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NOTES ON THE THEORY OF THE ACCELEROMETER.

by

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By Edward P. Warner.

The accelerometers which have been proposed for use on airplanes fall into two classes. The first class, represented by the R.A.E. instrument and the one on which the staff of the National Advisory Committee is now working, has a single deflecting member. This deflecting member is a semicircular glass fiber in the R.A.E. accelerometer, a flat steel spring in the Advisory Committee's. The second class, the only representative of which among airplane accelerometers is the one designed by Dr. Zahm, has a number of plungers held up against stops by springs of varying strength or by springs of the same strength and loaded with varying weights. Increasing acceleration causes the springs to leave their stops, one after another, and to make contact with a sheet of paper which moves below them with a normal clearance of only a few thousandths of an inch. The springs in the Zahm instrument are helical, but they might be flat without changing the operation of the instrument. An accelerometer of this kind has the advantage of a very quick response to shocks of brief duration, but it makes a discontinuous record, giving indication only of the values between which the acceleration lay at any given instant. If there are enough springs a close approximation to the curve of acceleration can be secured, but this is essentially inferior to an instrument giving a continuous record directly if any satisfactory results can be secured from such an instrument.

The best way of approaching the theory of the accelerometer is to discuss first the obvious inherent errors and the means for reducing them to a minimum and then to take up the equations of motion and deduce from them the lag in responding to shocks, the damping of the sensitive member, and the possible errors due to resonance with the vibrations of the airplane engine.

Flat spring accelerometers of either the first or the second class are subject to errors arising from accelerations along the axis of the spring superimposed on those which it is intended to measure at right angles to the axis and from angular accelerations of the airplane. Furthermore, all accelerometers are affected and give inaccurate readings if they are not placed exactly at the center of gravity of the airplane (assuming that it is the acceleration of the center of gravity which is wanted and not, for special reason, that of some other part of the machine). The errors from this cause, in turn, may be divided into two groups. First, if the instrument is placed in a plane passing through the center of gravity and perpendicular to the axis along which the acceleration is to be measured, but not at the center of gravity itself, any angular acceleration of the airplane will correspond to a linear acceleration of the accelerometer spring and will be recorded as such. Second, if the accelerometer lies on the axis of measurement any angular velocity about either of the other axes will produce a centripetal acceleration which will be recorded by the instrument.

Taking up these sources of error in the order in which they were first mentioned, the first is the effect of linear accelerations along the axis of the spring. In treating this it will be assumed, to start with, that the deflecting member is normally straight and that it is homogeneous in material, constant in section, and submitted to an unvarying load. This assumption is very nearly true for the N.A.C.A. instrument, a considerable distance from the truth for the R.A.E. type, but it will be shown later that the distribution of load along the spring has very little effect on the error due to longitudinal acceleration.

In a flat spring, let w be the weight per unit length, x the distance from the free end of the spring, and k_x the acceleration normal to the plane of the spring in terms of g (taking the "acceleration" as including the component of gravity normal to the plane of the spring, so that it corresponds to the total air force acting on the airplane and preventing it from falling with an acceleration g downwards). Then the bending moment due to normal acceleration is:

$$M_x = \frac{-k_x w x^2}{2}$$

Integrating twice, the deflection at any point is found to be:

$$y_x = \frac{k_x w}{6EI} \left(\ell^3 x - \frac{x^4}{4} \right)$$

As regards the bending due to longitudinal accelerations of the spring, the moment arm of an element of mass at any point with respect to an axis of bending at any other point is equal to the difference between the normal deflections at the two points. The bending moment is then:

$$M_y = \int_0^{x_0} k_y w(y_0 - y)dx$$

where k_y is the longitudinal acceleration (including any component of gravity acting along that axis) in terms of g. Substituting the expression given above for y_o and y,

$$M_y = \frac{k_x k_y w^2}{6EI} \int_0^{x_o} \left(\ell^3 x_o - \frac{x_o{}^4}{4} - \ell^3 x + \frac{x^4}{4} \right) dx =$$

$$\frac{k_x k_y w^2}{6EI} \left(\ell^3 x_o x - \frac{x_o{}^4 x}{4} - \frac{\ell^3 x^2}{2} + \frac{x^5}{20} \right)_0^{x_o} =$$

$$\frac{k_x k_y w^2}{6EI} \left(\ell^3 x_o{}^2 - \frac{x_o{}^5}{4} - \frac{\ell^3 x_o{}^2}{2} + \frac{x_o{}^5}{20} \right) =$$

$$\frac{k_x k_y w^2}{6EI} \left(\frac{\ell^3 x_o{}^2}{2} - \frac{x_o{}^5}{5} \right)$$

