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LONGITUDINAL OSCILLATION OF AN AIRPLANE.

PART I - PROBLEM AND METHOD.

By R. Fuchs and L. Hopf.

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Introduction.

All aerodynamical calculations, which form the basis of the designing and analysis of airplanes, are founded upon an assumption of uniform steady motion. There are, however, many problems of practical importance which cannot be disposed of by means of such calculations. To these belong all questions relating to maneuverability and to the maximum stresses undergone during flight. Such problems have hitherto been solved by applying established scientific principles relating to steady flight, as conformably to the facts as possible and, where this method was not practicable, by relying in individual cases, on the opinion of the pilot. If, however, aerodynamics is to afford a wider basis for the art of flying, it must elucidate the problems of accelerated and disturbed motion, phenomena in an accidental or intentional disturbance through the deflection of the rudder or elevator, or any other change in the conditions of flight.

* From Technische Berichte, Volume III, No. 7, pp. 317-330.

Our experimental knowledge is sufficiently extensive to afford, in many instances, the necessary basis for the mathematical analysis of these phenomena and, where such is not the case, the theoretical investigation of the problem can indicate the necessary experiments on models and actual flight.

The treatment of the whole problem may, at first, appear hopeless to the theorist. The problem is that of a body with six degrees of freedom moving in a fluid (air) and on which forces are acting, whose relation to the position of the body is only known empirically. Bryan's great service is the circumvention of these difficulties by the method of small oscillations and the opening of a way for the treatment of these problems, even though the method is restricted to simple conditions. The method of small oscillations is only applicable to conditions in the neighborhood of a known state of equilibrium and, as applied to the present instance, is as follows:

An airplane is in steady flight along a given line. The quantities which determine its position and conditions of flight (namely, velocity, angle of attack, slope of the flight path, angle of bank, rate of side-slipping and curvature of flight path) are all interdependent, when the engine generates a definite propeller thrust and the rudders have a definite position. The values of the above quantities are determined by the conditions of equilibrium in steady flight, which have never been satisfactorily discussed. If one or more of these variables have not the value required by the conditions of equilibrium, then the oscillation cannot be steady. It is simply assumed that none of these variables differs greatly from the value corresponding to equilibrium in steady flight.

All terms of the equations of motion, which are due to disturbances, are then expanded in powers of the small displacements and only the first term of each series is retained. The equations in this form are linear and easily solved. No further fundamental difficulties present themselves and only the mathematical work (which is sometimes very hard), has to be performed.

All previous calculations refer to small departures from rectilinear flight, for which there are two independent groups, each consisting of three equations of motion. Longitudinal and lateral oscillations take place, in this case, independently of one another. Changes in speed, in the slope of the flight-path and in the angle of attack produce no lateral or unsymmetrical motion oscillations. Banking, side-slipping and yawing, so long as they are small, have no effect on the symmetrical oscillations which are determined by the three above-mentioned variables. On the contrary, it is impossible to separate still further the lateral oscillations and treat rotations around the longitudinal axis X, (rolling), separately from rotation around the vertical axis Z, (yawing).

In practically all contributions which deal with the problem thus simplified, only the question of stability in rectilinear flight is raised. The actual course of a disturbance is not followed out, but only the question is raised as to whether an airplane will finally return to equilibrium from the disturbed condition due to the small variables introduced, or whether it has a tendency to diverge further from it. In the former case, the flight condition is termed stable, in the latter, unstable. The importance of and the effect due to aerodynamical quantities, as determined by the design of the airplane, are brought out by this procedure, but, on the contrary, it is not clear as to what significance is given the term stability and what qualities the airplane will exhibit in the hands of the pilot. Neither can it be maintained that the aerodynamic theory of stability has borne fruit in practice, except possibly in England, where it has been supplemented by systematic tests on models. So long as the stability of only one flight condition is examined, all the above-mentioned questions remain undecided. Above all, we do not understand how stability and maneuverability are mutually related and whether an airplane can be handled as well when constructed with a high degree of stability, and how an airplane reacts to an accidental disturbance.

Reissner* first recognized the need of investigating more deeply into the actual facts and of going beyond the question of mere possession of stability to describe the actual course of the disturbed motion. Reissner and his pupil Gehler** were the first

*Zeitschrift für Flugtechnik und Motorluftschiffahrt, 1910, Nos. 9 and 10.
**"Dissertation," published by R. Oldenbourg, Munich, 1913.

to work out the problem for lateral oscillations. Gehlen not only solves the equation of stability when longitudinal and lateral oscillations are mutually independent, but also determines the integration constants, from arbitrary initial values of a disturbance or of a movement of a rudder, and gives a complete description of the phenomena involved, namely: the angle of bank, the radius of the turn, and the drift of the airplane.

This analysis, which represents the utmost that can be attained by the method of small oscillations, still has the defect that it is restricted to small deviations from rectilinear flight. In curve-flight the problem is not one of small deviations only. The problems cannot be solved without taking into account, at the same time, the longitudinal oscillations, which are separated from the lateral oscillations in the method used for small oscillations. For instance, the question today depends on whether an airplane is ascending or descending in a turn. For finite oscillations, which may differ to any extent from rectilinear flight, even now the problem cannot be attacked, since not even the general case of steady motion which includes both rectilinear and curving flight, has yet been solved.

The question of longitudinal oscillation is different, since there are, of course, steady longitudinal oscillations with any desired velocity, angle of attack and slope of flight path, without lateral oscillations. Such a steady forward motion presents many points of technical importance, some of which we will touch upon. There is, for instance, the question of static stability, on which something has already been published in the Technische Berichte.* It does not follow, from static considerations, as to what magnitude of static stability should be chosen for an airplane, according to its purpose, and even analysis by the method of small oscillations furnishes no conclusion on this point. It is known that airplanes which are ordinarily unstable, may nevertheless fly well under special circumstances. It is also known that even with a stable airplane, conditions may arise out of which the airplane can only be rescued with difficulty. The best known example is "stalling," in which the airplane no longer obeys the elevator. Although the result is usually an unsymmetrical oscillation, it is generally begun by "stalling" (that is, a symmetrical condition of flight into which enter hitherto unexplained relations). Phenomena have also been observed in diving, which may give rise to danger. The problems of stressing are purely problems of longitudinal oscillation, since airplanes have hitherto only been tested under symmetrical loads. We have to determine what centrifugal forces appear in flattening out after a dive, or what lift coefficients and velocities combine, in passing from high speed at small angles of attack to low speed at large angles of attack.

There are, as yet, no analytical methods for longitudinal oscillations as valuable as those of Gehlen for lateral oscilla-

* Technische Berichte, Volume I, No. 1, p.16, and following; Volume I, No.4, p.108 and following; Volume II, No.1, p.33; Volume II, No.3, p.463 and following.

tions.* Instead of extending these calculations, based on the method of small oscillations, the range of which is difficult to perceive, and which lead to most complicated calculations, we have adopted another method which is not restricted to small oscillations. This was encouraged partly by the results of a number of purely practical numerical computations (according to which, approved airplanes appear to have almost neutral equilibrium) and partly by Lanchester's "Theory of Phugoids," which includes general longitudinal oscillations differing widely from steady flight.

Lift is considered as the only air force in the phugoid theory. Drag is neglected, thus eliminating all dissipative forces, and the principle of energy supplies a simple solution of the equations of motion. The angle of attack is further assumed to be invariable during the whole period of oscillation. In this way, all empirical relations are excluded and the whole motion may be analytically presented. The significance of these simplifications will be gone into in a subsequent section. However bold they may seem, the result nevertheless agrees with motions actually observed in flight. "Looping the loop" was recognized in the phugoid theory long before Pegoud. Paper darts, such as children play with, and gliding models, thrown into the air, describe motions which agree exactly with those required by the phugoid theory. Actual airplanes, when left to themselves, do not, however, fly in phugoids and the suppositions of the phugoid theory must, therefore, fail entirely in the domain of full-size

* Papers by Bryan and his pupils, which are difficult to obtain, appear, however, to deal with this subject.

controllable aircraft.

It has been pointed out by Von Karman and Trefftz* that the phugoid oscillation gives a solution for the ordinary equations of longitudinal oscillations when the static stability is infinite. An airplane then resists every change in its angle of attack and the most important assumption of the phugoid theory is satisfied. The possibility of controlling the airplane disappears completely, since transition from one condition of flight to another is inconceivable, without changing the angle of attack. Since no airplane can be built which will describe a phugoid, the phugoid theory is not suitable for the elucidation of all these relations. It fails to solve the most important problem of all, namely, that of controllability. Numerical calculations, in fact, lead to the anticipated result that the equilibrium of serviceable airplanes is not infinitely stable, but, on the contrary, is very small (positive or negative) and that airplanes, in the first approximation, are neutral (Technische Berichte, Volume II, No. 3, p.463).

As will be shown later, neutral equilibrium greatly simplifies the equations of oscillation. In the simplest case, indeed, it is not the angle of attack that remains unaltered, but the angle between the longitudinal axis of the airplane and the horizontal. The other equations can be easily solved, but not in a closed expression, as in the phugoid theory, since the forces only depend empirically on the now variable angle of attack. The solu-

* "Über Längsstabilität und Längsschwingungen von Flugzeugen" (Longitudinal stability and longitudinal oscillations of airplanes), Jahrbuch der Wissenschaftlichen Gesellschaft für Luftfahrt, Volume III, 1914-15, p.116.

tion, however, exhibits one very definite characteristic, namely, the oscillation which it represents has two distinct phases with respect to time, in that the forces at right angles to the flight-path first reach equilibrium and then (much more slowly), the forces in the direction of flight.

