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LIFT THEORY OF SUPPORTING SURFACES

Second Article

By

R. von Mises

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LIFT THEORY OF SUPPORTING SURFACES.*

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This continuation of my remarks on the hydrodynamic lift theory of supporting surfaces** relates chiefly to the task of finding suitable wing sections for given lift conditions. It is based on the formulas (set forth in my first article for sections in general) for lift magnitude and moment, which make it possible to obtain solutions of great simplicity and clearness. For instance, every section has a point for which the lift moment is independent of the incidence. The curve inclosing all the positions of the lift resultants is usually a parabola, which can be reduced to a cluster of rays. Within these limits, there are sections, with fixed pressure center, which are consequently removed from the general class of sections by a single condition.

In the second section, I complete and correlate the valuations, proceeding from the modern theory of functions (the circle of ideas of the "distortion laws"), for the parameter determining the lift, and explain their significance in the case of the delineation of the arc. The third and fourth sections treat of the application of the theory to the case of the Joukowski

* From "Zeitschrift für Flugtechnik und Motorluftschiffahrt," March 15, 1920, pp. 68-73, and March 31, 1920, pp. 87-89.

** First article, November 29, 1917, pp. 157-163. This article will henceforth be referred to as "I".

section construction and its extension by Von Karman and Trefftz (this publication, 1918, pp. 111-116). The fifth section outlines a common procedure, illustrated later by examples, which makes it possible to discover any desired number of parametric groups of sections, or of suitable delineation functions. In this connection, there is a considerable mathematical difficulty, the question of the criterion for the "smoothness" of the delineation, which was gotten around by a simple artifice adapted to its graphic execution, in the formulation of which I enjoyed the valuable assistance of Dr. L. Bieberbach in Frankfort. It, moreover, demonstrates in perfect harmony with experience, that the S-shaped upward bend of the section, near the following-edge, lessens the variation in the center of pressure, or may eliminate it altogether.

In a third and concluding article, I expect to take up the problem of finding the determining parameter for a given wing section.

1. Magnitude and Location of Lift.

In my first article, among other things, including the customary postulations of eddy-free horizontal motion, the following was demonstrated. Given any wing section containing a point or angle (Fig. 1). At the angle α with an x-axis rigidly connected with the section (angle of incidence), the section is struck by an air current which has the velocity u, and the specific mass μ that is $\frac{\gamma}{g}$, or approximately 1/8, under normal conditions.

There then hold, for the magnitude and moment of the lift (the resulting pressure perpendicular to the direction of the air current), simple formulas independent of the section shape, into which the section itself enters with only a few parameters. The equation for the power magnitude (Compare 29 in Art. I.) then reads:

$$A = 4 \pi \mu u^2 a \sin (\alpha + \beta) \quad . . . . (1)$$

with a and β as parameters, and the equation for the moment, with reference to a given origin M, whose coordinates thus belong also to the parameters of the section (Compare 37 in Art. I.), reads:

$$M_1 = 2 \pi \mu u^2 c^2 \sin 2 (\alpha + \gamma) \quad . . . . . (2)$$

with c^2 and γ as additional parameters. The origin for the moment to which equation (2) refers, was called the "middle point of the section" and that direction of the attacking air (determined by $\alpha = - \beta$ or by $\alpha = - \gamma$) for which respectively A or M_1 vanishes, was called respectively "first" or "second" axis of the section" (Art. I, section 7).

Still greater simplicity and clearness can be obtained by **choosing, instead of M, a new reference point F, which is** found in the following manner. From M we measure off a distance $MF = \frac{c^2}{a}$ on a line which is turned toward the x-axis in the positive direction by $2\gamma - \beta$, so that the second axis of the section bisects the angle between the first axis and MF. In Fig. 1, $\beta > 2\gamma$ was taken and accordingly $\beta - 2\gamma$ was represented as turning in the negative direction. The moment with reference to F, which we may now call M, is obtained by adding to M_1 the moment of any power A considered as affecting M, and hence with reference to the direction of rotation.

$$M = M_1 - A\frac{c^2}{a}\cos(\alpha - \beta + 2\gamma) =$$

$$2\pi\mu u^2 c^2 [\sin(2\alpha + 2\gamma) - 2\sin(\alpha + \beta)\cos(\alpha - \beta + 2\gamma)]$$

The second expression in the square brackets can be reduced, according to the "identity" $2\sin\varphi\cos\psi = \sin(\varphi + \psi) + \sin(\varphi - \psi)$, to $\sin(2\alpha + 2\gamma) + \sin(2\beta - 2\gamma)$, hence

$$M = -2\pi\mu u^2 c^2 \sin 2(\beta - \gamma) \quad . . . . . \quad (3)$$

This equation no longer contains the variable incidence α. We have hereby demonstrated the proposition that: <u>For every wing section there is a certain point for which the lift moment is independent of the incidence. We will call this point the "focus of the section."</u>

From (1) and (2) is calculated the lever arm h of the lift to

$$h = \frac{M}{A} = -\frac{c^2 \sin 2(\beta - \gamma)}{2a\sin(\alpha + \beta)} = -\frac{h_o}{\sin(\alpha + \beta)} \quad . . \quad (4)$$

when

$$h_o = \frac{c^2}{2a}\sin 2(\beta - \gamma) \quad . . . . \quad (4a)$$

is taken. Since the lever arm of F must be extended in the direction of the attacking air (Fig. 2), h_o represents the magnitude of its projection on the line perpendicular to the first axis. The projection of the lever arm on this line is therefore unchangeable, or: <u>The feet of the vertical lines, from the "sec-</u>

tion focus" to the attacking lines of the lift, lie in a straight line, which, at a distance h_o from F, is parallel to the first axis of the section and, moreover, bisects the distance MF of Fig. 1. Therewith we have reduced the determination of the attacking line for any given angle of incidence to the simplest imaginable construction.

It is a well known characteristic of ordinary parabolas, that the "foot-point-curve" for their focus coincides with the vertical tangent. Anyone who is not familiar with this fact may make the following calculation. For a system of coordinates with the origin at F and with F'F for the x-axis (Fig. 2), the equation for the attacking line A reads:

$$x \sin(\alpha + \beta) + y \cos(\alpha + \beta) = h = -\frac{h_o}{\sin(\alpha + \beta)}$$

and simplified:

$$-x \cos 2(\alpha + \beta) + y \sin 2(\alpha + \beta) = -2h_o - x.$$

Differentiated according to 2α:

$$x \sin 2(\alpha + \beta) + y \cos 2(\alpha + \beta) = 0$$

and with α eliminated from this and the preceding equation by squaring and adding, we obtain:

$$x^2 + y^2 = (2h_o + x)^2 \text{ or } y^2 = 4h_o(x + h_o) \quad . . (5)$$

Hence: The possible positions of the attacking lines of the lift generally inclose a parabola, whose focus is F, whose parameter is $2h_o$ and whose axis is perpendicular to the first axis of the section.

