TECHNICAL NOTES.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

No. 31.

CRIPPLING STRENGTH OF AXIALLY LOADED RODS.

By

Fr. Natalis.

Translated from
Technische Berichte, Volume III, No. 6,
by
F. W. Pawlowski,
University of Michigan.

October, 1921.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

TECHNICAL NOTE NO. 31.

CRIPPLING STRENGTH OF AXIALLY LOADED RODS.*

By

Fr. Natalis.

Let:

P_k = the load at the time of crippling in kg.

E = the modulus of elasticity in kg/cm²

F = the cross-section of the rod in cm²

l = the length of the rod in cm

J = the moment of inertia of the cross-section in cm⁴

$i = \sqrt{\frac{J}{F}}$ the radius of gyration of the cross-section in cm, $(J = i^2F)$.

$\frac{l}{i}$ = the slenderness ratio of the rod.

$k = \frac{P_k}{F}$ the mean unit compressive stress at the moment of crippling in kg/cm²

k_o = the ultimate compressive stress of the material in kg/cm

$M > 1$ the safety factor.

P = the allowable load in kg., $(P_k = mP)$.

According to Euler's formula, we have:

$$P_k = \frac{\pi^2 E\ J}{l^2};\ k = \pi^2 E \left(\frac{i}{l}\right)^2 \frac{k}{k_o} = \frac{\pi^2 E}{k_o} \left(\frac{i}{l}\right)^2 \qquad (1)$$

* Translated from Technische Berichte, Volume III, No. 6, pp. 207 - 217.

These formulas hold good only for slenderness ratios $\frac{l}{i} > 105$. However, a series of empirical formulas have been developed for $\frac{l}{i} < 105$. They govern, however, only a limited range of $\frac{l}{i}$ values, as for example, the formula of Tetmajer:

$$k = k_0 \left[1 - a\frac{l}{i} + b\left(\frac{l}{i}\right)^2\right], \text{ for } 10 < \frac{l}{i} < 105$$

and the formula of Ostenfeld:

$$k = k_0 \left[1 - c\left(\frac{l}{i}\right)^2\right], \text{ for } \frac{l}{i} < 125$$

both of which give too large dimensions. Further, the formula of Schwarz-Rankine:

$$k = \frac{k_0}{1 + a\left(\frac{l}{i}\right)^2} \tag{2}$$

If in this formula we put $a = \frac{k_0}{\pi^2 E}$, so that

$$k = \frac{k_0}{1 + \frac{k_0}{\pi^2 E}\left(\frac{l}{i}\right)^2}$$

then it covers the entire range of $\frac{l}{i}$ from 0 to ∞, and gives correct results for the extreme value of $\frac{l}{i} = 0$ and ∞, inasmuch as for cases in which

$$\frac{l}{i} = 0 : k = k_0$$

and when

$$\frac{l}{i} = \infty : k = \pi^2 E\left(\frac{i}{l}\right)^2,$$

as in Euler's formula, but for the intermediate values of $\frac{l}{i}$ it gives too large a factor of safety; for example, for

$$\frac{l}{i} = \sqrt{\frac{\pi^2 E}{k_o}} \qquad k_1 = 0.5\ k_o$$

The value $\frac{l}{i} = \sqrt{\frac{\pi^2 E}{k_o}}$ has for the later consideration a special significance, for it is the ordinate for the point of intersection of the straight lines $k = k_o$ and the Euler's curve $k = \pi^2 E \left(\frac{i}{l}\right)^2$

$$\frac{l_1}{i} = \sqrt{\frac{\pi^2 E}{k_o}} \text{ bzw. } \frac{l_1}{i}\sqrt{\frac{k_o}{\pi^2 E}} = 1 \qquad (3)$$

it is therefore an important unit of measure for the slenderness with respect to the characteristics k_o and E of the material.

In the case of $\frac{l_1}{i} = \sqrt{\frac{\pi^2 E}{k_o}}$, for which the Schwarz-Rankine formula gives $k_1 = 0.5\ k_o$, experimental investigations have determined a value of $k_1 = \frac{2}{3} k_o$ for both wood and steel.

Table 1 contains the values of $\frac{k}{k_o}$ ~~for~~ different values of $\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}}$, according to Schwarz-Rankine and Euler's formulas:

Table 1.

$\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}}$ . . . . . =	0.25	0.50	0.75	1.0	1.25	1.50	Slenderness of rod.
$\frac{k}{k_o} = \frac{1}{1 + \frac{k_o}{\pi^2 E}\left(\frac{l}{i}\right)^2}$ =	0.94	0.80	0.64	0.50	0.39	0.31	Schwarz-Rankine
$\frac{k}{k_o} = \frac{\pi^2 E}{k_o}\left(\frac{i}{l}\right)^2$ . . =	---	---	(1.79)	1.00	0.64	0.44	Euler

Table 1 (Contd.)

$\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}}$ .... =	1.75	2.00	2.25	2.50	2.75	3.0	Slenderness of rod.
$\frac{k}{k_o} = \frac{1}{1 + \frac{k_o}{\pi^2 E}\left(\frac{l}{i}\right)^2}$ =	0.25	0.20	0.17	0.14	0.12	0.10	Schwarz-Rankine
$\frac{k}{k_o} = \frac{\pi^2 E}{k_o}\left(\frac{i}{l}\right)^2$ =	0.33	0.25	0.20	0.16	0.13	0.11	Euler

In the following, a new formula will be derived. It corresponds to a curve of $\frac{k}{k_o}$ valid for the entire range of $\frac{l}{i}$ and coinciding at the beginning $\frac{k}{k_o} = 1$ with the Schwarz-Rankine and at the end with Euler's curves, and approaching closely during the whole range of the experimental investigations on strength of rods of different length and constant cross section.

As the formula should give same $\frac{k}{k_o}$ values for positive and negative l the slenderness ratio $\frac{l}{i}$ must appear in it only in even powers.

The formula must be therefore of the form:

$$\frac{k}{k_o} = \frac{1 + a\left(\frac{l}{i}\right)^2}{1 + b\left(\frac{l}{i}\right)^2 + c\left(\frac{l}{i}\right)^4} \tag{4}$$

For very small $\frac{l}{i}$ values, $c\left(\frac{l}{i}\right)^4$ becomes negligible compared to $b\left(\frac{l}{i}\right)^2$. If therefore, at the beginning the curve approaches the line $\frac{k}{k_o} = 1$, then

$$\frac{1 + a\left(\frac{l}{i}\right)^2}{1 + b\left(\frac{l}{i}\right)} = 1$$

necessitating $b = a$. If, on the other hand, for very large values of $\frac{l}{i}$, the curve is to agree with that of Euler, then the lower powers of $\frac{l}{i}$ must vanish. Therefore,

$$\frac{a\left(\frac{l}{i}\right)^2}{c\left(\frac{l}{i}\right)^4} = \frac{\pi^2 E}{k_o}\left(\frac{i}{l}\right)^2$$

whence $c = a\,\frac{k_o}{\pi^2 E}$. The formula becomes therefore now

$$\frac{k}{k_o} = \frac{1 + a\left(\frac{l}{i}\right)^2}{1 + a\left(\frac{l}{i}\right)^2 + a\,\frac{k_o}{\pi^2 E}\left(\frac{l}{i}\right)^4} \tag{5}$$

In order to determine now the value of a it will be assumed that the new curve cuts Euler's curve at the abscissa $\frac{l}{i} = n\sqrt{\frac{\pi^2 E}{k_o}}$, where n is any number greater than 2, $(n \geqq 2)$.