Our further considerations are based on this fact. The result is used suggestively, in an attempt to find an approximate solution of the non-neutral airplane having the above-described characteristic. It is assumed that the velocity changes more slowly than the other terms determining the conditions of flight. The method has in every case proved applicable. It leads to a step-by-step integration of the equations of oscillation from the original condition, but the steps are so long that rarely more than two are required and, within the range of each step, the individual quantities are obtained in closed form, as solutions of linear differential equations.

By this method, it is easy to represent the most important part (the initial stage) of the course of an oscillation without being limited to fixed conditions. A variable angle of the control surface, slow application of the controls or a back and forth motion of the controls can, in this way, be as easily expressed as any accidental external disturbances, gusts, etc. In the first part, the method will be worked out and the formulas given, while, in the following parts, definite problems will be treated.

I - Symbols Used in Oscillation Equations.

χ = Angle formed with horizontal by tangent to flight-path (Fig. 1);

θ = Angle between upper wing and horizontal (Fig. 1).

α = Angle of attack, angle between chord of upper wing and tangent to flight-path;

φ = Angle between propeller axis and flight-path hence $\alpha - \varphi = i_w$;

i_w = Angle of incidence between upper wing and propeller axis, $\chi = \theta - \alpha$ (Fig. 1);

W = Total weight of airplane in kilograms;

T = Propeller thrust in kilograms;

S = Area of supporting surface in m^2;

V = Resultant speed of airplane in meters per second, considered positive in the direction of flight;

λ = Specific weight of the air in kilograms per m^3;

g = Acceleration due to gravity = 9.81 meters per second^2

q = Dynamic pressure = $\frac{1}{2} \times \frac{\lambda}{g} \times V^2$;

C_L and C_D coefficients of lift and drag;

$L = q\ S\ C_L$ (lift); $D = q\ S\ C_D$ (drag);

M = Moment of forces of airplane about its center of gravity, measured in such a way that a moment is positive when it turns the nose of the airplane downward. The positive direction of the turning moment is, therefore, opposite to that of the angles θ, α, χ.

Forces in the Direction of Flight.- In the direction of flight there act: a component of the propeller thrust, $T \cos\varphi$, a component of the force of gravity, $-W \sin\chi$; and the air resistance, $-C_D\, q\, S$. We therefore obtain

$$\frac{W}{g}\frac{dV}{dt} = T \cos\varphi - W \sin\chi - \frac{1}{2} C_D \frac{\lambda}{g} S V^2 \qquad (I)$$

Forces at Right Angles to the Direction of Flight.- There act at right angles to the direction of flight: a component of the propeller thrust, $T \sin\varphi$; a component of the force of gravity, $-W \cos\chi$; and the lift, $C_L\, q\, S$. To this must be added, in curve-flight, the centrifugal force $\frac{W}{g} \times \frac{V^2}{r}$, in which r is the radius of curvature of the flight-path. To determine r, we have (Fig. 2) the equation

$$\frac{V\, d\, t}{r} = d\,\chi$$

The centrifugal force acts in the direction of gravity, when $d\,\chi$ is positive. If the direction of gravity is considered negative, the centrifugal force is written

$$-\frac{W}{g} V \frac{d\,\chi}{d\,t}$$

Since there are no components of velocity at right angles to the flight-path, we obtain

$$0 = \frac{-W}{g} V \frac{d\chi}{dt} + T \sin\varphi - W \cos\chi + \frac{1}{2}\frac{\lambda}{g} C_L S V^2 \qquad (II)$$

Moments.- If k is the radius of gyration of the airplane about its center of gravity, the positive direction of the moment being opposite to that of increasing θ, we have

$$\frac{W}{g} k^2 \frac{d^2\theta}{dt^2} = - M$$

In finding the value of the moment, it was borne in mind that M depends upon V, α and the angular velocity $\frac{d\theta}{dt}$. The resultant moment M, is principally made up* of the moments of the wings, together with that of the horizontal stabilizer and elevator. Assuming as usual, that the moment is proportional to the square of the velocity and is a linear function of α,

$$M = (m_0 + m_1 \alpha) V^2$$

and $m_1 V^2 = \frac{dM}{d\alpha}$, the so-called static stability. m_0 is determined by the position of the elevator at the time. This, in turn, fixes the angle of attack at which the moments are in equilibrium. It cannot, of course, be assumed that m_1 is the same for all conditions of flight. If, however, the equations are taken for successive intervals of time, within which the angle of attack does not vary too much, it may then be safely assumed that m_1 is constant for the duration of such an interval. An interval can, in any case, last only so long as the position of the elevator does not vary, that is, so long as m_0 has the same value.

When θ itself varies with the time, the moment is a function of this variation, $\frac{d\theta}{dt} = \dot{\theta}$. To obtain the differential

* Zur Berechnung der Längsmomente von Flugzeugen (Calculation of the longitudinal moments of airplanes), Technische Berichte, Volume II, No.3, pp.463-483.

coefficient of this relation, the damping, it must be remembered that any variation of θ affects α as well as V. The effect on V may be neglected, since it is very small, and even the alteration of α need only be introduced in the calculation for the moment of the horizontal stabilizer and elevator. An estimate indicates that the damping factors, which allow for the variation of V, only amount to $\frac{1}{40}$ and the damping effect of the wings to only $\frac{1}{20}$, the damping of the horizontal tail plane being taken as 1. If, therefore, M_H is the moment of the horizontal tailplane and elevator, it is only necessary, in the expression for M, to add

$$\frac{dM_H}{d\theta}\dot{\theta} = \frac{dM_H}{d\alpha}\frac{d\alpha}{d\theta}\dot{\theta}$$

In order to be able to express the ratio $\frac{d\alpha}{d\theta}$, let r_H indicate the distance of the middle of the horizontal tail surfaces from the center of gravity. By a rotation $d\theta$, around the axis, the horizontal tail surfaces are lowered by $r_H\, d\theta$. If the airplane simultaneously advances a distance of dx, the angle of attack increases by $\Delta\alpha$, so that

$$\tan\Delta\alpha = \frac{r_H d\theta}{dx} = \frac{r_H\dot{\theta}}{V}$$

The cosine of the small angle which the horizontal tailplane makes with the propeller axis, is here taken as unity. With the small size of the angle in question, the tangent may be replaced by the arc and we then obtain

$$\frac{d\alpha}{d\theta} = \frac{r_H}{V}$$

The equation of moments thus takes the form

$$\frac{d^2\theta}{dt^2} = (f - m\alpha)\ V^2 - d\ V\frac{d\theta}{dt} \qquad \text{(III)}$$

in which

$$m = \frac{g}{Wk^2}\ \frac{d}{d\alpha}\ \frac{M}{V^2},\ d = \frac{g}{Wk^2}\ r_H\ \frac{d}{d\alpha}\ \frac{M_H}{V^2} \qquad \text{(I)}$$

while f is a constant determined by the position of the elevator at the time.

In equations (I) and (II) we put $\lambda = \theta - \alpha$; further because of the small angles: $\cos\alpha = \cos\varphi = 1$; $\sin\alpha = \frac{\alpha}{57.3}$; and $\sin\varphi = \frac{\varphi}{57.3}$; all angles being measured in degrees. On putting $\frac{d\theta}{dt} = \gamma$, the differential equations become

$$\frac{dV}{dt} = \frac{Tg}{W} - g\ \sin\theta + \frac{g}{57.3}\ \cos\theta\ \alpha - \frac{1}{2}\ \frac{\lambda}{W}\ C_D\ S\ V^2 \qquad \text{(Ia)}$$

$$\frac{d\alpha}{dt} = \gamma - \frac{1}{V}\left[\frac{Tg}{W}\ (\alpha - i_W) + 57.3\ g\ \cos\theta + g\ \sin\theta\alpha\right] - \frac{57.3}{2}\ \frac{\lambda}{g}\ C_L S_V \qquad \text{(IIa)}$$

$$\frac{d\alpha}{dt} = \gamma \qquad \text{(IIIa)}$$

$$\frac{d\gamma}{dt} = (f - m\alpha)\ V^2 - d\ \gamma\ V \qquad \text{(IVa)}$$

A few additional remarks may here be made on the analytical expression for the propeller thrust. If we assume* that the thrust decreases as the square of the velocity (which is

* An exact basis for this law is unfortunately lacking, as yet. The attempts by Kann (Technische Berichte, Volume I, No.6, pp. 232-241) would here be too elaborate. The present assumption agrees approximately with the expression derived by Everling (Zeitschrift für Flugtechnik und Motorluftschiffahrt, 1916, p.127, equation (8).

only approximate) or, in other words, that the thrust is represented (within the range discussed) by the parabola

$$T = T_0 - \rho V^2 \qquad (2)$$

(in which, of course, T_0 is not exactly the thrust when standing still) then ρ is still a function of the density of the air. We will, however, assume that the altitude of the airplane does not change materially within the limits of one of the time intervals considered, and that the density of the air therefore remains approximately constant. In equation (IIa), T may be considered constant, because $\frac{\rho g}{W}(\alpha - i_w)V$ is significant in comparison with $\frac{57.3}{2}\frac{\lambda}{W}C_L\,S\,V$ and also because the propeller thrust plays no part in this equation. In order to adapt equation (2) to equation (Ia), we then write

$$T = T_o - \frac{1}{2}\frac{\lambda}{g} S\, V^2 \rho'$$

and consider ρ' combined with C_D, that is, when flying with the engine going, the coefficient of drag is correspondingly increased. T can then be considered invariable in the differential equations. Moreover, an increase in C_D must also be made for steep gliding flight, when the propeller is running light.

