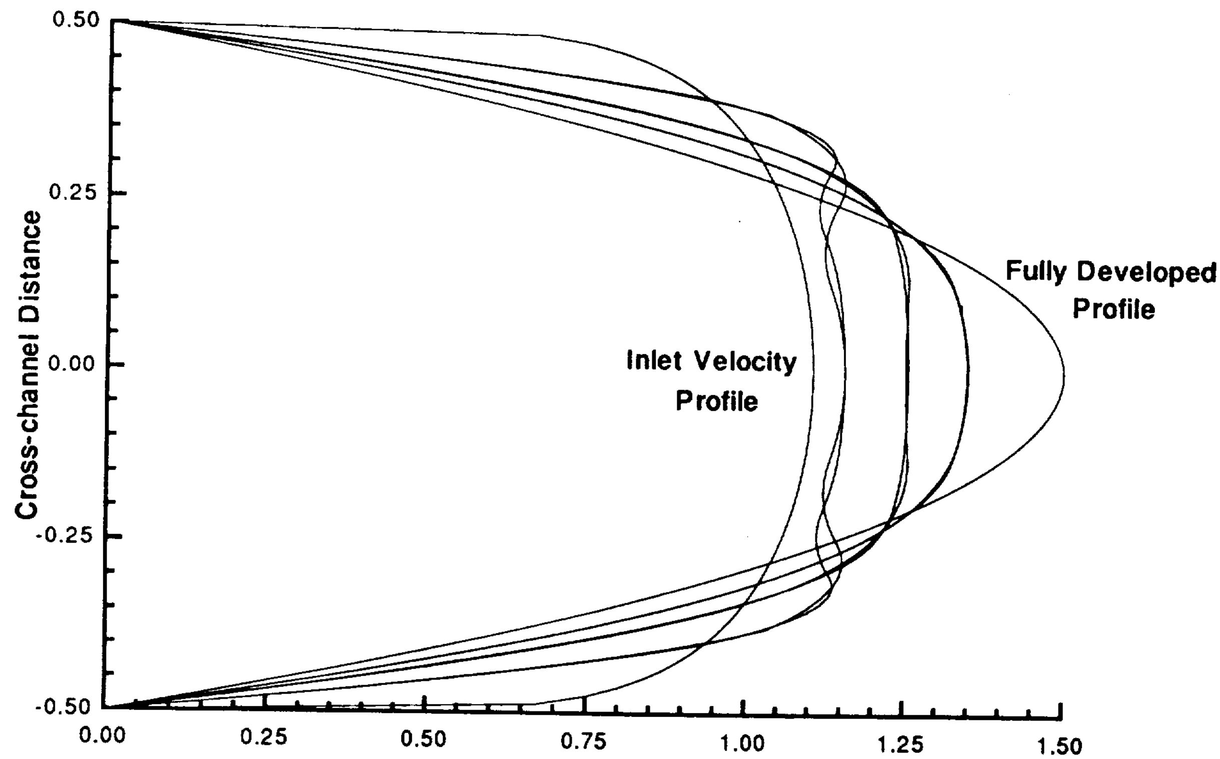

Figure 15. Comparison of predicted streamwise velocity at 1, 5, and 11 channel heights downstream from calculations on 39 X 8 and 39 X 10 grids. Re = 1000.