Then

$$\frac{k}{k_o} = \frac{\pi^2 E}{k_o}\left(\frac{i}{l}\right)^2 = \frac{\pi^2 E}{k_o}\;\frac{k_o}{n^2\,\pi^2 E} =$$

$$= \frac{1 + an^2\,\frac{\pi^2 E}{k_o}}{1 + an^2\,\frac{\pi^2 E}{k_o} + an^4\,\frac{k_o}{\pi^2 E}\left(\frac{\pi^2 E}{k_o}\right)^2}$$

$$\frac{1}{n^2} = \frac{1 + an^2\,\frac{\pi^2 E}{k_o}}{1 + an^2\,\frac{\pi^2 E}{k_o}\,(1 + n^2)}$$

$$1 + an^2\,\frac{\pi^2 E}{k_o}\,(1 + n^2) = n^2 + an^4\,\frac{\pi^2 E}{k_o}$$

$$an^2 \frac{\pi^2 E}{k_o} (1 + n^2 - n^2) = n^2 - 1$$

$$a = \frac{k_o}{\pi^2 E} \frac{n^2 - 1}{n^2}$$

The formula then becomes:

$$\frac{k}{k_o} = \frac{1 + \frac{n^2 - 1}{n^2} \frac{k_o}{\pi^2 E}\left(\frac{l}{i}\right)^2}{1 + \frac{n^2 - 1}{n^2} \frac{k_o}{\pi^2 E}\left(\frac{l}{i}\right)^2 + \frac{n^2 - 1}{n^2}\left(\frac{k_o}{\pi^2 E}\right)^2\left(\frac{l}{i}\right)^4} \qquad (6)$$

If, for example, the curve is to cut Euler's curve at the ordinate $\frac{l}{i} = 2\sqrt{\frac{\pi^2 E}{k_o}}$, then $n = 2$ and

$$\frac{k}{k_o} = \frac{1 + \frac{3}{4} \frac{k_o}{\pi^2 E}\left(\frac{l}{i}\right)^2}{1 + \frac{3}{4} \frac{k_o}{\pi^2 E}\left(\frac{l}{i}\right)^2 + \frac{3}{4}\left(\frac{k_o}{\pi^2 E}\right)^2\left(\frac{l}{i}\right)^4} \qquad (7)$$

It should be noted that the new formula will give larger values than Euler's for $\frac{l}{i}$ between $2\sqrt{\frac{\pi^2 E}{k_o}}$ and ∞. This is however unobjectionable, as the disagreement will not exceed 5%.

Further, it is evident that the point of intersection of the new curve with that of Euler's can be moved very far off, that is, n can be chosen very large without essentially diminishing the values of $\frac{k}{k_o}$ in the central region of the curve and that the latter then will agree still better with the test results. Further, if one considers that the new curve which cuts Euler's at $\frac{l}{i} = n\sqrt{\frac{\pi^2 E}{k_o}}$ will touch it at infinity, then the condition can be made, that also that also the first intersection point is moved off to infinity; in other words, that the new curve has three points in common with Euler's at infinity.

From this follows a simpler and for practical applications especially useful formula (for n up to infinity):

$$\frac{k}{k_o} = \frac{1 + \frac{k_o}{\pi^2 E}\left(\frac{l}{i}\right)^2}{1 + \frac{k_o}{\pi^2 E}\left(\frac{l}{i}\right)^2 + \left(\frac{k_o}{\pi^2 E}\right)^2\left(\frac{l}{i}\right)^4} = \frac{1 + A}{1 + A + A^2} \qquad (8)$$

$$P_k = k\,F = k_o F\,\frac{1 + A}{1 + A + A^2}$$

$$\text{where } A = \frac{k_o}{\pi^2 E}\left(\frac{l}{i}\right)^2$$

This formula is further distinguished by the fact that it contains no empirical constants but only the characteristics k_o and E of the material.

Table 2.

$\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}}$ . . . . =	0.25	0.50	0.75	1.00	1.25	1.50
$\frac{k}{k_o}$ according to Euler . . =	--	--	(1.79)	1.00	0.64	0.44
$\frac{k}{k_o}$ according to equation 7 (n = 2) =	0.995	0.963	0.858	0.700	0.545	0.415
$\frac{k}{k_o}$ according to equation 8 (n = ∞) =	0.995	0.955	0.835	0.667	0.513	0.392

Table 2 (Contd.)

$\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}}$ . . . . =	1.75	2.00	2.25	2.50	2.75	3.00
$\frac{k}{k_o}$ according to Euler . . =	0.33	0.25	0.20	0.16	0.13	0.11
$\frac{k}{k_o}$ according to equation 7 (n = 2) =	0.319	0.250	0.200	0.163	0.135	0.113
$\frac{k}{k_o}$ according to equation 8 ($n = \infty$) =	0.303	0.238	0.190	0.152	0.130	0.110

For different values of

$$\frac{l}{i}\sqrt{\frac{k}{\pi^2 E}} = \sqrt{\frac{k}{\pi^2 E} \times \frac{F l^2}{J}}$$

there are grouped in Table 2 the values of $\frac{k}{k_o}$.

Fig. 1 shows the curves of $\frac{k}{k_o}$ plotted from formula (8) and from these of Euler and Schwarz-Rankine. In order to verify the new formula a series of pine rods, 4 x 4 cm.^{2} and of different lengths was tested.

The material selected was as uniform as possible. Taking $F = 16$ cm.^{2}, $J = 21.3$ cm.^{4}, $i = 1.15$ cm, and $E = 130{,}000$ kg/cm.^{2}, the results of tests were computed in terms of $\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}}$ and $\frac{k}{k_o}$ and plotted in Fig. 1.

Table 3.

$\frac{l}{i}$	k kg/cm.^{2}	$\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}}$	$\frac{k}{k_o}$
0	525	0.000	1.000
7.5	523	0.152	0.996
10.1	524	0.204	0.997
16	508	0.324	0.968
20	479	0.405	0.913
29	476	0.587	0.906
39	430	0.790	0.820
46	362	0.932	0.690
(49.4)	(350)	(1.00)	(0.667)
54.5	309	1.10	0.589
63.5	244	1.29	0.465
71.5	218	1.45	0.416
(74)	(206)	(1.50)	(0.392)
80	187	1.62	0.357
87.5	145	1.78	0.276
(98.5)	(125)	(2.00)	(0.238)

In Table 3 there are included further (in brackets) the calculated results for $\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}} = 1.0$, 1.5 and 2.0.

As can be seen from Fig. 1, the test results agree well with the curve from formula (8). That the test results do not give an entirely smooth curve, is not surprising at all, as in such compression tests slight differences in material and its uniformity exert a considerable influence.

For a material of unknown properties it is sufficient to make two tests only, in order to determine the characteristics k_o and E: one compression test of a short rod giving the ultimate compression strength k_o and one bending test of a horizontal rod, freely supported at the ends and loaded in the center, giving $E = \frac{Q}{fJ} \; \frac{l^3}{48}$ from the known load Q in kg. and the observed deflection f in cm.

The results of another similar series of tests are given in Table 4. They refer to hollow square sectioned rods and are calculated for k_o = 525 kg/cm.^{2}, E = 130,000 kg/cm.^{2}, F = 7.94 cm.^{2}, J = 15.9 cm.^{4}, and i = 1.41 cm. Their dimensions and the curve of the test results in comparison to the curve given in formula (8) are shown in Fig. 2.

Table 4.

$\frac{l}{i}$	k kg/qcm	$\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}}$	$\frac{k}{k_o}$
0	525	0.000	1.000
13	519	0.264	0.989
31	448	0.628	0.855
(49.4)	(350)	(1.000)	(0.667)
52.2	407	1.057	0.775
(74)	(206)	(1.50)	(0.392)
77	237	1.540	0.452

From tests of drawn seamless steel tubes, manufactured according to the standards of the Army Air Service Inspection (Idflz)* Table 5 and Fig. 3 were established, showing that formula (8) holds also for other materials. In these tests specimens 30 mm. in diameter and of varying length were used; the wall thickness varied between 0.79 and 1.18 mm. The wall thickness of individual tubes was not quite uniform, varying, for example, from 1.02 to 1.18 mm. This should explain the irregularities of the curve in Fig. 3.

* "Idflz" means probably, "Inspektion der Flugzeugtruppen."

Table 5.

$\frac{l}{i}$	k kg/qcm.	$\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}}$	$\frac{k}{k_o}$
11.20	5225	0.182	1.005
13.15	5192	0.213	0.998
15.10	4786	0.245	0.920
18.03	5203	0.292	1.001
20.96	4930	0.340	0.948
24.86	4720	0.403	0.908
27.79	4918	0.450	0.946
32.66	4984	0.530	0.957
37.54	4500	0.608	0.865
42.42	4841	0.687	0.930
51.80	4850	0.840	0.932
61.56	3541	0.998	0.682
(61.70)	(3470)	(1.000)	(0.667)
71.32	3084	1.156	0.593
81.08	2698	1.314	0.519
90.83	2022	1.473	0.389
(92.55)	(2022)	(1.500)	(0.389)
100.6	1655	1.630	0.318

The value $k_o = \frac{5225 + 5192}{2}$ = approximately 5200 kg/cm.^{2} is the mean value obtained from the compression test of the two shorter tubes. The value E = 2,000,000 kg/cm.^{2} of the modulus of elasticity is the average of the bending tests of two tubes, which gave 2,041,900 and 2,008,370 kg/cm.^{2} respectively (average 2,025,000) and of two compression tests giving 1,990,000 and 1,970,000 kg/cm.^{2} respectively (average = 1,980,000). The calculations were also based on F = 0.911 cm.^{2}, J = 0.959 cm.^{4}, $i = \sqrt{\frac{J}{F}}$ = 1.025 cm., so that $\frac{l}{i}$ = 61.70 and $\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}} = 1$.

