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ON WAVEFORM MULTIGRID METHOD

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ABSTRACT

Waveform multigrid method is an efficient method for solving certain classes of time-dependent PDEs. This paper studies the relationship between this method and the analogous multigrid method for steady-state problems. Using a Fourier-Laplace analysis, practical convergence rate estimates of the waveform multigrid iterations are obtained. Experimental results show that the analysis yields accurate performance prediction.

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1. Introduction. Waveform relaxation, also called dynamic iteration or Picard-Linderlöf iteration [7], is a technique for solving ordinary differential systems of initial-value type [5] [8]. Its key idea is to solve an ordinary differential system in n variables by solving sequence of subsystems in fewer variables. This nature of the method allows independent integration with different timesteps for each of subsystems. Thus the method is usually considered in the context of parallel or multirate algorithms [2].

The convergence of the waveform relaxation methods has been analyzed by Miekkala and Nevanlinna [7]. They discussed this issue over infinite time interval and showed that, for linear heat equation with standard spatial discretization, the convergence rates of the waveform iterations are similar to those for the analogous steady-state problems. Therefore the convergence could be too slow for the waveform relaxation to be competitive with standard timestepping methods.

Multigrid techniques (in space) can be incorporated to accelerate the convergence. This was studied by Lubich and Ostermann [6] who compared the multigrid performance of waveform iteration with that of static iteration (i.e., the iteration for steady-state problems) for one-dimensional heat equation. Combining the analysis on the smoothing rate of high frequencies, they conjectured the waveform multigrid performance for two-dimensional case. They showed that the typical multigrid acceleration can be achieved with an estimated convergence rate, which is similar but not quite as good as the one for the steady-state problems.

A number of parallel waveform multigrid algorithms have been proposed and implemented [9] [11]. Analytical and experimental results have shown that the waveform multigrid methods can be implemented on a parallel computer with satisfactory efficiencies.

In this paper we study the relationship between the waveform multigrid method for time-dependent PDEs and the standard multigrid for the corresponding steady-state problems. This study is important since the steady-state multigrid has been investigated extensively while the properties of waveform multigrid algorithm are relatively unknown. We present a Fourier-Laplace analysis for obtaining practical convergence rate estimates of waveform multigrid iterations. The approachs used in this paper are simple, applicable to wide class of applications, and provide insight into the details of the basic interaction between the coarse grid correction and the waveform relaxation.

This paper is organized as follows. Section 2 briefly introduces the waveform relaxation method. In Section 3, the waveform relaxation and multigrid iteration are combined. A theorem is proved which indicates that the convergence rates of waveform multigrid are essentially the same as those for the analogous steady-state problems when number of smoothings is large. Section 4 gives the details of Fourier-Laplace analysis, which is used for obtaining exact convergence rates. As an example, analytical comparison of multigrid convergence rates in practical number of smoothings is made for a two-dimensional heat equation. Finally, the comparison to the measured convergence rates is presented in Section 5.

2. Waveform relaxation method. Waveform relaxation method was originally proposed for solving ordinary differential equations consisting of subproblems with few external variables in circuit simulation [5]. Unlike standard time stepping methods, it iteratively partitions a large system into loosely coupled subsystems and integrates each subsystem independently. Hence, it is well suited for parallel machines, especially massive parallel machines.

The method can be described as follows. Consider a linear initial value problem

$$\frac{du}{dt} + Au = f, \qquad t > 0, \quad u(0) = u_0. \tag{1}$$

Let A be split as $A = M - N$. Under certain conditions, Eq.(1) can be solved iteratively by

$$\frac{du^{(v)}}{dt} + Mu^{(v)} = Nu^{(v-1)} + f, \qquad t > 0, \quad u^{(v)}(0) = u_0, \tag{2}$$

which is equivalent to the integral equation

$$u^{(v)} = \mathcal{S}u^{(v-1)} + \phi, \tag{3}$$

where $\mathcal{S}$ is a linear integral operator on $L^p(R^+, C^n)$ $(1 \le p \le \infty)$ with kernel $k_s(t) = e^{-tM}N$ and ϕ is a function depending upon f and M.

