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A DEFLECTION FORMULA FOR SINGLE-SPAN BEAMS OF CONSTANT SECTION SUBJECTED TO COMBINED AXIAL AND TRANSVERSE LOADS

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SUMMARY

In this paper there is presented a deflection formula for single-span beams of constant section subjected to combined axial and transverse loads of the types commonly encountered in airplane design. The form of the equation is obtainable by dimensional analysis. Tables and curves of the nondimensional coefficients are appended to facilitate the use of the formula.

The equation is applied to the determination of the spring constant of a beam. Tables and curves are presented to show the variation of the spring constant with changes in the axial load and position along the beam.

INTRODUCTION

In reference 1 deflection formulas are presented for single-span beams of constant section subjected to axial compression and transverse loading. These formulas are considerably different for the various loading conditions treated and must be altered when the axial load is tension.

The purpose of this report is to present a simple formula that includes all the above-mentioned cases of reference 1 and is valid when the axial load is either tension or compression.

In reference 2 is presented a detailed study of the interaction between a lift strut and a wing spar when connected by a jury strut. Therein the authors state, "It would be very interesting to make a general study of the effect of varying the axial load upon the sign and magnitude of the spring constant . . . ." In reference 2 the term "jury strut" is applied to a member whose primary function is to provide an elastic support to a lift strut at some intermediate point and thereby to increase the critical load of the lift strut.

The deflection formula derived in this report makes it possible to make the suggested study and to show very clearly the effect of changes in the axial load. The general deflection formula presented herein is derived by the strain-energy method of analysis which is believed to lead to a more simple form of solution. A comprehensive treatment of this method has been presented by Timoshenko. (See reference 3.)

DERIVATION OF EQUATION FOR BEAM DEFLECTION

The following derivation applies to the case of a single-span beam of constant section subjected to the combination of loadings shown in figure 1. The loads, moments, and deflections are shown in the positive directions. The conventions adopted here are such that positive lateral loads and positive end moments produce positive deflections; also positive axial load increases deflections. It should be noted that these conventions differ from those of reference 1.

The deflection curve of the beam may be obtained by the addition of simple curves of sinusoidal form having different amplitudes and frequencies so that

$$y = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + \ . \ . \ . + a_n \sin \frac{n\pi x}{l} \qquad (1)$$

In order that this expression may exactly represent a particular deflection curve at every point of the beam, an infinite number of terms are, in general, required. Therefore the expression

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \qquad (2)$$

may be made to represent any deflection curve for a single-span beam by adjusting the values of the coefficients a_n. In order to evaluate the coefficients a_n, the changes of energy of the external loads and of the beam due to bending are determined for a small change da_n in any one of the coefficients a_n. For equilibrium, the changes in energy of the external loads must be equal and opposite to the change in internal energy of the beam. It is therefore necessary to find the energy changes during a small displacement $da_n \sin \frac{n\pi x}{l}$ from the equilibrium position. During this small displacement the increase in energy of the beam due to bending is $\frac{EI\pi^4}{2l^3} n^4 a_n da_n$ (reference 3, pp. 417-422).

The work done by the axial force $\bar{P}$ is

$$\bar{P} \frac{\pi^2}{2l} n^2 a_n da_n$$

The work done by the load W is

$$W da_n \sin \frac{n\pi d}{l}$$

The intensity of the trapezoidal loading is

$$w_o + cx$$

where c is the change in the intensity of loading per unit length of span. The work done by that loading is

$$\int_o^l (w_o + cx)\, da_n \sin \frac{n\pi x}{l}\, dx$$

$$= \left[\frac{w_o l}{n\pi} - \frac{w_o l}{n\pi}(-1)^n - \frac{c l^2}{n\pi}(-1)^n\right] da_n$$

The rotations of the ends of the beam are

$$\left[\frac{d\left(da_n \sin \frac{n\pi x}{l}\right)}{dx}\right]_{x=o} = \frac{n\pi}{l} da_n \qquad \text{at } x = o$$

and $$\left[\frac{d\left(da_n \sin \frac{n\pi x}{l}\right)}{dx}\right]_{x=l} = \frac{n\pi}{l} (-1)^n da_n \qquad \text{at } x = l$$

During these rotations the work done by the end moments

M_1 and M_2 is

$$\left[M_1 \frac{n\pi}{l} - M_2 \frac{n\pi}{l} (-1)^n\right] da_n$$

When the work done by the external forces is equated to the change in bending energy of the beam, the following equation for a_n is obtained.

$$a_n = \frac{2l}{\pi^2 \overline{P}} \frac{\overline{\alpha}}{n^2(n^2 - \overline{\alpha})} \left[\left(\frac{M_1}{l}\right)(n\pi) - \frac{M_2}{l}(n\pi \cos n\pi) + \right.$$

$$\left.(w_0 l)\left(\frac{1}{n\pi} - \frac{\cos' n\pi}{n\pi}\right) - (c l^2)\left(\frac{\cos' n\pi}{n\pi}\right) + W \sin \frac{n\pi d}{l}\right] \quad (3)$$

where

$$\overline{\alpha} = \frac{\overline{P}}{\left(\frac{\pi^2 EI}{l^2}\right)} = \frac{\overline{P}}{P_e} \quad (4)$$

and P_e is the critical Euler load for a pin-ended column of length l and bending rigidity EI. The axial load, $\overline{P}$, is positive when it is compression and negative when it is tension. By definition, $\overline{\alpha}$ has the same sign as $\overline{P}$. When the foregoing value of a_n is substituted in equation (2), the expression for the deflection curve becomes

$$y = \frac{l}{\overline{P}}\left[\overline{\beta}\left(\frac{M_1}{l}\right) + \overline{\gamma}\left(\frac{M_2}{l}\right) + \overline{\delta}(w_0 l) + \overline{\epsilon}(c l^2) + (\overline{\varphi}_1 + \overline{\varphi}_2) W\right] \quad (5)$$

The coefficients $\overline{\beta}$, $\overline{\gamma}$, $\overline{\delta}$, $\overline{\epsilon}$, $\overline{\varphi_1}$, and $\overline{\varphi_2}$ vary in form with changes in the sign of $\overline{\alpha}$. It is more convenient if these coefficients are replaced by the coefficients β, γ, δ, ϵ, φ_1, and φ_2 in which

$$\frac{\beta}{P} = \frac{\overline{\beta}}{\overline{P}},\ \frac{\gamma}{P} = \frac{\overline{\gamma}}{\overline{P}},\ \text{etc.}$$

where P is the absolute value of $\overline{P}$ and $\overline{\alpha}$ is replaced by its absolute value α. In order to differentiate between compression, tension, and no axial load the coefficients β, γ, etc., are given the subscripts c, t, and o. Equation (5) may then be written as

$$y = \frac{l}{P}\left[\beta_{(\ )}\left(\frac{M_1}{l}\right) + \gamma_{(\ )}\left(\frac{M_2}{l}\right) + \delta_{(\ )}(w_0 l)\right.$$

$$\left. + \epsilon_{(\ )}(c l^2) + (\varphi_{1(\ )} + \varphi_{2(\ )}) W\right] \quad (5a)$$

When the axial load is compression or tension, use the coefficients with subscripts c and t, respectively. When there is no axial load replace P by P_e and use the coefficients with subscript o. For axial compression, denoted by subscript c,

$$\beta_c = \frac{2\alpha}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{l}}{n(n^2-\alpha)} = \left\{ \frac{\sin\left[\sqrt{\alpha}\pi\left(1 - \frac{x}{l}\right)\right]}{\sin(\sqrt{\alpha}\pi)} - \left(1 - \frac{x}{l}\right) \right\}$$

$$\gamma_c = \frac{-2\alpha}{\pi} \sum_{n=1}^{\infty} \frac{\cos(n\pi)\,\sin \frac{n\pi x}{l}}{n(n^2 - \alpha)} = -\left[\frac{x}{l} - \frac{\sin\left(\sqrt{\alpha}\pi\,\frac{x}{l}\right)}{\sin(\sqrt{\alpha}\pi)}\right]$$

$$\delta_c = \frac{2\alpha}{\pi^3} \sum_{n=1}^{\infty} \frac{(1 - \cos n\pi)\,\sin \frac{n\pi x}{l}}{n^3\,(n^2 - \alpha)} =$$

$$\left\{\left(\frac{x}{2l}\right)\left(\frac{x}{l} - 1\right) + \frac{2\,\sin\left(\sqrt{\alpha}\,\frac{\pi}{2}\,\frac{x}{l}\right)\sin\left[\sqrt{\alpha}\,\frac{\pi}{2}\left(1 - \frac{x}{l}\right)\right]}{\pi^2\,\alpha\,\cos\left(\sqrt{\alpha}\,\frac{\pi}{2}\right)}\right\}$$

$$\epsilon_c = -\frac{2\alpha}{\pi^3} \sum_{n=1}^{\infty} \frac{\cos n\pi\,\sin \frac{n\pi x}{l}}{n^3(n^2 - \alpha)} =$$

$$-\left\{\frac{1}{\pi^2\alpha}\left[\frac{x}{l} - \frac{\sin\left(\sqrt{\alpha}\pi\,\frac{x}{l}\right)}{\sin(\sqrt{\alpha}\pi)}\right] + \frac{1}{6\alpha}\left(\frac{x}{l} - \frac{x^3}{l^3}\right)\right\}$$

(Dr. Kaplan of this Laboratory assisted in the evaluation of the foregoing infinite series)

and $$\varphi_{1_c} + \varphi_{2_c} = \frac{2\alpha}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{l}\,\sin \frac{n\pi d}{l}}{n^2\,(n^2 - \alpha)}$$

where $$\varphi_{1_c} = \left\{\frac{u}{2}\left(1 - \frac{u}{2}\right) - \frac{\cos\left[\sqrt{\alpha}\pi(u - 1)\right]}{2\sqrt{\alpha}\pi\,\sin(\sqrt{\alpha}\pi)}\right\}$$

where $$u = \left(\frac{d}{l} - \frac{x}{l}\right)$$

and $$\varphi_{2_c} = -\left\{\frac{v}{2}\left(1 - \frac{v}{2}\right) - \frac{\cos\left[\sqrt{\alpha}\pi(v - 1)\right]}{2\sqrt{\alpha}\pi\,\sin(\sqrt{\alpha}\pi)}\right\}$$

where $$v = \left(\frac{d}{l} + \frac{x}{l}\right)$$

For axial tension, denoted by subscript t, $\bar{\alpha}$ is negative and the coefficients take the following forms.

$$\beta_t = -\left\{\frac{\sinh\left[\sqrt{\alpha}\pi\left(1 - \frac{x}{l}\right)\right]}{\sinh\left(\sqrt{\alpha}\pi\right)} - \left(1 - \frac{x}{l}\right)\right\}$$

$$\gamma_t = \left[\frac{x}{l} - \frac{\sinh\left(\sqrt{\alpha}\pi\,\frac{x}{l}\right)}{\sinh\left(\sqrt{\alpha}\pi\right)}\right]$$

$$\delta_t = \left\{\left(\frac{x}{2l}\right)\left(\frac{x}{l} - 1\right) - \frac{2\sinh\left(\sqrt{\alpha}\,\frac{\pi}{2}\,\frac{x}{l}\right)\sinh\left[\sqrt{\alpha}\,\frac{\pi}{2}\left(1 - \frac{x}{l}\right)\right]}{\pi^2\,\alpha\,\cosh\left(\sqrt{\alpha}\,\frac{\pi}{2}\right)}\right\}$$

$$\epsilon_t = \left\{\frac{1}{\pi^2\alpha}\left[\frac{x}{l} - \frac{\sinh\left(\sqrt{\alpha}\pi\,\frac{x}{l}\right)}{\sinh\left(\sqrt{\alpha}\pi\right)}\right] - \frac{1}{6\alpha}\left(\frac{x}{l} - \frac{x^3}{l^3}\right)\right\}$$

$$\varphi_{1_t} = \left\{\frac{u}{2}\left(1 - \frac{u}{2}\right) + \frac{\cosh\left[\sqrt{\alpha}\pi\,(u - 1)\right]}{2\sqrt{\alpha}\pi\,\sinh\left(\sqrt{\alpha}\pi\right)}\right\}$$

where $u = \left(\frac{d}{l} - \frac{x}{l}\right)$

and $\varphi_{2_t} = \left\{\frac{v}{2}\left(1 - \frac{v}{2}\right) + \frac{\cosh\left[\sqrt{\alpha}\pi\,(v - 1)\right]}{2\sqrt{\alpha}\pi\,\sinh\left(\sqrt{\alpha}\pi\right)}\right\}$

where $v = \left(\frac{d}{l} + \frac{x}{l}\right)$

When there is no axial load, replace P by P_e, in equation (5a), and use the coefficients with subscript zero.

$$\beta_o = \frac{\pi^2}{6}\,\frac{x}{l}\left(1 - \frac{x}{l}\right)\left(2 - \frac{x}{l}\right)$$

$$\gamma_o = \frac{\pi^2}{6}\,\frac{x}{l}\left\{1 - \left(\frac{x}{l}\right)^2\right\}$$

$$\delta_o = \frac{\pi^2}{24}\left\{\frac{x}{l} - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4\right\}$$

$$\epsilon_o = \frac{\pi^2}{360}\left\{7\left(\frac{x}{l}\right) - 10\left(\frac{x}{l}\right)^3 + 3\left(\frac{x}{l}\right)^5\right\}$$

$$\varphi_{1_0} = \pi^2 \left\{\frac{1}{90} - \frac{u^2}{12} + \frac{u^3}{12} - \frac{u^4}{48}\right\}$$

where $u = \left(\frac{d}{l} - \frac{x}{l}\right)$

$$\varphi_{2_0} = -\pi^2 \left\{\frac{1}{90} - \frac{v^2}{12} + \frac{v^3}{12} - \frac{v^4}{48}\right\}$$

where $v = \left(\frac{d}{l} + \frac{x}{l}\right)$

The foregoing coefficients are given in tables I to IV and figures 5 to 16 for the range likely to be encountered in airplane design.

When more than one concentrated load acts on the beam, the deflection formula becomes

$$y = \frac{l}{P}\left[\beta_{(\ )}\left(\frac{M_1}{l}\right) + \gamma_{(\ )}\left(\frac{M_2}{l}\right) + \delta_{(\ )}\,(w_0 l) + \epsilon_{(\ )}\,(c l^2) + \Sigma\,(\varphi_{1(\ )} + \varphi_{2(\ )})\,\overline{W}\right] \qquad (6)$$

where the summation sign indicates that there must be one term of that form for each concentrated load. When the values of φ_1 and φ_2 are being chosen, the following rule must be observed. If the point whose deflection is being determined lies to the left of the concentrated load being considered, use the values of $\frac{x}{l}$ and $\frac{d}{l}$ as in the derivation. When the deflection is being determined for a point to the right of the load, however, replace $\frac{x}{l}$ and $\frac{d}{l}$ by $\left(1 - \frac{x}{l}\right)$ and $\left(1 - \frac{d}{l}\right)$, respectively, in computing u and v.

An inspection of the curves and tables shows that when the axial load is compression the deflections become excessive as α approaches unity, even for small lateral loads. A comprehensive treatment of critical loading conditions is given in reference 2.

DETERMINATION OF SPRING CONSTANT

The spring constant of a beam at any point is defined as the lateral force required at that point to produce unit deflection of that point. The spring constant K may be defined (fig. 2) mathematically as

$$K = \left[\frac{\partial W}{\partial y}\right]_{x=d} \tag{7}$$

Equation (5a) may be applied to figure 2 and is reducible to the form

$$y = \frac{W\,l}{P_e}\,\zeta \tag{8}$$

For axial compression

$$\zeta_c = -\frac{1}{\alpha}\left(\frac{x}{l}\right)\left(1 - \frac{x}{l}\right) + \frac{\sqrt{\alpha}\pi\left\{\cos\left[\sqrt{\alpha}\pi\left(\frac{2x}{l} - 1\right)\right] - \cos\left(\sqrt{\alpha}\pi\right)\right\}}{2\,\alpha^2\,\pi^2\,\sin\left(\sqrt{\alpha}\pi\right)}$$

and for axial tension

$$\zeta_t = \frac{1}{\alpha}\left(\frac{x}{l}\right)\left(1 - \frac{x}{l}\right) + \frac{\sqrt{\alpha}\pi\left\{\cosh\left[\sqrt{\alpha}\pi\left(\frac{2x}{l} - 1\right)\right] - \cosh\left(\sqrt{\alpha}\pi\right)\right\}}{2\,\alpha^2\,\pi^2\,\sinh\left(\sqrt{\alpha}\pi\right)}$$

When there is no axial load present

$$\zeta_o = \frac{\pi^2}{3}\,\frac{x^2}{l^2}\left(\frac{x}{l} - 1\right)^2$$

Therefore, the spring constant K may be written as

$$K = K_1\,\frac{P_e}{l} \tag{9}$$

where $$K_1 = \frac{1}{\zeta}$$

or, in the alternative form, which may be more convenient at times

$$K = K_2 \frac{P}{l} \tag{10}$$

where

$$K_2 = \frac{1}{\alpha \zeta}$$

The values of ζ, K_1, and K_2 are presented in tables V to VIII and in figures 17 to 24.