With $h_o = 0$, the parabola is reduced to the bundle of rays through F. This is the case of the vanishing variable pressure point. There is a fixed middle pressure point, through which the resultant lift always passes, the so-called "diving moment" (for $\alpha = -\beta$) being 0. It is noteworthy that the section with a fixed pressure center is separated from all other sections by only a single equation $h_o = 0$. Thereby the disappearance of h_o (as shown by equation 4) shows that $\beta = \gamma$, or that the first and second axes of the section are parallel. In the quantity h_o we have a definite measure of the pressure point variation (or migration). It is readily seen, for example, from equation (2), that the straight line drawn through the "center" M in the direction of the second axis and, barring exceptional cases, also the straight line drawn at right angles to it, touch the parabola*.

2. Section Parameters.

Only five of the six section parameters, considered in my first article, are found to be essential, since c^s and γ appear only in the expression $c^s \sin 2 (\beta - \gamma)$. The following five

* General principles concerning the location of lift resultants are also given by R. Grammel, in "Die hydrodynamischen Grundlagen des Fluges" (The hydrodynamic principles of flight), Braunschweig, 1917, p.13. Here the task is reduced to the determination of a "moment and center of gravity of the circulation." I cannot understand the deductions in this book. It can, however, be demonstrated that a "moment of the circulation," as a quantity independent of the integration method, can only exist, when the integration is made concerning closed stream (or level) line. Now, since the lift-producing current comes out of infinity and goes into infinity, these considerations are of little value.

quantities may be regarded asdetermining factors for the lift relations of a section: for the moment of the lift, the length a and the angle β (direction of the first axis); for the moment or location of the lift, the coordinates of the section-focus F and the length h_0 or, instead of the last, the location of the first axis (which passes through M) at the distance $2 h_0$ from F (base line of parabola). How these five quantities are to be ascertained for the given section follows from my first article, and I will indicate it briefly here.

If we regard the coordinates x and y of a point in the plane of the section as components of a complex quantity

$$z = z + iy,$$

then there is one, and only one, evolvable complex function ζ of z in the form (Equation 7 in Art. I).

$$\zeta = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots \ldots \ldots \quad (6)$$

through which any point of a circle is coordinated with every point of the section perimeter and any point outside the circle (7), with every point outside the section, in constant and definitely convertible manner. The radius a of the circle and the coordinates (contained in m) of its middle point M are through

$$|\zeta - m| = a \ldots \ldots \ldots \quad (7)$$

the coefficients of a, a ...... of (6), definitely determined, aslikewise every point B of the circle into which the section tip is drawn. Thereby the parameter values are found. The radius a is the parameter referred to above under the same designation. BM is the first axis of the section and, hence β is the angle formed by BM with its x-axis. The focus F is finally obtained, by making the vector MF equal the complex quantity $-\frac{a_2}{a}e^{-\beta i}$ (according to Equation 34 in Art. I and the above procedure).

In order to obtain a cursory survey of the parameter values and of the location of the invariable points and straight lines, we will next consider the arc as the approximation, or better, as the limit, of a wing section. Strictly considered, the "infinitely thin" arc does not fall under our conception of a section, because it does not possess simply one place of variable tangent direction, but two of them. We can only dispose of it in the sense of our consideration, by disregarding one edge (the right one, for example) and not letting ourselves be disturbed by the fact that the calculated current is unrealizable on this edge. The results are accordingly significant only in that they give the approximate position (of the focus, for example) for a very thin section, closely fitting the arc, which section has its tip in the left corner.

The complex process, which transforms a circle into a doubly traversed arc, is known and may be expressed in the form

$$z = \zeta + \frac{c^2}{\zeta} \quad \ldots\ldots\ldots\ldots (8)$$

when applied to a circle passing through the points $\zeta = \pm c$ (B, B_1 in Fig. 3). Fig. (8) comes under the form (6) desired by us since its solution according to ζ, by development according to the descending powers of z begins with the members.

$$\zeta = z - \frac{c^2}{z} \text{. . . . . (8')}$$

In the transformation, the end points $z = \pm 2c$ of the arc correspond to the points $\zeta = \pm c$ on the x-axis. To any given point A of the circle (Fig. 3) and to the A_1 on the symmetrical curve (Fahrstrahl) (so that $\measuredangle$ OA, x = $\measuredangle$ x, OA_1), there corresponds one and the same point A', which is obtained by the addition of the vectors OA and OA_1. For, according to the principle of the constant "power of a circle," the length of OA_1 is like that of OA_2, like c^2 through the length of OA, hence with OA = ζ, as the complex equation: $OA_1 = c^2 : \zeta$. That, in fact, the point A', found by this process, falls on an arc, is readily seen from the following.*

If the length OA is represented by ρ and the angle between OA and the x-axis by ψ, the coordinates of A' are according to the equations:

$$x = (\rho + \frac{c^2}{\rho}) \cos \psi, \; y = (\rho - \frac{c^2}{\rho}) \sin \psi,$$

from which we obtain:

$$x^2 \sin^2 \psi - y^2 \cos^2 \psi = 4 c^2 \sin^2 \psi \cos^2 \psi$$

On the other hand, the perpendicular MA_0 from the middle point M to OA shows that OA_0, half the difference between OA and OA_2, consequently of ρ and $\frac{c^2}{\rho}$, has the value $s \sin \psi$, when s = OM denotes the distance of the center of the circle from O. There follows, therefore, from the expression given for y:

$$y = 2 s \sin^2 \psi,$$

and if this value for $\sin^2 \psi$ is substituted in the preceding

* Another proof, more closely connected with the conceptions of complex calculation, follows from the deduction given in Sec. 4.

equation and simplified:

$$x + (y + \frac{c^2 - s^2}{s})^2 = (\frac{c^2 + s^2}{s})^2 \quad . . . . . \quad (9)$$

as circle equation for position of A'. Thereby, since $s > o$ and $\sin^2 \psi > o$, y can only have positive values. The arc has the camber 2s, the chord 4c, the radius $\frac{c^2}{s} + s$ which is readily constructed, and the half center angle $\sin \frac{2\,c\,s}{c^2 + s^2}$.

The task of finding the parameter (as also the inverse) for a given arc B' B", with chord 4c and camber 2s, as the limit of a section with its tip at B', is executed as follows. Divide the chord B' B_1' into four parts at B, O, B_1 and bisect the vertical line OS at M. Then B represents the tip and M the center of the section. Therewith the radius $a = \sqrt{s^2 + c^2}$ and the direction BM of the first axis with $tg\ \beta = s:c$ are found and the lift magnitude is thus determined. In order to obtain the moment-course, after the location of the first axis is already known, we only have to find F and, for this purpose, according to (8'), the distance $MF = c^2 : a$, under the angle $-\beta$, is measured off on the line MB_1. F is found constructively in the simplest way (Fig. 3) by erecting at B_1 the vertical line s perpendicular to the chord and projecting its terminal point on MB_1. The projected line contains, in addition to the focus F, also the center M' of the arc. The straight line BM is the base line of the lift-parabola and of the parameter h_o, equal to half the distance between F and BM.