Dropping the subscript, since x_o can have any value, and integrating twice we have:

$$y_y = \frac{k_x k_y w^2}{36E^2I^2} \left(\frac{\ell^3 x^4}{4} - \frac{x^7}{35} - \frac{4\ell^6 x}{5} \right.$$

Substituting ℓ for x in the expressions for y_x and y_y, it appears that the deflections of the free end of the spring due to accelerations in the two directions are

$$y_x = \frac{k_x w \ell^4}{8EI}$$

$$y_y = \frac{-9k_x k_y w^2}{560E^2I^2} \cdot \ell^7$$

$$\frac{y_y}{y_x} = \frac{-9w\ell^3 k_y}{70EI} = \frac{72}{70} \frac{k_y \cdot y_x}{k_x \cdot \ell}$$

The ratio $\frac{y_x}{k_x}$ is a constant for any given instrument, and is equal to the static deflection of the free end of the spring. The ratio of deflections is then

$$\frac{y_y}{y_x} = -\frac{36}{35} \frac{\delta_o}{\ell} \cdot k_y$$

$$M_x = -W.k_x.x$$

The deflection due to normal acceleration is found by integrating twice:

$$i = \frac{1}{EI}.W.k_x\left(\frac{-x^2}{2} + \frac{\ell^2}{2}\right)$$

$$y_x = \frac{1}{EI}.W.k\left(\frac{\ell^2 x}{2} - \frac{x^3}{6}\right)$$

The deflection at the free end, measured relative to the base, is then

$$y_{x_o} = \frac{W.k_x}{EI}\left(\frac{\ell^3}{3}\right)$$

Proceeding in the same manner as for a distributed load,

$$M_y = k_y W y_x = \frac{W^2 k_x k_y}{EI}\left(\frac{\ell^2 x}{2} - \frac{x^3}{6}\right)$$

$$i_y = \frac{W^2 k_x k_y}{E^2 I^2}\left(\frac{\ell^2 x^2}{4} - \frac{x^4}{24} - \frac{5\ell^4}{24}\right)$$

$$Y_y = \frac{W^2 k_x k_y}{E^2 I^2}\left(\frac{\ell^2 x^3}{12} - \frac{x^5}{120} - \frac{5\ell^4 x}{24}\right)$$

The deflection at the free end, due to longitudinal acceleration, is

$$Y_{y_o} = \frac{-W^2 k_x k_y \ell^5}{E^2 I^2} \cdot \frac{2}{15}$$

Then

$$\frac{Y_{y_o}}{Y_{x_o}} = -\frac{W k_y \ell^2}{EI} \cdot \frac{2}{5} = -\frac{k_y}{k_x} \cdot \frac{Y_{x_o}}{\ell} \cdot \frac{6}{5}$$

The error due to longitudinal acceleration in this case is therefore about 17% greater than in the case of a uniformly distributed loading. These cases are the extreme antitheses of each other and the true value of the error in either the R.A.E. or the N.A.C.A. instrument will lie somewhere between the two values found, as both these accelerometers have a tendency to concentrate the active mass near the free end of the springs.

ation of that machine when pulling out of a dive may be as great as 1.92g. The conditions assumed in this problem were unduly severe, and g may be taken as the maximum acceleration along the spring axis to which the accelerometer will be submitted. In the R.A.E. accelerometer an acceleration of this magnitude would produce an error of +5.43% or -4.69% in the determination of the normal acceleration, or a maximum error of .23g for the largest normal acceleration so far recorded. This is considerably greater than the sensitivity of the instrument, and shows that it is not safe to rely on its indications to within 0.1g.