The effect of changes in axial load upon the sign and magnitude of the spring constant may be seen by an inspection of curves of K_1 against α for various positions along the span.

PRACTICAL APPLICATIONS

Equation (6) may readily be used in preliminary or final design to compute the deflections of beams of constant section subjected to combined axial and transverse loads. Its form is simple and the tables and charts reduce the amount of computation to a minimum. It may also be used to compute the additional deflections and from them the additional bending moments due to the load in a jury strut.

When the deflection of any point on a beam is known, the bending moment at that point may be written as

$$M = M_0 \pm Py$$

where M_0 is the bending moment at that point neglecting the effect of beam deflection and $\pm Py$ is the bending moment due to the axial load P and the beam deflection y at the point. Equation (6) may be used to determine the value of y for the cases treated in this note.

Equation (10) has a special significance in jury-strut problems. In reference 2, equations are given for the required minimum value of the spring constant of a lift strut for various conditions. A lift strut will have minimum weight when its value of α is so chosen that it just develops the required spring constant. The tables and curves computed from equation (10) enable the designer to select the lightest strut for any particular case.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., July 15, 1935.

APPENDIX A

In order to demonstrate the application of the general equation to a specific case, the following problem is considered. Referring to figure 3, let it be required to determine (a) the deflection of point A due to the external loads, exclusive of the reaction of the jury strut (AB), (b) a suitable lift strut, and (c), the supporting force on the spar.

	Spar	Strut
M_1	= +33,300 in.-lb.	0
M_2	= 0	0
P	= 5170 lb. tension	5,450 lb. compression
I	= 150 in.^{4}	
$\left(\frac{x}{l}\right)_A$	= 0.30	0.30
E	= 1,300,000 lb./sq.in.	28,000,000 lb./sq.in.

$$P_e = \frac{9.88 \times 1300000 \times 150}{(150)^2} = 85600 \text{ lb.}$$

$$\alpha = \frac{P}{P_e} = \frac{5170}{85600} = 0.0604$$

$$w_0 = -10 \text{ lb./in.}$$

(a) The deflection at point A, due to the external loads exclusive of the jury-strut reaction, is given by

$$y_A = \frac{l}{P} \left[\beta_t \frac{M_1}{l} + \delta_t \, (w_o l)\right]$$

$$= \frac{150}{5170} \left[(0.0335) \left(\frac{+33300}{150}\right) + (0.0059) \; (-10 \times 150)\right]$$

$$= -0.041 \text{ in.}$$

β_t from figure 7; δ_t from figure 10

(b) From figure 3

$$\cos^2 \mu = \left(\frac{150}{158.11}\right)^2 = 0.9$$

The spring constant of the spar at point A is (equation (9))

$$K_{1_t} \frac{P_e}{l} = 7.2 \, \frac{85600}{150} = +4110$$

(K_{1_t} from fig. 22)

In the notation of this paper the maximum allowable negative value of the spring constant of the lift strut is (reference 2, equation (22)),

$$-K_{(spar)} \cos^2 \mu = K_{(strut)}$$

$$K_{(strut)} = -(4,110) \; (0.9) = -3,700$$

and from equation (10)

$$(K_2)_{strut} = \frac{-3700 \times 158.11}{5450} = -107.3$$

From figure 23 it may be seen that for $\frac{x}{l} = 0.3$ the maximum allowable value of α for the strut lies between $\alpha = 3.18$ and $\alpha = 3.20$. If a strut be chosen with this value, it will be in a condition of indifferent elastic stability. For positive stability the strut chosen should have a value of K_{2_c} algebraically greater than -107.3. The optimum strut will, in general, have a value of K_{2_c} given by the point on its curve at which the slope begins to change rapidly. (See reference 2.) In this example the optimum strut is at $\alpha = 2.8$, approximately, at which $K_{2_c} = -10.5$. When this value of $\alpha = 2.8$ has been chosen, the moment of inertia of the strut is obtained from

$$I = \frac{l^2}{\pi^2 E}\,\frac{P}{\alpha} = \frac{(158.11)^2}{(9.88)\ (28000000)} \times \frac{5450}{2.8} = 0.176 \text{ in.}^4$$

(c) The corrected value of the spring constant of this strut is (reference 2, equation (22))

$$K = \frac{5450}{158.11} \times \frac{1}{0.9}\ (-10.5) = -\ 402$$

In the notation of this paper the supporting load on the spar is (reference 2, equation (9))

$$W_e = -\ \frac{K_{(spar)}\ K_{(strut)}\ y_A}{K_{(spar)} + K_{(strut)}}$$

$$W_e = -\ (-0.041)\ \frac{4110\ (-402)}{4110\ -\ 402} = -\ 18.3 \text{ lb.}$$

The minus sign in the foregoing equation indicates that the supporting load acts in the same direction as the lateral load.

APPENDIX B

Beams with Restrained Ends

In the derivation of equation (6) the values of M_1 and M_2 were assumed to be known. It is possible, however, to apply the general equation to the solution of problems in which the end moments must first be determined.

Let it be required to determine the moments M_1 and M_2 for the problem in figure 4.

The deflection at any point is

$$y = \frac{l}{P}\left[-\ \frac{M_1}{l}\ \beta_c\ -\ \frac{M_2}{l}\ \gamma_c\ +\ \delta_c\ (w_o\, l)\right]$$

The slope at any point is

$$\frac{dy}{dx} = \frac{l}{P}\left[-\frac{M_1}{l}\frac{d\beta_c}{dx} - \frac{M_2}{l}\frac{d\gamma_c}{dx} + (w_o\, l)\frac{d\delta_c}{dx}\right]$$

The end moments are equal due to symmetry and, since the slopes at the ends are zero,

$$M_1 = M_2 = M = w_o\, l^2 \frac{\dfrac{d\delta_c}{dx}}{\dfrac{d\beta_c}{dx} + \dfrac{d\gamma_c}{dx}}$$

A close approximation may be obtained by substituting $\frac{\Delta\beta}{\Delta x}$, $\frac{\Delta\gamma}{\Delta x}$, and $\frac{\Delta\delta}{\Delta x}$ for $\frac{d\beta}{dx}$, $\frac{d\gamma}{dx}$, and $\frac{d\delta}{dx}$, respectively, the values of the former being readily obtainable from tables I and II. When the values thus obtained are substituted in the preceding equation,

$$M = w_o\, l^2 \frac{\dfrac{0.0203}{0.05}}{\dfrac{0.1282}{0.05} + \dfrac{0.0893}{0.05}} = 0.0934\ w_o\, l^2$$

In reference 4, equation (6.128), the exact solution for this case yields the result

$$M = 0.091\ w_o l^2$$

More complicated loading conditions, for which the exact solution is not available, may be solved with equal ease and accuracy.

APPENDIX C

Many instances occur in structural engineering in which continuous loading occurs over only a part of the span. A very close approximation of the deflections due to such a load distribution is obtainable by the use of equation (6). The following example will demonstrate the method of solution. Let it be required to determine the deflection at point B of the beam loaded as shown in figure 4a. The lateral load may be replaced by a number of small concentrated loads of magnitude $w\,dx$, where w varies from w_o to $2w_o$, and the deflection at point B obtained as a sum of the deflections due to the small concentrated loads $w\,dx$. When a concentrated load $w\,dx$ is acting on the beam, let the beam deflection be designated dy. Then from equation (6)

$$dy = \frac{l}{P}\left(\varphi_{1_c} + \varphi_{2_c}\right) w\,dx$$

and
$$y = \frac{l}{P}\int_{x=0.4l}^{x=0.6l} \left(\varphi_{1_c} + \varphi_{2_c}\right) w\,dx$$

$$= \frac{l}{P}\left\{\int_{x=0.4l}^{x=0.5l} \left(\varphi_{1_c} + \varphi_{2_c}\right) w\,dx + \int_{x=0.5l}^{x=0.6l} \left(\varphi_{1_c} + \varphi_{2_c}\right) w\,dx\right\}$$

If it were possible to express φ_{1_c}, φ_{2_c}, and w in terms of x the value of y could be accurately determined. This procedure, however, is usually either too difficult or impossible. A very close approximation may be obtained by substituting Δx for dx and replacing the integral by a summation of a finite number of terms.

At point A

$$u = \left(1 - \frac{d}{l}\right) - \left(1 - \frac{x}{l}\right) \qquad \text{(See p. 10.)}$$

$$= (1 - 0.4) - (1 - 0.5)$$

$$= 0.1$$

$$v = \left(1 - \frac{d}{l}\right) + \left(1 - \frac{x}{l}\right) \qquad \text{(See p. 10.)}$$

$$= (1 - 0.4) + (1 - 0.5)$$

$$= 1.1$$

See table IV for value of φ_{1_c} and φ_{2_c}

$$\varphi_{1_c} = 0.2309$$

$$\varphi_{2_c} = 0.0591$$

$$\varphi_{1_c} + \varphi_{2_c} = 0.2900$$

$$\left(\varphi_{1_c} + \varphi_{2_c}\right) w = 0.2900\, w_o$$

At point B

$$u = (1 - 0.5) - (1 - 0.5) = 0$$

$$v = (1 - 0.5) + (1 - 0.5) = 1.0$$

$$\varphi_{1_c} = 0.2400$$

$$\varphi_{2_c} = 0.0659$$

$$\varphi_{1_c} + \varphi_{2_c} = 0.3059$$

$$(\varphi_{1_c} + \varphi_{2_c})\ w = 0.3059\ (1.5\ w_o) = 0.4589\ w_o$$

At point C

$$u = \frac{d}{l} - \frac{x}{l} \qquad \text{(See p. 10.)}$$

$$= 0.6 - 0.5 = 0.1$$

$$v = \frac{d}{l} + \frac{x}{l}$$

$$= 0.6 + 0.5 = 1.1$$

$$\varphi_{1_c} = 0.2309$$

$$\varphi_{2_c} = 0.0591$$

$$\varphi_{1_c} + \varphi_{2_c} = 0.2900$$

$$(\varphi_{1_c} + \varphi_{2_c})\ w = 0.2900\ (2\ w_o) = 0.5800\ w_o$$

In this illustrative problem, let $\Delta x = 0.1l$

$$\int_{x=0.4l}^{x=0.5l} (\varphi_{1_c} + \varphi_{2_c})\ w\ dx = \frac{0.2900 + 0.4589}{2}\ w_o\ (0.1l)$$

$$= 0.0375\ w_o l \text{ (approx.)}$$

$$\int_{x=0.5l}^{x=0.6l} (\varphi_{1_c} + \varphi_{2_c})\ w\ dx = \frac{0.4589 + 0.5800}{2}\ w_o\ (0.1l)$$

$$= 0.0520\ w_o l \text{ (approx.)}$$

Therefore the approximate deflection of the beam at point

B is given by

$$y = \frac{l}{P}\left\{0.0375\ w_0 l + 0.0520\ w_0 l\right\}$$

$$= 0.0895\ \frac{w_0 l^2}{P}$$

If less accuracy is sufficient the trapezoidal loading in figure 4a may be replaced by a single concentrated load of magnitude

$$1.5\ w_0\ (0.2l) = 0.3\ w_0 l$$

located at the centroid of the trapezoid. The centroid is located at

$$x = 0.4l + 0.556\ (0.2l) = 0.5112l = 0.5l\ \text{(approx.)}$$

From equation (8) and table V the deflection at point B is given by

$$y = \frac{l\alpha}{P}\ (0.3\ w_0 l)\ (0.5098) = 0.0918\ \frac{w_0 l^2}{P}$$

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TABLE I

a / $\frac{x}{c}$ for β	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	$\frac{x}{c}$ for β / a
for γ	0.95	0.90	0.85	0.80	0.75	0.70	0.65	0.60	0.55	0.50	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10	0.05	for γ
Values of β_0 and γ_0																				
0	0.1585	0.2812	0.3880	0.4734	0.5597	0.5972	0.6174	0.6316	0.6310	0.6168	0.5905	0.5537	0.5063	0.4491	0.3856	0.3158	0.2148	0.1598	0.0820	0
Values of β_0 and γ_0																				
0.05	0.0079	0.0146	0.0201	0.0248	0.0282	0.0308	0.0324	0.0332	0.0332	0.0326	0.0312	0.0292	0.0267	0.0235	0.0204	0.0167	0.0128	0.0088	0.0043	0.05
.10	.0153	.0304	.0422	.0518	.0599	.0647	.0683	.0700	.0701	.0697	.0669	.0619	.0586	.0504	.0434	.0355	.0272	.0184	.0093	.10
.15	.0237	.0478	.0653	.0814	.0934	.1021	.1060	.1110	.1114	.1095	.1050	.0966	.0904	.0808	.0690	.0558	.0436	.0294	.0148	.15
.20	.0357	.0666	.0898	.1143	.1214	.1440	.1535	.1570	.1578	.1551	.1421	.1403	.1298	.1148	.0968	.0612	.0521	.0420	.0212	.20
.25	.0459	.0877	.1294	.1511	.1739	.1910	.2066	.2090	.2104	.2071	.1996	.1878	.1726	.1540	.1327	.1090	.0834	.0554	.0285	.25
.30	.0593	.1111	.1554	.1985	.2219	.2442	.2595	.2632	.2715	.2667	.2671	.2424	.2229	.1992	.1717	.1412	.1061	.0731	.0369	.30
.35	.0752	.1374	.1990	.2332	.2766	.3061	.3260	.3365	.3399	.3325	.3240	.3058	.2816	.2618	.2173	.1788	.1118	.0945	.0468	.35
.40	.0869	.1676	.2357	.2939	.3399	.3758	.4011	.4160	.4309	.4163	.4085	.3805	.3506	.3138	.2711	.2351	.1711	.1158	.0594	.40
.45	.1070	.2085	.2856	.3588	.4126	.4551	.4901	.5098	.5162	.5114	.4963	.4686	.4394	.3876	.3350	.2760	.2117	.1434	.0725	.45
.50	.1299	.2451	.3440	.4301	.5011	.5656	.5967	.6218	.6310	.6261	.6073	.5755	.5316	.4769	.4127	.3402	.2611	.1768	.0893	.50
.55	.1533	.2917	.4140	.5193	.6086	.6755	.7263	.7574	.7707	.7661	.7442	.7062	.6553	.5956	.5090	.4191	.3216	.2183	.1101	.55
.60	.1944	.3550	.5010	.6300	.7380	.8940	.8874	.9280	.9468	.9452	.9159	.8712	.8058	.7284	.6367	.5191	.4100	.2705	.1568	.60
.65	.2251	.4275	.6106	.7704	.9048	1.015	1.094	1.147	1.171	1.166	1.139	1.084	1.005	.9044	.7645	.6484	.4984	.3322	.1708	.65
.70	.2741	.5260	.7561	.9559	1.126	1.284	1.368	1.438	1.472	1.471	1.438	1.359	1.271	1.146	.9947	.8294	.6398	.4326	.2171	.70
.75	.3430	.6636	.9644	1.212	1.433	1.612	1.750	1.843	1.890	1.893	1.855	1.768	1.644	1.485	1.289	1.067	.8214	.5574	.2819	.75
.80	.4467	.8669	1.250	1.593	1.899	2.155	2.321	2.450	2.519	2.539	2.477	2.389	2.206	1.958	1.734	1.436	1.105	.7614	.3798	.80
.85	.6169	1.201	1.740	2.296	2.648	3.000	3.273	3.485	3.571	3.591	3.525	3.377	3.158	2.848	2.470	2.065	1.586	1.077	.5449	.85
.90	.9616	1.862	2.710	3.480	4.155	4.721	5.167	5.464	5.656	5.711	5.617	5.359	5.053	4.558	3.975	3.299	2.543	1.750	.8759	.90
.95	1.963	3.838	5.612	7.265	8.670	9.894	10.859	11.65	11.96	12.08	11.91	11.45	10.71	9.709	8.476	7.040	5.455	3.697	1.871	.95
1.00	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	1.00
Values of β_t and γ_t																				
0.05	0.0074	0.0135	0.0188	0.0229	0.0258	0.0282	0.0296	0.0303	0.0300	0.0294	0.0281	0.0263	0.0239	0.0213	0.0183	0.0149	0.0114	0.0077	0.0039	0.05
.10	.0142	.0261	.0359	.0437	.0496	.0537	.0564	.0576	.0578	.0569	.0554	.0498	.0454	.0404	.0345	.0283	.0219	.0147	.0073	.10
.15	.0207	.0381	.0521	.0632	.0717	.0776	.0812	.0827	.0829	.0808	.0784	.0718	.0650	.0577	.0494	.0404	.0308	.0208	.0106	.15
.20	.0268	.0492	.0672	.0815	.0920	.0995	.1040	.1058	.1059	.1045	.0974	.0908	.0828	.0733	.0628	.0613	.0392	.0264	.0135	.20
.25	.0324	.0595	.0812	.0984	.1111	.1198	.1250	.1270	.1260	.1236	.1165	.1088	.0988	.0875	.0749	.0612	.0466	.0314	.0159	.25
.30	.0381	.0696	.0947	.1143	.1290	.1389	.1447	.1468	.1464	.1412	.1349	.1249	.1156	.1006	.0860	.0703	.0535	.0361	.0182	.30
.35	.0433	.0789	.1074	.1294	.1466	.1586	.1629	.1650	.1654	.1594	.1505	.1400	.1272	.1135	.0951	.0785	.0598	.0405	.0206	.35
.40	.0484	.0879	.1194	.1438	.1613	.1732	.1800	.1821	.1801	.1744	.1654	.1538	.1396	.1254	.1054	.0860	.0655	.0441	.0222	.40
.45	.0533	.0963	.1307	.1570	.1760	.1888	.1969	.1980	.1966	.1891	.1786	.1665	.1511	.1354	.1139	.0929	.0707	.0477	.0239	.45
.50	.0578	.1045	.1414	.1695	.1900	.2035	.2109	.2128	.2099	.2039	.1921	.1762	.1616	.1426	.1217	.0992	.0755	.0508	.0256	.50
.55	.0622	.1123	.1518	.1817	.2032	.2174	.2249	.2267	.2234	.2157	.2041	.1868	.1714	.1512	.1290	.1051	.0799	.0538	.0271	.55
.60	.0665	.1198	.1616	.1932	.2158	.2305	.2383	.2398	.2361	.2277	.2153	.1994	.1806	.1591	.1357	.1106	.0841	.0566	.0284	.60
.65	.0705	.1270	.1710	.2042	.2277	.2429	.2507	.2521	.2480	.2390	.2257	.2086	.1890	.1665	.1419	.1155	.0878	.0591	.0297	.65
.70	.0745	.1340	.1801	.2146	.2390	.2547	.2625	.2637	.2591	.2494	.2354	.2177	.1968	.1735	.1476	.1201	.0913	.0614	.0309	.70
.75	.0783	.1406	.1887	.2245	.2496	.2658	.2737	.2748	.2695	.2593	.2445	.2260	.2041	.1797	.1529	.1244	.0945	.0636	.0320	.75
.80	.0821	.1470	.1971	.2342	.2599	.2764	.2843	.2850	.2794	.2686	.2530	.2337	.2110	.1855	.1579	.1285	.0976	.0656	.0330	.80
.85	.0857	.1535	.2052	.2434	.2700	.2866	.2945	.2949	.2889	.2774	.2611	.2410	.2175	.1912	.1626	.1322	.1004	.0675	.0339	.85
.90	.0892	.1592	.2128	.2525	.2795	.2963	.3040	.3042	.2978	.2856	.2686	.2478	.2235	.1964	.1669	.1356	.1030	.0692	.0348	.90
.95	.0920	.1645	.2205	.2609	.2887	.3056	.3141	.3130	.3061	.2935	.2755	.2540	.2291	.2012	.1710	.1389	.1054	.0709	.0356	.95
1.00	.0961	.1709	.2276	.2691	.2974	.3145	.3220	.3215	.3140	.3009	.2826	.2602	.2344	.2048	.1748	.1420	.1077	.0724	.0364	1.00
1.25	.1114	.1967	.2605	.3060	.3362	.3539	.3602	.3580	.3482	.3322	.3111	.2865	.2566	.2248	.1906	.1545	.1145	.0786	.0396	1.25
1.50	.1261	.2198	.2890	.3376	.3686	.3859	.3915	.3874	.3754	.3570	.3335	.3051	.2734	.2391	.2025	.1636	.1240	.0832	.0417	1.50
1.75	.1377	.2403	.3141	.3649	.3967	.4133	.4175	.4116	.3974	.3768	.3507	.3209	.2865	.2499	.2112	.1706	.1292	.0866	.0434	1.75
2.00	.1492	.2588	.3358	.3890	.4210	.4368	.4394	.4316	.4158	.3928	.3646	.3324	.2968	.2584	.2182	.1762	.1330	.0892	.0448	2.00