In Fig. 4 there are represented several arcs, which are developed from the same circle by choosing different starting points O, with their axes, parabolas, etc. If the curve becomes O, then the "straight line" section coincides with its first axis. The focus is situated on this line at 3/4 of the distance from the left end and the parabola is reduced to the bunch of rays, which means that all attacking lines pass through F. The greater the curvature, just so much farther F moves from the section surface toward the interior, but always remains at about 3/4 of the chord. In Fig. 5 the so-called "pressure point migration" is represented, with reference to the chords for the curvature relations 0, 0.1, 0.2, and 0.3, in the customary form. We can see how it increases with increasing curvature. Since there is an endless number of different parabolas, all possible cases within the series of arcs are in a certain sense exhausted.

For given section forms, the parameter values, first known from the theory of "consistent imitation," are subjected to the following inequalities, which furnish practically valuable reference points for their determination. If we supplement the data in my first article by a restriction for the location of F from the same source (Bieberbach's "Law of Surfaces") and, further, by the adoption of an old law (from Landau and Toeplitz) as well as of a continuation according to my first article, of the hydrodynamic theory by Frank and Lowner and, lastly, of a new work by G. Pick on "Consistent Imitation," we then obtain the following determination methods.

1. For the value of the radius a: The radius of the curve of a section determined by (6) is at least equal to 1/4* and at most to 1/2** of the longest diameter of the section. For the lift coefficient, with reference to the greatest width b, there are obtained therefrom the inequalities

$$\pi \sin (\alpha + \beta) \leqq \frac{A}{\mu u^2 b} \leqq 2 \pi \sin (\alpha + \beta) \quad (10)$$

In the case of very narrow sections, the lower limit is only slightly exceeded.

2. For the location of the center M: The center of the curve of a section ("section center") is located so that the whole sections falls within the circle described about it with the radius 2a***. Furthermore, M lies within the smallest convex space**** inclosing the section (like the center of gravity of a surface on the section perimeter). The first condition in the case of very narrow sections is exactly fulfilled, that is, the ends of the section lie near the perimeter.

3. For the location of the focus F: The above defined focus of the section (focus of parabola inclosing the lift attacking lines) lies within the perimeter (Bieberbach, a.e. O., prin-

* This statement is contained in the first of the principles mentioned under 2.

** Landau and Toeplitz, Arch. f. Math. and Phys. II (1906), pp. 302-307; compare also Frank and Lowner, "A unten a.O."

*** L. Bieberbach, Sitzungsber. d. Berliner Akad. XXXVIII (1916) pp. 940-955, principle V.

**** Frank and Lowner, Math. Zeitschr. 3 (1919), pp. 78-86. Here the following is demonstrated: If the surface is considered as uniformly covered with substance and if, in the transformation, each point is allowed to retain its mass, the center of gravity then remains at M.

ciple I). In the case of narrow sections, it is not far from the edge of the circle. Its distance from the circumference is greater than $r^2 : a$, when r denotes the radius of the circle inscribed in the section (G. Pick, Sitzungsber, d. Wiener Akadamie, math. naturw. Klasse, Abt. IIa, 126 (1917), pp. 247-263, equation 4).

According to the results obtained by L. Bieberbach and G. Pick, the first principles stated under 1 and 2 can be made still broader: not only the section itself, but every circle which is created by concentric duplication of a circle lying entirely within the section, falls in the duplication of the perimeter (Pick, a.a.O. eq.9). Further, when r denotes the radius of a circle entirely within the section and d the distance of its center from any given point of the section perimeter, then

$$a > \frac{1}{4} \frac{(d+r)^2}{d} \quad \ldots\ldots\ldots\ldots \quad (11)$$

while the first statement under 1 only claims substantially that $a > 1/4\,(d+r)$. (Pick, a.a.O. eq. V. and Bieberbach, Math. Annal 77 (1916), pp. 153-172). Therefrom, for example, the conclusion may be drawn that a thickening of the front end (leading edge) of a section raises the lower limit of the lift coefficient. In Fig. 6, where b denotes the greatest width of the section, it is possible to construct, in an easily understood manner, for the radius of the rounded front edge, the length $b' = (d + r)^2 : d$, which is greater than b and holds good for a $1/4\, b'$.

The parameter value and therewith the lift magnitude and location are naturally only incompletely determined by all these principles. The exact determination, for one of its form according to the previously given section, requires, each time, the solution of the problem of "consistent imitation," that is, the determination of the function (6) belonging to the section. My next article will take up this task. In the present article, I will only take up the inverse task of determining sections for given parameter values. We will first consider the known examples theoretically, that is, sections defined by their delineation functions.

3. The Joukowski Section Form.

The Joukowski section form is known to be produced, when the transformation (8) is applied to a circle K which passes through the point $\zeta = -c$ (B in Fig. 7) and embraces the point $\zeta = c$, near its edge (B_1 in Fig. 7). Since a circle (with center M_0, dashed in Fig. 7), passing through both points and hence transformable by (8) into an arc, can be so placed that it touches the given circle at $\zeta = -c$, then the image of the latter, in the vicinity of $\zeta = -c$ must appear as the left end of the arc. The Joukowski section has, on the left, a point, which stands at an angle of arc sin $\frac{2cs}{c^2+s^2}$ to the direction of the chord, in which s denotes the segment OM_0 of the straight line BM on the axis of ordinates. Moreover, the section approaches the arc of a circle more closely, the nearer the point B_1 is to the circumference.

For the construction of the section, we note that when the point A, as the end point of the vector $OA = \zeta$, runs through a circle K, also the end point A_1 of the vector $OA_1 = c^- : \zeta$ must move on a circle K_1. The circle K_1 proceeds from K, through the transformation of "reciprocal radii" and the corresponding reflection on the x-axis, and touches K at the point B. Consequently, the center M_1 of K_1 is found immediately, by taking the direction OM_1 on the left symmetrically to OM and locating M_1 on BM. The point A_1 (corresponding to A) of the section perimeter is found, as in the case of the arc formation, as the sum of the related vectors OA and OA_1, which again lie symmetrically with the x-axis, but now have their end points on different circles. In order to construct the complete section, after finding K_1, it is only necessary to draw through O a bunch of symmetrical lines, for example, all lines at intervals of 30°, and thus obtain as many points of the section as desired, by simply drawing parallel lines or intersections* (Fig.7). Only the points falling on the axes must be taken by the compasses.