The general convergence results of the scheme (2) over entire interval $t \ge 0$ were given by [7]. In their work, the Laplace transformation was applied to the time variable t and it showed that the convergence rate for scheme (2) was given by the spectral radius of $\mathcal{S}$, derived as

$$\rho(\mathcal{S}) = \max_{Re z \ge 0} \rho(S(z)), \qquad S(z) : \text{Laplace transform of } \mathcal{S}. \tag{4}$$

We shall follow their approach and concentrate on the damped Jacobi relaxation and the red-black Gauss-Seidel relaxation for linear equations with time-independent coefficients of the form

$$\frac{du}{dt} + Lu = f, \qquad t > 0, \quad u(0) = u_0, \tag{5}$$

where L is a linear elliptic operator. Let L_h be a discrete approximation of L and assume it can be written as

$$L_h = d \begin{bmatrix} I & -B \\ -R & I \end{bmatrix}. \tag{6}$$

Then the damped Jacobi relaxation and the red-black Gauss-Seidel relaxation have the Laplace transform

$$S_J(z) = (zI + M_J)^{-1}N_J = \frac{d}{z + d/\omega} \begin{bmatrix} (\frac{1}{\omega} - 1)I & B \\ R & (\frac{1}{\omega} - 1)I \end{bmatrix}, \tag{7}$$

$$M_J = \frac{d}{\omega}I, \qquad N_J = M_J - L_h, \qquad 0 < \omega \le 1;$$

$$(8) \qquad S_{GS}(z) = (zI + M_{GS})^{-1} N_{GS} = (\frac{d}{z+d})^2 \begin{bmatrix} 0 & \left(\frac{z+d}{d}\right) B \\ 0 & RB \end{bmatrix},$$

$$M_{GS} = d \begin{bmatrix} I & 0 \\ -R & I \end{bmatrix}, \qquad N_{GS} = M_{GS} - L_h.$$

In the following sections, the subscripts of the matrices will be dropped when the context is clear.

3. Waveform versus steady-state multigrid method. The convergence of the waveform relaxation method can be accelerated if the multigrid technique is incorporated in space. In this section, we shall show that, the convergence rate of waveform multigrid iteration converges to that of steady-state multigrid iteration as the number of smoothings increases. In next section, a Fourier-Laplace analysis will be introduced to show that the performance of the waveform multigrid iteration for time-dependent PDEs is virtually as good as the standard multigrid iteration for the corresponding steady-state problems for practical number of smoothings.

The multigrid method is adapted to the waveform iteration for a time-dependent PDE in the following way. First, the equation is discretized in space to give a semi-discrete problem. Next, the multigrid iteration with waveform relaxation is applied to the space variables. As an example, a two grid V-cycle for Eq.(5) is illustrated as follows:

- Perform v_1 pre-smoothings:

$$\frac{du^{(v)}}{dt} + Mu^{(v)} = Nu^{(v-1)} + f, \qquad t > 0, \quad u^{(v)}(0) = u_0,$$

  where $L_h = M - N$, $v = 1, 2, \cdots, v_1$, $u^{(0)} = u^{(0)}(t)$ is given.
- Restrict the defect from grid h to grid H:

$$(9) \qquad d_h := \frac{du^{(v_1)}}{dt} + L_h u^{(v_1)} - f, \qquad d_H := I_h^H d_h.$$

- On the coarse grid, solve

$$\frac{dv}{dt} + L_H v = d_H, \qquad v(0) = 0.$$

- Correct

$$\bar{u} = u^{(v_1)} - I_H^h v$$

  where I_H^h is a suitable interpolation from grid H to grid h.
- Perform v_2 post-smoothings on $\bar{u}$.