TABLE II

α \ $\frac{x}{l}$	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
					Values of δ_o					
0	0.0205	0.0403	0.0591	0.0761	0.0916	0.1405	0.1148	0.1224	0.1270	0.1285
					Values of δ_c					
0.05	0.0011	0.0021	0.0031	0.0040	0.0048	0.0055	0.0060	0.0065	0.0067	0.0068
.10	.0023	.0045	.0065	.0085	.0102	.0116	.0128	.0136	.0141	.0143
.15	.0036	.0071	.0104	.0134	.0161	.0184	.0202	.0216	.0224	.0224
.20	.0051	.0100	.0147	.0191	.0228	.0261	.0287	.0306	.0317	.0322
.25	.0068	.0134	.0196	.0254	.0305	.0348	.0382	.0408	.0423	.0430
.30	.0087	.0172	.0253	.0326	.0392	.0447	.0492	.0525	.0544	.0551
.35	.0110	.0216	.0317	.0410	.0492	.0562	.0618	.0659	.0684	.0693
.40	.0136	.0268	.0393	.0508	.0609	.0697	.0766	.0816	.0848	.0858
.45	.0167	.0328	.0481	.0622	.0748	.0855	.0940	.1002	.1040	.1053
.50	.0203	.0401	.0588	.0760	.0913	.1044	.1148	.1225	.1271	.1287
.55	.0249	.0490	.0719	.0929	.1116	.1277	.1404	.1498	.1555	.1574
.60	.0305	.0601	.0882	.1143	.1369	.1567	.1724	.1838	.1909	.1931
.65	.0376	.0744	.1091	.1412	.1696	.1939	.2134	.2277	.2363	.2392
.70	.0473	.0935	.1371	.1773	.2132	.2438	.2682	.2862	.2971	.3008
.75	.0607	.1199	.1761	.2277	.2740	.3132	.3446	.3677	.3818	.3865
.80	.0812	.1599	.2350	.3038	.3654	.4178	.4601	.4908	.5095	.5158
.85	.1147	.2265	.3328	.4305	.5200	.5960	.6519	.6956	.7222	.7312
.90	.1820	.3595	.5279	.6834	.8217	.9398	1.035	1.104	1.147	1.161
.95	.3840	.7586	1.114	1.443	1.735	1.985	2.186	2.333	2.422	2.453
1.00	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
					Values of δ_t					
0.05	0.0010	0.0019	0.0028	0.0036	0.0044	0.0050	0.0055	0.0058	0.0060	0.0061
.10	.0019	.0037	.0054	.0069	.0083	.0095	.0105	.0111	.0115	.0117
.15	.0027	.0053	.0077	.0099	.0119	.0137	.0150	.0160	.0166	.0168
.20	.0034	.0068	.0099	.0127	.0152	.0175	.0191	.0204	.0211	.0214
.25	.0041	.0081	.0118	.0153	.0183	.0209	.0230	.0245	.0254	.0257
.30	.0047	.0094	.0137	.0176	.0211	.0241	.0265	.0283	.0293	.0296
.35	.0053	.0105	.0154	.0198	.0238	.0271	.0298	.0317	.0329	.0332
.40	.0059	.0116	.0170	.0218	.0262	.0298	.0328	.0349	.0363	.0367
.45	.0064	.0126	.0184	.0238	.0284	.0324	.0356	.0379	.0393	.0398
.50	.0069	.0135	.0198	.0255	.0305	.0348	.0383	.0407	.0422	.0427
.55	.0073	.0144	.0211	.0272	.0326	.0371	.0408	.0434	.0450	.0455
.60	.0077	.0152	.0223	.0287	.0344	.0392	.0431	.0458	.0475	.0481
.65	.0081	.0160	.0234	.0302	.0361	.0412	.0452	.0482	.0499	.0505
.70	.0085	.0167	.0245	.0315	.0378	.0430	.0472	.0503	.0521	.0528
.75	.0088	.0174	.0255	.0328	.0393	.0448	.0492	.0523	.0543	.0550
.80	.0092	.0181	.0265	.0341	.0408	.0465	.0510	.0543	.0563	.0570
.85	.0095	.0187	.0273	.0352	.0422	.0481	.0528	.0561	.0582	.0589
.90	.0098	.0193	.0282	.0363	.0435	.0495	.0544	.0579	.0600	.0607
.95	.0101	.0198	.0290	.0374	.0447	.0510	.0559	.0595	.0617	.0625
1.00	.0104	.0204	.0298	.0384	.0459	.0523	.0575	.0612	.0634	.0641
1.25	.0115	.0227	.0331	.0427	.0510	.0581	.0637	.0678	.0703	.0711
1.50	.0125	.0245	.0358	.0461	.0552	.0628	.0688	.0732	.0759	.0768
1.75	.0133	.0261	.0381	.0490	.0585	.0666	.0730	.0776	.0804	.0814
2.00	.0139	.0274	.0400	.0514	.0614	.0700	.0765	.0813	.0842	.0852

TABLE III

z/l \ x/l	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	x/l / z/l
Values of ϵ_0																				
0	0.0096	0.0189	0.0279	0.0363	0.0438	0.0504	0.0559	0.0601	0.0629	0.0643	0.0641	0.0623	0.0590	0.0541	0.0479	0.0401	0.0313	0.0214	0.0109	0
Values of ϵ_0																				
0.05	0.0006	0.0011	0.0015	0.0018	0.0023	0.0027	0.0030	0.0032	0.0034	0.0035	0.0035	0.0033	0.0031	0.0029	0.0025	0.0019	0.0015	0.0010	0.0005	0.05
.10	.0011	.0021	.0031	.0041	.0049	.0056	.0063	.0067	.0070	.0071	.0071	.0069	.0066	.0059	.0053	.0045	.0035	.0023	.0011	.10
.15	.0017	.0033	.0049	.0064	.0075	.0089	.0099	.0106	.0111	.0113	.0113	.0110	.0104	.0095	.0084	.0070	.0055	.0038	.0019	.15
.20	.0024	.0046	.0070	.0091	.0110	.0128	.0140	.0151	.0157	.0160	.0160	.0156	.0147	.0134	.0119	.0099	.0077	.0053	.0026	.20
.25	.0032	.0064	.0094	.0122	.0147	.0169	.0187	.0201	.0210	.0214	.0213	.0207	.0195	.0179	.0158	.0132	.0103	.0070	.0035	.25
.30	.0041	.0082	.0121	.0157	.0189	.0218	.0241	.0259	.0270	.0275	.0274	.0266	.0252	.0230	.0203	.0169	.0132	.0090	.0046	.30
.35	.0052	.0103	.0152	.0196	.0236	.0274	.0305	.0325	.0340	.0346	.0345	.0334	.0315	.0288	.0254	.0212	.0165	.0113	.0058	.35
.40	.0065	.0128	.0189	.0245	.0296	.0340	.0376	.0403	.0421	.0429	.0427	.0414	.0390	.0357	.0314	.0263	.0204	.0140	.0071	.40
.45	.0080	.0158	.0232	.0301	.0364	.0418	.0462	.0497	.0517	.0526	.0523	.0507	.0478	.0437	.0384	.0321	.0249	.0171	.0087	.45
.50	.0098	.0193	.0286	.0369	.0446	.0511	.0560	.0606	.0633	.0644	.0639	.0619	.0584	.0533	.0469	.0392	.0304	.0208	.0106	.50
.55	.0120	.0237	.0348	.0452	.0545	.0626	.0691	.0741	.0772	.0785	.0780	.0755	.0711	.0649	.0571	.0477	.0369	.0253	.0128	.55
.60	.0148	.0292	.0429	.0555	.0671	.0770	.0850	.0911	.0950	.0966	.0958	.0927	.0873	.0796	.0699	.0584	.0453	.0309	.0157	.60
.65	.0183	.0362	.0533	.0691	.0836	.0955	.1054	.1129	.1177	.1196	.1186	.1148	.1079	.0984	.0863	.0720	.0559	.0381	.0193	.65
.70	.0231	.0457	.0672	.0870	.1049	.1205	.1328	.1421	.1481	.1504	.1491	.1441	.1355	.1235	.1083	.0904	.0700	.0478	.0242	.70
.75	.0298	.0588	.0865	.1121	.1351	.1548	.1709	.1828	.1904	.1932	.1914	.1849	.1738	.1583	.1388	.1157	.0896	.0611	.0310	.75
.80	.0399	.0787	.1156	.1499	.1806	.2068	.2282	.2440	.2539	.2577	.2551	.2463	.2315	.2106	.1846	.1537	.1190	.0812	.0411	.80
.85	.0586	.1119	.1644	.2131	.2569	.2939	.3276	.3464	.3605	.3654	.3616	.3489	.3276	.2979	.2609	.2173	.1681	.1146	.0581	.85
.90	.0908	.1783	.2619	.3390	.4085	.4677	.5155	.5508	.5726	.5806	.5740	.5534	.5192	.4720	.4131	.3438	.2658	.1811	.0917	.90
.95	.1433	.2780	.5558	.7197	.8653	.9904	1.091	1.165	1.211	1.237	1.212	1.168	1.095	.9960	.8704	.7239	.5594	.3810	.1929	.95
1.00	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	1.00
Values of ϵ_1																				
0.05	0.0005	0.0009	0.0013	0.0017	0.0021	0.0024	0.0027	0.0028	0.0030	0.0031	0.0031	0.0030	0.0028	0.0026	0.0025	0.0019	0.0017	0.0010	0.0005	0.05
.10	.0009	.0017	.0025	.0033	.0040	.0046	.0051	.0055	.0057	.0059	.0058	.0057	.0054	.0049	.0044	.0037	.0029	.0019	.0010	.10
.15	.0012	.0025	.0036	.0047	.0057	.0065	.0072	.0078	.0082	.0084	.0084	.0081	.0076	.0071	.0063	.0055	.0041	.0028	.0015	.15
.20	.0016	.0031	.0046	.0060	.0075	.0084	.0095	.0100	.0104	.0107	.0107	.0104	.0099	.0091	.0081	.0067	.0053	.0036	.0018	.20
.25	.0019	.0038	.0055	.0072	.0087	.0100	.0112	.0120	.0125	.0129	.0129	.0125	.0119	.0109	.0096	.0081	.0064	.0044	.0023	.25
.30	.0022	.0043	.0064	.0083	.0100	.0115	.0128	.0139	.0145	.0148	.0148	.0144	.0136	.0126	.0111	.0094	.0075	.0050	.0026	.30
.35	.0024	.0048	.0071	.0093	.0112	.0129	.0144	.0156	.0163	.0166	.0166	.0162	.0154	.0141	.0125	.0105	.0082	.0055	.0029	.35
.40	.0027	.0053	.0078	.0102	.0123	.0142	.0158	.0170	.0179	.0183	.0183	.0179	.0170	.0156	.0138	.0118	.0091	.0062	.0032	.40
.45	.0029	.0058	.0085	.0111	.0134	.0155	.0172	.0185	.0194	.0199	.0199	.0194	.0185	.0170	.0151	.0127	.0099	.0068	.0035	.45
.50	.0031	.0062	.0091	.0119	.0144	.0166	.0184	.0199	.0209	.0214	.0214	.0209	.0198	.0182	.0162	.0136	.0106	.0073	.0037	.50
.55	.0033	.0066	.0097	.0126	.0153	.0177	.0196	.0211	.0222	.0228	.0228	.0222	.0211	.0195	.0172	.0145	.0114	.0078	.0040	.55
.60	.0035	.0070	.0103	.0134	.0162	.0186	.0207	.0223	.0235	.0241	.0241	.0235	.0224	.0206	.0183	.0154	.0120	.0083	.0042	.60
.65	.0037	.0073	.0107	.0140	.0170	.0196	.0217	.0234	.0246	.0253	.0253	.0247	.0235	.0217	.0193	.0163	.0127	.0087	.0044	.65
.70	.0039	.0076	.0112	.0146	.0177	.0204	.0227	.0245	.0257	.0264	.0264	.0258	.0245	.0227	.0201	.0169	.0132	.0091	.0046	.70
.75	.0040	.0079	.0117	.0152	.0184	.0212	.0236	.0255	.0268	.0275	.0275	.0269	.0256	.0236	.0209	.0176	.0138	.0095	.0048	.75
.80	.0041	.0082	.0121	.0157	.0190	.0220	.0245	.0264	.0278	.0286	.0285	.0279	.0265	.0245	.0217	.0183	.0143	.0099	.0050	.80
.85	.0043	.0085	.0125	.0163	.0197	.0227	.0253	.0273	.0287	.0295	.0295	.0288	.0275	.0253	.0225	.0190	.0149	.0102	.0053	.85
.90	.0044	.0087	.0128	.0167	.0202	.0234	.0260	.0281	.0296	.0304	.0304	.0298	.0284	.0262	.0232	.0196	.0154	.0106	.0054	.90
.95	.0045	.0089	.0132	.0172	.0208	.0240	.0267	.0289	.0304	.0312	.0313	.0305	.0292	.0269	.0239	.0202	.0158	.0109	.0056	.95
1.00	.0046	.0092	.0135	.0176	.0214	.0246	.0275	.0296	.0312	.0320	.0321	.0314	.0300	.0277	.0246	.0207	.0163	.0112	.0057	1.00
1.25	.0051	.0101	.0150	.0195	.0236	.0273	.0304	.0328	.0347	.0356	.0357	.0350	.0334	.0308	.0274	.0232	.0182	.0125	.0064	1.25
1.50	.0055	.0109	.0161	.0209	.0254	.0294	.0327	.0354	.0373	.0384	.0385	.0378	.0361	.0334	.0298	.0252	.0198	.0137	.0070	1.50
1.75	.0058	.0115	.0170	.0221	.0268	.0310	.0345	.0373	.0395	.0407	.0409	.0408	.0384	.0355	.0317	.0268	.0211	.0146	.0075	1.75
2.00	.0060	.0120	.0177	.0231	.0280	.0324	.0362	.0392	.0413	.0426	.0429	.0421	.0403	.0374	.0334	.0283	.0223	.0154	.0079	2.00