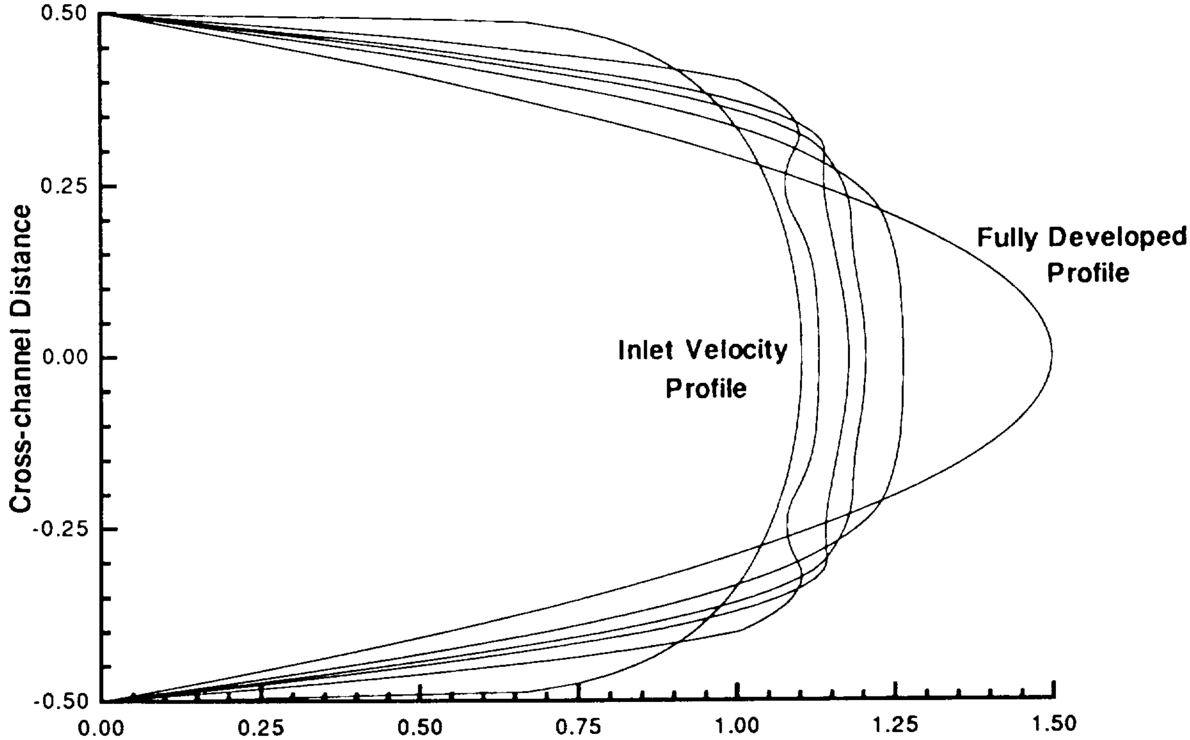

Figure 16. Predicted streamwise velocity profile at 1, 3, 5, and 11 channel heights downstream. Re = 2000. 39 X 10 grid.

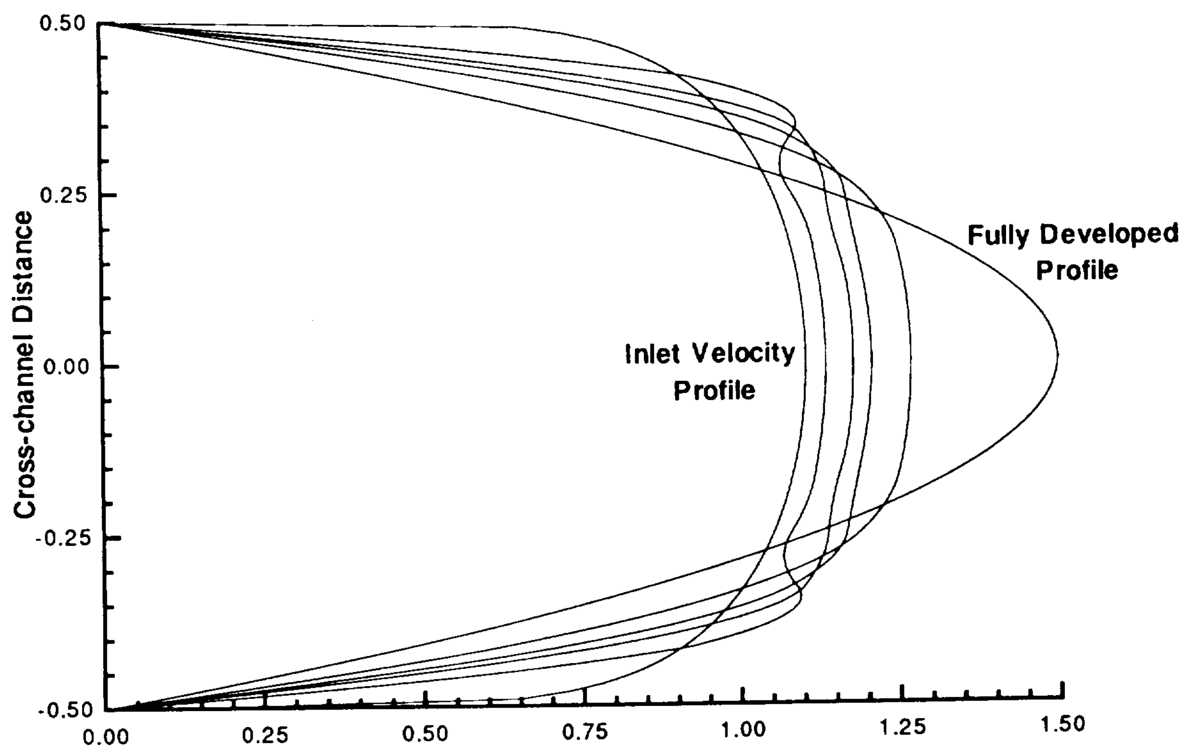

Figure 17. Predicted streamwise velocity profile at 1, 3, 5, and 11 channel heights downstream. Re = 2000. 39 X 12 grid.