It is often useful to know the tangent direction of a system of points for constructing the curve. I will here give the following very simple construction without demonstration. Erect at A and A_1 the perpendiculars to the lines OA and OA_1 and mark on them the points N and N_1 so that ON is parallel to MA and ON_1 is parallel to $M_1 A_1$. Then $N N_1$ gives the direction of the normal perimeter at the point A'.**

* E. Trefftz gave a similar method of construction. See this publication, Vol. IV, (1913), p.131.

** The proof follows from the general theory of "geometrical differentiation," which shows how to find the tangents to any curve defined by point construction and which I developed in Zeitschr. f. Math. in phys. 52 (1905), pp. 44-85.

For a Joukowski section obtained in this way from its "image circle" (Bildkreis) K, the lift parameter is immediately deducible. The radius of K is the length in formula 1, and BM is its direction and location with reference to the first axis, hence the angle of this line with the x-axis of the angle β. The second axis has the x-direction, since, on account of the real value of $c^2 = - a_1$, the angle γ vanishes. The focus F is found by taking $MF = c^2 : a$ on the line going from M to the second intersection point of K with the x-axis. The best way is to take on this line (as also on the line MD drawn through M parallel to the x-axis) the distance MC = MC' = c and then draw CF parallel to DC'. The base line of the parabola is BM and therefore the parameter h_0 equals half the distance between F and BM.

If it is desired to follow up the connection between the pressure point migration and the shape of the section, a good way is to imagine all those sections which are derived from different circles through this transformation process (with fixed c). Thereby the "first axis" BM remains unchanged, as likewise the direction MF, while the distance between F and the axis varies according to the ratio $c^2 : a$, hence diminishing with the increase in the length of the radius. Since the section becomes thicker with increasing <u>a</u> (the circumference receding farther and farther from the point B_1), it is evident that a thickening of the section lessens the pressure point migration and vice versa We have seen above that, in the arc, the distance of the focus from the axis grows with increasing curvature. We may now say, therefore, that the curvature and thickening of the section exert a compensating effect with reference to the pressure point migration. In Fig. 8 there is drawn a series of three Joukowski sections which (even in their location) belong to the same lift parabola. It is seen that, while the thickness varies greatly, the curvature (underneath) only varies slightly.

Joukowski sections with fixed center of pressure are obtained only when the curvature (underneath) is zero. These are the club-shaped cross-sections which come into consideration for stays, etc., but not for supporting-sections.

4. Enlarged Joukowski Sections.

Th. v. Karman and E. Trefftz (this publication, Vol. IX, 1918, pp. 111-116) have, in connection with a remark of Kutta (Sitzungsber d. bayer. Akad. d. Wiss. Math.-Physik. Klasse 1911, p. 77), somewhat extended the transformation (8) applied by Joukowski to the construction of the section and confirmed, for their case, by direct calculation, the general formulas repeated in (1) and (2) for lift magnitude and moment. Their expression for the moment is considerably more complicated, simply on account of an unfavorable choice of their reference point.

Equation (8), by once adding 2c and once subtracting 2c and finding the quotient, may be brought to the form

$$\frac{z - 2c}{z + 2c} = \left(\frac{\zeta - c}{\zeta + c}\right)^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (12)$$

in which it is seen, even without the calculation given in section 2, that there corresponds to the circle going through B and B_1 in the ζ - plane (Fig. 9) a doubly intersected arc through the points $z = \pm 2c$ (B' and B'_1 in Fig. 9). Then, when A is anypoint (designated by ζ) of the first-named circle, the complex numbers $\zeta \pm c$ are represented by the vectors BA respectively. B_1 A, and when A' is coordinated with A in conformity with (8) and (12), the numbers $z \pm 2c$ are represented by B'A' respectively. B'_1 A! Since the quotient of two complex numbers in the angle calculation has the angle of both vectors representing the numbers, then (12) signifies that $\measuredangle\ B'A'B'_1 = 2 \measuredangle\ BAB_1$. Now an arc is known to be characterized by the property of possessing a constant angle at the circumference, so that the arc BAB' is transformed into an arc $B'A'B'_1$ of double the angle at the circumference. The lower arc BB_1 of the original circle has the angle at the circumference which supplements the upper angle to 180°, so that the double angles accordingly combine to give 360° and lead to the same arc over $B'B'_1$.

Equation (12) can be changed so that the reciprocal relation of certain arcs remains the same, without, however, the same arc's corresponding twice to the circle BB_1 . If we write, namely,

$$\frac{z - c'}{z + c'} = \left(\frac{\zeta - c}{\zeta + c}\right)^n \qquad (13)$$

in which c' denotes a still-to-be-determined constant, and n one of two different exponents, then to each arc through BB_1 there is a corresponding arc through $z = \pm c'$ with the n-fold angle at the circumference. The arcs, which correspond to both parts of a circle through BB_1, consequently have angles at the circumference which differ by $n\pi$ and therefore intersect, if n is a little larger or smaller than 2, at the acute angle $\pm(n-2)\pi$. The constant c' must be determined from the condition that (13) assumes the form established in equation (6), namely, that the infinity in z and ζ are identical. If the numerator and denominator are shortened in (13) by z resp. ζ and developed according to I/z resp. I/ζ, the first members read

$$I - \frac{2c'}{z} \;\ldots\ldots = I - \frac{2nc}{\zeta}$$

so that $c' = nc$ must be written in order that $z = \zeta$ to infinity. In fact, with this value of c (and with ζ developed according to the falling powers of z, the solution of the problem gives

$$\zeta = z - \frac{n^2 - I}{3}\,\frac{e^2}{z} \;\ldots\ldots \quad \ldots\ldots \qquad (14)$$

Thus the coefficient a_1 of I/z is real, so that $\gamma = 0$, that is, the "second axis" of the section has the direction of the x-axis.

The crescent, into which the circle through BB_1 is converted by the transformation (13), represents no section in our sense, because it presents not one but two "singular" locations. But we can, in a very similar manner to that of the Joukowski transformation, obtain a correct section, by applying (13) to a circle which contains only B on its circumference and B_1 inside near the circumference. These are the sections which Von Karman and Trefftz considered. In Fig. 10, a section of this kind is drawn, with $n = 1.95$ and with the aid of a calculated point. The shape of the perimeter is, in a general way, similar to the Joukowski section, excepting that the left end doesnot represent a point but a very acute angle, even with the angle $-(n-2)\pi = 0.05\pi = 9^{\circ}$.

The construction of the section, with the small values of $n - 2$, is not very simple. It is best accomplished with the aid of two groups of circles, as suggested by Karman and Trefftz.

Of the parameters, which determine the lift, the length $\underline{a}$, as the radius of the base circle; then the "first axis," according to the location and direction of BM; and, lastly, the direction BO of the second axis (real a_1) are given directly. In order to find the focus F, we must take the length

$$MF = -\frac{a_1}{a} = \frac{n^2 - 1}{3}\ \frac{c^2}{a}$$

from the center M at the angle $-\beta$ thus on the straight line, which goes from M to the second intersection point of the circle with the x-axis. Thereby the lift relations are fully determined. Reversed, we can see how (with the assumption of a straight line BM as base line and a point F as focus of the lift parabola for the given $\underline{a}$, β and n) to find the location of M and B and from them to construct the section. In Fig. 10, the lift parabola is shown for the traced section.