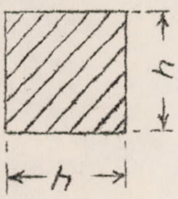
In the foregoing calculations, besides k_o and E, the value of F, J and l are assumed to be known and from them k and $P_k = mP$ are calculated; frequently, however, k_o and E also $P_k = mP$ and l

are given and F and J are to be calculated.

In order to simplify such calculations, Tables 6 to 8 can be used.

Table 6.

Solid Square Cross-Section Pine.



$$F = h^2 \text{ cm.}^2; \; J = \frac{h^4}{12} \text{ cm}^4; \; \frac{J}{F} = \frac{h^2}{12} \text{cm}^2; \; i = \frac{h}{\sqrt{12}} = \frac{h}{3.47} \text{ cm.};$$

$$k_0 = 525 \frac{\text{kg}}{\text{cm}^2}; \; E = 130000 \frac{\text{kg}}{\text{cm}^2}$$

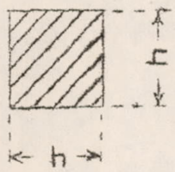
$$mP = 525\,F \frac{1 + 4.09 \frac{F}{J}\left(\frac{l}{100}\right)^2}{1 + 4.09 \frac{F}{J}\left(\frac{l}{100}\right)^2 + \left[4.09 \frac{F}{J}\left(\frac{l}{100}\right)^2\right]^2} \text{ kg.}$$

h	F	J	$\frac{J}{F}$	i	l =						
cm	cm²	cm⁴	cm²	cm	0	20	40	60	80	100	120
					$\frac{mP}{1000}$ =						
2	4	1.33	0.33	0.577	2.10	1.81	0.91	0.45	0.263	0.171	0.119
3	9	6.75	0.75	0.865	4.72	4.55	3.38	2.06	1.27	0.846	0.595
4	16	21.3	1.33	1.15	8.40	8.28	7.23	5.33	3.65	2.53	1.82
5	25	52.1	2.08	1.44	13.1	13.0	12.2	10.1	7.73	5.70	4.24
6	36	108	3.00	1.73	18.9	18.9	18.2	16.3	13.4	10.6	8.20
7	49	200	4.08	2.02	25.7	25.7	25.2	23.5	20.6	17.1	13.9
8	64	340	5.33	2.31	33.6	33.6	33.2	31.7	28.9	25.2	21.2
9	81	547	6.75	2.60	42.5	42.5	42.3	40.9	38.4	34.5	30.2
10	100	833	8.43	2.88	52.5	52.4	52.1	51.1	48.8	45.2	40.6

Table 6 (Contd.)

Solid Square Cross-Section Pine.

$$F = h^2 \text{ cm}^2;\ J = \frac{h^4}{12} \text{ cm}^4;\ \frac{J}{F} = \frac{h^2}{12} \text{cm}^2;\ i = \frac{h}{\sqrt{12}} = \frac{h}{3.47} \text{ cm.}$$



$$k_o = 525 \ \frac{\text{kg.}}{\text{cm}^2};\ E = 130{,}000 \ \frac{\text{kg.}}{\text{cm}^2}$$

$$mP = 525\ F \ \frac{1 + 4.09 \ \frac{F}{J} \left(\frac{l}{100}\right)^2}{1 + 4.09 \ \frac{F}{J}\left(\frac{l}{100}\right)^2 + \left[4.09 \ \frac{F}{J} \left(\frac{l}{100}\right)^2\right]^2} \text{ kg.}$$

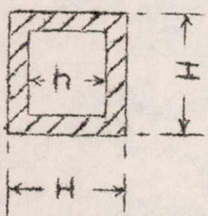
h	F	J	$\frac{J}{F}$	i	l =							
cm	cm^2	cm^4	cm^2	cm	140	160	180	200	220	240	260	280
					$\frac{mP}{1000}$ =							
2	4	1.33	0.33	0.577	0.087	0.067	0.053	0.043	0.035	0.030	0.025	0.022
3	9	6.75	0.75	0.865	0.444	0.338	0.266	0.216	0.180	0.151	0.128	0.111
4	16	21.3	1.33	1.15	1.37	1.06	0.836	0.680	0.563	0.473	0.404	0.349
5	25	52.1	2.08	1.44	3.22	2.53	2.02	1.64	1.36	1.15	0.980	0.843
6	36	108	3.00	1.73	6.41	5.08	4.12	3.38	2.81	2.37	2.02	1.76
7	49	200	4.08	2.02	11.2	9.02	7.41	6.10	5.13	4.35	3.72	3.23
8	64	340	5.33	2.31	17.6	14.6	12.1	10.1	8.54	7.31	6.27	5.45
9	81	547	6.75	2.60	25.8	21.9	18.5	15.6	13.3	11.5	9.89	8.65
10	100	833	8.43	2.88	35.7	30.9	26.6	22.8	19.6	17.0	14.8	13.0

Table 7.

Hollow Square Cross-Section Pine.

$$F = H^2 - h^2 \text{ cm}^2; \quad J = \frac{H^4 - h^4}{12} \text{ cm}^4; \quad \frac{J}{F} = \frac{H^2 + h^2}{12} \text{ cm}^2;$$

$$i = \sqrt{\frac{H^2 + h^2}{12}} \text{ cm};$$



$$k_o = 525 \text{ kg/cm}^2; \quad E = 130{,}000 \text{ kg/cm}^2$$

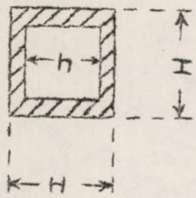
$$mP = 525\,F \frac{1 + 4.09 \frac{F}{J}\left(\frac{l}{100}\right)^2}{1 + 4.09 \frac{F}{J}\left(\frac{l}{100}\right)^2 + \left[4.09 \frac{F}{J}\left(\frac{l}{100}\right)^2\right]^2} \text{ kg.}$$

H	h	F	J	$\frac{J}{F}$	i	l =							
cm	cm	cm²	cm⁴	cm²	cm	0	20	40	60	80	100	120	140
						$\frac{mP}{1000}$ =							
4	2.8	8.16	16.2	1.98	1.41	4.28	4.26	3.96	3.26	2.46	1.80	1.32	1.01
5	3.6	12.0	38.2	3.17	1.78	6.30	6.29	6.09	5.48	4.58	3.66	2.86	2.24
6	4.4	16.6	76.8	4.62	2.15	8.71	8.70	8.56	8.07	7.23	6.15	5.07	4.14
7	5.2	22.0	141	6.42	2.54	11.5	11.5	11.4	11.1	10.3	9.25	8.04	6.82
8	6.0	28.0	233	8.34	2.89	14.7	14.7	14.6	14.4	13.7	12.6	11.4	10.0
9	6.8	34.8	366	10.6	3.26	18.3	18.3	18.2	18.0	17.5	16.5	15.2	13.7
10	7.6	42.2	455	13.2	3.64	22.2	22.2	22.1	21.9	21.5	20.6	19.5	18.0

Table 7 (Contd.)

Hollow Square Cross-Section Pine.

$$F = H^2 - h^2 \text{ cm}^2; \; J = \frac{H^4 - h^4}{12} \text{ cm}^4; \frac{J}{F} = \frac{H^2 \quad h^2}{12} \text{ cm}^2;$$



$$i = \sqrt{\frac{H^2 + h^2}{12}} \text{ cm};$$

$$k_o = 525 \text{ kg/cm}^2; \; E = 130{,}000 \text{ kg/cm}^2$$

$$mP = 525\, F \frac{1 + 4.09 \frac{F}{J} \left(\frac{l}{100}\right)^2}{1 + 4.09 \frac{F}{J}\left(\frac{l}{100}\right)^2 + \left[4.09 \frac{F}{J}\left(\frac{l}{100}\right)^2\right]^2} \text{ kg.}$$

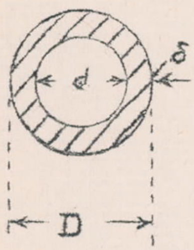
H	h	F	J	$\frac{J}{F}$	i	$l =$						
cm	cm	cm^2	cm^4	cm^2	cm	160	180	200	220	240	260	280
						$\frac{mP}{1000} =$						
4	2.8	8.16	16.2	1.98	1.41	0.791	0.631	0.514	0.424	0.360	0.309	0.265
5	3.6	12.0	38.2	3.17	1.78	1.80	1.44	1.18	0.989	0.834	0.715	0.618
6	4.4	16.6	76.8	4.62	2.15	3.39	2.79	2.32	1.95	1.66	1.42	1.23
7	5.2	22.0	141	6.42	2.54	5.74	4.84	4.08	3.47	2.97	2.56	2.24
8	6.0	28.0	233	8.34	2.89	8.65	7.45	6.40	5.50	4.76	4.14	3.63
9	6.8	34.8	368	10.6	3.26	12.2	10.7	9.43	8.23	7.21	6.33	5.59
10	7.6	42.2	455	13.2	3.64	16.4	14.7	13.2	11.7	10.3	9.17	8.13

In Table 8, only one value of wall thickness was taken for each tube diameter. As F and J are nearly proportional to δ for thin walled tubes, therefore the strength of the tubes is practically proportional to the thickness of the wall and can be easily estimated for other thicknesses from the values of the table.