II. Airplane with Neutral Equilibrium.

When static stability is zero and the moment does not change with the angle of attack,

$$m = \frac{g}{Wk^2}\frac{1}{V^2}\frac{dM}{d\alpha} = 0$$

In order that the airplane may be in equilibrium and that the moments set up by the wings and the tail may balance, it is necessary for the position of the elevators to be chosen so that $f = 0$. The airplane is then in neutral equilibrium and equation (IVa) becomes

$$\frac{d\gamma}{dt} = - d\gamma \quad V, \quad \text{also} \quad \gamma = C \, e^{-\int_0^t d \, V \, dt} \tag{1}$$

Assuming that when $t = 0$, $\gamma = \frac{d\theta}{dt} = 0$, it follows that $C = 0$, that is, $\gamma = 0$. In this case, therefore, the angle which the upper wing makes with the horizontal remains unaltered.

Equations (IIIa) and (IVa) are now eliminated and the variations of V and α are determined by

$$\frac{dV}{dt} = \frac{Tg}{W} - g \sin \theta_o + \frac{g}{57.3} \cos \theta_o \, \alpha - \frac{1}{2} \frac{\lambda}{W} S \, C_D \, V^2, \tag{2}$$

$$\frac{d\alpha}{dt} = \frac{57.3 \, g \cos \theta_o}{V} - \frac{57.3}{2} \frac{\lambda}{W} S \, C_L \, V^2 \tag{3}$$

The terms $\frac{Tg}{W}(\alpha - i_W) + g \sin \theta_o \, \alpha$ of equation (IIa) are omitted in equation (3), since they are very small in comparison with $57.3 \, g \cos \theta_o$. This can always be done when θ_o does not approach $\pm 90^\circ$.

From equations (2) and (3), V and α must now be calculated as functions of t. It is convenient, in many cases, to plot simultaneous values of α and V as coordinates. The following rule is important for the discussion of the resulting V,α curve. For each θ_o, there is, in the V,α plane, a curve, along which there is equilibrium of the forces in the direction of flight. This curve is obtained by putting $\frac{dV}{dt} = 0$ in equation (2) and then expressing V as a function of α. For each θ_σ, there is also a curve, along which there is equilibrium of the forces at right angles to the direction of flight. This curve is derived from equation (3) by putting $\frac{d\alpha}{dt} = 0$. There is complete equilibrium of the forces at the point of intersection of these two curves. Figs. 3 to 5 show these curves for three different values of θ_o, so that there is equilibrium when $\alpha = 3^o$ in Fig. 3, $\alpha = 10^o$ in Fig. 4, and $\alpha = 15^o$ in Fig. 5. We have taken $W = 1{,}530$ kg; $S = 41.3\ m^2$; $T = 485 - 0.05 \times \frac{1}{2} \times \frac{\lambda}{g} \times S\,V^2$; $\frac{\lambda}{\lambda_\sigma} = 0.81$; $\frac{1}{2} \times \frac{\lambda_\sigma}{g} = \frac{1}{15.2}$. The values for C_L and C_D are taken from the polar diagram of the Dfw C V. The two equilibrium curves intersect in the V, α plane in a second point, in addition to the point for which they are calculated. In the later investigations of the so-called "stalled" condition, the importance is shown of the question as to whether this second intersection is at a greater or a smaller angle of attack than stalling. We must, therefore, compute immediately at what value of θ_o the two curves of equilibrium just touch each other. If we write the equations, obtained by putting $\frac{dV}{dt} = 0$ and $\frac{d\alpha}{dt} = 0$ in equations (2) and (3), in the following form

$$V = s(\theta_o, \alpha),\ V = \chi(\theta_o, \alpha),$$

we obtain, for the desired point of contact,

$$\frac{ds}{d\alpha} = \frac{d\chi}{d\alpha}, \quad s = \chi \ . \tag{4}$$

From the second of these equations we obtain, for each α, the corresponding θ_o of complete equilibrium. We further find

$$\left(\frac{ds}{d\theta_o} - \frac{d\chi}{d\theta_o}\right)\frac{d\theta_o}{d\alpha} + \frac{ds}{d\alpha} - \frac{d\chi}{d\alpha} = 0 \tag{5}$$

and, consequently, $\frac{d\theta_o}{d\alpha} = 0$. If, therefore, θ_o is plotted against α for all positions of equilibrium, this curve must have a maximum value for the point of contact.

From equation (4) it follows that

$$\frac{d}{d\alpha}\left(\frac{C_D}{C_L}\right) = 1. \tag{6}$$

The numerical factor 57.3 must not be forgotten, when α is expressed in degrees.

III. Analytical Calculation of Neutral Equilibrium in Flight.

In Figs. 3 to 5, using the data for the airplane Dfw C V, in addition to the curves of equilibrium for several solutions of differential equations, simultaneous values of V and α are plotted by means of a numerical integration. From all these curves, it will be seen that (for a point V, α at some distance from the curve $\frac{d\alpha}{dt} = 0$ of the equilibrium of forces at right angles to the direction of flight) the corresponding integral curve always runs almost parallel to the α axis, so that variations in α correspond to much smaller variations of V. In other words, the forces at right angles to the direction of flight attain a state of equilibrium much more rapidly than those in the direction of flight. The reason is that much smaller values appear on the right side of equation 2 (II) than of equation 3 (III). Only when we approach the curve of equilibrium $\frac{d\alpha}{dt} = 0$, does the order of magnitude of $\frac{dV}{dt}$ and $\frac{d\alpha}{dt}$ become the same.

Accordingly, in the analytical calculation of the variations in the velocity and in the angle of attack, each step will be divided into separate intervals of time.

Case A.- Let us suppose that the equilibrium of an airplane has been disturbed, so that, at the beginning of the unsteady flight thus initiated, the velocity V_o and the angle of attack α_o determine a point far removed from the V, α curve $\frac{d\alpha}{dt} = 0$. The return toward neutral equilibrium must first of all be examined, as to the time when the forces at right angles to the line of flight are approximately in equilibrium. As a first approximation, we put $V = V_o$ and then determine, from equation 3 (II)

$$\frac{d\alpha}{dt} = \frac{57.3\ g \cos\theta_o}{V_o} - \frac{57.3}{2}\ \frac{\lambda}{W}\ S\ V_o\ C_L \qquad (1)$$

In carrying out this integration, only the time intervals will be considered, during which C_L may be regarded as a linear function of α. This is possible with C_L through a wide range, unless we come quite near the maximum lift. In the neighborhood of this maximum value the range becomes smaller and the time intervals of the integration must be taken correspondingly smaller.

If we put for C_L in equation (1)

$$C_L = C_{Lo} + C_{L1}\alpha \tag{2}$$

and further, for shortness

$$\epsilon = \frac{57.3}{2} \frac{\lambda}{W} S, \tag{3}$$

equation (1) then takes the form

$$\frac{d\alpha}{dt} = \frac{57.3 \ g \cos\theta_o}{V_o} - \epsilon C_{Lo} V_o - \epsilon C_{L1} V_o \alpha \tag{1a}$$

The solution of this equation, which has the value for $t = 0$, is

$$\alpha = L + (\alpha_o - L)e^{-\epsilon C_{L1} V_o t} \tag{4}$$

when

$$L = \frac{57.3 \cos\theta_o}{\epsilon C_{L1} V_o^2} - \frac{C_{Lo}}{C_{L1}}$$

In order to make a second approximation for V, this value is put for α, together with $V = V_o$, on the right-hand side of equation 2 (II).

$$\frac{dV}{dt} = \frac{Tg}{W} - g \sin\theta_o + \frac{g \cos\theta_o}{57.3}\alpha - \frac{\epsilon}{57.3} C_D V_o^2 \tag{5}$$

and takes, for simplicity in the time interval under consideration, as a linear function of

$$C_D = C_{Do} + C_{D1} \tag{6}$$

A less simple expression offers no difficulty, but makes the result less concise, without affecting it materially. This gives

$$\frac{dV}{dt} = \frac{Tg}{W} - g \sin\theta_o + \left(\frac{g \cos\theta_o}{57.3} - \frac{\epsilon C_{D1} V_o^2}{57.3}\right) L - \frac{\epsilon C_{Do} V_o^2}{57.3} +$$

$$+ \left(\frac{g \cos\theta_o}{57.3} - \frac{\epsilon C_{D1} V_o^2}{57.3}\right)(\alpha_o - L) e^{-\epsilon C_{L1} V_o t} \qquad (7)$$

We find, by integration, the following solution which has the value $V = V_o$ for $t = 0$:

$$V = V_o - P + Nt + P e^{-\epsilon C_{L1} V_o t} \qquad (8)$$

when

$$P = \left(\frac{C_{D1} V_o}{57.3 \ C_{L1}} - \frac{g \cos\theta_o}{57.3 \ \epsilon C_{L1} V_o}\right)(\alpha_o - L),$$

$$N = \frac{Tg}{W} - g \sin\theta_o + \left(\frac{g \cos\theta_o}{57.3} - \frac{\epsilon C_{D1} V_o^2}{57.3}\right) L - \frac{\epsilon C_{Do} V_o^2}{57.3} .$$

Example: In the above numerical example, $S = 41.3 \ m^2$; $W = 1530$ kg; $T = 485 - 0.05 \times \frac{1}{2} \times \frac{\lambda}{g} S V^2$,

$$\frac{\lambda}{\lambda_o} = 0.81, \ \frac{1}{2}\frac{\lambda_o}{g} = \frac{1}{15.2}$$

To the values of C_D in the polar diagram (Dfw C V), there must be added, in accordance with the rule for T, the amount 0.05 and also the coefficient of structural drag 0.0336. We may then, according to the dimensions of the model, put

$$C_{Lo} = 0.325, \ C_{L1} = 0.0672; \ C_{Do} = 0.115, \ C_{D1} = 0.00562$$

For $\theta_o = 7^o$, there is equilibrium when $\alpha = 3^o$.