If we wish to estimate the dependence of the pressure point migration on the incidence of the section, we must vary the value of N, without varying the circle. There then remain stationary both the "first axis" BM and the line MB_1, on which F is located, and the parameter h_o of the lift parabola increases in proportion to MF and consequently to $n^2 - 1$. But now it must be remembered that the length of the section increases approximately with n, because with $c' = nc$, the abscissas of the single points of the figure vary in this proportion. Hence, the comparative change in the pressure point migration is determined by $\frac{n^2 - 1}{n}$ and it will be advantageous to let n fall below the Joukowski value. With an incidence of 12%, we have $n = 2 - 1/15$ and therefrom the change of the pressure point migration $\frac{n^2 - 1}{n}$, corresponding to an improvement of 5.4%.

In connection with the above, we may now conclude that lessening the curvature, increasing the thickness and increasing the incidence exert a favorable influence on the pressure point motion. Sections with fixed center of pressure are obtained here, however, as in the original Joukowski case, only with the curvature zero, namely, with symmetrical "stay" sections.

5. General Procedure.

The Joukowski section construction, including the above mentioned extension, can only be considered as a very special example of sections, for which the general laws of lift hold good, as set forth in section 1. Three characteristics (curvature, thickness and incidence) can indeed be varied at will, but in contrast with the possible multiplicity of shapes, only a very limited number has been obtained. As sections with fixed pressure center, for example, there have been obtained only the symmetrical shapes unsuited for supporting surfaces. Finally, its whole development, however valuable and fruitful it may have been for building up the theory, shows, in its relation to the special arc portrayal a one-sidedness which is not grounded in the nature of the problem and from which we must be completely freed, if we are to obtain the full value of the theory.

We start out with the supposition that the reverse of each delineation of the form (6) applied to a circle (7), if the coefficients a_1, a_2, a_3, ... only fulfill certain conditions, must lead to the section of a supporting surface. These conditions are: 1. The circle (7) must be converted into a simple, closed, double-point free curve, which answers the common form requirements which may be made of a supporting-surface section; 2. The portrayal must in the whole outside space of the circle, be simple, that is, reversible; 3. To one point of the circle there must correspond, in any given case, a point or angle of previously determined opening; 4. The lift-determining parameters must have approximately predetermined values.

The last point is most simply executed, after the preceding one. For, when the circle, whose image the section is to be, and, on it, the point B, which is to be converted into the section tip, is arbitrarily chosen, there only remains the location of the focus, for determining the lift. This location depends, however, according to section 2, only on the first coefficient a_1 which is thus alone partially determined. With regard to condition 3, it need only be borne in mind that at the chosen point B, the deduction $\frac{dz}{d\zeta}$ must vanish, which signifies an easily prescribed condition for the coefficients, especially when this disappearance is of a definite order. The real deciding difficulty lies in condition 2, since it forms a known and not yet fully solved problem of the theory of functions, for recognizing the reversibility of the delineation on the properties of the coefficients. We can, nevertheless, easily get around this difficulty, and indeed in connection with the consideration of condition 1. Naturally, one can not make sure of the shape of a section, without actually drawing it once. From the nature of the thing, it is not possible to give definite rules, without first being obliged to try out one or another hypothesis. Therefore it would be of no great practical importance for us to be able to know positively, in advance, that the construction will give a double-point line. Our solution now consists therein that we, in the decision on the reversibility of the diagram, proceed on the assumption that the section perimeter, and hence the representation of K, has already been drawn and found to be free from the double-point. The following statement (whose simplification, as compared with my original proposition, is due to a friendly suggestion of Prof. Bieberbach) therefore holds good. As the reverse of (6) let

$$z = \zeta + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} + \ldots \qquad \ldots\ldots\ldots (15)$$

a representation, which converts the circle $\zeta - m = a$ into a simple closed double-point free line. The zero points of $\frac{dz}{d\zeta}$ may all lie within the circle, excepting the one that lies on the circumference itself. Then the representation in the whole outside space of the circle is reversible, when the origin $\zeta = 0$ falls inside the circle.* As to how the principle is applied will be immediately evident in the description of the whole construction process.

We will confine ourselves to the case in which the incidence is zero at the rear end of the section, which accordingly constitutes a true point. In this case, the deviations of the entire section, in comparison with one having an angle of 6 to 12°, are practically negligible. Moreover, our method can be employed, even for the more common assumption, as soon as, instead of ζ and z, the corresponding powers of these variables or their combinations are properly applied.

The coordinates are so placed in the ζ plane that the circle-point B, which is converted into the section tip, has the negative abscissa -c and the ordinate zero, as in the Joukowski ex-

* The principle may be made intelligible as follows: From the given condition, in harmony with the fact that in infinity z is opposed to ζ, it follows that a sufficiently large circle K', concentric with K, is formed with a double-point-free circumference P' of any desired size, in which some point outside of K corresponds to every point outside of P'. Now, any desired point z, which lies between the section outline P and the outer circumference P' can always be converted into a point in the space outside of P'. Thereby the number of the points, coordinated with z outside of K, cannot be changed, because there are no branching points of (15) in the space outside of K and because no passage of the image point through K can take place, so long as z does not pass through P. An exact proof for the given case can be fig-

ample. The circle K which is to produce the section, must therefore, pass through this point $\zeta = -c$, while inclosing the point $\zeta = 0$. The delineation function $f(\zeta)$, which above was a binomial (8), we now write in $n + 1$ terms.

$$z = \zeta + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} + \dots \frac{c_n}{\zeta^n} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (16)$$

The few limitations, to be made for the $c_1 \dots c_n$, still leave sufficient play for their application to the most widely differing requirements. We must next find the deduction from (16)

$$\frac{dz}{d\zeta} = I - \frac{c}{\zeta^2} - \frac{2c}{\zeta^3} - \dots - \frac{nc_n}{\zeta^{n+1}} \quad . \; . \; . \; . \; . \; . \quad (17)$$

The right side is a product of $n + 1$ factors of the form $(1 - \frac{v}{\zeta})$, in which one of the v's must have the value $-c$, while the total of the v's is subject to the limitation, that their sum vanishes, so that, in the product, the member with $\frac{I}{\zeta}$ drops out. Accordingly, we proceed so that we (instead of immediately choosing the quantities $c_1 \dots c_n$) adopt the points v_1, $v_2 \dots v_n$ within the circle K with the contraction.

$$v_1 + v_2 + \dots v_n = c \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (18)$$

(Fig. 11 with $n = 2$) and from these chosen v's construct the polynomial (17), as follows:

$$(I + \frac{c}{\zeta})(I - \frac{v_1}{\zeta})(I - \frac{v_2}{\zeta}) \dots\dots (I - \frac{v_n}{\zeta}) -$$

$$I - \frac{c_1}{\zeta^2} - \dots \frac{nc_n}{\zeta^{n+1}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (19)$$

(Continued from p.24)
ured out by analogy with Bieberbach's demonstration (Funktionentheorie, Sammlg. Goschen, No.768, p.56) of a similar proposition.