TABLE IV

u \ u and v	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	u and v / u
v	2.0	1.9	1.8	1.7	1.6	1.5	1.4	1.3	1.2	1.1	1.0	v
Values of ψ_{1_0} and $-\psi_{2_0}$												
0	0.1096	0.1038	0.0851	0.0582	0.0265	-0.0059	-0.0354	-0.0606	-0.0798	-0.0919	-0.0959	0
Values of φ_{1_0} and $-\varphi_{2_0}$												
0.05	-0.8408	-0.8418	-0.8422	-0.8436	-0.8455	-0.8468	-0.8485	-0.8498	-0.8509	-0.8514	-0.8517	0.05
.1	-.3330	-.3337	-.3353	-.3357	-.3371	-.3406	-.3438	-.3467	-.3489	-.3505	-.3507	.1
.2	-.0595	-.0611	-.0659	-.0735	-.0801	-.0879	-.0953	-.1018	-.1067	-.1098	-.1108	.2
.4	.1112	.1059	.0963	.0783	.0583	.0374	.0173	-.0002	-.0137	-.0228	-.0251	.4
.6	.2400	.2309	.2050	.1593	.1255	.0790	.0325	-.0079	-.0392	-.0591	-.0659	.6
.8	.5155	.4946	.4525	.3385	.3228	.0975	-.0261	-.0580	-.2225	-.2774	-.2954	.8
.9	1.038	.9845	.8496	.6426	.5855	.1034	-.1764	-.4270	-.6360	-.7517	-.7953	.9
.95	2.050	1.953	2.485	1.348	.7010	.1056	-.4875	-1.081	-1.442	-1.713	-1.805	.95
.975	4.071	3.874	3.308	2.430	1.325	.1059	-1.110	-2.205	-3.075	-3.653	-3.825	.975
1.0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	1.0
1.025	-4.054	-3.959	-3.247	-2.335	-1.177	.1099	1.307	2.551	3.487	4.031	4.288	1.025
1.05	-1.998	-1.897	-1.601	-1.155	-.5471	.1098	.7098	1.357	1.842	2.148	2.254	1.05
1.1	-.9881	-.9309	-.7794	-.5594	-.2539	.1114	.4595	.7741	1.035	1.155	1.244	1.1
1.2	-.4599	-.4482	-.3652	-.2381	-.0752	.1140	.3048	.4797	.6198	.7106	.7418	1.2
1.4	-.2073	-.1945	-.2535	-.0845	.0085	.1174	.2507	.3358	.4219	.4778	.4971	1.4
1.6	-.1148	-.1058	-.0601	-.0518	.0355	.1187	.2088	.2904	.3592	.4044	.4302	1.6
1.8	-.0644	-.0598	-.0414	-.0050	.0496	.1185	.1945	.2682	.3296	.3707	.3960	1.8
2.0	-.0311	-.0289	-.0159	.0068	.0571	.1168	.1851	.2550	.3138	.3529	.3698	2.0
2.25	0	-.0007	.0042	.0297	.0591	.1155	.1772	.2441	.3094	.3480	.3551	2.25
2.5	.0252	.0226	.0200	.0295	.0675	.1062	.1680	.2359	.2958	.3398	.3540	2.5
2.75	.0581	.0449	.0334	.0319	.0608	.0957	.1564	.2294	.2951	.3422	.3592	2.75
3.0	.0881	.0702	.0457	.0307	.0377	.0751	.1398	.2199	.2972	.3529	.3732	3.0
3.25	.1238	.1046	.0626	.0241	.0128	.0427	.1127	.2050	.3045	.3755	.4051	3.25
3.5	.1980	.1851	.0877	.0051	-.0297	-.0238	.0594	.1863	.3250	.4266	.4655	3.5
3.75	.4065	.3540	.1539	-.0543	-.2032	-.2251	-.1060	.1232	.3858	.5976	.6647	3.75
3.9	1.016	.8872	.3429	-.2405	-.5980	-.6512	-.5980	-.0642	.5703	1.077	1.269	3.9
3.95	2.058	1.854	.8591	-.5858	-1.518	-1.858	-1.421	-.3799	.8858	1.902	2.290	3.95
4.00	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	4.00
Values of Ψ_{1_t} and $-\Psi_{2_t}$												
0.1	-0.6634	-0.5520	-0.4856	-0.4133	-0.3511	-0.2979	-0.2556	-0.2239	-0.2005	-0.1867	-0.1820	0.1
.2	-.4015	-.3078	-.2251	-.1557	-.0959	-.0461	-.0050	.0249	.0458	.0598	.0641	.2
.3	-.3098	-.2155	-.1364	-.0678	-.0100	.0378	.0761	.1054	.1261	.1385	.1425	.3
.4	-.2513	-.1666	-.0895	-.0235	.0385	.0795	.1155	.1445	.1641	.1758	.1797	.4
.5	-.2304	-.1382	-.0608	.0065	.0526	.1044	.1398	.1667	.1858	.1969	.2006	.5
.6	-.2067	-.1159	-.0389	.0244	.0776	.1208	.1551	.1810	.1993	.2101	.2137	.6
.7	-.1882	-.1008	-.0247	.0384	.0903	.1324	.1657	.1909	.2085	.2189	.2224	.7
.8	-.1722	-.0888	-.0129	.0461	.1000	.1410	.1738	.1979	.2150	.2251	.2285	.8
.9	-.1586	-.0779	-.0035	.0577	.1075	.1477	.1793	.2031	.2198	.2297	.2329	.9
1.0	-.1597	-.0694	.0044	.0646	.1135	.1529	.1838	.2071	.2234	.2330	.2363	1.0
1.25	-.1488	-.0530	.0192	.0775	.1245	.1622	.1918	.2138	.2295	.2385	.2415	1.25
1.5	-.1301	-.0411	.0296	.0863	.1318	.1689	.1965	.2178	.2327	.2415	.2445	1.5
1.75	-.1206	-.0320	.0375	.0926	.1370	.1722	.1997	.2204	.2349	.2434	.2462	1.75
2.0	-.1192	-.0247	.0437	.0978	.1409	.1752	.2019	.2221	.2363	.2446	.2474	2.0

TABLE V

α \ x/l	0.1	0.2	0.3	0.4	0.5
		Values of ζ_0			
0	0.0266	0.0842	0.1451	0.1895	0.2056
		Values of ζ_c			
0.2	0.0318	0.1025	0.1788	0.2355	0.2563
.4	.0401	.1324	.2349	.3124	.3411
.6	.0566	.1914	.3461	.4654	.5098
.8	.1053	.3671	.6782	.9235	1.016
.9	.2023	.7179	1.342	1.841	2.030
.95	.3963	1.420	2.671	3.676	4.059
.975	.7816	2.814	5.313	7.328	8.096
1.0	∞	∞	∞	∞	∞
1.025	-.7662	-2.786	-5.297	-7.336	-8.115
1.05	-.3779	-1.381	-2.636	-3.657	-4.049
1.1	-.1842	-.6814	-1.309	-1.825	-2.023
1.2	-.0872	-.3305	-.6456	-.9081	-1.010
1.4	-.0383	-.1542	-.3129	-.4494	-.5032
1.6	-.0216	-.0943	-.2009	-.2961	-.3342
1.8	-.0128	-.0633	-.1439	-.2190	-.2497
2.0	-.0071	-.0437	-.1086	-.1724	-.1989
2.25	-.0018	-.0262	-.0787	-.1344	-.1583
2.5	.0025	-.0125	-.0567	-.1082	-.1311
2.75	.0068	.0005	-.0379	-.0883	-.1117
3.0	.0117	.0148	-.0192	-.0717	-.0970
3.25	.0189	.0342	.0034	-.0556	-.0856
3.5	.0315	.0679	.0399	-.0357	-.0764
3.75	.0674	.1623	.1364	.0060	-.0689
3.9	.1725	.4379	.4131	.1143	-.0649
3.95	.3491	1.466	1.023	.2917	-.0638
4.0	∞	∞	∞	∞	-.0625
		Values of ζ_t			
0.1	0.0248	0.0774	0.1330	0.1730	0.1874
.2	.0232	.0720	.1225	.1588	.1719
.3	.0218	.0662	.1137	.1468	.1587
.4	.0206	.0630	.1060	.1366	.1475
.5	.0196	.0595	.0996	.1289	.1379
.6	.0187	.0563	.0938	.1201	.1295
.7	.0179	.0535	.0887	.1133	.1220
.8	.0171	.0510	.0842	.1072	.1154
.9	.0165	.0487	.0802	.1018	.1094
1.0	.0159	.0467	.0764	.0969	.1040
1.25	.0146	.0423	.0686	.0865	.0926
1.5	.0135	.0387	.0623	.0781	.0837
1.75	.0126	.0357	.0570	.0712	.0761
2.0	.0119	.0333	.0527	.0656	.0700

TABLE VI

α \ x/l	0.1	0.2	0.3	0.4	0.5
	Values of $\alpha\zeta_c$				
0.2	0.0064	0.0205	0.0358	0.0471	0.0513
.4	.0160	.0530	.0940	.1250	.1364
.6	.0340	.1148	.2077	.2792	.3059
.8	.0843	.2937	.5426	.7388	.8128
.9	.1821	.6461	1.208	1.657	1.827
.95	.3765	1.349	2.537	3.493	3.857
.975	.7627	2.744	5.180	7.145	7.893
1.0	∞	∞	∞	∞	∞
1.025	-.7854	-2.856	-5.430	-7.520	-8.318
1.05	-.3968	-1.451	-2.767	-3.840	-4.251
1.1	-.2026	-.7495	-1.441	-2.008	-2.226
1.2	-.1047	-.3966	-.7747	-1.090	-1.212
1.4	-.0537	-.2159	-.4381	-.6292	-.7045
1.6	-.0345	-.1509	-.3214	-.4738	-.5347
1.8	-.0230	-.1139	-.2589	-.3942	-.4495
2.0	-.0141	-.0874	-.2172	-.3448	-.3978
2.25	-.0040	-.0590	-.1772	-.3024	-.3561
2.5	.0062	-.0312	-.1417	-.2705	-.3278
2.75	.0187	.0013	-.1043	-.2430	-.3071
3.0	.0352	.0444	-.0577	-.2151	-.2911
3.25	.0614	.1112	.0111	-.1807	-.2783
3.5	.1103	.2377	.1395	-.1250	-.2675
3.75	.2526	.6086	.5114	.0226	-.2582
3.90	.6727	1.708	1.611	.4458	-.2531
3.95	1.379	5.792	4.042	1.152	-.2520
4.0	∞	∞	∞	∞	-.2500
	Values of $\alpha\zeta_t$				
0.1	0.0025	0.0077	0.0133	0.0173	0.0187
.2	.0046	.0144	.0245	.0318	.0344
.3	.0065	.0199	.0341	.0440	.0476
.4	.0082	.0252	.0424	.0546	.0590
.5	.0098	.0297	.0498	.0644	.0689
.6	.0112	.0338	.0563	.0721	.0777
.7	.0125	.0375	.0621	.0793	.0854
.8	.0137	.0408	.0674	.0858	.0923
.9	.0148	.0439	.0722	.0916	.0985
1.0	.0159	.0467	.0764	.0969	.1040
1.25	.0182	.0529	.0857	.1081	.1157
1.5	.0203	.0581	.0934	.1171	.1256
1.75	.0220	.0624	.0997	.1246	.1332
2.0	.0238	.0666	.1054	.1312	.1400

TABLE VII

α \ x/l	0.1	0.2	0.3	0.4	0.5
			Values of K_{1_0}		
0	37.59	11.88	6.892	5.277	4.863
			Values of K_{1_c}		
0.2	31.48	9.756	5.593	4.246	3.902
.4	24.92	7.553	4.257	3.201	2.932
.6	17.66	5.225	2.889	2.149	1.962
.8	9.494	2.724	1.474	1.083	.984
.9	4.943	1.393	.745	.543	.493
.95	2.523	.704	.374	.272	.246
.975	1.279	.355	.188	.136	.124
1.0	0	0	0	0	0
1.025	-1.305	-.359	-.189	-.136	-.123
1.05	-2.646	-.724	-.379	-.273	-.247
1.1	-5.429	-1.468	-.764	-.548	-.494
1.2	-11.46	-3.026	-1.549	-1.101	-.990
1.4	-26.08	-6.485	-3.196	-2.225	-1.987
1.6	-46.36	-10.60	-4.979	-3.377	-2.992
1.8	-78.25	-15.80	-6.951	-4.566	-4.005
2.0	-141.4	-22.88	-9.210	-5.800	-5.027
2.25	-568.2	-38.17	-12.70	-7.442	-6.319
2.5	400.0	-80.0	-17.64	-9.240	-7.627
2.75	--	212.8	-26.37	-11.32	-8.956
3.0	--	--	-51.95	-13.95	-10.30
3.25	--	--	292.4	-17.99	-11.68
3.5	--	--	--	-28.00	-13.08
3.75	--	--	--	165.6	-14.52
3.90	--	--	--	--	-15.41
3.95	--	--	--	--	-15.67
4.0	--	--	--	--	-16.00
			Values of K_{1_t}		
0.1	40.32	12.92	7.519	5.780	5.336
.2	43.20	13.88	8.163	6.297	5.817
.3	45.81	15.11	8.795	6.812	6.301
.4	48.57	15.87	9.434	7.321	6.780
.5	51.02	16.81	10.04	7.758	7.252
.6	53.53	17.76	10.66	8.326	7.722
.7	55.99	18.69	11.27	8.826	8.197
.8	58.34	19.60	11.88	9.328	8.666
.9	60.72	20.52	12.47	9.823	9.141
1.0	62.97	21.42	13.09	10.32	9.615
1.25	68.59	23.64	14.58	11.56	10.80
1.5	73.96	25.81	16.05	12.80	11.95
1.75	79.68	28.03	17.54	14.05	13.14
2.00	84.17	30.05	18.97	15.24	14.29

TABLE VIII

α \ x/l	0.1	0.2	0.3	0.4	0.5
	Values of K_{2c}				
0.2	156.3	48.78	27.93	21.23	19.49
.4	62.50	18.87	10.64	8.000	7.331
.6	29.41	8.711	4.815	3.582	3.269
.8	11.86	3.405	1.843	1.354	1.230
.9	5.491	1.548	.828	.604	.547
.95	2.656	.741	.394	.286	.259
.975	1.312	.364	.193	.140	.127
1.0	0	0	0	0	0
1.025	-1.273	-.350	-.184	-.133	-.120
1.05	-2.520	-.689	-.361	-.260	-.235
1.1	-4.936	-1.334	-.694	-.498	-.449
1.2	-9.551	-2.521	-1.291	-.918	-.825
1.4	-18.62	-4.632	-2.283	-1.589	-1.419
1.6	-28.99	-6.627	-3.111	-2.111	-1.870
1.8	-43.48	-8.780	-3.862	-2.537	-2.225
2.0	-70.92	-11.44	-4.604	-2.900	-2.514
2.25	-250.0	-16.95	-5.643	-3.307	-2.808
2.5	161.3	-32.05	-7.057	-3.697	-3.051
2.75	--	769.2	-9.588	-4.115	-3.256
3.0	--	--	-17.33	-4.649	-3.435
3.25	--	--	90.00	-5.534	-3.593
3.5	--	--	--	-8.000	-3.738
3.75	--	--	--	44.25	-3.873
3.9	--	--	--	--	-3.951
3.95	--	--	--	--	-3.968
4.0	--	--	--	--	-4.000
	Values of K_{2t}				
0.1	400.0	129.9	75.19	57.80	53.48
.2	217.4	69.44	40.82	31.45	29.07
.3	153.8	50.25	29.33	22.73	21.01
.4	122.0	39.68	23.58	18.31	16.95
.5	102.0	33.67	20.08	15.53	14.51
.6	89.29	29.59	17.76	13.87	12.87
.7	80.00	26.67	16.10	12.61	11.71
.8	72.99	24.51	14.84	11.65	10.83
.9	67.57	22.78	13.85	10.92	10.15
1.0	62.89	21.41	13.09	10.32	9.615
1.25	54.94	18.90	11.67	9.251	8.643
1.5	49.26	17.21	10.71	8.540	7.962
1.75	45.45	16.03	10.03	8.026	7.508
2.0	42.02	15.01	9.488	7.622	7.143

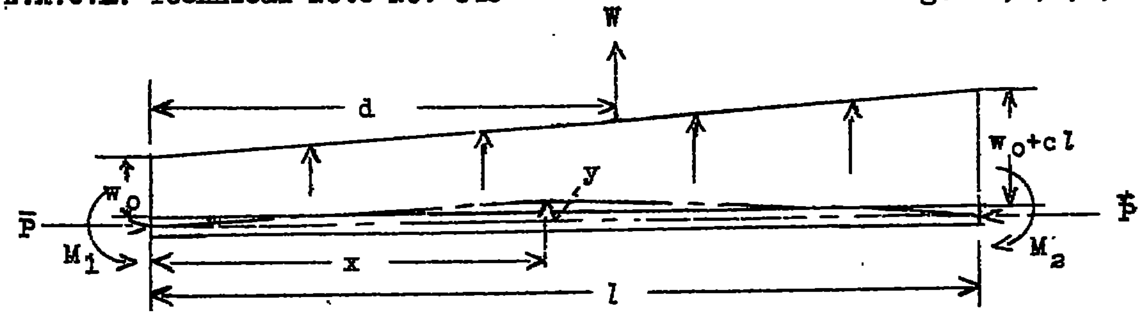


Figure 1.