In the N.A.C.A. accelerometer as originally designed (not yet tried out) ℓ is 12.7 cm., and δ_0 is 0.006 cm. The maximum error due to a longitudinal acceleration of g under these conditions would be 0.05%, the plus and minus errors being practically identical. This is 0.002g for the maximum normal acceleration. It is evident that in this instrument, if it is satisfactory in other respects, the errors due to accelerations at right angles to the one to be measured will be small enough to be neglected with perfect safety.

Instruments such as the one designed by Dr. Zahm, using helical springs, are of course quite free from any errors due to longitudinal accelerations.

As has already been noted, the assumption of a uniform distribution of weight along the spring does not accord closely with the facts. If it be assumed, as an alternative, that all the weight is concentrated at the free end of the spring, the bending moment due to the normal acceleration is:

Since any change in deflection perpendicular to the plane of the spring gives rise to a change in the deflection due to forces acting parallel to that plane, there is a secondary effect which modifies y_y. If, for example, k_y acts towards the free end of the spring, so that y_y and y_x are in the same direction, the increase of y due to the addition of y_y will itself produce a further increase, and the total effect of longitudinal acceleration will be greater than that given by the first approximation written above. If the two deflections are opposed, on the other hand, the actual value of y_y will be less than that given by the approximate formula. These effects can be allowed for by substituting y_t, the total deflection, for y_x in the above equation, writing

$$\frac{y_y}{y_t} = \frac{y_y}{y_x \pm y_y} \cong - \frac{36}{35} \cdot \frac{\delta_o}{\rho} \cdot k_y$$

$$\frac{y_y}{y_x} = \frac{-\frac{36}{35} \frac{\delta_o}{\rho} \cdot k_y}{1 \pm \frac{36}{35} \cdot \frac{\delta_o}{\rho} \cdot k_y}$$

In the glass fiber instrument devised and used at the Royal Aircraft Establishment ρ is 1.3 cm. and δ_o ranges from .05 to .08 cm. For an acceleration of g along the axis of the spring $\frac{y_y}{y_x}$ would therefore be .05. The accelerometer may be placed with the axis of the spring coinciding either with the X or the Y-axis of the airplane. The accelerations along the Y-axis certainly never exceed g, whereas the computation of the behavior of a JN2 during a loop* shows that the longitudinal deceler-

* Forces in Dive and Loop: Bulletin Airplane Engineering Department, U.S.A., June, 1918.

Effect of Angular Accelerations.

The effect of angular acceleration appears in two ways. In the first place, the spring, no matter where it may be placed, is affected by the angular acceleration as such. Secondly, if the origin of co-ordinates in the spring does not coincide with the center of gravity of the airplane an angular acceleration about the C.G. will give a linear acceleration to the spring.

The origin will be taken at the base of the spring as a first assumption, being shifted later to a more convenient and logically chosen location. An angular acceleration of k_a radians per sec. per sec., the base of the spring being assumed to remain stationary, imposes upon every element of length Δ x a load

$$\frac{X.k_a.W.\ \Delta x}{g}$$

where w is the weight per unit length. The shear at a distance x from the base is, integrating from the free end of the spring to the point in question,

$$S_a = \int_{\ell}^{x} \frac{x.k_a.W.dx}{g} = -\frac{k_a.W}{g}\left(\frac{\ell^2 - x^2}{2}\right)$$

and the bending moment is

$$M_a = \frac{-k_a.W}{g}\left(\frac{\ell^2 x}{2} - \frac{x^3}{6} - \frac{\ell^3}{3}\right)$$

Integrating twice more,

$$i_a = \frac{k_a.W}{gEI}\left(\frac{-\ell^2 x^2}{4} + \frac{x^4}{24} + \frac{\ell^3 x}{3}\right)$$

$$Y_a = \frac{k_a.W}{gEI}\left(\frac{x^5}{120} + \frac{\ell^3 x^2}{6} - \frac{\ell^2 x^3}{12}\right)$$

The deflection at the free end is then

$$Y_{a_0} = \frac{k_a \cdot W}{gEI} \left(\frac{11\,\ell^5}{120}\right) = \frac{11}{15} \times \frac{\ell}{g} \times k_a \times \delta_0$$

The direct error arising from angular accelerations is therefore directly proportional to the length of the spring, and the R.A.E. instrument, with its very short spring would seem to have a marked advantage in this particular. It is, however, evident that a judicious location of the origin of co-ordinates with respect to the C.G. of the airplane will introduce linear accelerations, resulting from angular accelerations, which will counterbalance the direct effect of the accelerated rotational motion.