The values of the individual quantities $c_1 \ldots c_n$ are obtained by putting like powers on both sides, for example:

$$-c_1 = v_1 v_2 + \ldots v_{n-1} v_n - c(v_1 + v_2 + \ldots v_n) =$$

$$= \Sigma v_l v_k - c^2 \quad \ldots \ldots \ldots \ldots \quad (20)$$

Hereby the following is accomplished. The point R is converted into a tip and all other junction points lie within the circle. Then the delineation function has the form (16), that is, the coefficient of the linear member is 1 and that of the constant is zero. If the amount of the pressure point migration, and hence of the parameter h_o of the lift parabola, is predetermined, then the quantities $v_1 \ldots v_n$ must only be subjected to the second restriction, in order that the complex quantity $\frac{c_1}{a} c^{-\beta i}$ (in which a represents the radius, β the "angle-argument" of the distance $-c + m$ from B to the center M), drawn from the center M to F, may have the desired distance $2h_o$ from BM.

Naturally, it is not necessary to take the circle K and distribute the points $v_1 \ldots v_n$ within it in advance. It is better, namely, when h_o is predetermined, to make first an assumption for the first values of $n - 1$ of the v, then to calculate the nth value from (18), and then c_1 from (20) and lastly choose the circle so that it will inclose the v's and will satisfy the conditions with respect to the focus F. If, for instance, we wish to obtain a section with a fixed pressure center, then it is only necessary to locate the center M of the circle, so that BM will have the direction of $\sqrt{c_1}$ (that is, $\beta = \gamma$, with the first and second axes parallel).

After the circle K and all the v's have been determined, then the section outline can be located point for point, as the representation of K. Thereby it is to be recommended, namely, for the first trial, to represent only a few points of the circle (measured from O) about 30° apart. The individual "summands" of (15) are comparatively easy to determine and to combine, partly by calculation and partly by construction, as we will further illustrate by examples.

The chief practical difficulty always consists in finding such values for the v's that utilizable shapes will result from them. To this end, a certain skill can be acquired through the execution of many examples. A reference point may be suggested by the remark that the crowding of the v's in the vicinity of the zero point (excepting the one which on account of "18", must lie at $\zeta = c$) leads to the Joukowski forms.

6. Examples. Sections with Pressure Center.

As the first example, we will construct a section of the form $n = 3$, without any previous instructions in regard to the lift values. We accordingly choose (Fig. 12), on the real axis, two points B and O, at a distance $c = 0.4$ (a length of 100 mm. was taken as the unit in the original figure) and a circle K going through B and inclosing O with the center M and radius $a = 0.44$. The angle included between BM and the axis is the angle β of Section 1. BM is the first axis of the section and M its "center". We choose the three v-values as follows:

$$v_1 = c, \; v_2 = - v_3 = \frac{c}{2} e^{\frac{\pi i}{4}}$$

so that they satisfy equation (18) and, as shown by Fig. 12, the end points V_1. V_2 and V_3 of the vectors $OV_1 = v_1$ etc. lie within K. Equation (19) is now converted into

$$(I - \frac{c^2}{\zeta^2})(I - \frac{v_2^2}{\zeta^2}) = I - \frac{c_1}{\zeta^2} - \frac{2c_2}{\zeta^3} - \frac{3c_3}{\zeta^4}$$

from which proceeds

$$c_1 = c^2 + v_2^2 = c^2 (I + \frac{i}{4}), \; c_2 = 0, \; c_3 = - \frac{c^2 v_2^2}{3} \quad \frac{c^4 i}{I2}$$

Hence the transformation equation reads:

$$z = \zeta + \frac{c_1}{\zeta} + \frac{c^3}{\zeta^3} = \zeta + \frac{c^2 \quad I + \frac{i}{4}}{\zeta} - \frac{c^4 \; i}{I2 \; \zeta^3}$$

in which $c = 0.4$ is to be introduced. The solution of this equation is best accomplished, partly by calculation and partly by construction.

We next determine the value of c_1 and the direction of $\sqrt{c_1}$. For this purpose we took, in Fig. 12, $BC = - \frac{c}{4}i$, so that $OC = - \frac{c_1}{c}$, and bisected the angle BOC, whereby the angle γ of Section 1 and therewith the direction of the second axis of the section was found. Counting from this second axis, the radial lines ζ were drawn at every 30° toward the points A of the circle K (only one is actually traced, simply the terminal points of the others being marked). The end points of the radial lines $c_1 : \zeta$ lie, according to the known characteristic of transformation through "reciprocal radii," again on a circle K_1. Its center M_1 lies on a straight line OM_1, which is situated symmetrically to OM with reference to the perpendicular to the second axis. One point of K_1 is C and another one can easily be obtained from the proportion

$$\zeta : c = OC : \frac{c_1}{\zeta}$$

After K_1 has been obtained, the points $\zeta + \frac{c_1}{\zeta}$ can be located by combining the symmetrically situated vectors OA and OA_1, just as in the Joukowski case. In Fig. 12, the construction is completely carried out for one point and the rest of the 12 points of K_1 are simply indicated, so far as the size relations allow. Now each point A' must be shifted $c_3 : \zeta^3$. The direction of c_3 is that of the negative imaginary axis. Hence the shifted direction is obtained approximately for the point with the amplitude $\gamma - 30^\circ$, by taking the angle $180 - 3\gamma$ from the real axis. The construction is easily completed for the remaining points since the direction changes in each instance by $3 \times 30 = 90^\circ$. On account of its smallness, the length of the displacement is best determined by calculation. It is $c_3 = 0.00213$ and hence for the just named point A, with $\zeta = 0.514$, the length $A'A'' = 0.0156$ and we must take $A'A'' = 1.56$ mm. according to the chosen unit of measure. Thus 12 circumference points are definitely determined in Fig. 12, as also, in very simple fashion, both the points, right and left which proceed from the intersection points of K with the real axis.

If we draw through M the parallel to the second axis and transfer its angle with the first axis to the lower side of this parallel, we thus obtain the direction MF toward the focus of the section, on which $M = c_1 : a = 0.375$ is to be taken. With F as the focus and BM as the base line, the lift parabola is determined. It is seen that the parabola is very flat (somewhat similar to that for the much thicker Joukowski section of Fig. 6) and that the pressure point migration is small, which is connected with the S-shaped upward bend of the rear end. Had we wished to obtain a section with fixed pressure center, it would only have been necessary, after determining c_1, to draw the line BM parallel to $\sqrt{c_1}$, which would have hardly changed the shape of the section. We will demonstrate this by a second example.