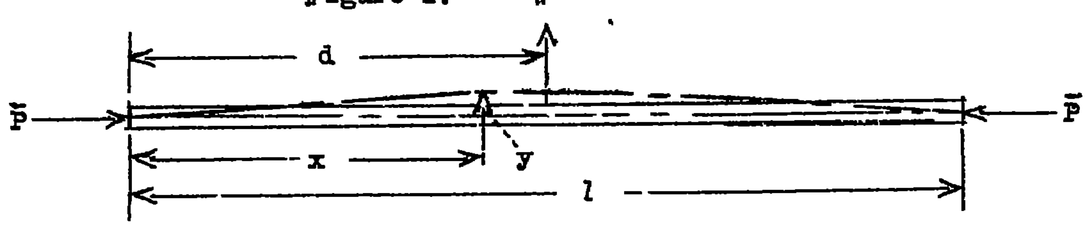


Figure 2.

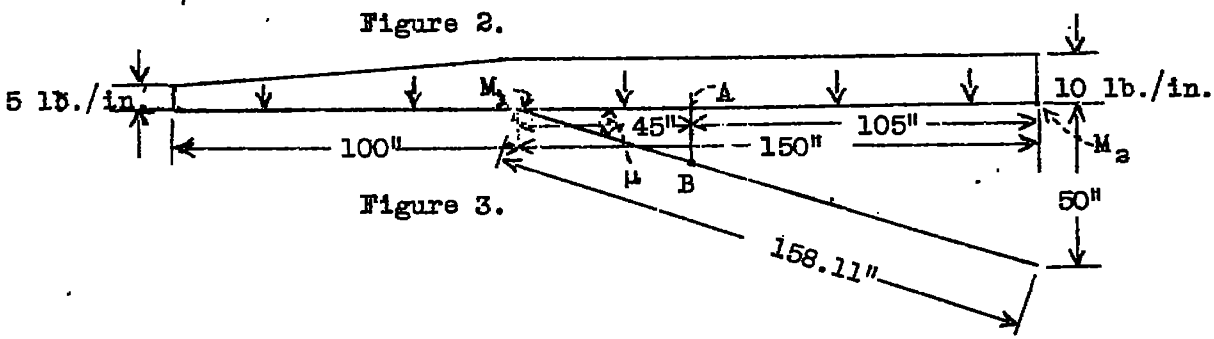
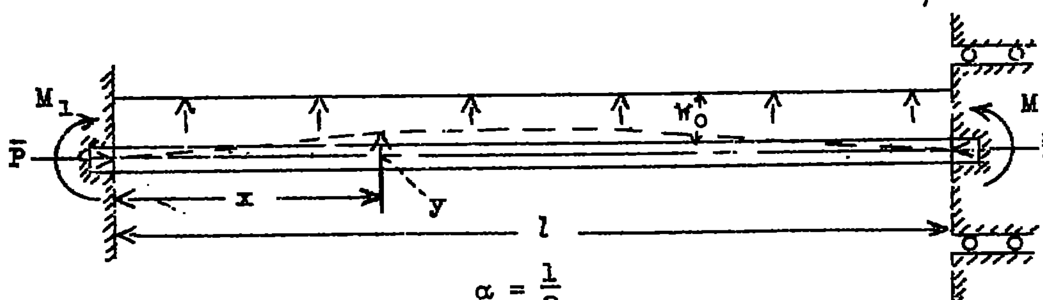
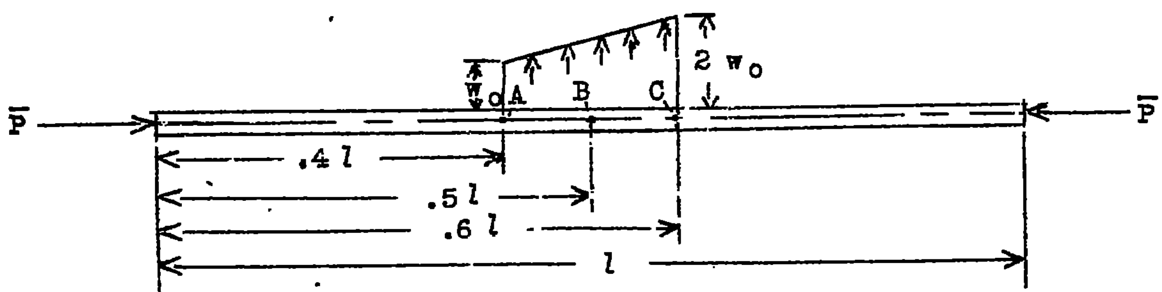


Figure 3.



$\alpha = \frac{1}{2}$

Figure 4.



$\alpha = 0.6$

Figure 4a.

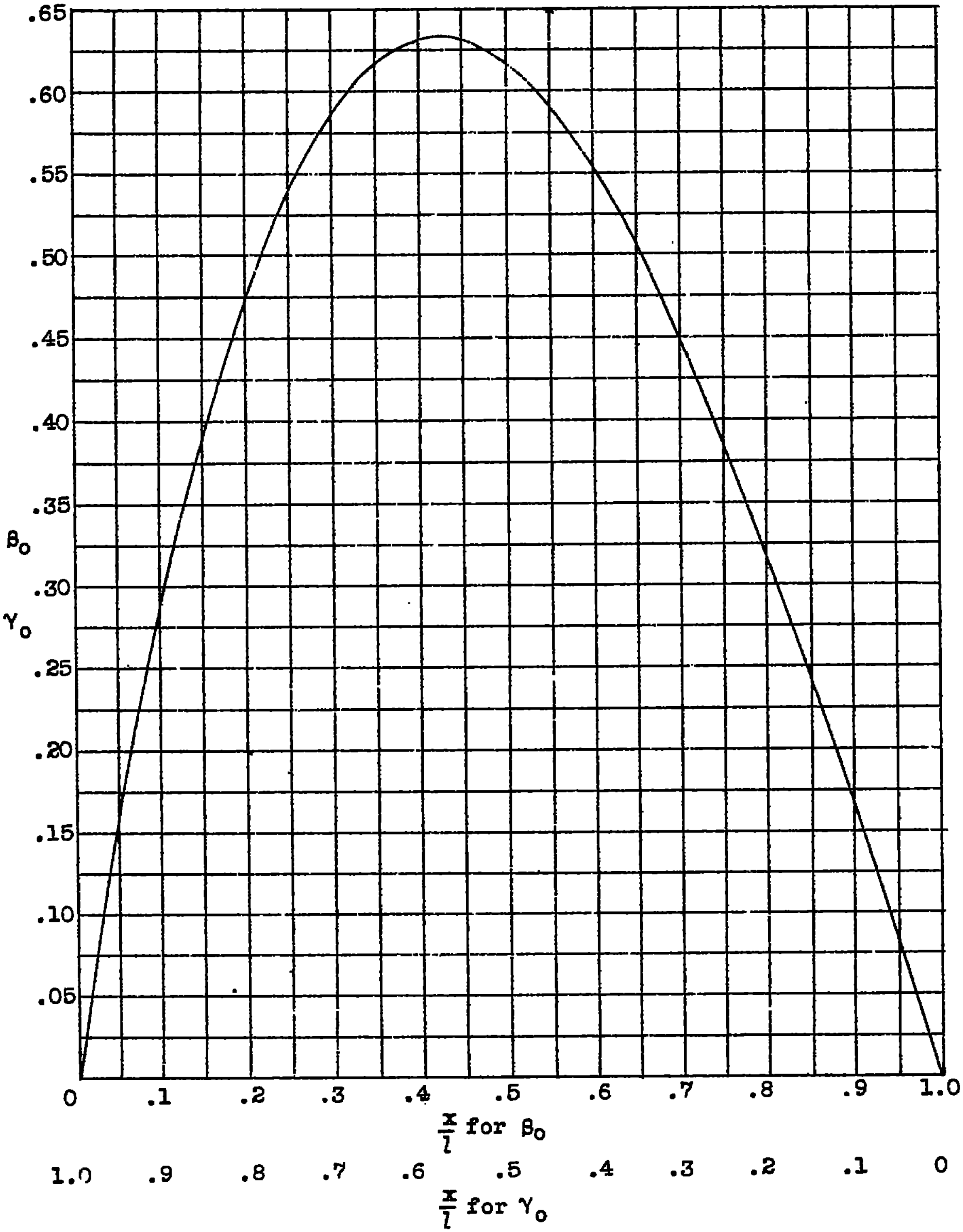


Figure 5.- Plot of β_o and γ_o against $\frac{x}{l}$.

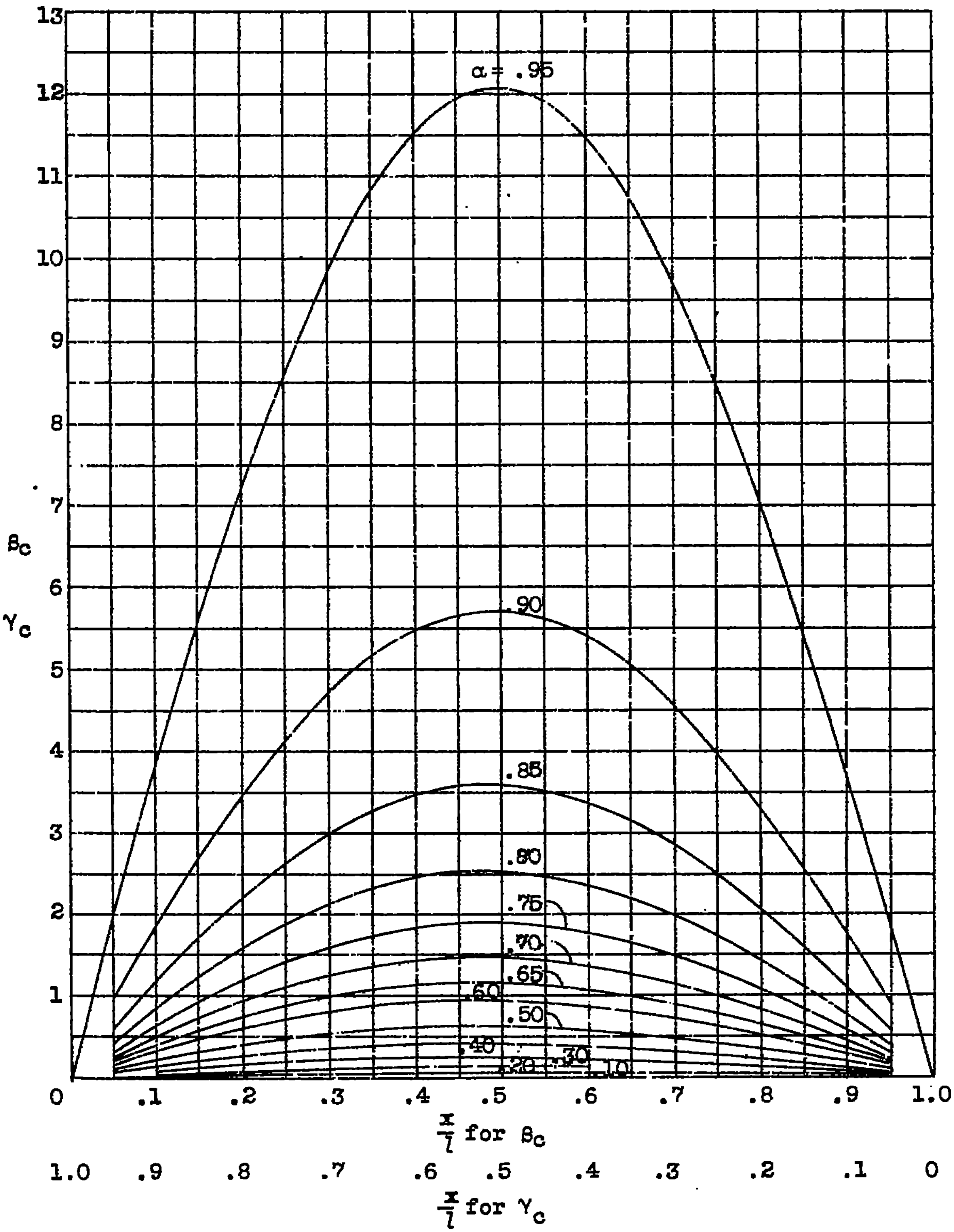


Figure 6.- Plot of β_c and γ_c against $\frac{x}{l}$.

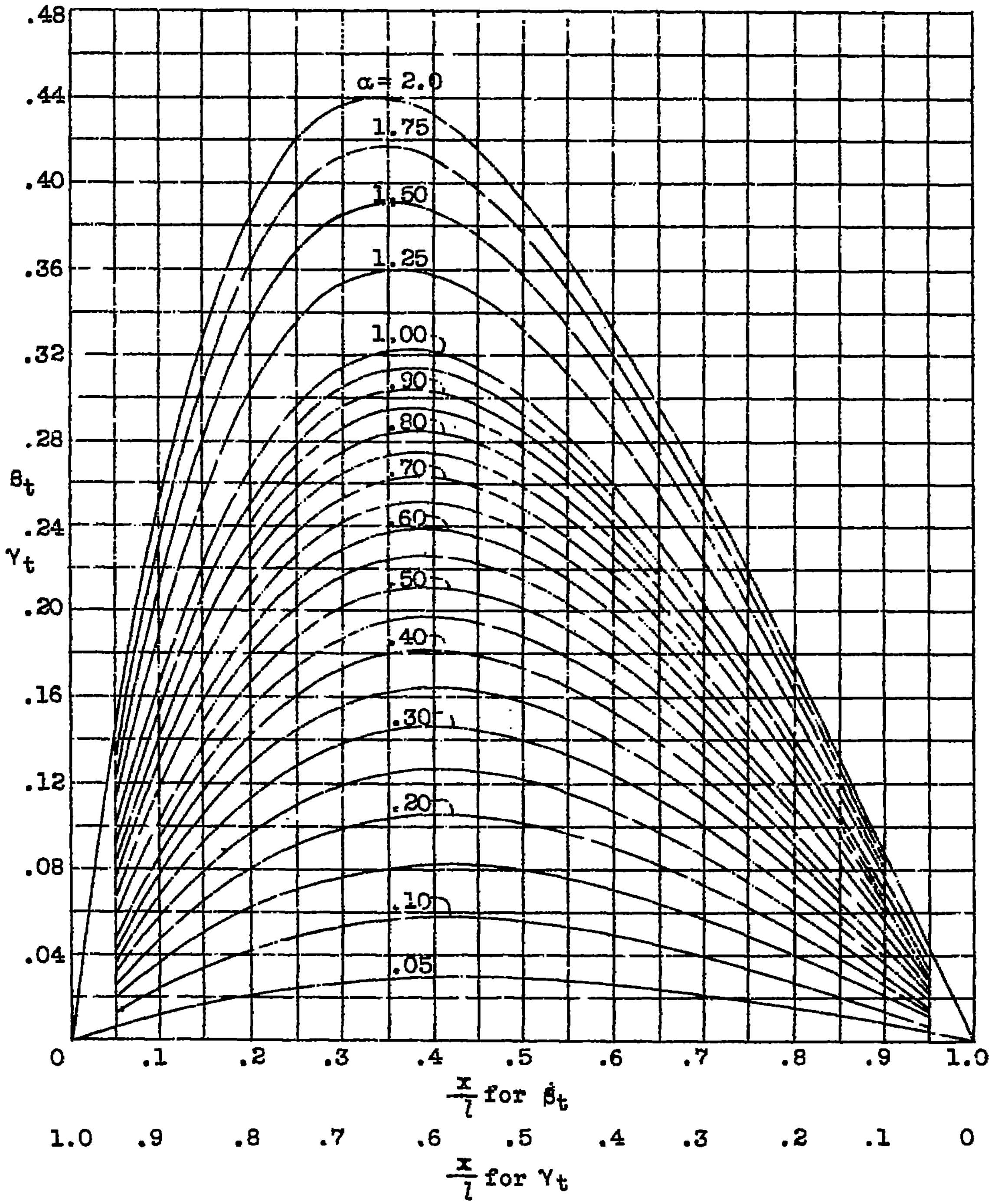


Figure 7.- Plot of β_t and γ_t against $\frac{x}{l}$.