We obtain:

$$\alpha = \frac{10300}{V_o^2} - 4.84 + \left(\alpha_o - \frac{10300}{V_o^2} + 4.84\right) e^{-0.0543 \ V_o t} \qquad (9)$$

$$V = V_o - \left(0.00146\ V_o - \frac{3.13}{V_o}\right)\left(\alpha_o - \frac{10300}{V_o^2} + 4.84\right) +$$

$$+ \left(0.27 + \frac{1760}{V_o^2} - 0.00124\ V_c^2\right) t + \qquad (10)$$

$$+ \left(0.00146\ V_o - \frac{3.13}{V_o}\right)\left(\alpha_o - \frac{10300}{V_o^2} + 4.84\right) e^{-0.0543\ V_o t}$$

These expressions will be discussed later in a numerical example.

Case B.- If the point determined by V_o and c_o in the V, α plane is very close to the V, α $\left(\frac{d\alpha}{dt} = 0\right)$ curve at the commencement of the motion, we then find, in contrast to Case A, that the variation of V is of the same order as that of α and we must consider the two equations together

$$\frac{dV}{dt} = \frac{Tg}{W} - g\ \sin\theta_o + \frac{g}{57.3}\cos\theta_o\ \alpha -$$

$$\frac{\epsilon}{57.3}\left(C_{Do} + C_{D1}\alpha\right) V^2 = \chi\ (V, \alpha), \qquad (11)$$

$$\frac{d\alpha}{dt} = \frac{57.3\ g\ \cos\theta_c}{V} - \epsilon\left(C_{Lo} + C_{L1}\alpha\right) V = s(V, \alpha) \qquad (12)$$

If a solution commences close to the equilibrium curve $\frac{d\alpha}{dt} = 0$, as assumed here, simultaneous values of V, α will remain close to it throughout. To make this clear, we will consider the solution again in time intervals, within which $\chi\ (V, \alpha)$ and $s(V, \alpha)$ may be regarded as linear (which is obviously always possible) and we will suppose the final values V, α of any time interval, to be the initial values V_o, α_o of the succeeding interval. Examples show that, in practice, only one or two time intervals are required.

For a single time interval, we have

$$\frac{dV}{dt} = \chi(V_0, \alpha_0) + p_1(V - V_0) + q_1(\alpha - \alpha_0). \tag{11a}$$

$$\frac{d\alpha}{dt} = s(V_0, \alpha_0) + p_2(V - V_0) + q_2(\alpha - \alpha_0). \tag{12a}$$

At the beginning of the first interval, $s(V_0, \alpha_0)$ is almost 0 and we have

$$p_1 = \left(\frac{d\chi}{dV}\right)_0 = -\frac{2\epsilon}{57.3}(C_{D0} + C_{D1}\,\alpha_0)\,V_0,$$

$$q_1 = \left(\frac{d\chi}{d\alpha}\right)_0 = \frac{g\cos\theta_0}{57.3} - \frac{\epsilon C_{D1} V_0^2}{57.3}$$

$$p_2 = \left(\frac{ds}{dV}\right)_0 = -\epsilon C_{L1}\,\alpha_0 - \frac{57.3\,g\cos\theta_0}{V_0^2}$$

$$q_2 = \left(\frac{ds}{d\alpha}\right)_0 = -\epsilon C_{L1} V_0.$$

Equations (11a) and (12a) are solved by putting

$$\alpha = \alpha_0 + L + c_1 e^{r_1 t} + c_2 e^{r_2 t}, \tag{13}$$

$$V = V_0 + B + d_1 e^{r_1 t} + d_2 e^{r_2 t} \tag{14}$$

and we find

$$L = \chi(V_0, \alpha_0)\frac{p_2}{p_1 q_2 - p_2 q_1}, B = -\chi(V_0, \alpha_0)\frac{q_2}{p_1 q_2 - p_2 q_1} \tag{15}$$

r_1 and r_2 are the roots of the equation

$$r^2 - r(p_1 + q_2) + p_1 q_2 - p_2 q_1 = 0 \tag{16}$$

The factors c_1, c_2, d_1, d_2 follow from

$$c_1 + c_2 = -L, \; r_1 c_1 = q_2 c_1 + p_2 d_1, \; r_1 d_1 = q_1 c_1 + p_1 d_1; \quad (17)$$

$$d_1 + d_2 = -B, \; r_2 c_2 = q_2 c_2 + p_2 d_2, \; r_2 d_2 = q_1 c_2 + p_1 d_2. \quad (18)$$

Equation (16) has real negative roots, so long as the initial velocity does not fall to stalling speed. In the former example, when V_o, α_o lies close to $X(V_o, \alpha_o) = 0$, we have

$$r^2 + \left(5.68 \times 10^{-2} V_o + \frac{1.64 \cos\theta_o}{V_o}\right) r + 1.35 \times 10^{-4} V_o^2 + \\ + \frac{1.93 \times 10^2 \cos^2\theta_o}{V_o^2} = 0,$$

and the roots become complex, only when V_o falls below $23.4\sqrt{\cos\theta_c}$.

The oscillations from the instant we approach the line of equilibrium of forces normal to the direction of flight, can now be surveyed in detail. If we again consider simultaneous values of V, α then the V, α curve can only reach the line of equilibrium for the perpendicular forces, when V and α rise or fall together, since, at the instant of crossing the line of equilibrium $\frac{d\alpha}{dt} = 0$, the curve runs in the direction of the V-axis. Should α, for instance, rise and V fall, then $\frac{d\alpha}{dt}$ would first be positive, then zero and then again positive, that is $\frac{d\alpha}{dt}$ must have a minimum value and, at the same time, become zero at the instant when the line of equilibrium is reached. The expressions $\frac{d\alpha}{dt} = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}$ and $\frac{d^2\alpha}{dt} = c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}$ cannot be zero together, when r_1 and r_2 have different values. If, therefore, α increases and V decreases, or α decreases and V increases, then the V, α curve will certainly remain permanently close to the line of equilibrium; but will only reach it after a very long time (theoretically $t = \infty$). When α and V increase or decrease simultaneously, the line of equilibrium will be crossed once and thereafter the V, α curve will again remain in close proximity to the line of equilibrium.

The motion can be understood better from a numerical example.

If $W = 1530$ kg, $S = 41.3$ m^2,

$$C_L = 0.325 + 0.0672\alpha ,$$

$$C_D = 0.115 + 0.00562\,\alpha$$

then equilibrium exists when $\alpha = 3^{\circ}$, $\theta_o = 7^{\circ}$ and $V = 36.2$ m per second.

Let the equilibrium be so disturbed that, at the beginning of the unsteady flight, $\alpha_o = 5^{\circ}$, $V_o = 43.2$ m per second (Fig. 6) are values which give points lying far from the line of equilibrium of the normal forces. In the first part of the oscillation, the solution therefore corresponds to Case A:

$$\alpha = 0.67 + 4.33e^{-2.35\,t} \tag{19}$$

$$V = 43.2 - 1.11t + 0.0407e^{-2.35t} \tag{20}$$

The calculated values of V and α have been plotted in Fig. 6 as functions of each other and in Fig. 7, singly, as functions of t. The result is, moreover, compared with a very careful numerical integration (dash lines) and the excellent agreement between the curves shows that the analytical calculation is very exact.

The equilibrium curve is reached in 1.12 seconds, when $V = 42$ m per second and $\alpha = 1^\circ$. Now take case B and the values become

$$\alpha = 2.65 - 1.77e^{-0.152(t-1.12)} + 0.12e^{-2.27(t-1.12)} \tag{21}$$

$$V = 36.05 + 5.95e^{-0.152(t-1.12)} \tag{22}$$

Equations (21) and (22) can be used for the whole course up to $t = \infty$, since, for $t = \infty$, they give $\alpha = 2.65^\circ$ and $V = 36.05$ m per second, which, therefore, come very close to the coordinate values $\infty = 3^\circ$ and $V = 36.2$ m per second. In order to estimate the time it actually takes to restore equilibrium, it must be borne in mind that the term containing $e^{-2.27(t-1.12)}$, (which from the first is vanishingly small in equation (22) and therefore can be entirely omitted) diminishes rapidly. The term $5.95e^{-0.152(t-1.12)}$ has the value 0.1 after 28 seconds. It may, therefore, be said that, with the given disturbance, equilibrium is practically reached in about half a minute

IV. The General Case. Discussion of the Constants. Analytical Treatment.

The general equations for the velocity V, the angle of attack α, the angle of inclination of the upper wing to the horizontal θ, and the angular velocity $\gamma = \frac{d\theta}{dt}$ are

$$\frac{dV}{dt} = \frac{Tg}{W} - g \sin \theta + \frac{g}{57.3} \cos \theta \, \alpha - \frac{\epsilon}{57.3} C_D V^2,$$

$$\epsilon = \frac{57.3}{2} \frac{\lambda}{W} S, \qquad \text{(Ia)}$$

$$\frac{d\alpha}{dt} = \gamma - \frac{1}{V} \left[\frac{Tg}{W} (\alpha - i_W) + 57.3 \, g \cos \theta + \right.$$

$$\left. + g \sin \theta \, \alpha \right] - \epsilon \, C_L \, V, \qquad \text{(IIa)}$$

$$\frac{d\theta}{dt} = \gamma \qquad \text{(IIIa)}$$

$$\frac{d\gamma}{dt} = (f - m\alpha)V^2 - d \, \gamma \, V. \qquad \text{(IVa)}$$

Among the coefficients appearing in these equations, some are always invariable even under different flight conditions (permanent constants), while others vary under different flight conditions and can only be regarded as constant within a given time interval (temporary constants).