Similar to the Full-Approximation-Scheme (FAS) formulation of multigrid, one can formulate a coarse grid problem as

$$\frac{du_H}{dt} + L_H u_H = -I_h^H d_h + \frac{d}{dt}(I_h^H u_h^{(v_1)}) + L_H I_h^H u_h^{(v_1)},$$

$$u_H(0) = I_h^H u_0$$

and the correction step as

$$\bar{u} = u_h^{(v_1)} + I_H^h(u_H - I_h^H u_h^{(v_1)})$$

in order to handle non-linear problems. This however should be used in an FMG algorithm where the problems is solved first on coarse levels to obtain a good initial approximation to fine ones.

The error $e^{(i)} = u^{(i)} - u$ of a complete two grid V-cycle iteration described above satisfies

$$e^{(i)} = \mathcal{V} e^{(i-1)} \tag{10}$$

for an integral operator $\mathcal{V}$, which has the Laplace transform (see [6])

$$V(z) = S(z)^{v_2}(I - I_H^h(z + L_H)^{-1} I_h^H(z + L_h))S(z)^{v_1}, \quad Rez \geq 0, \tag{11}$$

where $S(z)$ is the Laplace transform of the smoother being used. In order to indicate the dependency of $\mathcal{V}$ and $V(z)$ on the number of smoothings $v = v_1 + v_2$, we use the notation $\mathcal{V}(v)$ and $V(z,v)$ whenever it is necessary.

Assuming that all the entries of $V(z)$ are rational functions of z vanishing at infinity with poles having negative real part, and taking $\mathcal{V}$ as an operator on $L^p(R^+, C^n)$ $(1 \leq p \leq \infty)$, the spectral radius $\rho(\mathcal{V}) = \lim_{k\to\infty} ||\mathcal{V}^k||^{1/k}$ satisfies ([6])

$$\rho(\mathcal{V}) = \max_{Rez \geq 0} \rho(V(z)). \tag{12}$$

Note that $S(0)$ and $V(0)$ are respectively, the smoothing and multigrid two-grid cycle operators for the corresponding steady-state problem (or static problem) $L_h u = f$.

Theorem 1. *The spectral radius for the waveform two-grid V-cycle operator $\mathcal{V} = \mathcal{V}(v)$ for equation (5) satisfies*

$$\lim_{v\to\infty} |\rho(\mathcal{V}(v)) - \rho(V(0,v))| = 0 \tag{13}$$

for either damped Jacobi or red-black Gauss-Seidel smoothings.

Proof. We shall prove the case of the red-black Gauss-Seidel relaxation only. The proof for the damped Jacobi relaxation follows similarly. Let $CG(z)$ denote the Laplace transform of coarse grid correction operator,

$$\begin{aligned} CG(z) &= I - I_H^h(z + L_H)^{-1} I_h^H(z + L_h) \\ &= CG(0) + CG_\Delta(z) \end{aligned} \tag{14}$$

with

$$CG_\Delta(z) = zI_H^h(z+L_H)^{-1}(L_H^{-1}I_h^H L_h - I_h^H). \tag{15}$$

Since the spectral radius

$$\rho(S^{v_2}(z)CG(z)S^{v_1}(z)) = \rho(CG(z)S^v(z)), \quad v = v_1 + v_2, \text{ for all } z,$$

for simplicity, we rewrite the two-grid operator (11) as

$$\begin{aligned} V(z,v) &= CG(z)S^v(z) \\ &= CG(0)S^v(z) + CG_\Delta(z)S^v(z). \end{aligned} \tag{16}$$

As noted in Section 2, the red-black Gauss-Seidel smoother (8)

$$\begin{aligned} S^v(z) &= \left(\tfrac{d}{z+d}\right)^{2v} \begin{bmatrix} 0 & \left(\tfrac{z+d}{d}\right) BC^{v-1} \\ 0 & C^v \end{bmatrix}, \qquad C = RB, \\ &= \left(\tfrac{d}{z+d}\right)^{2v} S^v(0) + \tfrac{z}{d}\left(\tfrac{d}{z+d}\right)^{2v} \begin{bmatrix} 0 & BC^{v-1} \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{17}$$