Table 8.

Seamless Steel Tubes, Army Air Service Inspection Specification.

$$F = \frac{\pi}{4}\left(D^2 - d^2\right) = \pi\,\delta\,(D - \delta)\,\text{cm}^2; \quad J = \frac{\pi}{64}\left(D^4 - d^4\right)\text{cm}^4;$$



$$\frac{J}{F} = \frac{D^2 + d^2}{16}\ \text{cm}^2; \quad i = \frac{1}{4}\sqrt{D^2 + d^2}\ \text{cm};$$

$$k_o = 5200\ \text{kg/cm}^2; \quad E = 2{,}000{,}000\ \text{kg/cm}^2$$

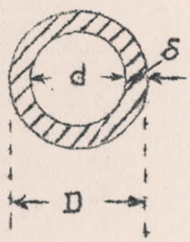
$$mP = 5200\,F\,\frac{1 + 2.63\,\frac{F}{J}\left(\frac{l}{100}\right)^2}{1 + 2.63\,\frac{F}{J}\left(\frac{l}{100}\right)^2 + \left[2.63\,\frac{F}{J}\left(\frac{l}{100}\right)^2\right]^2}\ \text{kg.}$$

D	d	δ	F	J	$\frac{J}{F}$	i	$l =$					
cm	cm	cm	cm^2	cm^4	cm^2	cm	0	25	50	75	100	125
							$\frac{mP}{1000} =$					
2.0	1.8	0.1	0.597	0.270	0.453	0.673	3.10	2.83	1.67	0.889	0.523	0.338
2.5	2.3	0.1	0.754	0.544	0.723	0.850	3.92	3.76	2.74	1.65	1.02	0.673
3.0	2.8	0.1	0.911	0.959	1.050	1.025	4.74	4.63	3.64	2.60	1.70	1.15
3.5	3.3	0.1	1.068	1.545	1.444	1.203	5.55	5.49	4.83	3.67	2.55	1.78
4.0	3.8	0.1	1.225	2.331	1.90	1.38	6.37	6.32	5.84	4.75	3.54	2.57
4.5	4.2	0.15	2.050	4.854	2.37	1.54	10.6	10.6	10.0	8.60	6.73	5.08
							$l =$					
							0	150	175	200	225	250
							$\frac{mP}{1000} =$					
5.0	4.7	0.15	2.286	6.726	2.95	1.72	11.9	5.10	3.97	3.17	2.54	2.08
5.5	5.2	0.15	2.521	9.027	3.58	1.89	13.1	6.47	5.13	4.13	3.36	2.76
6.0	5.6	0.2	3.644	15.34	4.20	2.05	18.9	10.4	8.60	6.80	5.55	4.60
7.0	6.6	0.2	4.273	24.72	5.79	2.41	22.2	14.7	12.3	10.2	8.54	7.16
8.0	7.6	0.2	4.901	37.30	7.60	2.76	25.5	19.0	16.5	14.1	12.1	10.3
9.0	8.6	0.2	5.529	53.55	9.68	3.11	28.7	23.3	20.9	18.4	16.0	14.0
10.0	9.6	0.2	6.158	73.95	12.0	3.47	32.0	27.5	25.2	22.7	20.2	17.9

Table 8 (Contd.)

Seamless Steel Tubes, Army Air Service Inspection Specification.

$$F = \frac{\pi}{4}\left(D^2 - d^2\right) = \pi\,\delta\,(D - \delta)\ \text{cm}^2;\quad J = \frac{\pi}{64}\left(D^4 - d^4\right)\text{cm}^4$$



$$\frac{J}{F} = \frac{D^2 - d^2}{16}\ \text{cm}^2;\quad i = \frac{1}{4}\sqrt{D^2 + d^2}\ \text{cm}:$$

$$k_o = 5200\ \text{kg/cm}^2;\quad E = 2{,}000{,}000\ \text{kg/cm}^2$$

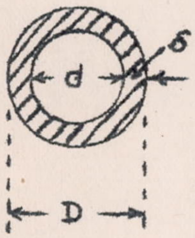
$$mP = 5200\,F\,\frac{1 + 2.63\,\frac{F}{J}\left(\frac{l}{100}\right)^2}{1 + 2.63\,\frac{F}{J}\left(\frac{l}{100}\right)^2 + \left[2.63\,\frac{F}{J}\left(\frac{l}{100}\right)^2\right]^2}\ \text{kg.}$$

D	d	δ	F	J	$\frac{J}{F}$	i	l =				
cm	cm	cm	cm²	cm⁴	cm²	cm	150	175	200	225	250
							$\frac{mP}{1000}$ =				
2.0	1.8	0.1	0.597	0.270	0.453	0.673	0.237	0.174	0.134		
2.5	2.3	0.1	0.754	0.544	0.723	0.850	0.473	0.348	0.268		
3.0	2.8	0.1	0.911	0.959	1.050	1.025	0.828	0.609	0.469	0.372	0.301
3.5	3.3	0.1	1.068	1.545	1.444	1.203	1.29	0.972	0.751	0.596	0.485
4.0	3.8	0.1	1.225	2.331	1.90	1.38	1.89	1.43	1.12	0.894	0.725
4.5	4.2	0.15	2.050	4.854	2.37	1.54	3.78	2.94	2.30	1.85	1.51
							l =				
							275	300	325	350	375
							$\frac{mP}{1000}$ =				
5.0	4.7	0.15	2.386	6.726	2.95	1.72	1.73	1.46	1.24	1.08	0.945
5.5	5.2	0.15	2.521	0.027	3.58	1.89	2.30	1.94	1.67	1.44	1.26
6.0	5.6	0.2	3.644	15.34	4.20	2.05	3.87	3.28	2.82	2.43	2.13
7.0	6.6	0.2	4.273	24.72	5.79	2.41	6.06	5.18	4.47	3.89	3.40
8.0	7.6	0.2	4.901	37.30	7.60	2.76	8.82	7.59	6.60	5.74	5.08
9.0	8.6	0.2	5.529	53.55	9.68	3.11	12.1	10.5	9.20	8.06	7.21
10.0	9.6	0.2	6.158	73.95	12.0	3.47	15.7	13.9	12.2	10.8	9.65

Table 8 (Contd.)

Seamless Steel Tubes, Army Air Service Inspection Specification.

$$F = \frac{\pi}{4}(D^2 - d^2) = \pi\,\delta\,(D - \delta)\ \text{cm}^2;\quad J = \frac{\pi}{64}(D^4 - d^4)\,\text{cm}^4;$$



$$\frac{J}{F} = \frac{D^2 + d^2}{16}\ \text{cm}^2;\quad i = \frac{1}{4}\sqrt{D^2 + d^2}\ \text{cm}:$$

$$k_o = 5200\ \text{kg/cm}^2;\quad E = 2{,}000{,}000\ \text{kg/cm}^2$$

$$mP = 5200\,F\,\frac{1 + 2.63\,\frac{F}{J}\left(\frac{l}{100}\right)^2}{1 + 2.63\,\frac{F}{J}\left(\frac{l}{100}\right)^2 + \left[2.63\,\frac{F}{J}\left(\frac{l}{100}\right)^2\right]^2}\ \text{kg.}$$

D cm	d cm	δ cm	F cm²	J cm⁴	J/F cm²	i cm	l = 275	300	325	350	375
							mP/1000 =				
2.0	1.8	0.1	0.597	0.270	0.453	0.673					
2.5	2.3	0.1	0.754	0.544	0.723	0.850					
3.0	2.8	0.1	0.911	0.959	1.050	1.025					
3.5	3.3	0.1	1.068	1.545	1.444	1.203	0.400	0.339			
4.0	3.8	0.1	1.225	2.331	1.90	1.38	0.605	0.508	0.432	0.375	0.326
4.5	4.2	0.15	2.050	4.854	2.37	1.54	1.25	1.06	0.904	0.778	0.682
							l = 400	425	450	475	500
							mP/1000 =				
5.0	4.7	0.15	2.286	6.726	2.95	1.72	0.830	0.736	0.658	0.590	0.534
5.5	5.2	0.15	2.521	9.027	3.58	1.89	1.11	0.983	0.882	0.791	0.712
6.0	5.6	0.2	3.644	15.34	4.20	2.05	1.88	1.66	1.49	1.34	1.21
7.0	6.6	0.2	4.273	24.72	5.79	2.41	3.00	2.67	2 39	2.15	1.94
8.0	7.6	0.2	4.901	37.30	7.60	2.76	4.47	3.99	3.58	3.22	2.91
9.0	8.6	0.2	5.529	53.55	9.68	3.11	6.36	5.66	5.09	4.60	4.15
10.0	9.6	0.2	6.158	73.95	12.0	3.47	8.58	7.67	6.95	6.30	5.68

GRAPHICAL DETERMINATION OF CRIPPLING LOAD.

