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REMARKS ON THE PRESSURE DISTRIBUTION OVER THE SURFACE OF AN

ELLIPSOID, MOVING TRANSLATIONALLY THROUGH A PERFECT FLUID.

By Max M. Munk.

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ELLIPSOID, MOVING TRANSLATIONALLY THROUGH A PERFECT FLUID.

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Summary

This note, prepared for the National Advisory Committee for
Aeronautics, contains a discussion of the pressure distribution
over ellipsoids when in translatory motion through a perfect fluid.
An easy and convenient way to determine the magnitude of the veloc-
ity and of the pressure at each point of the surface of an ellip-
soid of rotation is described.

The knowledge of such pressure distribution is of great prac-
tical value for the airship designer. The pressure distribution
over the nose of an airship hull is known to be in such good agree-
ment with the theoretical distribution as to permit basing the com-
putation of the nose stiffening structure on the theoretical dis-
tribution of pressure.

References

1. Horace Lamb - "Hydrodynamics," Chapter V.

2. Max M. Munk - "The Aerodynamic Forces on Airship Hulls,"
      N.A.C.A. Technical Report No. 184.

3. R. Jones & D. H. Williams - "The Distribution of Pressure
      Over the Surface of Airship Model N 721,"
      Reports and Memoranda No. 600.

Experiments have shown that the knowledge of the pressure dis-
tribution over the surface of ellipsoids, moving translationally
through a perfect fluid, is often of considerable practical inter-
est. This pressure distribution is of a simple description; it
can easily and quickly be determined by analytical methods. To
the best of my knowledge this has never been brought out clearly
in any publication. The mathematical theory of the flow created
by an ellipsoid is given by H. Lamb in his "Treatise on Hydrody-
namics," Chapter V. It requires considerable mathematical train-
ing to grasp the full meaning of the results as given by Lamb.
E. G. Gallop (Ref. 3) has given some comments on the nature of the
resulting distribution of the velocity and pressure, for the spec-
ial case of an ellipsoid of revolution. A part of this holds true
for all ellipsoids, including those with three different principal
axes. Mr. Gallop does not, however, for the special case of sphe-
roids make the distribution of the pressure sufficiently plain for
immediate computation or for the practical application of this
interesting analysis.

The knowledge of one simple lemma on the potential flow around
ellipsoids, implicitly contained in Lamb's result (Third Edition,
Equation (114 (8)),) is sufficient for the deduction of all the
following theorems and for the determination of the pressure dis-
tribution. This lemma is:

If an ellipsoid is moving with uniform velocity parallel to
one of its principal axes, say parallel to the x-axis, the velocity

potential at any point of the surface can be written in the form

$$\Phi = A'x \qquad\qquad (1)$$

where A' is constant for a given flow and a given ellipsoid.
This theorem is the key to all the relations referring to the distribution of velocity and of pressure.

If the velocity of flow is not parallel to a principal axis, but has components in the direction of each of them, the resulting flow is the superposition of three flows analogous to the one just considered. Hence, at all points of the surface, the potential is a linear function of the Cartesian coordinates x, y, and z again, and can be written in the form

$$\Phi = A'x + B'y + C'z \qquad\qquad (2)$$

where the coordinate axes are chosen to coincide with the axes of the ellipsoid. Hence the curves of equal potential Φ are situated on parallel planes.

Now, suppose first the ellipsoid to be at rest and the fluid to be moving relative to it, as in a wind tunnel or as with an airship moored in a gale. The change from the ellipsoid moving through the fluid otherwise at rest to the fluid passing by the stationary ellipsoid does not affect the validity of Equation (2) except giving the constants A', B', and C' other values, say A'', B'', and C''. In the latter case (the body at rest) the velocity of the fluid at all points of the surface is parallel to the surface. Consider first the elements of surface containing a line

element at right angles to the planes of constant potential, i.e.,
at the points of the ellipsoid where the plane $A''x + B''y + C''z = 0$
meets the surface. It is apparent from (2) that at all these points
the velocity has the components A'', B'', and C''. This is evidently
the maximum velocity.

At all other points of the ellipsoid the elements of surface
are inclined towards the direction of maximum velocity, say by the
angle ϵ. Then the elements of distance on the surface, Δs, be-
tween curves of equal potential are increased in the ratio $\dfrac{1}{\cos \epsilon}$,
when compared with the actual distances between the planes of equal
potential. Accordingly, the velocity, being equal to $\dfrac{d\Phi}{ds}$ is de-
creased inversely, its magnitude is $A'' \cos \epsilon$. It will be noted
in particular that the velocity is equal at surface elements which
are inclined by the same angle ϵ. It is equal to the projection
of the maximum velocity at right angles to the surface element.
Hence the velocity cannot exceed the one rightly denoted by "maxi-
mum velocity," having the components A'', B'', and C''.