From the practical standpoint, the considerable reduction in the thickness of the rear end of the section just considered is very noticeable. On account of the requisite thickness of the rear spar, such forms cannot ordinarily be used. We obtain another perimeter which, to a certain degree, realizes the opposite extreme of retaining a nearly uniform thickness almost to the rear edge, through the following assumption. Still simpler than in the first example $n = 2$ and our statement now reads

$$z = \zeta + \frac{c_1}{\zeta} \quad \frac{c_2}{\zeta^2}$$

Both v-values which according to (18) must be given the sum c, we choose as

$$v_1 = -\frac{c}{2} e^{\frac{\pi i}{3}}, \quad v_2 = c - v_1 .$$

By substitution in (19) we get

$$c_1 = c^2 - v_1 v_2, \quad c_2 = -\frac{c}{2} v_1 v_2 ; \quad v_1 v_2 = -\frac{c^2}{8} (1 + 3\sqrt{3 i}).$$

In a similar manner to the above, the radial line $BC = v_1 \; v_2 : c$ is drawn in Fig. 13, so that OC represents its length and direction according to $-c_1 : c$. By bisecting the angle formed by OC with the real axis, the direction of the "second axis" of the section is found, which is, now that we wish to obtain a section with a pressure center, also the direction of the "first axis." We therefore draw through B the parallel to the bisector of the angle and choose on it the center M of the circle K, with the radius BM = a = 0.5. It is seen that K incloses the endpoints V_1 and V_2 of the vectors v_1, v_2 and O. We now construct, just as in the first example, an auxiliary circle K_1, which contains the end points A_1 of the vectors $c_1 : \zeta$ and derives the end points A' of the vectors $\zeta + \frac{c_1}{\zeta}$ from the 12 points A of the original circle K, with the aid of A_1. In order to obtain the ultimate section points A", we must still make the displacements $c_2 \zeta^2$. Note here that CB has the direction of c_2 and that from this, therefore, by the subtraction of 2γ plus entire multiples of 60°, all the shifted directions are obtained. The length of the displacements are determined either with the abacus, whereby the value $c_2 = -0.0211$ is to be used, or by calculation by means of the proportion

$$\frac{c_2}{\zeta^2} : \frac{c_1}{\zeta} = \frac{c_2}{c_1} : \zeta$$

Furthermore, it is only necessary to draw a circle with O for its center and the radius $\frac{c_2}{c_1}$, in order to obtain (Fig. 13) throug HH' parallel to OA' the desired length in A'H'. The pressure center lies on the first axis at a distance of

$$MF = c_1 : a = \frac{c^2}{8} \sqrt{108} : 0.5 = 0.415.$$

It is seen that this section also, like the first one, has an S-shaped curve which shows plainly on the pressure side. In contrast, however, with the first example, the thickness of the section is much more uniform. Intermediate forms, between the two given here, can readily be obtained, by varying the value of v, whereby the pressure point migration can be either entirely eliminated, as in the second, or largely, as in the first example. It is worth noting, further, that both sections are almost the same with respect to the lift magnitude, since the determining ratio 4a : b (b = greatest width) for the lift coefficient is, in the first case 176 : 163 and in the second case 200:183 accordingly about 1.09 in both cases. The theoretical lift coefficient is therefore $1.09\pi \sin \alpha_1$ in which α_1 is the effective angle of incidence $\alpha + \beta$. (According to Bieberbach's estimate, the coefficient is larger than $\pi \sin \alpha_1$).

In the third example, a section is drawn in Fig. 14 on the same plan as the last mentioned, but with other values of v_1 and v_2, namely,

$$v_1, v_2 = \frac{c}{2} (I \pm \sqrt{I + 0.16 i})$$

which might do for propeller blades. This section also has, like Fig. 13, a fixed pressure center, but shows a leaning toward the Joukowski shape. The point V_1 is still nearer 0. In any event, Fig. 13 and 14 illustrate the manifold variations in shape, even with the maintenance of the pressure-center condition $h_0 = 0$ and although we have confined ourselves to the simplest case of the general method with $n = 2$. On the construction of Fig. 14 there is nothing new to say. The values of $\frac{c_2}{\zeta^2}$ were here determined by calculation. The numerical values for the lift coefficient and the focal distance are here

$$c_1 = c^2 (1 + 0.4\ i),\quad c_2 = 0.2\ c^2\ i;$$
$$\zeta\ A = 1.1\ \pi\ \sin(\alpha + \beta),\quad MF = 0.375$$

If the method were carried beyond $n = 2$, the attainable shapes would then become so numerous that it would be impossible to give even a glimpse of them within the narrow limits of this fundamental exposition.

Translated by N. A. C. A.

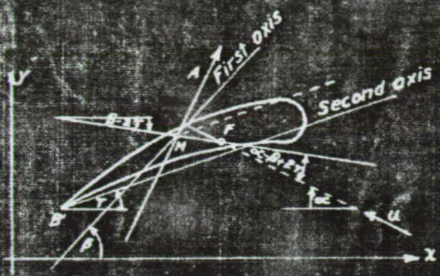

Fig.1.

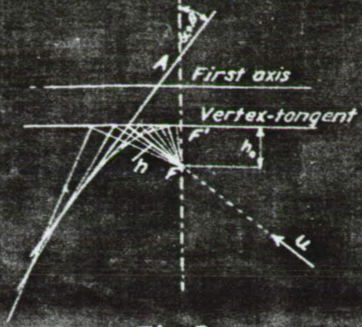

Fig.2.

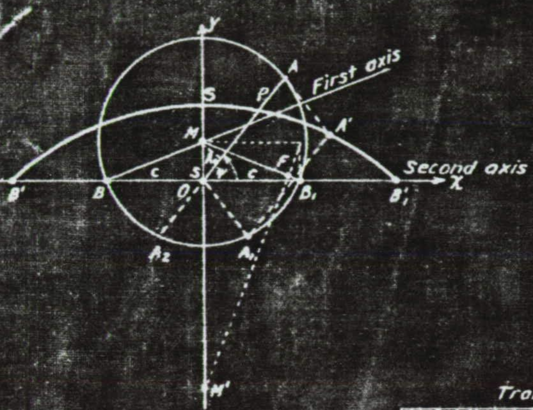

Fig.3.

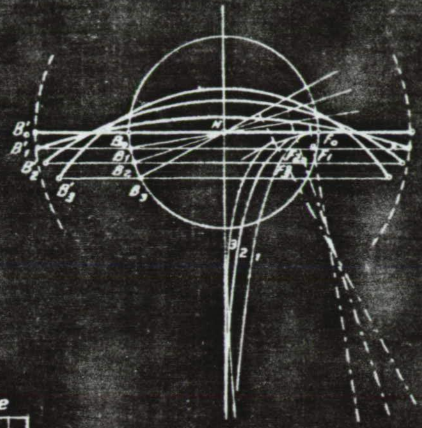

Fig.4.