In formula $mP = kF = k_oF\frac{k}{k_o}$, k_o and F are known and $\frac{k}{k_o}$ depends on $\frac{l}{i}$ and $\frac{E}{k_o}$ only. During the derivation of formula (3) it was pointed out that the expression $\frac{l}{i} = \sqrt{\frac{k_o}{\pi^2 E}} = 1 = \overline{OA}$ (See Fig. 4) is an important unit of measurement for the slenderness of the rod and makes it possible to read off the values of $\frac{k}{k_o}$ for all values of $\frac{l}{i}$ and for all materials from a single curve.

If the crippling load for any value of $\frac{l}{i}$ is to be determined, it is necessary first of all to multiply $\frac{l}{i}$ by $\sqrt{\frac{k_o}{\pi^2 E}}$. Along the ordinate AB corresponding to the abscissa 1.0 a scale for $\frac{l}{i}$ is provided and from the origin O radial lines are drawn for different values of $\frac{E}{k_o}$. For instance, the line for $\frac{E}{k_o} = 250$ (wood) agrees with the experimental results in Table 3, in which

$\frac{E}{K_o} = \frac{130,000}{525} = 248$ = approximately 250.

Table 9.

$\frac{E}{k_o} =$	200	250 (Holz)	300	350	385 (Stahl)	400	450
$\frac{l}{i}^1 =$	44.4	49.7	54.6	58.9	61.7	62.8	66.8

If, for example, $\frac{l}{i} = 39 = \overline{AC}$ and if the line CDE is drawn, then $\overline{OE} = c \times \overline{AC}$, where c is a constant. The radial line OD must therefore have a slope corresponding to $c = \sqrt{\frac{k_o}{\pi^2 E}}$. The intersection point F of line OD and the ordinate AB gives the value of $\frac{l}{i}$ for which $\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}} = 1$; i.e., $\overline{AF} = \frac{l}{i} = \sqrt{\frac{E}{k_o}}$

For $\frac{E}{k_o} = 250$, therefore $\frac{l}{i} = 49.7$.

For other $\frac{E}{k_o}$ the values are given in Table 9, from which the radial lines for the characteristics of different materials can be easily drawn in Fig. 4.

If the value of $\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}} = \overline{OE}$ is once determined by means of a proper radial line for a value of $\frac{l}{i} = \overline{AC}$ the line DGH gives immediately the value $\frac{k}{k_o}$, as for instance $\frac{k}{k_o} = .81$ in the example, this value is then to be inserted in the equation

$mP = k_o F \frac{k}{k_o} = .81\ k_o F$.

For another value of $\frac{l}{i} = 74 > \overline{AF}$, corresponding to $\frac{l}{i}\sqrt{\frac{k_o}{\pi^2 E}} = 1.5$ the line JKLM should be drawn, giving $\frac{k}{k_o} = \overline{OM} = .392$, i.e. $mP = .392\ k_o F$.

When the load mP, instead of the cross section of the rod, is known, the process must be repeated in order to determine the cross sectional dimensions.

THE ELASTIC CURVE AND THE LATERAL DEFLECTION OF THE ROD.

The differential equation of the elastic line, (Fig. 5), from which Euler's formula is derived is:

$$\frac{d^2 y}{d x^2} = \frac{M}{E J} = -\frac{P}{E J} y, \tag{9}$$

where P is any load at all.

This equation is satisfied by

$$y = B \sin w x + C \cos w x, \tag{10}$$

in which B, C and w are constants and are still to be determined. $x = 0$ gives $y = 0$, therefore, $C = 0$. As, however, y must also $= 0$ for $x = l$, therefore $B \sin wl = 0$, hence $wl = \pi$, 2π or 3π ...... The values 2π, 3π, etc., correspond to a rod bent into several waves by fourfold, ninefold, ... loads, and therefore are not to be considered.

Hence, we have:

$$w = \frac{\pi}{l} \text{ and } y = B \sin \pi \frac{x}{l} \tag{11}$$

The elastic curve is therefore a sine curve.

As further, for $x = \frac{l}{2}$, the deflection $y = a$, then $B = a$,

$$y = a \sin \pi \frac{x}{l} \text{ and} \tag{12}$$

$$\frac{d^2y}{dx^2} = - a \frac{\pi^2}{l^2} \sin \pi \frac{x}{l} = - \frac{\pi^2}{l^2} y.$$

On the other hand, according to equation (9)

$$\frac{d^2y}{dx^2} = - \frac{P}{EJ} y = - \frac{\pi^2}{l^2} y.$$

Then there follows:

$$P_k = \frac{\pi^2 E J}{l^2} \tag{13}$$

In other words, if a deflection of the rod takes place at all $\frac{d^2y}{dx^2} > 0$ then it can occur only under the effect of a definite load P_k.

However, the maximum deflection a for $x = \frac{l}{2}$ cannot be calculated from equation (12) as the equation $a = a \sin \frac{\pi}{l} \frac{l}{2} = a \sin \frac{\pi}{2}$ can be satisfied for any value of a. Hence, it follows first, that in case of $P < P_k$ there is no deflection at all; second, that for a definite load $P_k = \frac{\pi^2 EJ}{l^2}$, the equilibrium of the inner and outer forces is present for any value of a, and third, that in case of $P > P_k$ the deflection a will continue to increase until the rod will break and $a = \infty$.

This is, however, not to be taken literally, for the deflection of a rod in proportion to its length cannot exceed certain practical limits. The conclusions of this investigation apply therefore to comparatively small deflections of the rod and to rods which originally were perfectly straight, symmetrical and of uniform material. With these reservations, the formula and the conclusions drawn therefrom are unobjectionable.

That the deflection, according to Euler, for the load $P_k = \frac{\pi^2 EJ}{l^2}$ can assume any value, seems astonishing at first, but is easily explained by the following consideration.

If, at a certain definite deflection a, there is an equilibrium between the outer forces and the inner bending moments, and if the deflection is artificially increased to $n\,a$, then all the inner stresses and simultaneously the outer bending moments for every cross-section will be increased n times. The rod is therefore in an indifferent equilibrium. If P_k is somewhat diminished, the rod will come back to its original form, if P_k be slightly increased, the original small deflection will continue to grow larger until the rod breaks.

In contrast to the above is the state of equilibrium of a horizontal rod supported at the ends and loaded transversally to the axis in which case there is a definite deflection for every load.

The difference in the two cases is due to the fact that, in the transversally loaded rod the bending moments are independent of the deflections while in the axially loaded rods they are proportional to the deflection.

The conclusion drawn from the discussion of Euler's formula that an axially loaded rod should not deflect any measurable amount and that no gradual increase in deflection should correspond to the gradually increasing load, is, however, not generally confirmed by experiments. For instance, Fig. 6 shows the lateral deflection in relation to the axial load for two steel tubes.

No. I, 304 cm. long, 65 mm. diameter and 1.46 mm. wall thickness.

" II, 294 cm. long, 80.1 mm. diameter and 1.98 mm. wall thickness.

It follows from the curves that, although the deflections grow rapidly with the load, the crippling of the rod does not occur very suddenly.

This contradiction might be attributed to the fact that the modulus of elasticity E does not remain quite constant as the stress increases or that either the cross-sections of the rod are not perfectly symmetrical or the material is not uniform, or also, that the rod was not perfectly straight before the load was applied. Finally, it is not quite correct to assume that for considerable deflections the arc element ds of the rod is equal to dx, using $\frac{d^2y}{dx^2}$ instead of $\frac{d\left(\frac{dy}{dx}\right)}{ds}$ in the derivation of Euler's formula.