Let $\bar{z} := z/d$, we have

$$V(z,v) = \frac{1}{(\bar{z}+1)^{2v}} V(0,v) + \Delta(\bar{z},v)$$

with

$$\Delta(\bar{z},v) = F^{(v)}(\bar{z})\tilde{\Delta}(\bar{z},v), \tag{18}$$

$$F^{(v)}(\bar{z}) = \frac{\bar{z}}{(\bar{z}+1)^{2v}},$$

$$\tilde{\Delta}(\bar{z},v) = CG(0)\begin{bmatrix} 0 & BC^{v-1} \\ 0 & 0 \end{bmatrix} + I_H^h\left(\bar{z} + \frac{1}{d}L_H\right)^{-1}\left(L_H^{-1}I_h^H L_h - I_h^H\right)\begin{bmatrix} 0 & (1+\bar{z})BC^{v-1} \\ 0 & C^v \end{bmatrix}.$$

Given $\epsilon > 0$, there is a matrix norm $||\cdot||_\epsilon$, such that (see [4], pp.297)

$$\rho(V(z,v)) \le ||\frac{1}{(\bar{z}+1)^{2v}} V(0,v)||_\epsilon + ||\Delta(\bar{z},v)||_\epsilon$$

and

$$||\frac{1}{(\bar{z}+1)^{2v}} V(0,v)||_\epsilon < \rho(\frac{1}{(\bar{z}+1)^{2v}} V(0,v)) + \epsilon.$$

Therefore

$$\rho(\mathcal{V}(v)) \le \max_{Re\bar{z}\ge 0} \frac{1}{|\bar{z}+1|^{2v}} \cdot \rho(V(0,v)) + \epsilon + \max_{Re\bar{z}\ge 0} ||\Delta(\bar{z},v)||_\epsilon,$$

which leads to

$$0 \le \rho(\mathcal{V}(v)) - \rho(V(0,v)) \le \epsilon + \max_{Re\bar{z}\ge 0} ||\Delta(\bar{z},v)||_\epsilon. \tag{19}$$

The matrix function $\tilde{\Delta}(\bar{z},v)$ is analytic in $Re\bar{z} > 0$, continuous on $Re\bar{z} = 0$, and $\tilde{\Delta}(\infty,v)$ is bounded for all $v \ge 1$ (because $S^v(0) \to 0$ as $v \to \infty$ [7]), so does its ϵ-norm $||\tilde{\Delta}(\bar{z},v)||_\epsilon$, thus

$$\max_{Re\bar{z}\ge 0} ||\tilde{\Delta}(\bar{z},v)||_\epsilon \le c(\epsilon), \tag{20}$$

where $c(\epsilon)$ is a function of ϵ independent of $\bar{z}$ and v. Combining Eq.(18) and Eq.(20), we have

$$\max_{Re\bar{z}\ge 0} ||\Delta(\bar{z},v)||_\epsilon \le \max_{Re\bar{z}\ge 0} |F^{(v)}(\bar{z})| \cdot c(\epsilon) = \left|F^{(v)}(\pm i\frac{1}{\sqrt{2v-1}})\right| \cdot c(\epsilon)$$

$$\to |F^{(v)}(0)| \cdot c(\epsilon) = 0, \text{ as } v \to \infty. \qquad \square$$

4. Fourier-Laplace analysis. Fourier analysis has been used to provide exact convergence rate of multigrid iteration for some steady-state problems [1]. In order to analyze the convergence rate of waveform multigrid iteration, first, the Laplace transform is introduced in time variable to convert a m-dimensional semi-discrete linear differential equation with time-independent coefficients

$$u_t + L_h u = f \tag{21}$$

$$u(0) = 0$$

to equivalent algebraic equations defined over the right half complex plane

$$(z + L_h)\hat{u}(z) = \hat{f}(z), \qquad Re z \ge 0. \tag{22}$$

Then multigrid method applied in space only is analyzed by considering its action on each of these equations. This is done using Fourier analysis for each equation in (22), and combining the results to obtain the convergence rate of waveform multigrid iteration for Eq.(21). In this section a Fourier-Laplace analysis is applied to the waveform two grid V-cycle iteration described in Section 3.