To the former class belong:

1. Total weight of airplane, neglecting variation in weight due, for instance, to consumption of fuel;

2. Supporting surface;

3. Angle between upper wing and propeller axis;

4. Acceleration due to gravity, g;

5. Damping coefficient, $d = \frac{g}{Wk^2} \, r_H \, \frac{d}{d\alpha} \frac{M_H}{V^2}$

The temporary constants are:

1. Propeller thrust and air density.

For the propeller thrust, the expression

$$T = T_o - \frac{1}{2}\frac{\lambda}{g} S V^2 \rho'$$

was introduced. T, λ and ρ' are constant within a given time interval, but all three quantities may vary in different time intervals.

2. The static stability $m = \frac{g}{Wk^2}\frac{1}{V^2}\frac{dM}{d\alpha}$ is regarded as constant within a given time interval, but it is possible to use different values for m in different time intervals, when passing from one state of flight to another.

3. As already stated, f varies with the position of the elevator. If, for instance, at the beginning of the oscillation, the moments of the wings and of the tail balance at an angle of attack of 3^o and the elevator is then turned so they balance at 9^o, we will have $f = 3$ m for the first interval and $f = 9$ m for the second.

4. The coefficients C_L and C_D are here introduced as linear functions of α: $C_L = C_{Lo} + C_{L1}\alpha$, $C_D = C_{Do} + C_{D1}\alpha$. In this connection C_{Lo}, C_{L1}, C_{Do}, C_{D1} are assumed to be constant within any given time interval. These coefficients will, of course, vary in the different time intervals, if α increases or decreases.

If we now undertake the solution of the general equations by numerical processes, with given values of the permanent and temporary constants, starting from a definite instant, $t = 0$, with arbitrary initial values of V_o, α_o, θ_o, γ_o, it will always be found that at first the velocity only changes slowly, in comparison with the angles. This fact offers a very easy way for analytical treatment, by assuming in the first approximation, as in the case of neutral equilibrium, that $V = V_o$ constant. Equation (Ia) drops out and we have only to solve equations (IIa), (IIIa), and (IVa), in which V is put equal to V_o. These equations are, however, all linear, when C_L and C_D are linear functions of α, which is a great advantage in working out the problem. Moreover, these three equations are reducible to two, provided a certain correction is introduced for diving. The values found for V_o, α, θ, γ, are then put into equation (Ia) and we obtain, by simple integration, a second approximation for V, which, together with the previous values for α, θ and γ, presents an excellent solution for a definite time interval. These analytical expressions, as shown by comparison with solutions by means of fixed coefficients, give the actual path of flight very well for an interval of about two seconds. If it is desired to follow, during an extended period, until equilibrium is reached, the nonsteady flight caused by any disturbance on a stable airplane, by the same methods as for a neutral airplane, the above calculation can be used in conjunction with the method of small oscillations.

During the first seconds, we calculate by the above method and thereby determine the course of V, α, θ, γ. When α has completed its large variations and has substantially reached its equilibrium value, we calculate the further course, up to equilibrium, by the method of small oscillations, making use of all four equations and proceeding from the final values of the present method. Such an example is worked out in No. VII.

V. Problems of the General Case.

1. Let an airplane be in equilibrium, with all permanent and temporary constants known, and let the equilibrium be disturbed by some cause, such as a gust, so that the velocity is changed to V_0, the angle of attack to α_0, the inclination with the horizontal to θ_0, and the angular velocity to γ_0. What is the course of the non-steady motion now set up? More especially, how does a stable airplane return to equilibrium?

2. Let an airplane be in equilibrium and a deflection be imparted to the elevator. The values in the state of equilibrium V, α, θ, $\gamma = 0$ are to be taken as initial values. In the differential equations, however, a value of f is to be put, corresponding with the new position of the elevator. Again a non-steady motion sets in, which has to be followed. It must be especially investigated, as to how this deflection of the elevator affects stable $(m > 0)$, neutral $(m = 0)$ and unstable $(m < 0)$ airplanes. By the method of subdivision into time intervals, it is always possible to give successively different elevator settings. At the

moment of setting the elevator, we must start from the initial values of V, α, θ, γ, but, on the other hand, we must introduce, into the differential equations, the particular value of f which corresponds to the new elevator setting.

3. Let an airplane be in flight under engine power. At a given moment, the engine is shut off and a new setting is simultaneously given the elevator. The values for V, α, θ and γ, during engine-driven flight, stand as initial values. The differential equations must, however, be those of gliding flight; that is, $T = 0$, and in the expression $C_D = C_{Do} + C_{D1}\alpha$, for the corresponding α position, ρ', defined by $T = \frac{1}{2}\frac{\lambda}{g} S V^2\rho'$ must be omitted and, in its place, an amount put, which corresponds to the drag of the propeller revolving slowly. The reverse takes place when passing from gliding flight to power flight.

4. The stalled condition can very well be treated by the present method, since, precisely in this condition, the velocity changes very slowly. All the phenomena peculiar to stalling can, therefore, be represented by the general formulas given below, by putting the initial values characteristic of this condition (large angle of attack and low speed) in the differential equations for such values of $C_L = C_{Lo} + C_{L1}\alpha$, $C_D = C_{Do} + C_{D1}\alpha$ as correspond to the α position.

5. This method also suffices admirably for the treatment of diving flight, since the velocity in this case has been found to change but slowly with variations in the coefficients, variation

of f due to change of the elevator position and variation of T in passing from engine-driven flight to gliding flight.

In all these cases, we have to deal with the following mathematical problem. A system of four differential equations is given, with definite values of the permanent and temporary constants. Solutions for the four variables are sought as functions of the time with initial values of V_o, α_o, θ_o, γ_o, for $t = 0$. If these solutions are to hold for a fairly long period, the same problems must be solved for consecutive time intervals, the final values of V, α, θ, γ, of the one time interval being the initial values of the next. The general analytical method of working out these problems is given below and explained by examples. The actual solution of the above special problems is reserved.

VI. Application of the Analytical Process to the General Problem.

In all non-steady flights, the velocity V varies but slowly in comparison with the angles α and θ. In solving equations (Ia), (IIa), (IIIa), and (IVa), it is, therefore, assumed, in the first approximation, that $V = V_o$ is constant. We then have to deal with the following equations:

$$\frac{d\alpha}{dt} = \frac{57.3\ g \cos\theta}{V_o} - \epsilon(C_{Lo} + C_{L1}\ \alpha)V_o + \gamma \;. \qquad (23)$$

$$\frac{d\theta}{dt} = \gamma. \qquad (24)$$

$$\frac{d\gamma}{dt} = (f - m\alpha)V_o^2 - d\ V_o\ \gamma \qquad (25)$$

in which

$$\Sigma = \frac{57.3}{2}\ \frac{\lambda}{W}\ S \qquad (26)$$

Equation (Ia) comes first into consideration in seeking a second approximation for V. At first, the terms $-\frac{Tg}{W}(\alpha - i_W) + g \sin\theta\alpha$ are again neglected in comparison with $57.3\ g \cos\theta$. If, for instance, T = 485 kg, W = 1530 kg, and $\theta = 20^\circ$ then

$$-\frac{Tg}{W}(\alpha - i_W) + g \sin\theta\alpha = 0.24\alpha + 3.11\ i_W;$$

$$57.3\ g \cos\theta\ \alpha = 530\,\alpha;$$

and this neglect is, therefore, justified. When θ approaches -90° in diving flight, then these neglected terms again come into consideration. The correction, which then becomes necessary, will be discussed later.