The last point seems, however, unessential as the practically admissible deflection of the rod is slight. Also the variation of

E is inconsiderable. On the contrary, some bends in the rod before the test or nonuniformities of the wall thickness (for example, in a tube) or of the material, are of essential importance.

It is evident that even a perfectly straight rod will break only at a certain definite deflection a. This deflection for large values of $\frac{l}{i}$, at which the compressive stress is negligible in comparison with the bending stress can be calculated in the following manner:

$P_k = \frac{\pi^2 EJ}{l^2}$ $a P_k = k_o \frac{J}{e}$, where e is the distance of the outmost fiber from the centerline of the cross-section.

$$a = k_o \frac{J}{e} \frac{l^2}{\pi^2 E J} = \frac{k_o}{\pi^2 E} \frac{l^2}{e} \tag{14}$$

For small values of $\frac{l}{i}$, besides of the bending stress, the compressive stress must be considered. As both stresses taken together should not exceed k_o, the bending stress alone equals $k_o - k$.

In Fig. 1, $\frac{k}{k_o}$ is represented by the distance of the $\frac{k}{k_o}$ value from the x - axis and $\frac{k_o - k}{k}$ by the distance from the line for $\frac{k}{k_o} = 1$.

Therefore,

$$P_k = k F \text{ and } a P_k = (k_o - k) \frac{J}{e}$$

$$a = \frac{k_o - k}{k} \frac{J}{F_e} = \frac{k_o - k}{k} \frac{i^2}{e} \tag{15}$$

This equation gives the same values of a as equation (14) if, as for long rods, k in the numerator can be neglected and

P_k is taken equal to $k\,F = \frac{\pi^2 EJ}{l^2}$.

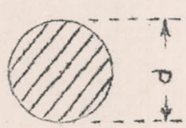
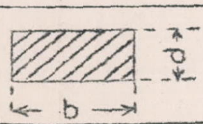
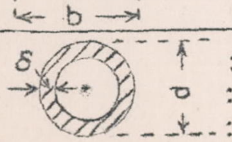
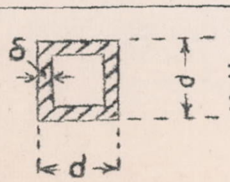
Subtracting in equation $a = \left(\frac{k_o}{k} - 1\right)\frac{i^2}{e}$ the value of $\frac{k_o}{k}$ from equation (8).

$$= \left(\frac{1 + \frac{k_o}{\pi^2 E}\left(\frac{l}{i}\right)^2 + \left(\frac{k_o}{\pi^2 E}\right)^2 \left(\frac{l}{i}\right)^4}{1 + \frac{k_o}{\pi^2 E}\left(\frac{l}{i}\right)^2} - 1 \right) \frac{i^2}{e} \qquad (16)$$

$$a = \frac{\left(\frac{k_o}{\pi^2 E}\,\frac{l^2}{i^2}\right)^2}{1 + \frac{k_o}{\pi^2 E}\,\frac{l^2}{i^2}}\,\frac{i^2}{e} \qquad (17)$$

Equations (15) and (17) lead further to the following important consideration. If, for a certain value of $\frac{l}{i}$ the ratio $\frac{k}{k_o}$ is determined according to equation (8) or Fig. 4, then the greatest admissible deflection a is proportional to $\frac{i^2}{e}$. Table 10 gives $\frac{i^2}{e}$ for some solid and thin walled hollow sections.

Table 10.

Cross-section	$i^2 = \frac{J}{F}$	$\frac{i^2}{e}$
	$\frac{\pi d^4 4}{64\pi d^2} = \frac{d^2}{16}$	$\frac{d^2/16}{d/2} = \frac{d}{8} = 0.125\ d$
	$\frac{b\ d^3}{12\ b\ d} = \frac{d^2}{12}$	$\frac{d^2/12}{d/2} = \frac{d}{6} = 0.167\ d$
	$\frac{1}{8}\ \frac{\pi\ \delta d^3}{\pi\ \delta\ d} = \frac{d^2}{8}$	$\frac{d^2/8}{d/2} = \frac{d}{4} = 0.250\ d$
	$\frac{\frac{1}{6}\ \delta\ d^3 + \frac{1}{2}\delta\ d^3}{4\ \delta\ d} = \frac{d^2}{6}$	$\frac{d^2/6}{d/2} = \frac{d}{3} = 0.333\ d$

In an airplane lateral deflection of rods can be produced by vibration or other external causes, and Table 10 shows that the hollow sectioned axially-loaded rods can be allowed to deflect twice as much as the solid sectioned rods. The square section is 4/3 times more advantageous than the circular one, due to the larger cross-sectional area.

Now, examining the case of an eccentric load, considering at the same time, instead of a straight rod, a rod having an initial bend b, before the load is applied, (See Fig. 7). The curve of the initial bend is unessential. To simplify the calculations, it can therefore be assumed that the curve of the initial bend is similar to the elastic curve of the rod deflected under the load. If then b denotes the initial bend and a the additional elastic deflection,

$$y_1 = \frac{b}{a+b} y; \; d\,y_1 = \frac{b}{a+b} d\,y; \; d^2 y_1 = \frac{b}{a+b} d^2 y.$$

As further

$$\frac{d^2\,y}{d\,x^2} - \frac{d^2 y_1}{d\,x^2} = - \frac{P}{E\,J} y,$$

where P is any load, then

$$\frac{d^2\,y}{d\,x^2} = - \frac{a+b}{a} \frac{P}{E\,J} y. \tag{18}$$

This equation is satisfied by

$$y = (a+b) \sin \pi \frac{x}{l} \tag{19}$$

$$\frac{d^2 y}{d\,x^2} = -\,(a + b)\,\frac{\pi^2}{l^2}\,\sin \pi\,\frac{x}{l} = -\,\frac{\pi^2}{l^2}\;y \tag{19a}$$

$$y_1 = b \sin \pi\,\frac{x}{l} \quad \frac{d^2 y_1}{d\,x^2} = -\,b\,\frac{\pi^2}{l^2}\,\sin \pi\,\frac{x}{l} = -\,\frac{\pi^2}{l^2}\;y_1 . \tag{19b}$$

With reference to equation (18) we have therefore:

$$\frac{a + b}{a}\;\frac{P}{EJ} = \frac{\pi^2}{l^2};\;\frac{a}{a + b} = \frac{P\,l^2}{\pi^2 E\,J}\;.$$

The elastic deflection however is not $a + b$ but a only and it depends upon b and P.

$$a = \frac{b\,P\,l^2}{\pi^2 E\,J - P\,l^2} = \frac{b}{\dfrac{\pi^2 E\,J}{P\,l^2} - 1} \tag{20}$$

In the case of an eccentrically loaded rod, the deflection a depends therefore, upon P. For $P = o$, $a = o$ and for

$P = P_k = \dfrac{\pi^2 E\,J}{l^2}$ $a = \infty$.

$\dfrac{\pi^2 E\,J}{l^2} = P_k$ is Euler's crippling load. Taking therefore $P = \gamma\,P_k$, it will be found that

$$a = b\,\frac{\gamma}{\gamma - 1} \tag{21}$$

For different values of γ the values of $\frac{a}{b}$ $\frac{\gamma}{\gamma - 1}$ are given in Table 11.

Table 11.

γ	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
$\frac{a}{b}$	∞	9.0	4.0	2.33	1.5	1.0	0.667	0.428	0.250	0.111	0.000

The eccentricity b cannot be easily measured as it is composed not only of the real eccentricity of the load, but also of the lack of uniformity of the wall thickness and the material. If, however, this seeming (total effective) eccentricity be calculated for one of the experimental results in Fig. 6 (curve 1 gives $b = 1.1$ cm. and curve II as 1.27 cm.) and then for the various loads the corresponding deflections a figured out (in Fig. 6 the various cases are marked by circles) it will be seen that the calculations agree surprisingly well with the test results. This proves the statement that the regular increase of deflection with the load is due mainly to lack of symmetry and uniformity of the rod, both in dimensions and material, and that the effect can be reproduced by the assumption of an eccentric load.

DETERMINATION OF ADMISSIBLE LOAD WITH REGARD TO ECCENTRICITY.