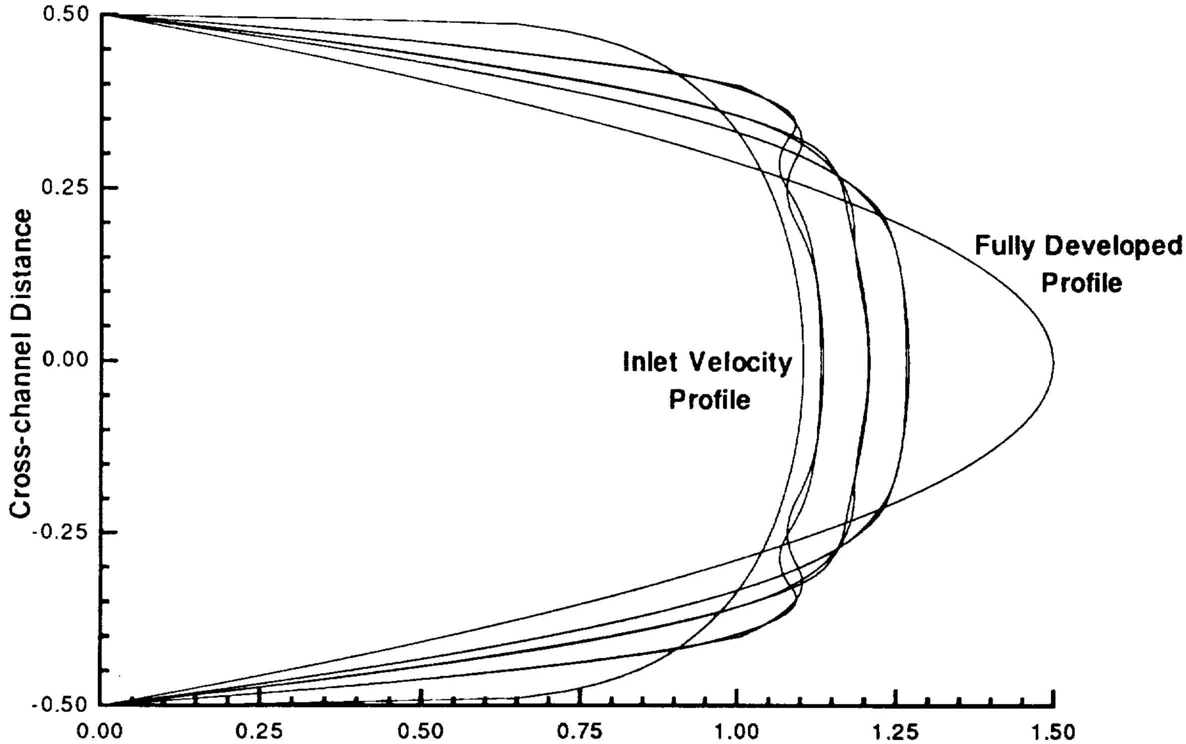

Figure 18. Comparison of predicted streamwise velocity at 1, 5, and 11 channel heights downstream from calculations on 39 X 10 and 39 X 12 grids. Re = 2000.

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13. ABSTRACT *(Maximum 200 words)*

A new numerical method is developed for the solution of the two-dimensional, steady Navier-Stokes equations. The method that is presented differs in significant ways from the established numerical methods for solving the Navier-Stokes equations. The major differences are the following: First, the focus of the present method is on satisfying flux conservation in an intergral formulation, rather than on simulating conservation laws in their differential form. Second, the present approach provides a unified treatment of the dependent variables and their unknown derivatives. All are treated as unknowns together to be solved for through simulating local and global flux conservation. Third, fluxes are balanced at cell interfaces without the use of interpolation or flux limiters. Fourth, flux conservation is achieved through the use of discrete regions known as conservation elements and solution elements. These elements are not the same as the standard control volumes used in the finite volume method. Fifth, the discrete approximation obtained on each solution element is a functional solution of both the integral and differential form of the Navier-Stokes equations. Finally, the method that is presented is a highly localized approach in which the coupling to nearby cells is only in one direction for each spatial coordinate, and involves only the immediately adjacent cells. A general third-order formulation for the steady, compressible Navier-Stokes equations is presented, and then a Newton's method scheme is developed for the solution of incompressible, low Reynolds number channel flow. It is shown that the Jacobian matrix is nearly block diagonal if the nonlinear system of discrete equations is arranged approximately and a proper pivoting strategy is used. Numerical results are presented for Reynolds numbers of 100, 1000, and 2000. Finally, it is shown that the present scheme can resolve the developing channel flow boundary layer using as few as six to ten cells per channel width, depending on the Reynolds number.

14. SUBJECT TERMS

Newton's method; Numerical; Flux; Space-time; Conservative; Navier-Stokes; Solution element; Conservation element; Steady

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