Let us begin with the error formula (10)

$$e^{(i)} = \mathcal{V}e^{(i-1)}.$$

Its Laplace transform is

$$\hat{e}^{(i)}(z) = V(z)\hat{e}^{(i-1)}(z), \qquad Re z \ge 0. \tag{23}$$

Let

$$\theta = (\theta^1, \theta^2, \cdots, \theta^{2^m}), \qquad \theta^j = \theta^1 (\text{mod } \pi), \quad j = 2, \cdots 2^m,$$

and $X(\theta)$, $\tilde{V}(\theta, z)$ denote the Fourier modes on grid h and the symbol of $V(z)$, i.e.,

$$X(\theta) = [exp(i\theta^1 x/h), \cdots, exp(i\theta^{2^m} x/h)]$$

$$\tilde{V}(\theta, z) = [\hat{V}_{j,k}(\theta, z)]_{2^m \times 2^m}$$

satisfying

(24) $$V(z)X(\theta) = X(\theta)\tilde{V}(\theta, z) \qquad \text{for all } \theta.$$

Then

(25) $$\rho(V(z)) = \sup_{\theta} \rho(\tilde{V}(\theta, z)) = \max_{0 \le |\theta^1| \le \frac{\pi}{2}} \rho(\tilde{V}(\theta, z)).$$

The symbol matrix $\tilde{V}(\theta, z)$ is of order 2^m, a tiny matrix comparing to $V(z)$. Its spectral radius can be calculated accurately for each given θ and z. Therefore $\rho(\mathcal{V})$, the convergence rate of the waveform two grid iteration, is obtained by computing $\rho(\tilde{V}(\theta, z)$ over $0 \le |\theta^1| \le \frac{\pi}{2}$ and $Rez = 0$:

$$\rho(\mathcal{V}) = \max_{Rez=0} \max_{0 \le |\theta^1| \le \frac{\pi}{2}} \rho(\tilde{V}(\theta, z)).$$

Example. Consider the heat equation on a square $\Omega = (0, \pi) \times (0, \pi)$ with Dirichlet boundary conditions

$$u_t - \Delta u = f, \quad (t, x) \in (0, \infty) \times \Omega$$

$$u = g, \quad (t, x) \in [0, \infty) \times \partial\Omega, \qquad u(0, x) = u_0(x), \quad x \in \Omega.$$

Let L_h correspond to the five point Laplacian

(26) $$L_h := \frac{1}{h^2} \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}.$$

The Laplace transform of the two grid iteration operator $\mathcal{V}$ is given by (see Eq.(16))

(27) $$V(z) = CG(z)S^{\nu}(z)$$

with

$$CG(z) = I - I_H^h (z + L_H)^{-1} I_h^H (z + L_h)$$

and the corresponding smoother $S(z)$. Since the space spanned by the Fourier mode $X(\theta)$ is invariant under each of operators in Eq.(27), we have

$$\text{(28)} \quad \tilde{V}(\theta, z) = (I - \tilde{I}_H^h(\theta)(z + \tilde{L}_H(2\theta))^{-1}\tilde{I}_h^H(\theta)(z + \tilde{L}_h(\theta)))\tilde{S}^v(\theta, z), \quad Re z \geq 0,$$

where $\sim$'s indicate the matrix symbols and $H = 2h$ is assumed. For this particular example, the fine grid operator is represented as

$$\tilde{L}_h(\theta) = \begin{bmatrix} \hat{L}_h(\theta^1) & & \\ & \ddots & \\ & & \hat{L}_h(\theta^4) \end{bmatrix}$$

with

$$\theta = (\theta_1, \theta_2), \qquad \hat{L}_h(\theta) = \frac{4(sin^2(\theta_1/2) + sin^2(\theta_2/2))}{h^2},$$