1. For relatively slender rods, in which the mean compressive stress k is negligible in comparison with the bending stress $k_o - k$, the following relations are approximately true:

$$P(a + b) = k_o \frac{J}{e} \tag{22}$$

$$a = \frac{k_o}{P} \frac{J}{e} - b \quad \text{and with equation (20)} \tag{23}$$

$$\frac{k_o}{P} \frac{J}{e} - b = \frac{b\,P\,l^2}{\pi^2 E\,J - P\,l^2}$$

$$P\left(k_o \frac{J}{e}\, l^2 + b\,\pi^2 E\,J\right) = k_o \frac{J}{e}\, \pi^2 E\,J$$

$$P = \frac{\pi^2 E\,J}{l^2} \; \frac{1}{1 + b\,\dfrac{\pi^2 E}{k_o}\,\dfrac{e}{l^2}} \tag{24}$$

2. For stouter rods, where the mean compressive stress should be considered:

$$P\,(a + b) = (k_0 - k)\,\frac{J}{e} \qquad (25)$$

Substituting $k = \frac{P}{F}$ and from the equation (20)

$$a = \frac{b\,P\,l^2}{\pi^2 E\,J - P\,l^2} \text{ and it will be found:}$$

$$P\,b\,\frac{\pi^2 E\,J}{\pi^2 E\,J - P\,l^2} = k_0\,\frac{J}{e} - \frac{P}{F}\,\frac{J}{e}$$

$$P^2 - P\left(\frac{\pi^2 E\,J}{l^2} + k_0 F + \frac{\pi^2 E\,Feb}{l^2}\right) = -\,\frac{k_0 F\,\pi^2 E\,J}{l^2}$$

$$P = \frac{1}{2}\,\frac{\pi^2 E\,J}{l^2}\left[\left(1 + \frac{k_0 F\,l^2}{\pi^2 E\,J} + \frac{Feb}{J}\right)\right.$$

$$\left.(\pm)\sqrt{\left(1 + \frac{k_0 F\,l^2}{\pi^2 E\,J} + \frac{Feb}{J}\right)^2 - \frac{4\,k_0 F\,l^2}{\pi^2 E\,J}}\,\right]. \qquad (26)$$

This formula has two roots and if applied to the steel tube No. II in Fig. 6, where l = 294 cm.; F = 4.9 cm^2; J = 37.3 cm^4; k_0 = 5200 kg/cm^2; E = 2,000,000 kg/cm^2 and b = 1.27 cm., the buckling load will be found either P_1 = 33,275 kg., and the corresponding elastic deflection according to equation (20) a_1 = - 1,715 cm. or P_2 = 6,635 kg. and a_2 = 4.14 cm.

P_1 and a_1 do not practically come into question as the deflection a_1 is in a direction opposing the initial bend b and k would be greater than k_0. Therefore, only the negative sign of the root in equation (26) should be used.

Equation (26) can be used in order to determine the buckling load P according to the idea of Mueller-Breslau* assuming eccentricity of the rod. The additional deflection a corresponding to the load P is given by equation (20); the total deflection is therefore:

$$a + b = b \frac{P l^2}{\pi^2 E J - P l^2} + b = b \frac{\pi^2 E J}{\pi^2 E J - P l^2} \qquad (27)$$

and the bending stress is obtained from equation (25) to

$$k - k_o = \frac{P (a + b) e}{J} \qquad (28)$$

The foregoing calculation becomes very clear when the moments M_a and M_i of the outer and inner forces are drawn in relation to the deflection a, for example, for the weakest section, as in Fig. 7.

$$M_a = P (a + b) \qquad (29)$$

$$M_i = - \frac{d^2y}{dx^2} - \frac{d^2y_1}{dx^2} E J \qquad (30)$$

in which, according to equations (19a) and (19b)

for $x = \frac{l}{2}$ $\quad \frac{d^2y}{dx^2} - \frac{d^2y_1}{dx^2} = - a \frac{\pi^2}{l^2}$

and

$$M_i = a \frac{\pi^2 E J}{l^2} \qquad (31)$$

The moments M_a and M_i are represented in Fig. 7 by full lines. Their point of intersection corresponds to a deflection a at which the inner and the outer forces are in equilibrium. A smaller load $P_1 < P$ would have a corresponding moment line M_{a1},

*Mueller (Breslau), Die neueren Methoden der Festigkeitlehre. Published by A. Kroener, Leipzig, 1913, Chap. VI, p.360.

dotted and a smaller a similarly for $P_2 = P$, there will be a line M_{a2} and a greater a and $a + b$. If P increases so much that the line M_{a3} becomes parallel to M_i, a will be equal to ∞ and $P = P_k = \frac{\pi^2 E J}{l^2}$. If P is increased still further, then the line M_{a4} will intersect the line M_i produced so that a becomes negative. This value of $-a$ corresponds to the unused root of equation (26).

The foregoing ought to prove that although the bend $a + b$, in the case of an eccentrically loaded rod, grows rapidly with the increase of the load, the break does not occur suddenly. On the other hand, it seems possible to eliminate in a compression test, by means of a proper arrangement, the always present slight eccentricity in which all the ununiformities of wall thickness and material are included. Then for a definite load (namely, Euler's crippling load $P_k = \frac{\pi^2 E J}{l^2}$), there must be an equilibrium for any deflection. This means that the curves I and II, Fig. 6, should be vertical lines P, having the same value for any deflection.

This apparatus may, for example, consist of two spherical compression blocks, with respect to which the ends of the tube, protected by end plates, may be laterally displaced by means of adjusting screws until the crippling load reaches a maximum.

The author had such a testing device made. The tubes to be tested are closed on both ends by plane pressure plates, and in the machine are placed two compression blocks whose knife edges, rounded off to 5 mm. radius, are adjusted accurately parallel. On both sides of the knife edges are adjusting screws, by means of which the ends of the tube may be adjusted laterally with reference to the knife edges, thereby avoiding the turning of the tube about its axis, as well as permanent deflections. In the experiments, tests were made upon the strut, a steel tube of 8 mm. diameter, 2 mm. wall thickness, and 3030 mm. length, with apparent eccentricities varying from +12 mm. to -2.6 mm. While the calculation gives Euler's buckling load $P = \frac{\pi^2 \times 2,150,000 \times 37.3}{303^2} = 8400$ kg., it was found that the tube could stand a load of 9500 kg. at an eccentricity e = -2.6 mm. At 9600 kg., without any increase of the load, the deflection rose immediately from -1.2 mm. to a very high value, that would have led to collapse (See Table 12).

TABLE 12.

Deflections "a" in mm. at Various Eccentricities.

Eccentricity C in mm.	Loads in kg. 1000	2000	3000	4000	5000	6000	7000	8000	8500	9000	9500	9600
12	2.1											
10	1.7											
8	1.4											
6	1.1	2.8										
4	0.9	2.4										
2	0.5	1.3	2.4	4.0								
0	0.1	0.3	0.9	1.6	2.7	4.5						
-2.5	0.0	0.0	0.0	0.0	0.4	0.1	0.2	0.6				
-2.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.1	-0.3	-1.2	Buckling.

It follows from these tests that a perfectly straight rod, in agreement with Euler's theory, does not deflect at all under loads below Euler's crippling load, and that after their ultimate load is reached, there is an equilibrium between the outer and the inner forces for any deflection within the elastic limit of the material.

SUMMARY.

The formulas hitherto employed for calculation of rods subject to compression, are usually of value only for stout or for slender rods. They do not cover, as a rule, the whole range of rod lengths, or they give too great a safety factor for short and moderately large rods. Therefore, a new empirical formula, equation (8), has been developed, that holds good for any length and any material of the rod, and agrees well with the results of extensive strength tests. To facilitate the calculations, three tables are included, giving the crippling load for solid and hollow sectioned wooden rods of different thickness and length, as well as for steel tubes manufactured according to the standards of Army Air Service Inspection (Idflz). Further, a graphical method of calculation of the breaking load is derived (Fig. 4) in which a single curve is employed for determination of the allowable fiber stress.

Finally, the theory is discussed of the elastic curve for a rod subject to compression, according to which no deflection occurs, and the apparent contradiction of this conclusion by test results is attributed to the fact that the rods under test are not perfectly straight, or that the wall thickness and the material are not uni-

form. Under the assumption of an eccentric rod, having a slight initial bend according to a sine curve, a simple formula for the deflection is derived, which shows a surprising agreement with test results. From this a further formula is derived for the determination of the allowable load on an eccentric rod. The resulting relations are made clearer by means of a graphical representation of the relation of the moments of the outer and inner forces (Fig. 7) to the deflection, and through the determination of equilibrium between moments.

Translated by F. W. Pawlowski,
University of Michigan.