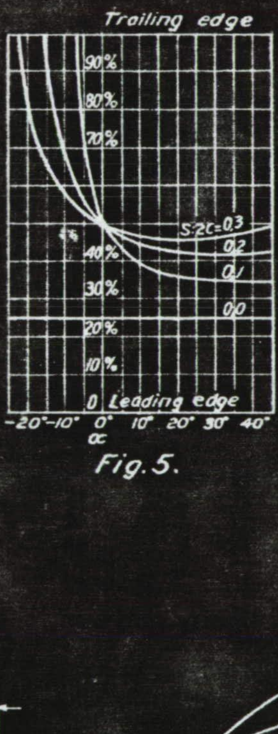

Fig.5.

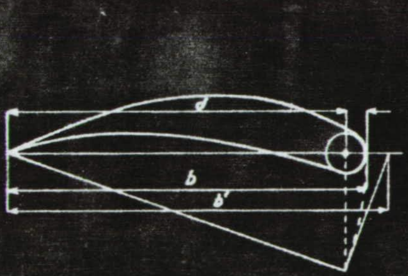

Fig.6.

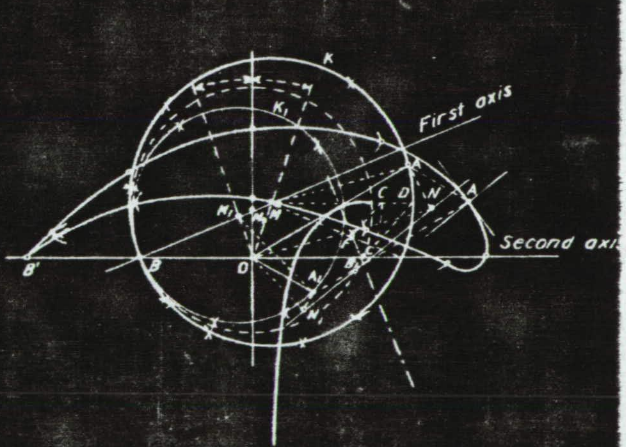

Fig.7.

Figs.1,2,3,4,5,6,

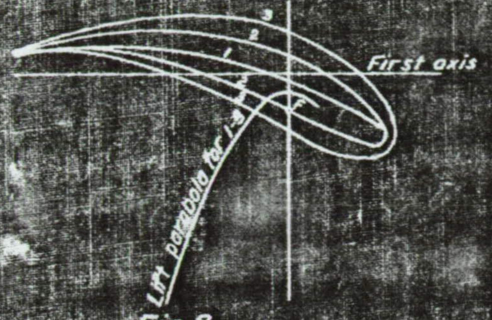

Fig.8.

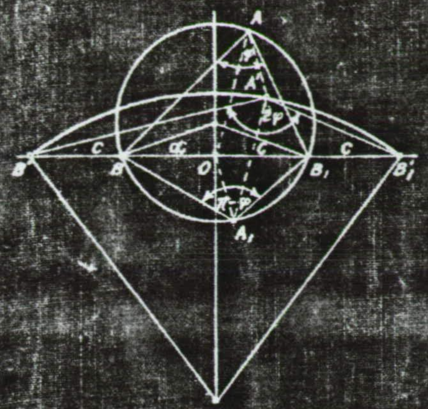

Fig.9.

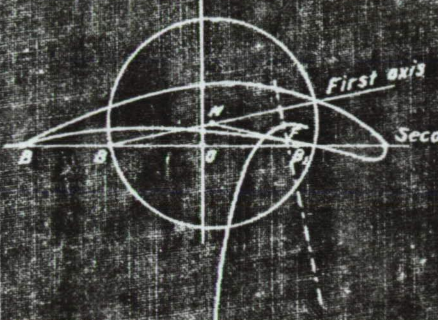

Fig.10

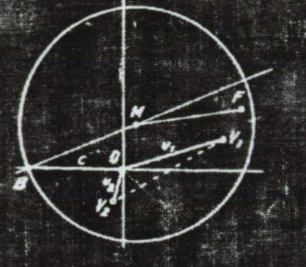

Fig.11.

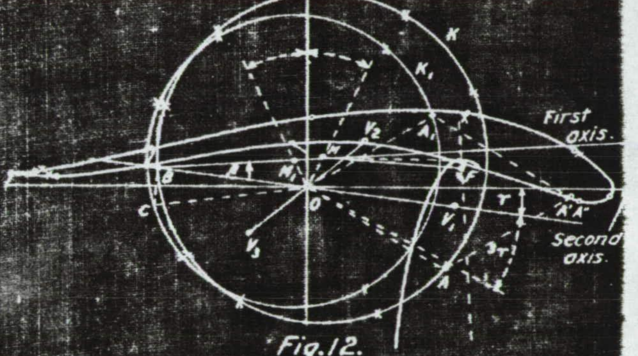

Fig.12.

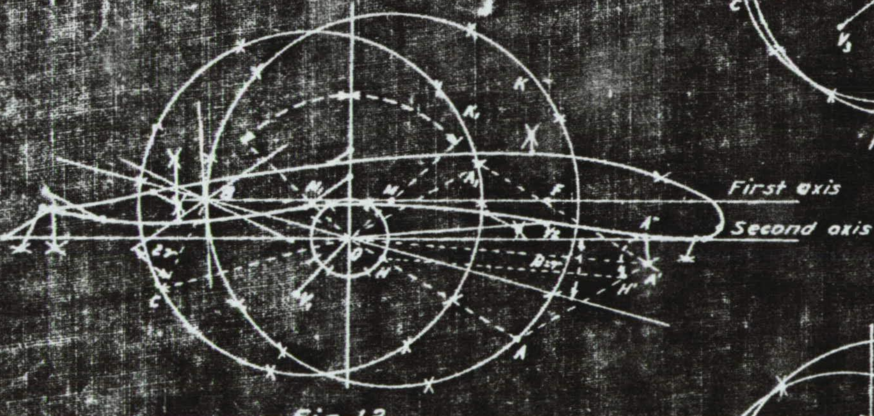

Fig.13.

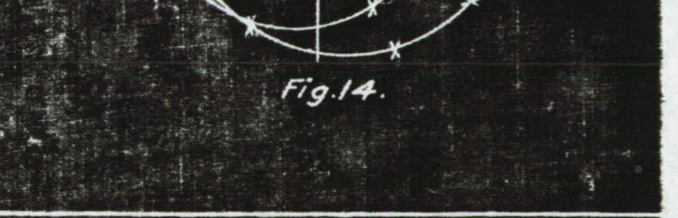

Fig.14.

Figs.8,9,10,11,12,13,