the coarse grid operator as

$$\tilde{L}_H(2\theta) = [\hat{L}_H(2\theta^1)].$$

The bi-linear interpolation is chosen for I_H^h and the restriction operator $I_h^H = (I_H^h)^T$. Their matrix symbols are

$$\tilde{I}_h^H(\theta) = [\hat{I}_h^H(\theta^1), \cdots, \hat{I}_h^H(\theta^4)], \qquad \hat{I}_h^H(\theta) = (\frac{1 + cos\theta_1}{2})(\frac{1 + cos\theta_2}{2});$$

$$\tilde{I}_H^h(\theta) = (\tilde{I}_h^H(\theta))^T.$$

Smoothers for the damped Jacobi and the red-black Gauss-Seidel relaxation are represented as (see Eq.(7)-(8))

$$\tilde{S}_J(\theta, z) = \begin{bmatrix} \hat{S}(\theta^1, z) & & \\ & \ddots & \\ & & \hat{S}(\theta^4, z) \end{bmatrix},$$

$$\hat{S}(\theta, z) = \frac{1}{z\omega + d}(d - \omega\hat{L}_h(\theta)), \quad d = \frac{4}{h^2}, \quad 0 < \omega \leq 1;$$

$$\tilde{S}_{GS}(\theta, z) = \begin{bmatrix} \hat{S}_a(\theta, z) & 0 \\ 0 & \hat{S}_b(\theta, z) \end{bmatrix}$$

$$\hat{S}_a(\theta, z) = \frac{a}{2}\begin{bmatrix} 1 + a & -(1 + a) \\ 1 - a & -(1 - a) \end{bmatrix}, \quad \hat{S}_b(\theta, z) = \frac{b}{2}\begin{bmatrix} 1 + b & -(1 + b) \\ 1 - b & -(1 - b) \end{bmatrix},$$

TABLE 1
Damped Jacobi Waveform Multigrid ($\omega = 2/3$)

v	1	2	3	4	5	6	7	8	9	10
$\rho(\mathcal{V}(v))$	.6626	.4390	.2909	.1927	.1486	.1269	.1124	.1004	.0908	.0838
$\rho(V(0,v))$	.6626	.4390	.2909	.1927	.1332	.1152	.1003	.0881	.0793	.0728

TABLE 2
Red-Black Gauss-Seidel Waveform Multigrid

v	1	2	3	4	5	6	7	8	9	10
$\rho(\mathcal{V}(v))$	.2439	.1567	.1132	.0861	.0737	.0614	.0548	.0473	.0430	.0390
$\rho(V(0,v))$	.2439	.1517	.1046	.0765	.0669	.0543	.0474	.0404	.0368	.0336

$$a = \frac{1}{2(1+z/d)}(cos\theta_1 + cos\theta_2), \qquad b = \frac{1}{2(1+z/d)}(-cos\theta_1 + cos\theta_2).$$

Tables 1 and 2 present computed $\rho(\mathcal{V}(v))$, the spectral radius of waveform multigrid operator, and $\rho(V(0,v))$ of corresponding steady-state multigrid operator for the damped Jacobi and the red-black Gauss-Seidel relaxation. As it shows, the estimated convergence rates of waveform multigrid iteration are virtually as good as those for steady-state problems. Since extensive research has been done in multigrid methods for steady-state problems [1] [3] [10], this analytic comparison is very useful in predicting the performance of waveform multigrid methods. Although our discussion was done only for the model heat equation, one may expect the same performance of the waveform multigrid iteration as that of steady-state one in wide classes of applications. The treatment of general problems, e.g., non-constant coefficients or general domains, is done in a similar way using frozen coefficients argument which is applicable for smooth coefficient problems. Its theoretical rigorous justification is involved and needs the use of pseudo-difference calculus.

5. Comparison and Conclusion. We have shown that the estimated convergence rate of waveform multigrid iteration is almost undistinquishable from that of steady-state one. Now, we compare the experimental results of the convergence rate to the analytic ones discussed in last section.