The normal acceleration required to produce a deflection equivalent to that produced by the angular acceleration k_a would be of the magnitude

$$k_x = \frac{11}{15} \times \ell \times k_a$$

where k_x is expressed in terms of ft. per sec. per sec. It is then evident that, if the center of gravity of the airplane lies in the plane of the spring and $\frac{11}{15}$ of its length from its base, there will be no deflection of the free end of the spring due to angular accelerations, the two manners in which the effects of such accelerations appear just cancelling each other. If the weight is concentrated at the tip of the spring, instead of being uniformly distributed along its whole length, the free end should obviously be at the C.G. Compromising between the two conditions it may be said that, for the accelerometers now in use, the location of the mounting should be such that the C.G. lies from 75% to 80% of the way out along the spring.

If the C.G. is not at the point thus defined there are, as has already been pointed out, two possible sorts of error. The first of these is the error due to angular acceleration when the C.G. lies in the axis of the spring. By properly choosing the origin in the spring the direct effect of angular acceleration can be eliminated, and the total effect can be reduced to that of a linear acceleration given (in terms of g) by the expression:

$$k_x = \frac{k_a \times d}{g}$$

where d is the distance from the center of gravity of the airplane to a point in the spring and 75% of its length from the base.

The analysis of the "loop problem" already mentioned, showed that, under the conditions assumed, the angular acceleration about the Y-axis has a maximum value of 10.5 radians per sec. per sec. at the instant when the elevator was pulled up, that it falls to 2.4 radians per sec. per sec. in 0.3 second, and that it never rises above 1.0 rad. per sec. per sec. after 0.9 sec. until the loop is completed. The assumption, made in this analysis, that the elevator is pulled up instantaneously is, of course, much too severe, and it is probable that 6.5 rad. per sec. per sec. is the largest acceleration in pitch that an airplane would ever have to undergo. Experiments on the rolling moment due to the ailerons suggest that the acceleration about the X-axis has a maximum value, on small and medium-sized airplanes, of about 5 rad. per sec. per sec.

If d_x, d_y, and d_z be the projections on the three axes of the distance from the origin of co-ordinates in the spring to the center of gravity of the airplane, it is evident that the maximum error due to acceleration in roll is 0.15g when d_y is 1 foot, and that the similar error arising from the pitching motion is 0.20g when d_x is 1 foot. Corrections can be made, using estimated values for the angular accelerations, which can be relied upon to reduce these errors by about 60%. In order that the normal acceleration, thus corrected, may be accurate within 0.05g, the value of d_x must be less than $7\frac{1}{2}$" and d_y must not exceed 10".

The other source of error is the centripetal acceleration due to angular velocity. In the usual case, where the accelerations along the Z-axis are being measured, centripetal accelerations arise whenever there is any rolling or pitching motion if the active mass of the instrument is above or below the C.G. The acceleration is, of course,

$$k_x = \frac{\omega^2 d_z}{g}$$

The loop analysis showed a maximum angular velocity of 1.81 radians per sec. occurring 0.4 sec. after the elevator was pulled up. Since the theoretical time for completing the loop was about a third less than actual measurement shows to be required, this maximum is about 50% too high, and the true maximum may be taken as 1.2 rad. per sec. Since it is reported that an airplane can be rolled onto its back in 3 sec., the maximum rolling velocity must be about 1.5 rad. per sec. The error when d_z is 1 foot would then be 0.07g. As in the case of angular accelerations, approximate corrections can be made, and the error reduced by at least 50%. To keep the corrected value of the normal acceleration within 0.05 g of the truth, d_z must not exceed 16". This condition is easy to realize, and it will usually be found that the best place for mounting an accelerometer is directly above or below the C.G.

The dynamics of the accelerometer will be treated in a subsequent report.