It further appears that, except in diving with large oscillations of θ, $57.3 \cos\theta$ can be replaced by $57.3 \cos\theta_0$, in which θ_0 is the initial value of θ, since (with the variations in θ considered here) $57.3 \cos\theta$ only changes by a small percentage, which (as comparison with numerous exact calculations always reaffirms) does not materially affect the result. We have, therefore, only two equations to deal with:

$$\frac{d\alpha}{dt} = \frac{57.3\ g \cos\theta_0}{V_0} - \epsilon\, C_{Lo}\, V_0 - \epsilon\, C_{L1}\, V_0\, \alpha + \gamma \qquad (27)$$

$$\frac{d\gamma}{dt} = (f - m\alpha)\, V_0^2 - d\, V_0 \gamma \qquad (28)$$

from which θ is at once given by

$$\frac{d\theta}{dt} = \gamma \qquad (29)$$

The solution of equations (27) and (28) is obtained by the values

$$\alpha = L + p_1 e^{r_1 t} + p_2 e^{r_2 t}$$

$$\gamma = B + q_1 e^{r_1 t} + q_2 e^{r_2 t}$$

On putting

$$N = m + \epsilon\, d\, C_{L1} \tag{30}$$

we find

$$L = \frac{d}{N V_o^2}\left[57.3\ g \cos\theta_o - \epsilon\, C_{Lo} V_o^2 \right] + \frac{f}{N} \tag{31}$$

$$B = -\frac{m}{N V_o}\left[57.3\ g \cos\theta_o - \epsilon\, C_{Lo} V_o^2 \right] + \frac{\epsilon\, C_{L1} V_o f}{N} \tag{32}$$

The values of p_1, p_2, q_1, q_2, r_1, r_2, are determined by

$$\alpha_o - L = m_1 + m_2;\ p_1 r_1 = -\epsilon\, C_{L1} V_o\, p_1 + q_1;$$

$$p_2 r_2 = -\epsilon\, C_{L1} V_o\, p_2 + q_2;$$

$$\gamma_o - B = n_1 + n_2;\ q_1 r_1 = -m V_o^2\, p_1 - d V_o\, q_1;$$

$$q_2 r_2 = -m V_o^2\, p_2 - d V_o\, q_2.$$

We therefore obtain for r_1 and r_2 the quadratic equation

$$r^2 + r V_o (d + \epsilon\, C_{L1}) + N V_o^2 = 0, \tag{33}$$

from which we obtain

$$r_1 = -R_2 V_o + V_o \sqrt{R_1^2 - m},$$

$$r_2 = -R_2 V_o - V_o \sqrt{R_1^2 - m} \tag{34}$$

when

$$R_1 = \frac{1}{2}(\epsilon\, C_{L1} - d),\ R_2 = \frac{1}{2}(\epsilon\, C_{L1} + d) \tag{35}$$

Hence

$$\alpha = L + e^{-R_2 V_0 t}\left[a_1 \cos V_0 t\sqrt{m - R_1^2} + \right.$$

$$\left. + \frac{a_2}{\sqrt{m - R_1^2}} \sin V_0 t\sqrt{m - R_1^2}\right], \tag{36}$$

$$\gamma = B + e^{-R_2 V_0 t}\left[e_1 \cos V_0 t\sqrt{m - R_1^2} + \right.$$

$$\left. + \frac{e_2}{\sqrt{m - R_1^2}} \sin V_0 t\sqrt{m - R_1^2}\right], \tag{37}$$

$$\theta = \theta_0 - c_1 + Bt + e^{-R_2 V_0 t}\left[c_1 \cos V_0 t\sqrt{m - R_1^2} + \right.$$

$$\left. + \frac{c_2}{\sqrt{m - R_1^2}} \sin V_0 t\sqrt{m - R_1^2}\right], \tag{38}$$

$$\chi = \theta - \alpha = \theta_0 - c_1 - L + Bt +$$

$$+ e^{-R_2 V_0 t}\left[s_1 \cos V_0 t\sqrt{m - R_1^2} + \frac{s_2}{\sqrt{m - R_1^2}} \right.$$

$$\left. \sin V_0 t\sqrt{m - R_1^2}\right] \tag{39}$$

We have, at the same time,

$$a_1 = \alpha_0 - L,\ a_2 = - R_1(\alpha_0 - L) + \frac{1}{V_0}(\gamma_0 - B), \tag{40}$$

$$e_1 = \gamma_0 - B,\ e_2 = - m V_0 (\alpha_0 - L) + R_1 (\gamma_0 - B), \tag{41}$$

$$c_1 = \frac{1}{N V_0}[m V_0(\alpha_0 - L) - \epsilon C_{L1} (\gamma_0 + B)], \tag{42}$$

$$c_2 = \frac{1}{N V_0}[m V_0 R_2(\alpha_0 - L) + (m - \epsilon C_{L1} R_1)(\gamma_0 - B)],$$

$$s_1 = - \frac{\epsilon C_{L1}}{N V_0}[V_0 d (\alpha_0 - L) + \gamma_0 - B], \tag{43}$$

$$s_2 = \frac{\epsilon C_{L1}}{N V_0}[V_0(m + d R_1)(\alpha_0 - L) - R_2 (\gamma_0 - B)].$$

These formulas remain unchanged, when $m - R_1^2 < 0$, excepting that it is necessary to replace $m - R_1^2$ by $R_1^2 - m$ and the trigonometrical functions by the corresponding hyperbolic functions.

In the case $N = 0$, which is not specially notable in its characteristics, the formulas break down. If, in such an instance, we take

$$K = \frac{d}{2\,R_2 V_o^2}\left[57.3\ g\cos\theta_o - \epsilon\, C_{Lo}\, V_o^2\right] + \frac{f}{2\,R_2}, \qquad (44)$$

$$A = \frac{\gamma_o - \epsilon\, C_{L1}\, V_o\, \alpha_o}{2\,R_2\,V_o} + \frac{\epsilon\, C_{L1}\, K - f}{2\,R_2 d},$$

they then become

$$\alpha = \alpha_o + K\,V_o\,t + A\,(1 - e^{-2R_2V_ot}), \qquad (45)$$

$$\gamma = \gamma_o + \epsilon\, C_{L1}\, K\, V_o^2 t - A d\, V_o\,(1 - e^{-2R_2V_ot}), \qquad (46)$$

$$\theta = \theta_o + (\gamma_o - A d\, V_o)t + \frac{1}{2}\,\epsilon\, C_{L1}\, K\, V_o^2\, t^2 +$$

$$+ \frac{Ad}{2R_2}(1 - e^{-2R_2V_ot}), \qquad (47)$$

$$\chi = \theta_o - \alpha_o + (\gamma_o - A d\, V_o - K\, V_o)\,t +$$

$$+ \frac{1}{2}\,\epsilon\, C_{L1}\, K\, V_o^2 t^2 - \frac{\epsilon\, C_{L1} A}{2\,R_2}(1 - e^{-2R_2V_ot}). \qquad (48)$$

A second approximation for V is obtained from

$$V = V_o + \int_o^t dt \left[\frac{Tg}{W} - g\sin\theta_o + \frac{g\,\theta_o\cos\theta_o}{57.3}\right.$$

$$+ \frac{g\sin\theta_o\,\alpha_o\,\theta_o}{57.3^2} - \frac{\epsilon\, C_{Do}\, V_o^2}{57.3} + \frac{\alpha}{57.3}(g\cos\theta_o - \epsilon\, C_{D1}\, V_o^2) -$$

$$\left. - \frac{\theta}{57.3}\, g\cos\theta_o + \frac{g\sin\theta_o\,\alpha_o}{57.3}\right] \qquad (49)$$

in which the values found above for α and θ must be inserted.

For judging the course of flight, the angle χ, between the tangent to the path and the horizontal, is of special importance. If we wish to know how the course of flight is influenced by the setting of the elevator, we must consider $\frac{d\chi}{df}$, that is, the variation of χ with respect to f, the variables which fix the elevator setting. For this we find

$$\frac{d\chi}{df} = \frac{C_{L1}\,\epsilon}{N}\Bigg[-2\,R_2 + N\,V_0 t + e^{-R_2 V_0 t}\left(2\,R_2 \cos V_0 t\sqrt{m - R_1^2} + \frac{R_1^2 + R_2^2 - m}{\sqrt{m - R_1^2}}\sin V_0 t\sqrt{m - R_1^2}\right)\Bigg] \qquad (50)$$

or, when $N = 0$,

$$\frac{d\chi}{df} = \frac{C_{L1}\,\epsilon}{8\,R_2^3}\left[1 + 2\,R_2^2\,V_0^2\,t^2 - 2\,R_2\,V_0 t - e^{-2R_2 V_0 t}\right] \qquad (51)$$

VII. Examples.

Taking the same data as above, $S = 41.3\ m^2$, $W = 1530$ kg, $T = 485$, $\frac{\lambda}{\lambda_o} = 0.81$ (at an altitude of 2000 m),

$$\frac{1}{2}\,\frac{\lambda_o}{g} = \frac{1}{15.2},\ C_{Lo} = 0.325,\ C_{L1} = 0.0672;\ C_{Do} = 0.115,$$

$$C_{D1} = 0.00562;$$

the figures correspond approximately to the Dfw C V. Let $m = +\,0.00191$. (This value was found in calculating the moments for the Dfw C V, though with the negative sign) and let $d = 0.0238$ (also the same as for the Dfw C V).

1. The state of equilibrium (V_g = 36.2 m per second, $\alpha_g = 3^{\circ}$, $\theta_g = 7^{\circ}$ and $\gamma_g = 0$) is so disturbed that the velocity increases to 43.1 m per second and the angle of attack to 6.9°. The non-steady motion, which now sets in, is examined and it is thus determined in what way the airplane returns to equilibrium. In the general formulas, we must put

$$V_o = 43.1,\ \alpha_o = 6.9,\ \theta_o = \theta_g = 7,\ \gamma_o = \gamma_g = 0.$$

Since the moments are assumed to be in equilibrium when $\alpha = 3^{\circ}$, we must put

$$f - 0.00191 \times 3 = 0, \quad \text{whence} \quad f = 0.00573$$

We then find, for the first few seconds:

$$\alpha = 2.07 + 6.15\, e^{-1.68t} \cos(101.2t + 36.7)^{\circ},$$

$$\theta = 2.69 + 3.24t + 4.8e^{-1.68t} \cos(101.2t - 27.4)^{\circ},$$

$$\dot{V} = 42.8 - 0.32t - 0.277t^2 + 0.314e^{-1.68t}$$

$$\cos(101.2t + 7.8)^{\circ}.$$

After two seconds, we obtain

$\alpha = 1.9^{\circ}$, $\theta = 9^{\circ}$, $V = 41.1$ m per second, $\gamma = 3^{\circ}$ per second.