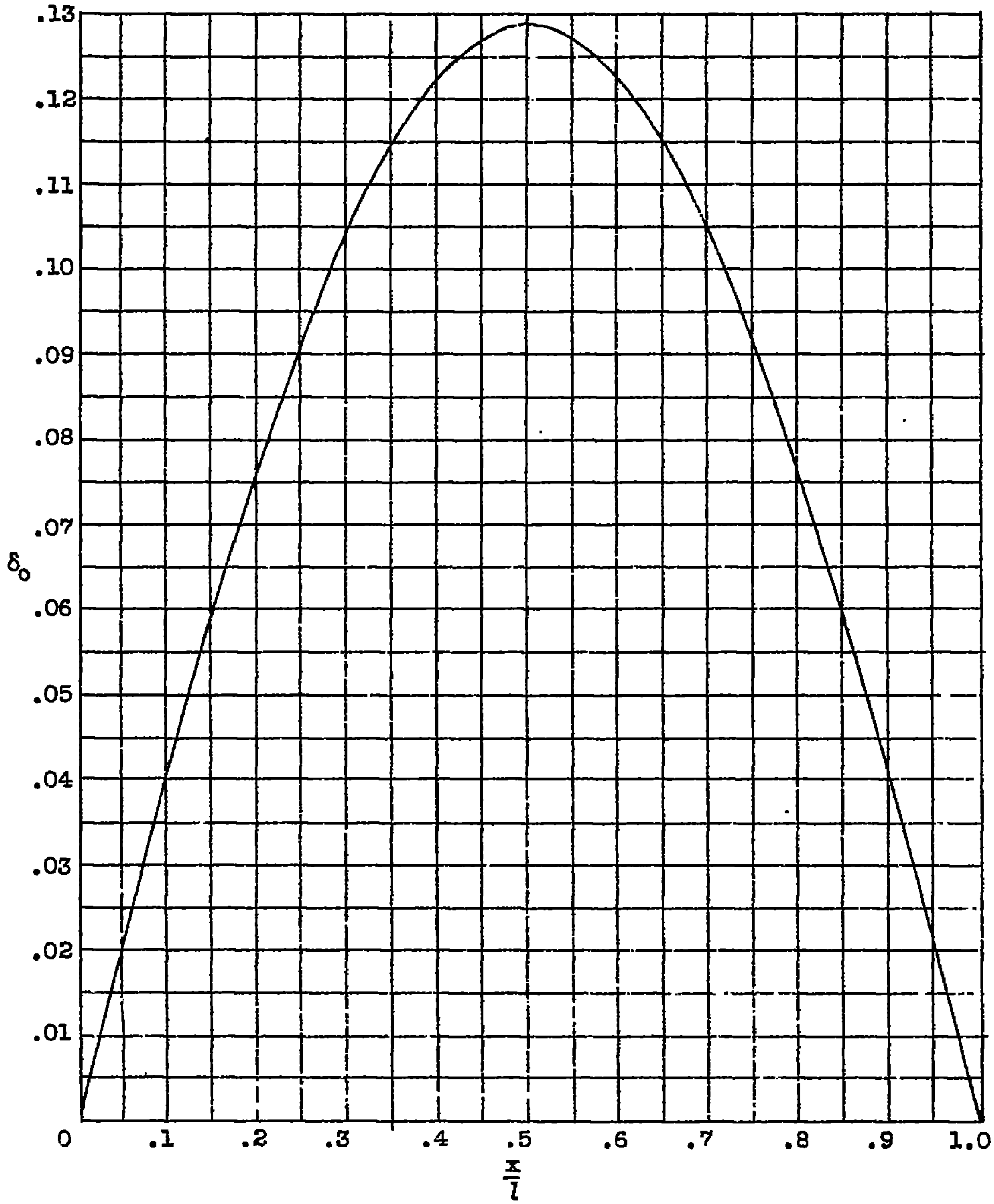


Figure 8.- Plot of δ_o against $\frac{x}{l}$.

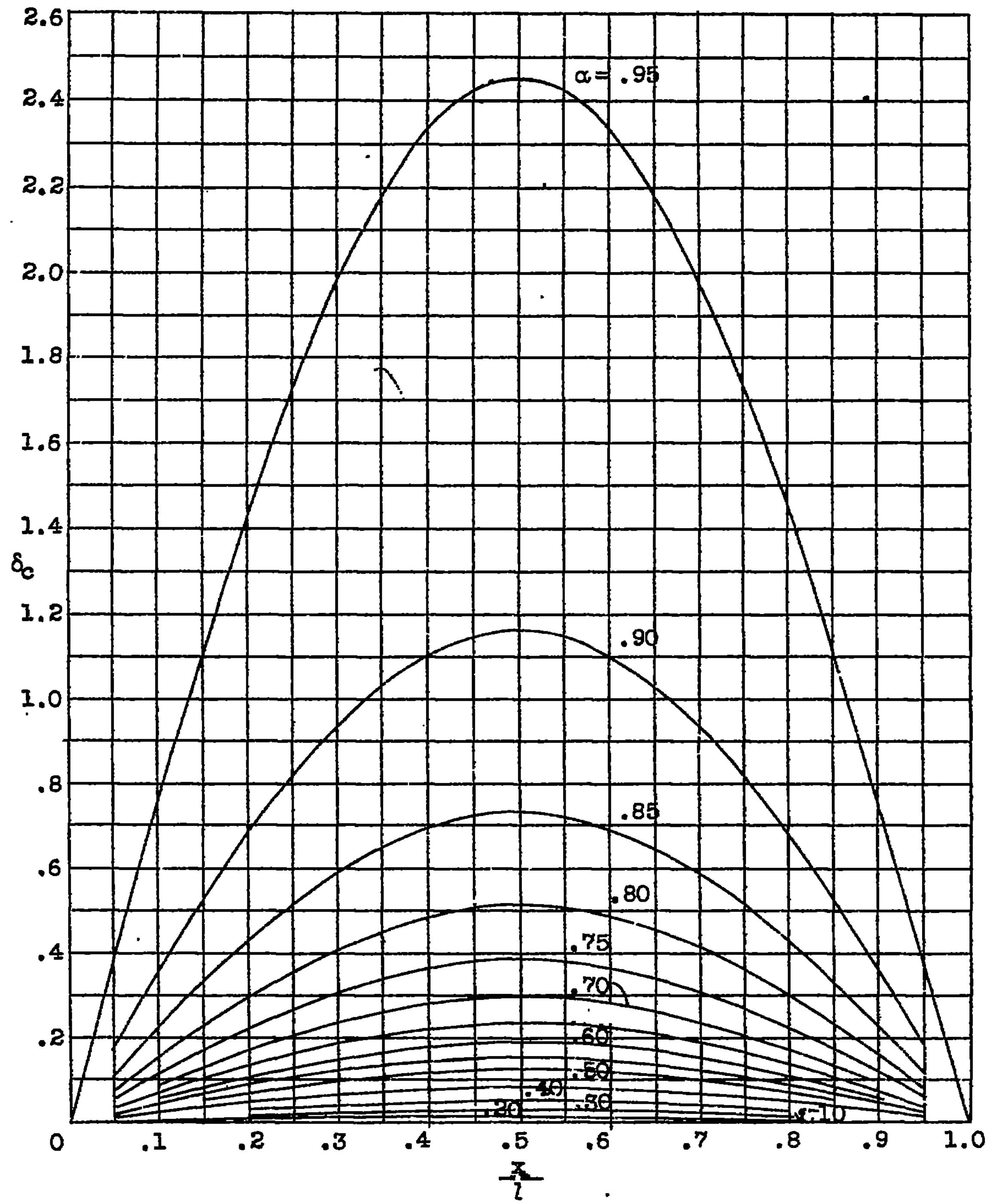


Figure 9.- Plot of δ_c against $\frac{x}{l}$.

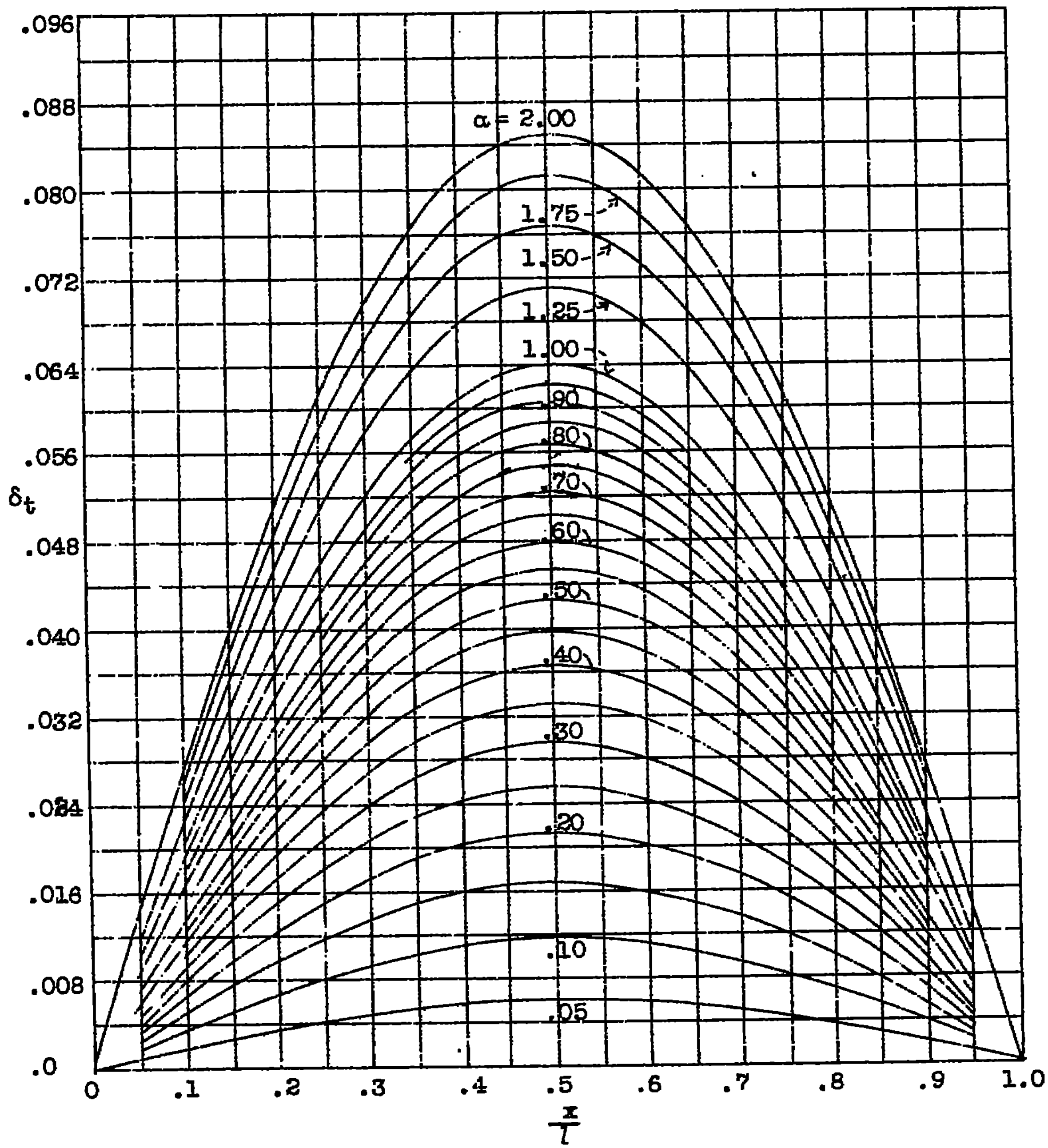


Figure 10.- Plot of δ_t against $\frac{x}{l}$.