Returning to the case when the direction of flow is parallel
to a principal axis, it can be shown that the maximum velocity A''
stands in a very simple relation to the kinetic energy of the flow,
and hence to the apparent additional mass of the ellipsoid. We
have now to suppose the fluid to be at rest and the ellipsoid to
move, say with the velocity U, parallel to a principal axis,
e.g., the x-axis. The kinetic energy of the flow set up is equal
to $-\dfrac{1}{2} \rho \int \Phi \dfrac{d\Phi}{dn} dS$, i.e. the volume of fluid displaced by an

element of the surface per unit of time, multiplied by the potential at the point of displacement and by $\frac{\rho}{2}$ where ρ denotes the density of the fluid. Now, the volume displaced by a surface element per unit time is equal to the projection of this element perpendicular to the direction of x, multiplied by the velocity U. The potential being $A'x$ the integrand becomes $A' Ux \frac{\rho}{2} dy\, dz$. $\int xdy\, dz$ is the volume of the ellipsoid, hence the integral gives $\overline{\text{Volume}}\, A'U\, \frac{\rho}{2}$. This is the kinetic energy, usually expressed by $\overline{\text{Volume}}\, k_1\, U^2 \frac{\rho}{2}$, where k_1 denotes the factor of apparent mass. It follows that

$$\frac{A'}{U} = k_1 .$$

A'', referring to the case when the ellipsoid is stationary, is connected with A' by the equation

$$A'' = A' + U_1$$

as the latter flow results from the former by the superposition of the constant velocity U. Hence it appears that

$$\frac{A''}{U} = k_1 + 1 = A .$$

A is the maximum velocity corresponding to a flow having unit velocity along the x-axis. It is a constant for a given ellipsoid.

It equals the sum of 1 and of the factor of apparent mass k_1 as is confirmed for two special cases, where the factor A is well known. With a sphere, the maximum velocity is 1.5 times the velocity of flow, and the additional apparent mass is one-half the mass

of the displaced fluid. With a circular cylinder, moving at right angles to its axis, the maximum velocity is twice the velocity of flow, and the apparent additional mass is equal to the mass of the displaced fluid.

The ratios $A = \frac{A''}{U}$, $B = \frac{B''}{V}$, $C = \frac{C''}{W}$ are independent of the velocities U, V, and W and hence only depend on the ratio of the three semi-axes of the ellipsoid a, b, and c. Lamb gives the method to compute them. Compute first the integral

$$\alpha = a\, b\, c \int_0^\infty \frac{dx}{(a^2 + x)\sqrt{(a^2 + x)(b^2 + x)(c^2 + x)}}$$

and the analogous integrals β and γ for the axes b and c. The factors of apparent mass are then

$$k_1 = \frac{\alpha}{2 - \alpha}, \text{ etc.}$$

There are no tables for A, B, C, or for k_1, k_2, k_3, published yet. The integrals for α etc., can be numerically evaluated in each case, and I will assume at present that A, B, and C, are therefore known. For the special case $b = c$, that is, for ellipsoids of revolution, k_1 and k_2 have been computed for a series of elongation ratios, and are reprinted in a small table in Ref. 3. They are connected by the relation $k_1 = \frac{1 - k_2}{2\, k_2}$. The determination of the velocity at any point is thus reduced to a simple geometric problem. The maximum velocity, whose components are AU, BV, and CW, has to be projected onto the plane tangent

to the ellipsoid at the point considered, i.e. it has to be multiplied by the cosine of the angle between the normals to the surface at this point and at the point where the velocity is a maximum.

In the most interesting case of an ellipsoid of revolution, this can be done analytically in a very convenient way. The formula is most easily arrived at by the application of elementary vector analysis. First compute the component of the maximum velocity in a direction normal to the surface at a given point. The longitudinal component of the maximum velocity is AU, and the lateral component of the maximal velocity is BV.

Let the angle between the normal and the longitudinal axis be η, and let the dihedral angle between the plane containing this axis and the line of velocity of flow and the plane containing the axis and the point in question be β. Then $\cos\eta$ is the longitudinal component and $\sin\eta \cos\beta$ the lateral component of the normal of unit length. Hence the component of the maximum velocity in a direction perpendicular to the element of surface is

$$V_2 = (1 + k_1) U \cos\eta + (1 + k_2) V \sin\eta \cos\beta.$$

Let V_1 denote the component parallel to the surface element. Then

$$V_1^2 + V_2^2 = V_{max}^2, \quad \text{and hence}$$

$$V_1 = \sqrt{V_{max}^2 - V_2^2}$$

$$= \sqrt{(1+k_1)^2 U^2 + (1+k_2)^2 V^2 - [(1+k_1)U \cos\eta + (1+k_2)V \sin\eta \cos\beta]^2}$$

This is the desired formula for the velocity of flow along the surface. The pressure is computed directly from the velocity at the points of the ellipsoid, now supposed to be stationary in the flowing fluid. For this is a steady flow, and hence Bernouilli's equation for the pressure holds true, viz.: $p + \frac{1}{2} \rho V_1^2 = $ const. That is, the pressure is equal to an arbitrary constant pressure minus $V_1^2 \frac{\rho}{2}$, where V_1 denotes the velocity. The points of greatest velocity are those of smallest pressure or of greatest suction. The curves of equal velocity are also the curves of equal pressure.

In practice, we are chiefly interested in rather elongated ellipsoids of rotation, and the angle α between the principal axis and the direction of motion is small. With very elongated ellipsoids, k_1 is about 1 and k_2 is very small. Hence A is about 2 and B is about 1, and the angle between the direction of the line of maximum velocity and the axis is about twice as large as the angle between the direction of motion and the axis. The maximum velocity is always greater than the velocity of motion. The difference between the largest negative pressure and the pressure in the undisturbed atmosphere is

$$V^2 \frac{\rho}{2} \left\{ (1 + k_1)^2 \cos^2 \alpha + (1 + k_2)^2 \sin^2 \alpha - 1 \right\}$$