These values (f remaining unaltered) can be inserted anew in the formula, as α_o, θ_o, V_o, γ_o, since there is no new setting of the elevator. We thus obtain for the further course (from t = 2):

$$\alpha = 2.27 - 0.518e^{-1.6(t-2)} \sin(224.7 - 96.5t)^{\circ},$$

$$\theta = 9.37 + 2.4(t - 2) - 0.485e^{-1.6(t-2)}$$

$$\cos(96.5t - 234.2)^{\circ},$$

$$V = 41.1 - 1.03(t - 2) - 0.205(t - 2)^{2} -$$

$$- 0.0239e^{-1.6(t-2)} \cos(96.5t - 172.5).$$

The numerical values given by these formulas have been plotted in Fig. 8, that is, from $t = 0$ to $t = 2$, by the first group of formulas and from $t = 2$ to $t = 4$, by the second group.

The continuous lines have been calculated from the formulas while the dotted lines are those obtained by numerical integration. The agreement is excellent.

In Fig. 9 the same curves (dash) are shown once more from $t = 0$ to $t = 2$; while the course from $t = 2$ to $t = 13$ (also dash) has been calculated by the method of small oscillations, in the neighborhood of the position of equilibrium, with initial values corresponding to $t = 2$. For comparison, the result of the numerical integration is also shown (by continuous lines).

2. Let the airplane be in equilibrium. Then $V_g = 36.2$ m per second, $\alpha_g = 3^{\circ}$, $\theta_g = 7^{\circ}$, $\gamma_g = 0$. Let it be given such an elevator setting that the moments are in equilibrium only at $\alpha = 9^{\circ}$. The non-steady motion set up in this way is to be followed. We again insert $m = + 0.00191$ and obtain:

$$V_o = 36.2,\ \alpha_o = 3^{\circ},\ \theta_o = 7^{\circ},\ \gamma_o = 0,$$

$$f - m\,9 = 0,\ f = 0.0172.$$

We find:

$$\alpha = 6.58 - 4.93e^{-1.41t}\cos(85.3t - 43.4)^\circ,$$

$$\theta = 5.87 + 7t - 3.82e^{-1.41t}\sin(85.3t - 17.3)^\circ,$$

$$V = 36.2 + 0.42t - 0.6t^2 - 0.286e^{-1.41t}\sin(85.3t - 0.6)^\circ .$$

The results of this calculation, from $t = 0$ to $t = 2$, are plotted in Fig. 10.

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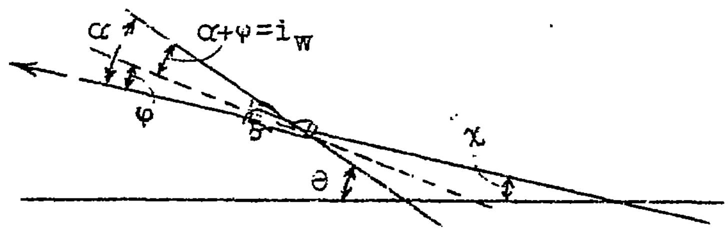
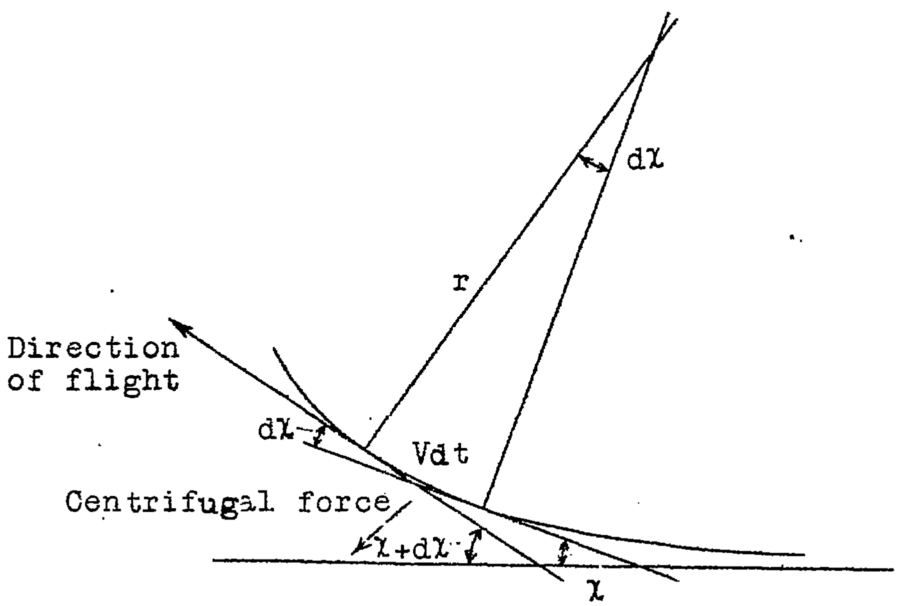
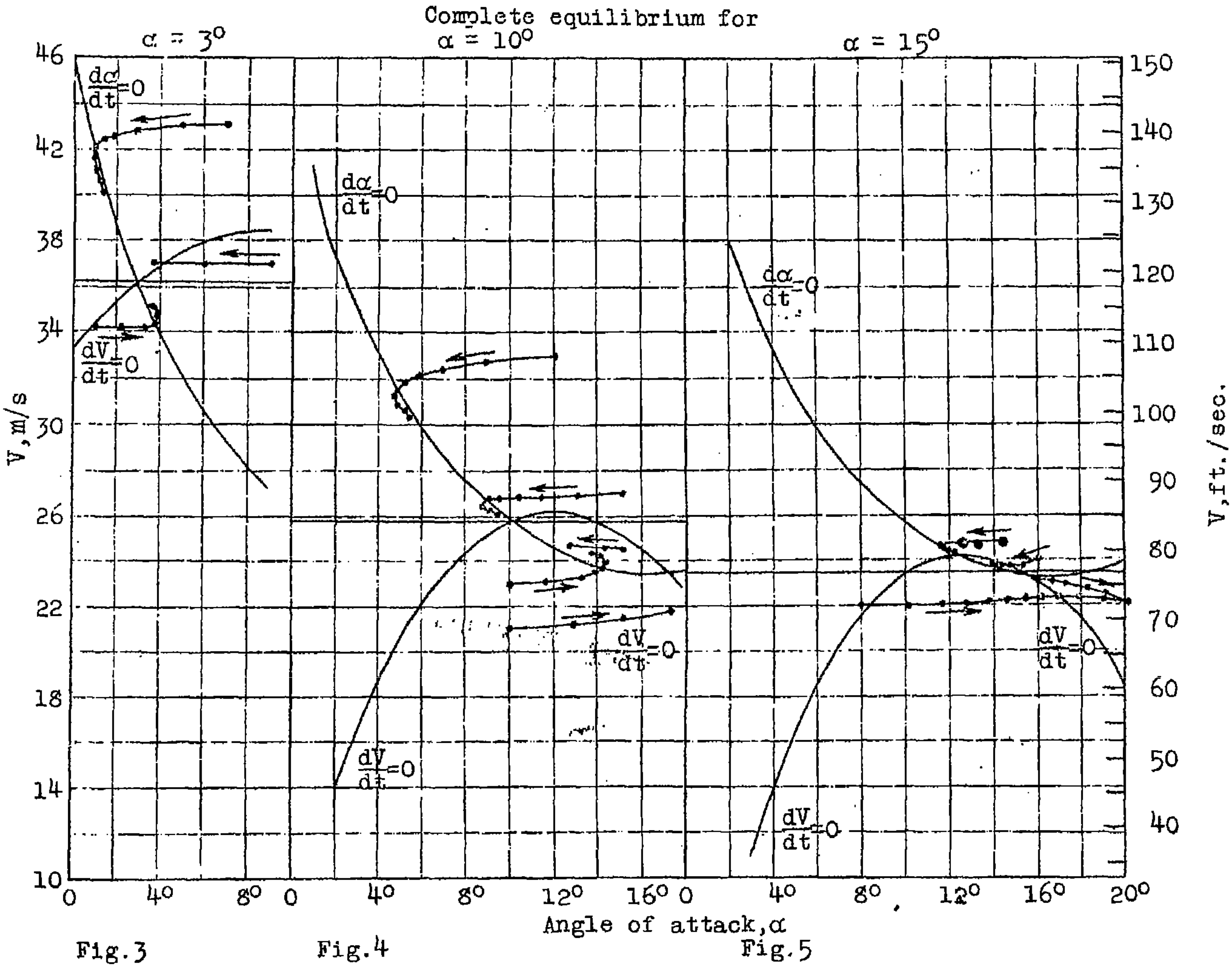
Figs.1,2.





Figs.1,2.



Figs. 3,4,5. Lines of equilibrium of the forces in the direction of flight and at right angles thereto.