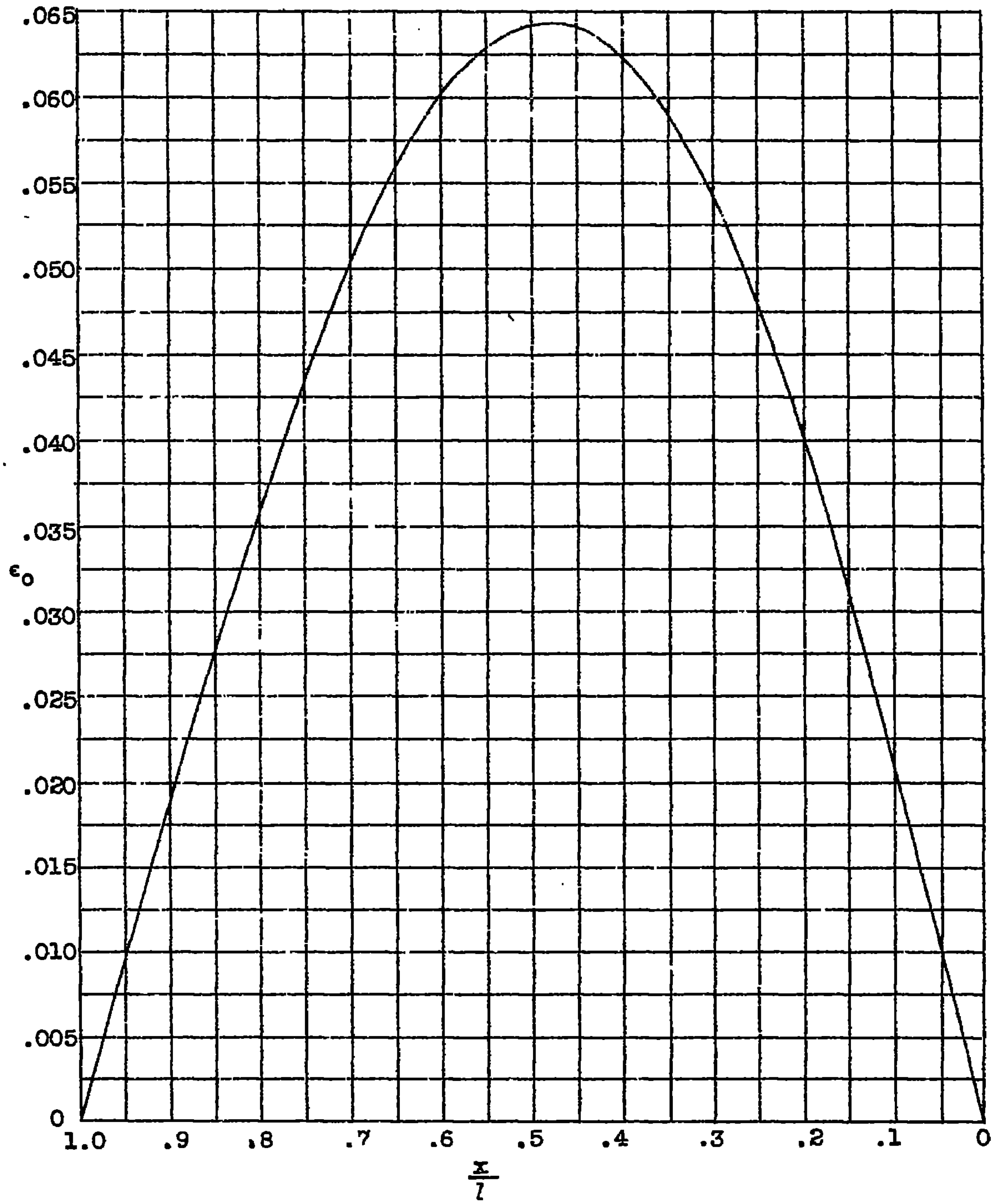


Figure 11.- Plot of ϵ_0 against $\frac{x}{l}$

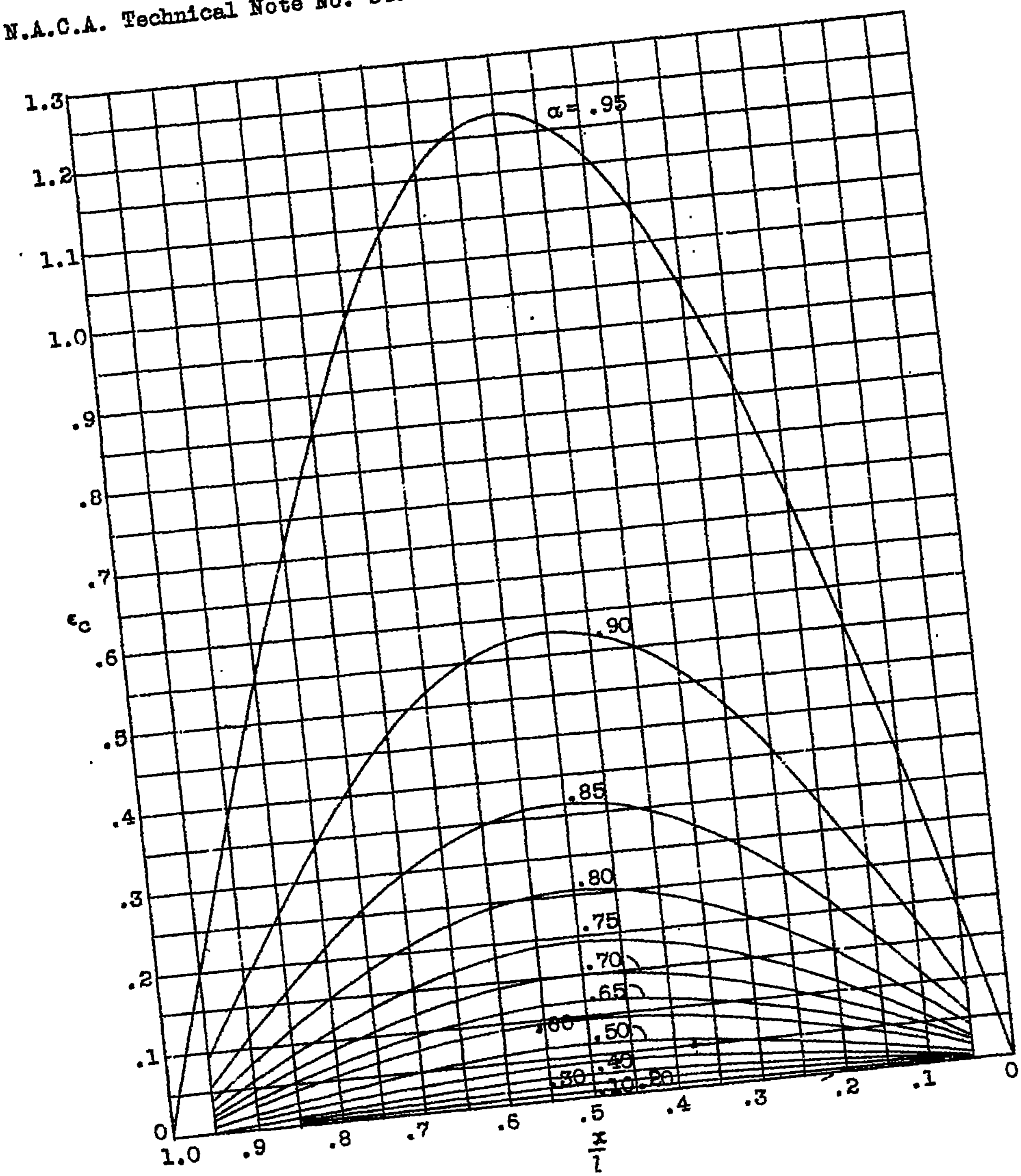


Figure 12.- Plot of ϵ_c against $\frac{x}{l}$.

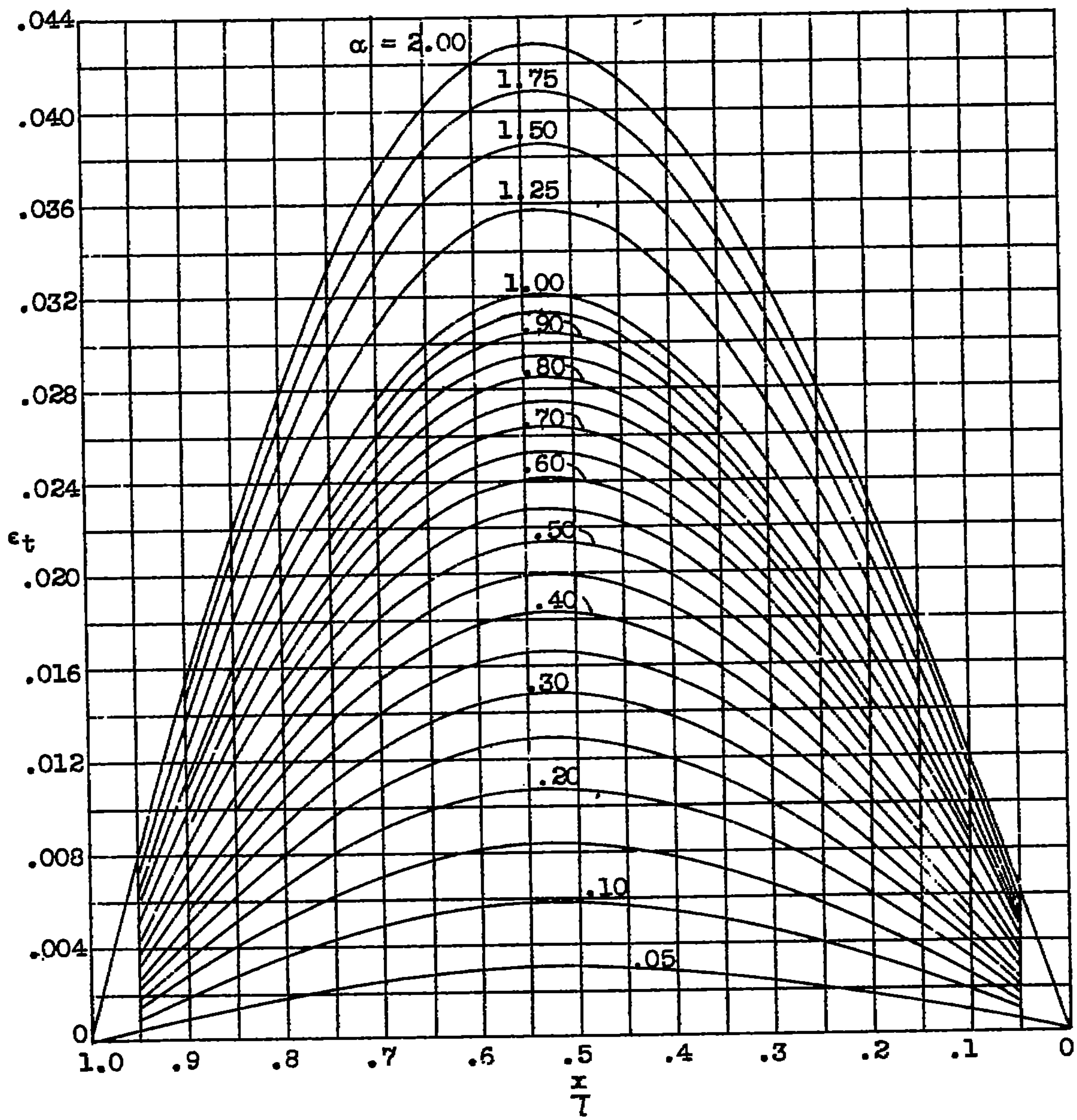


Figure 13.- Plot of ϵ_t against $\frac{x}{l}$.

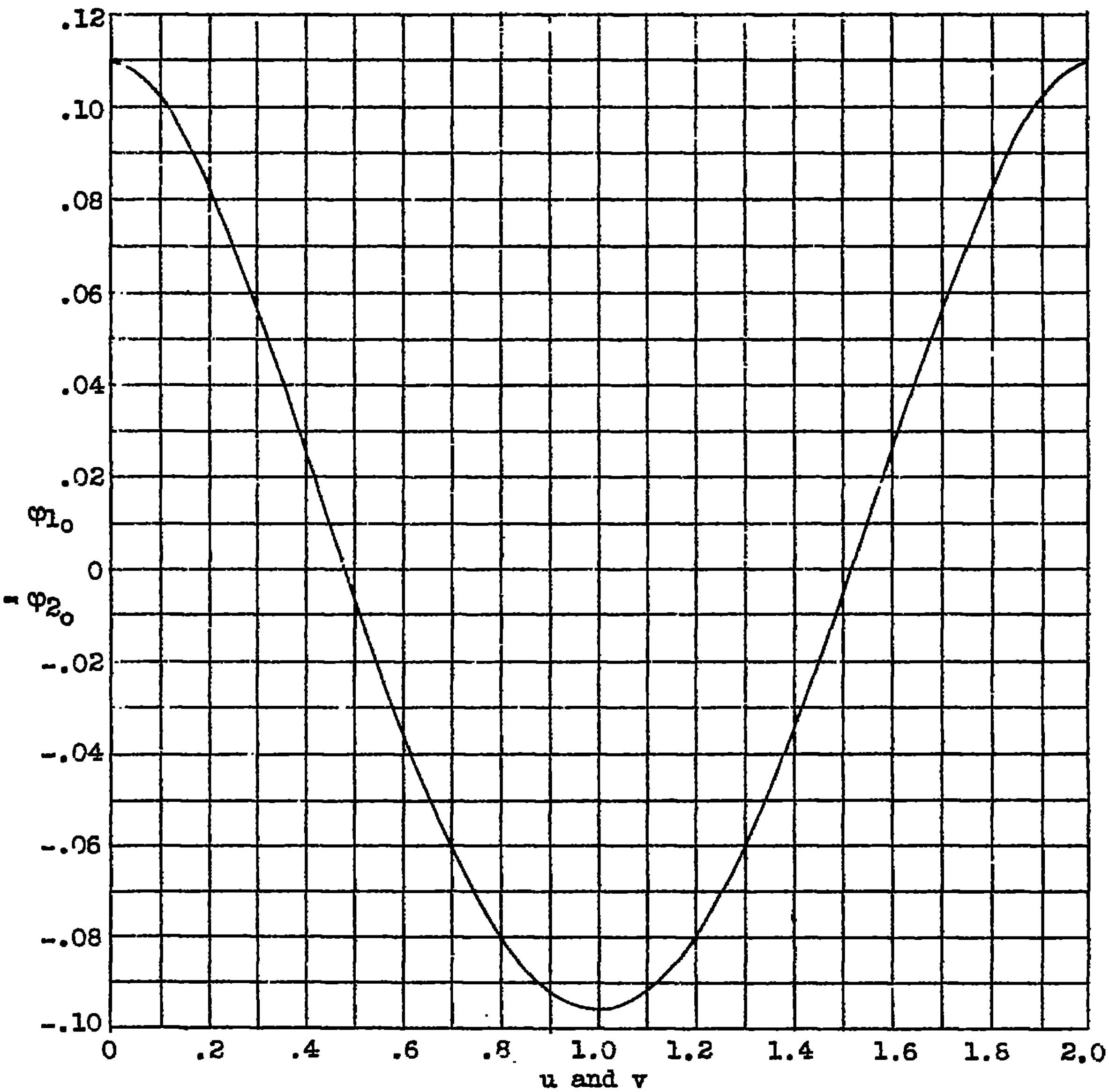


Figure 14.- Plot of φ_{1_o} and $-\varphi_{2_o}$ against u and v.

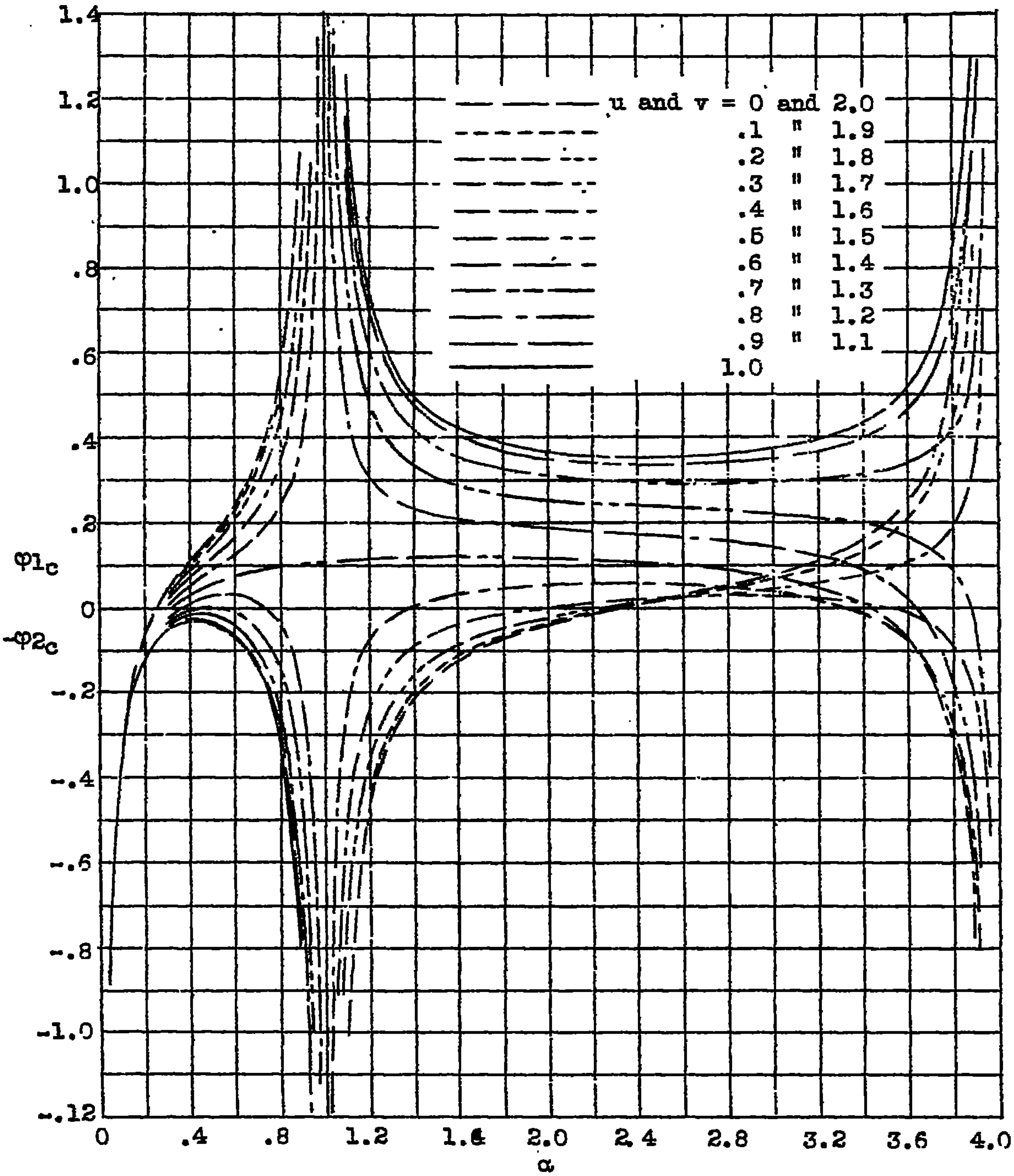


Figure 15.- Plot of φ_{1_c} and $-\varphi_{2_c}$ against α.