Consider the two-dimensional heat equation on $\Omega = (0,\pi) \times (0,\pi)$

$$u_t - \Delta u = f, \quad (t,x) \in (0, tf] \times \Omega,$$

$$u(t,x) = 0, \quad (t,x) \in [0, tf] \times \partial\Omega,$$

$$u(0,x) = u_0(x), \quad x \in \Omega,$$

where u_0 was generated randomly to excite all possible Fourier modes. The space derivatives were discretized by central differences with uniform fine grid size h_x. For time integration, as in [7], trapezoidal rule was used. The step size was always chosen

TABLE 3
Damped Jacobi Waveform Multigrid

v	analytic $\rho(\mathcal{V}(v))$	measured $\rho(\mathcal{V}(v))$	
		worst case	avareage case
1	.6626	.6657	.6403
2	.4390	.4401	.4160
3	.2909	.2964	.2693
4	.1927	.1890	.1836

TABLE 4
Red-Black Gauss-Seidel Waveform Multigrid

v	analytic $\rho(\mathcal{V}(v))$	measured $\rho(\mathcal{V}(v))$		$\frac{1}{2}\sqrt{\eta_0(2v-1)}+O(h_x^2)$
		worst case	avareage case	
1	.2439	.2560	.2282	.2500+$O(h_x^2)$
2	.1567	.1137	.0740	.1625+$O(h_x^2)$
3	.1132	.0761	.0524	.1295+$O(h_x^2)$
4	.0861	.0670	.0437	.1110+$O(h_x^2)$

as $h_t = .01$ on $[0, tf]$. Note, the efficiency of time integration is not the concern of this paper.

The experiments were run for two-grid V-cycle with v damped Jacobi ($\omega = 2/3$) or red-black Gauss-Seidel relaxations. The fine grid mesh size in space was $h_x = \pi/n$, $n = 8, 16, 32, 64$. Although the spectral radius $\rho(\mathcal{V}(v))$ was studied on entire time interval $t \in [0, \infty)$, finite interval $[0, tf]$ had to be used in experiments. For each set of tests, we used both $tf = 1$ and $tf = 10$. However, we found that the measured convergence rates did not depend on the size of time interval.

To measure the spectral radius $\rho(\mathcal{V}(v))$, we used the asymptotic ratio of the defects (see Eq.(9))

$$\frac{\max_t ||d_h^{(i+1)}||_2}{\max_t ||d_h^{(i)}||_2}.$$

Using the matrix split of L_h, the calculation of the derivatives in the defects can be avoided. Because of extensive computations involved, $||d_h^{(i)}||_2$ was evaluated only at $t = tf$ since the convergence of the waveform iteration is determined by the error at $t = tf$.

Tables 3 and 4 present the comparison results. They show that the analytic convergence rates obtained via Fourier-Laplace analysis almost coincide with the real ones, a result evidenced for the steady-state problems [1]. As a reference, Table 4 also lists values of $\frac{1}{2}\sqrt{\eta_0(2v-1)}$, the bounds of $\rho(\mathcal{V}(v))$ proved for one-dimensional problem by Lubich and Ostermann [6]. They conjectured the bounds for that of two-dimensional heat equation as $\frac{1}{2}\sqrt{\eta_0(2v-1)} + O(h_x^2)$. Observing our analysis in Section 4, with standard discretization of L, nine-point restriction and bi-linear interpolation, $\rho(\mathcal{V}(v))$ is independent of the grid size h_x. The major difference of our approach to the one in [6] is that, instead of using eigenvectors of L_h, we used Fourier modes, which are much easier to manipulate, extendible to high-dimensional problems and wide class of applications. Most important of all, this approach is able to give the exact convergence rates for special model problems and sharp estimates for general problems.

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Waveform multigrid method is an efficient method for solving certain classes of time-dependent PDEs. This paper studies the relationship between this method and the analogous multigrid method for steady-state problems. Using a Fourier-Laplace analysis, practical convergence rate estimates of the waveform multigrid iterations are obtained. Experimental results show that the analysis yields accurate performance prediction.

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