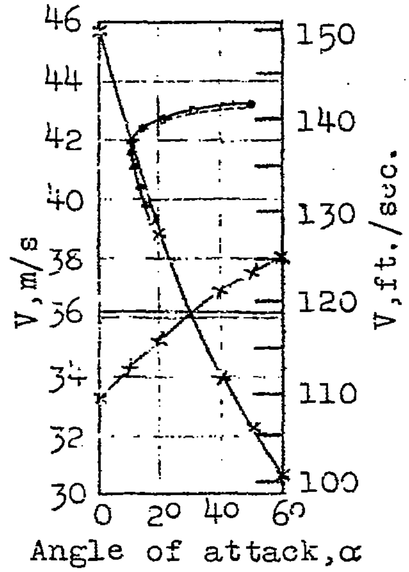


Fig.6

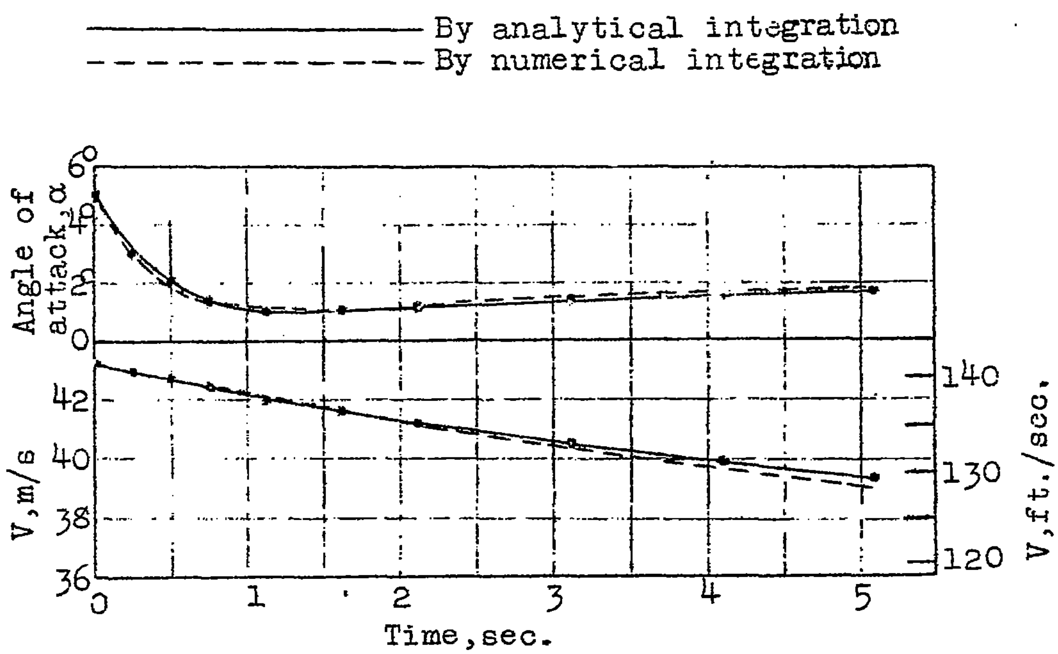


Fig.7 Return to equilibrium of an airplane with zero pitching stability.

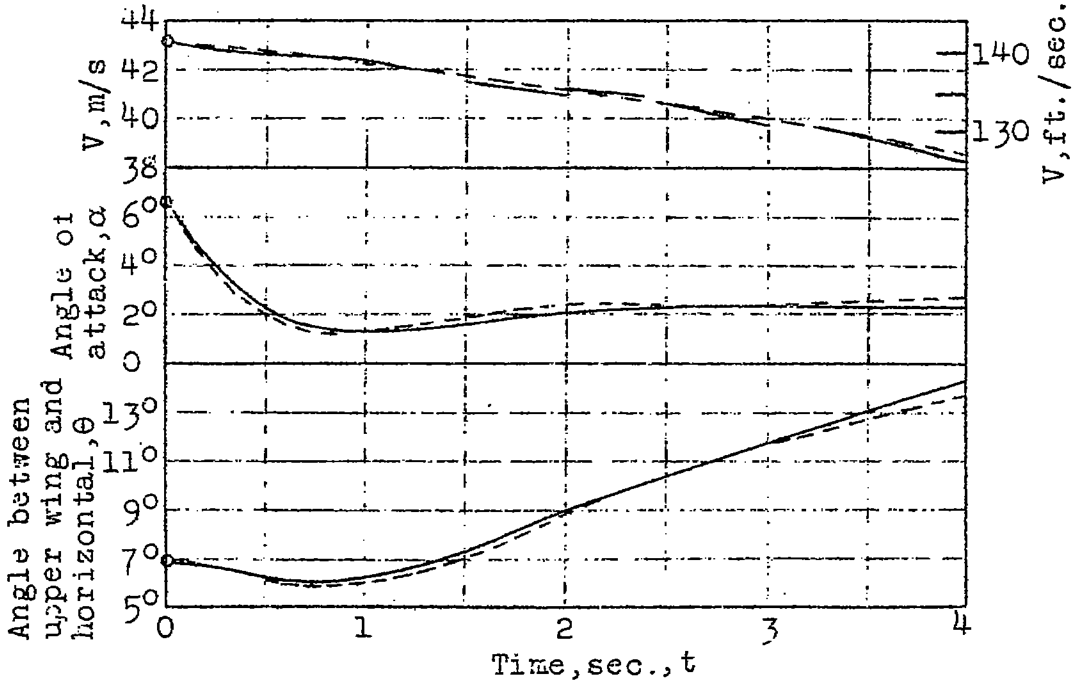


Fig.8 Return to a condition of equilibrium after a disturbance during the first four seconds.

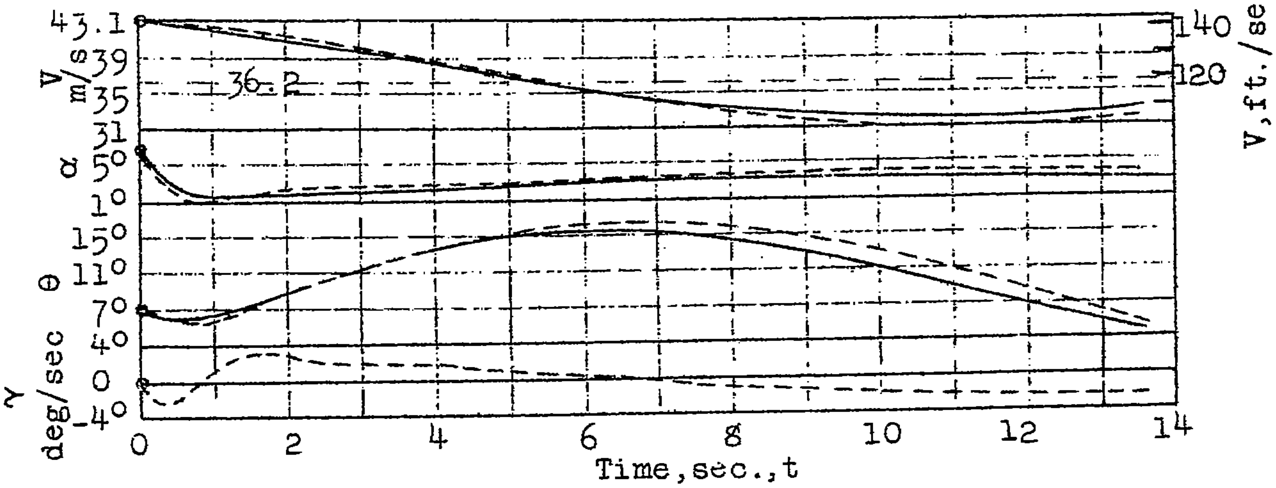


Fig.9 Return to equilibrium after a disturbance.

Fig.10

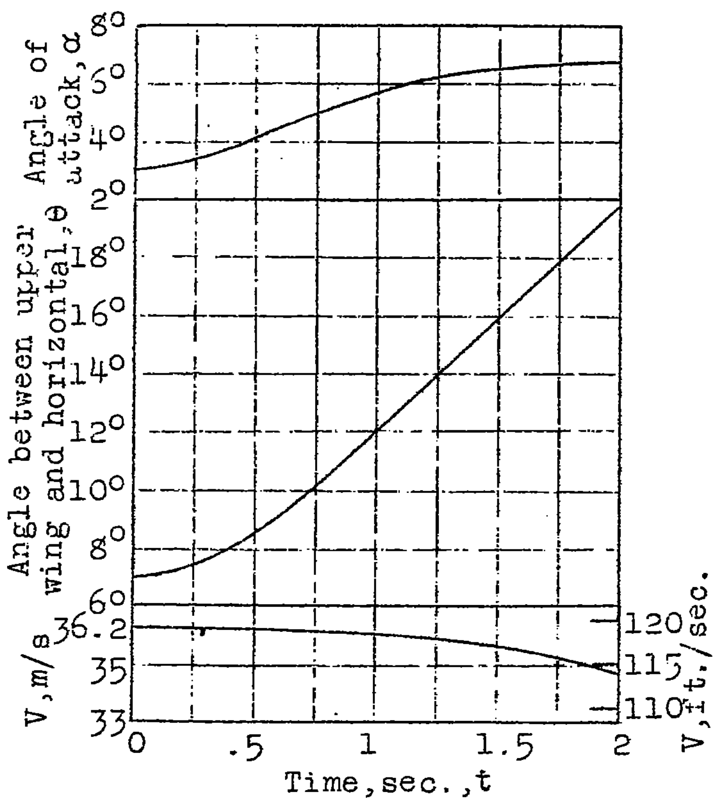


Fig.10 Transition from angle of attack, $\alpha = 3^{\circ}$ to $\alpha = 9^{\circ}$ produced by a deflection of the elevator.