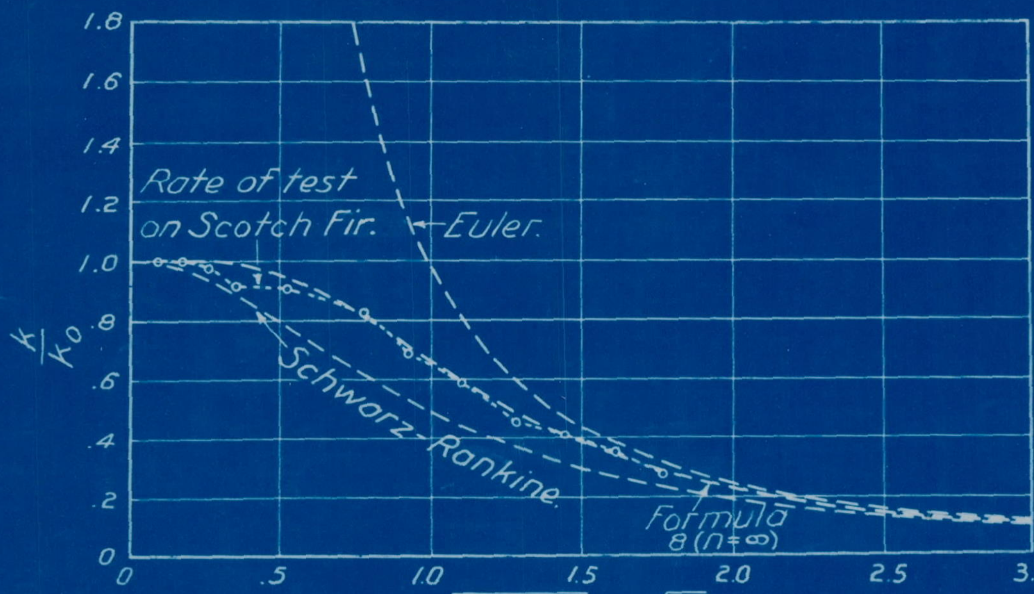


Fig. 1.

$$\sqrt{\frac{k_0 F l^2}{\pi^2 E J}} = \frac{l}{i}\sqrt{\frac{k_0}{\pi^2 E}} = \rightarrow$$

Comparison of the formulas of Euler, Schwarz-Rankine and Natalis with the Test Results of Pine Rods.

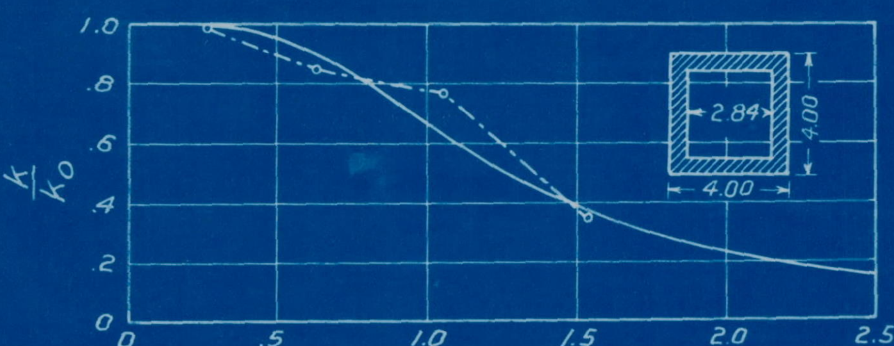


Fig. 2.

$$\frac{l}{i}\sqrt{\frac{k_0}{\pi^2 E}} = \rightarrow$$

Tests of the Hollow Section Pine Rods.

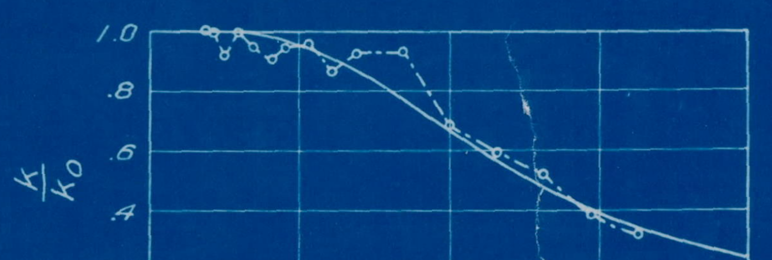


Fig. 3

$$\frac{l}{i}\sqrt{\frac{k_0}{\pi^2 E}} = \rightarrow$$

Tests of Seamless Steel Tubes.

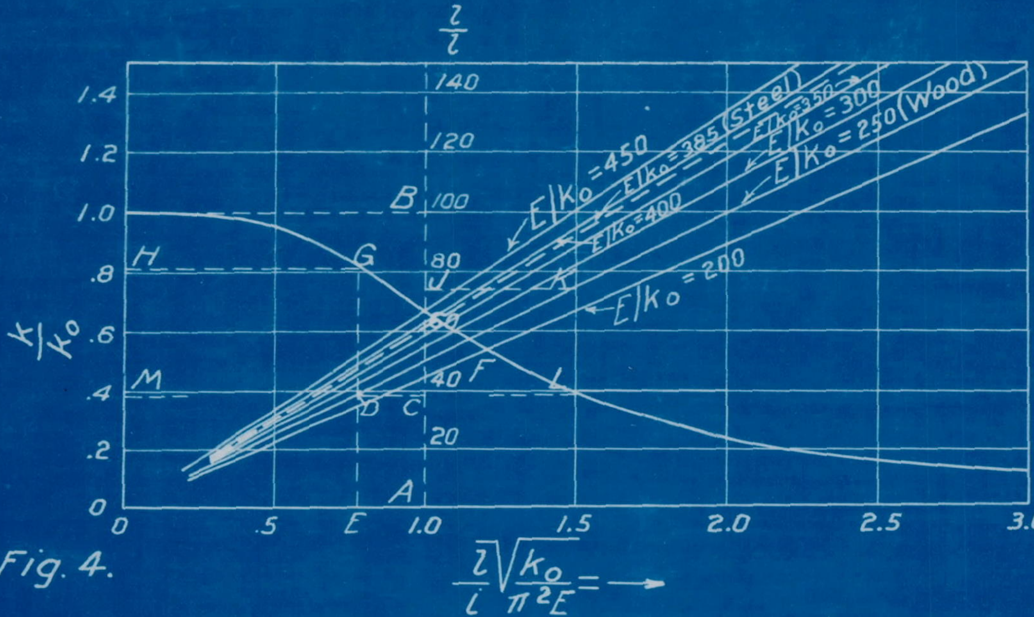


Fig. 4.

Graphical determination of Crippling Load.

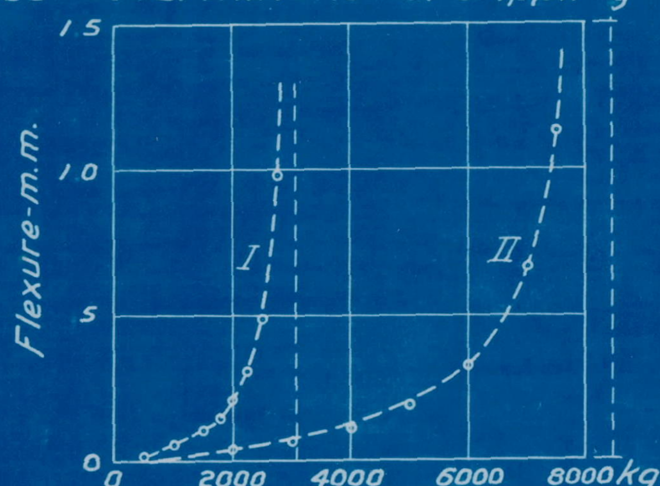


Fig. 6. Deflection of Rods in relation to the Load. The dotted lines (the asymptotes) represent the buckling load.

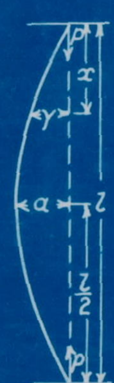


Fig. 5.

Elastic Line of a Straight Rod.

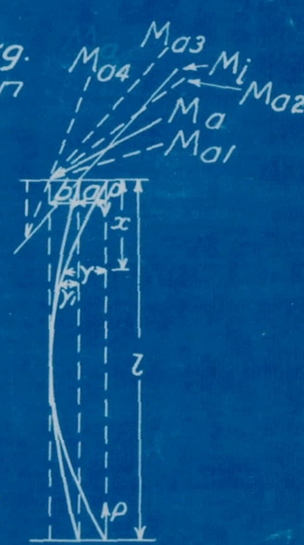


Fig. 7.

Elastic Line of a Slightly Deflected rod.