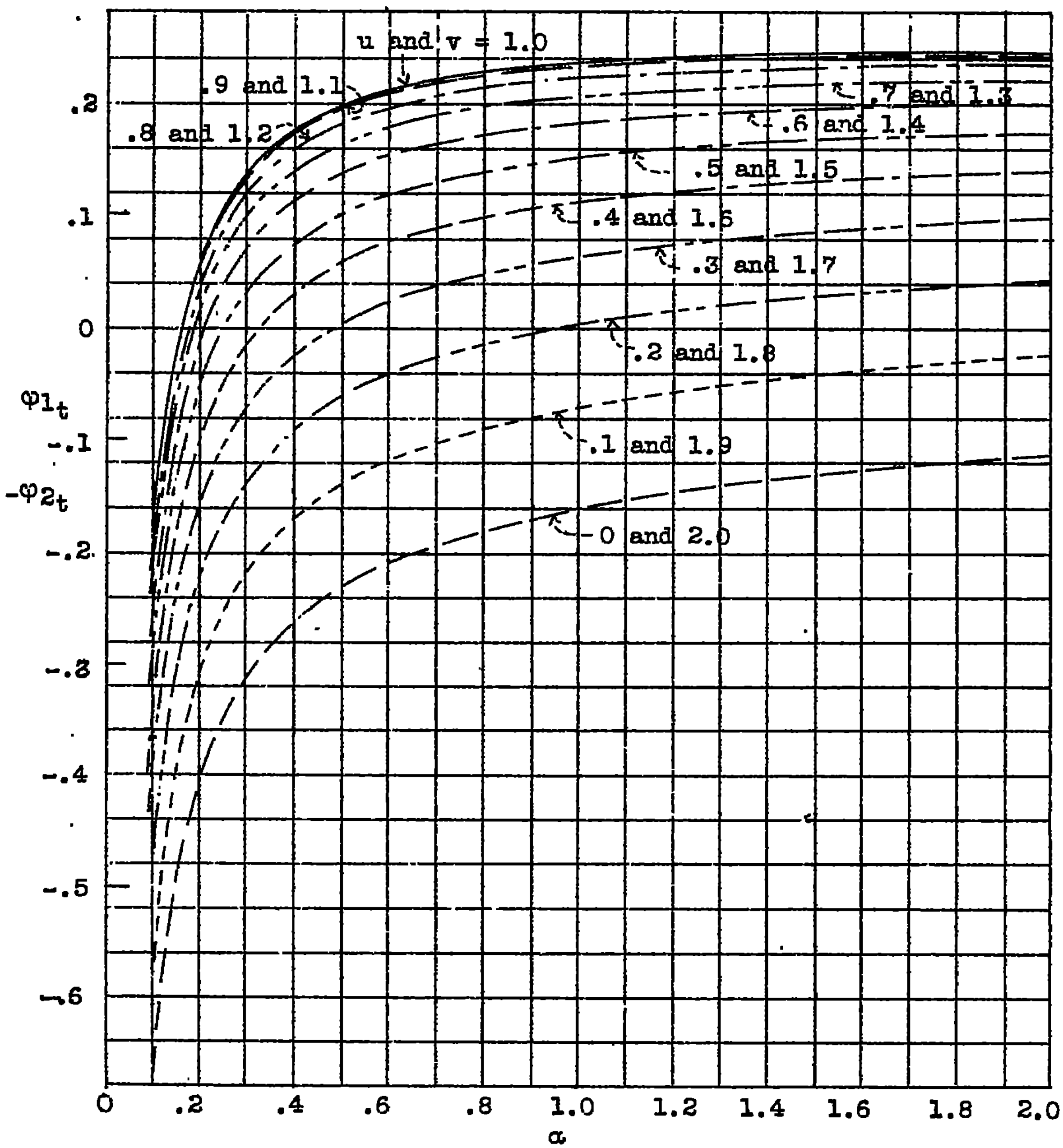


Figure 16.- Plot of φ_{1_t} and $-\varphi_{2_t}$ against α.

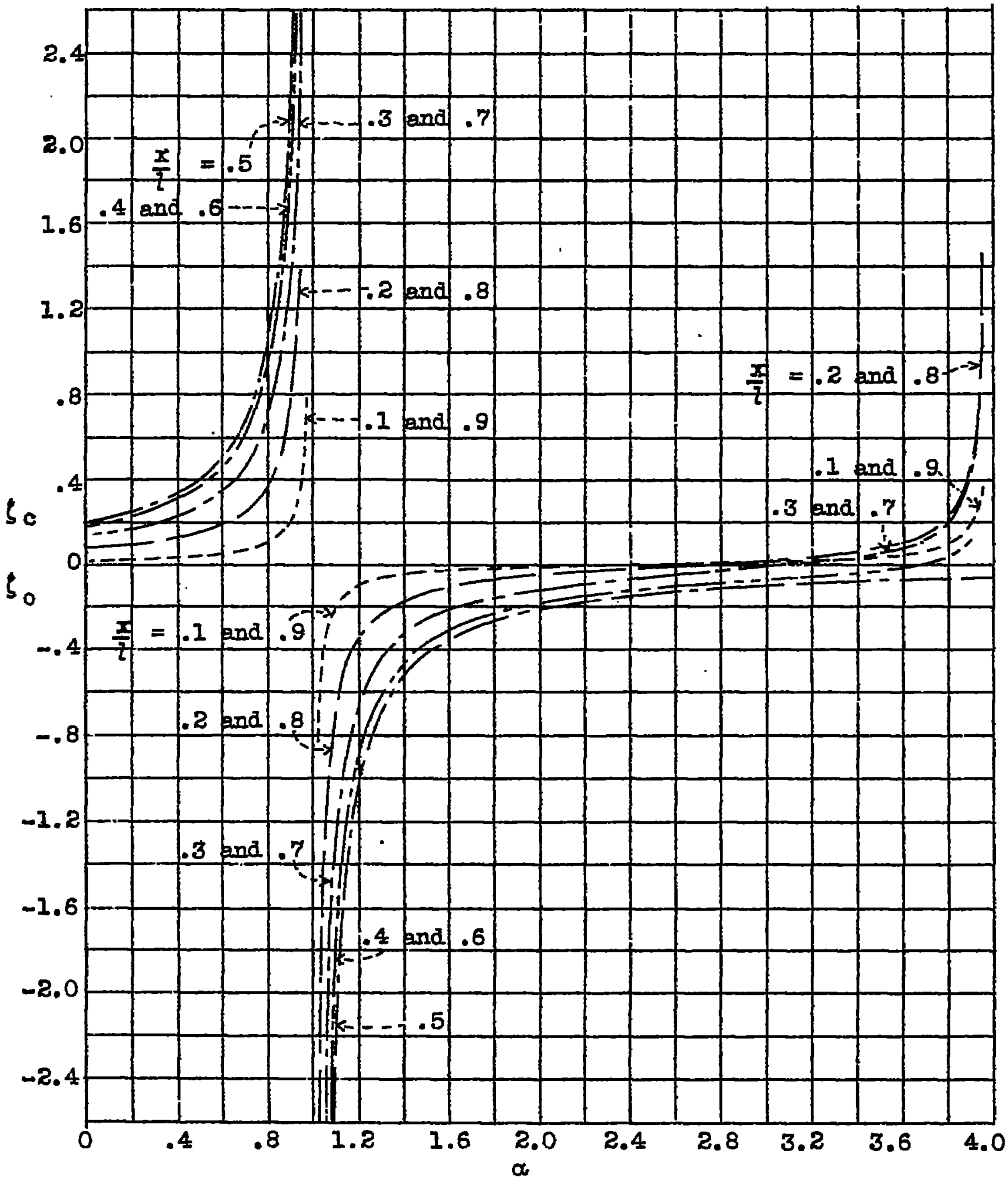


Figure 17.- Plot of ζ_c and ζ_o against α.

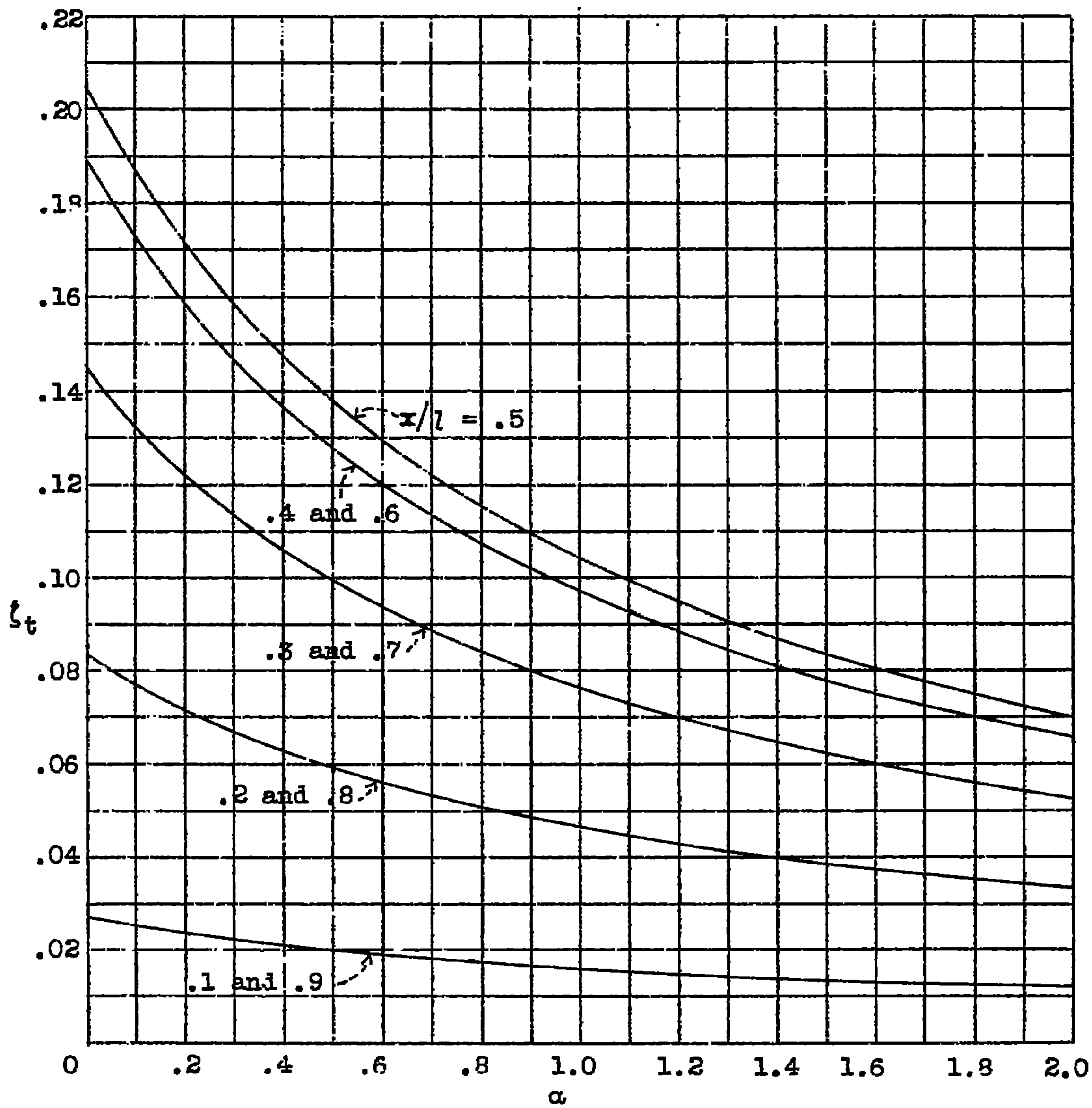


Figure 18.- Plot of ζ_t against α.

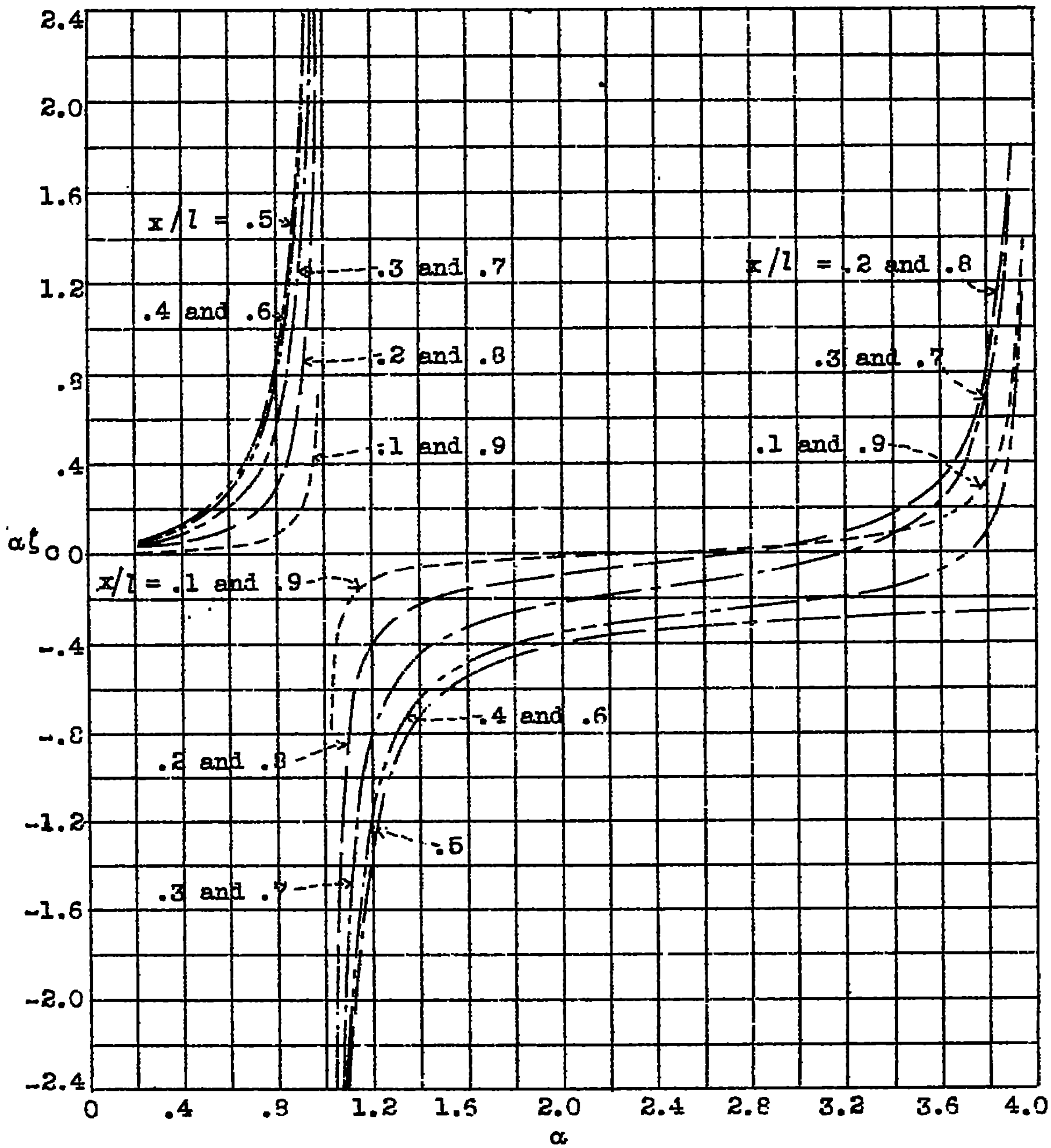


Figure 19.- Plot of $\alpha\zeta_0$ against α.

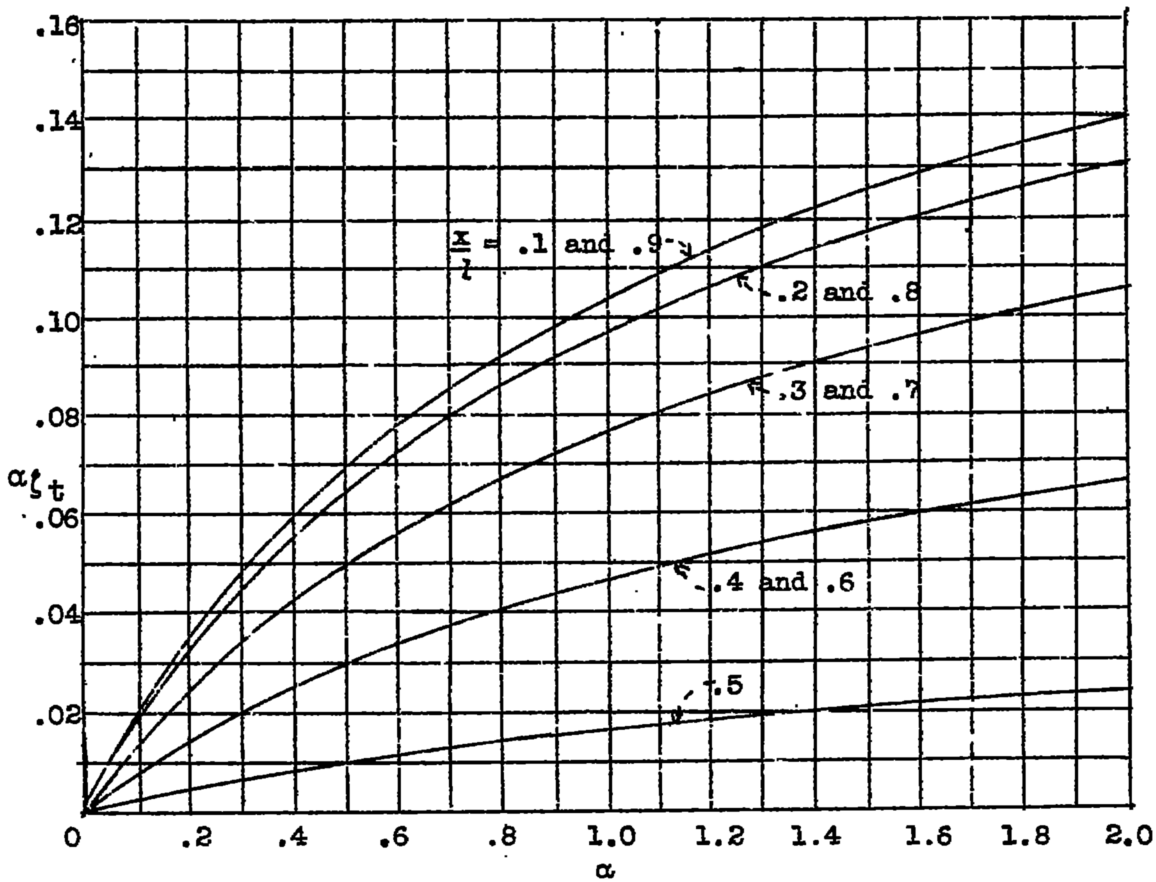


Figure 20.- Plot of α_{ζ_t} against α.

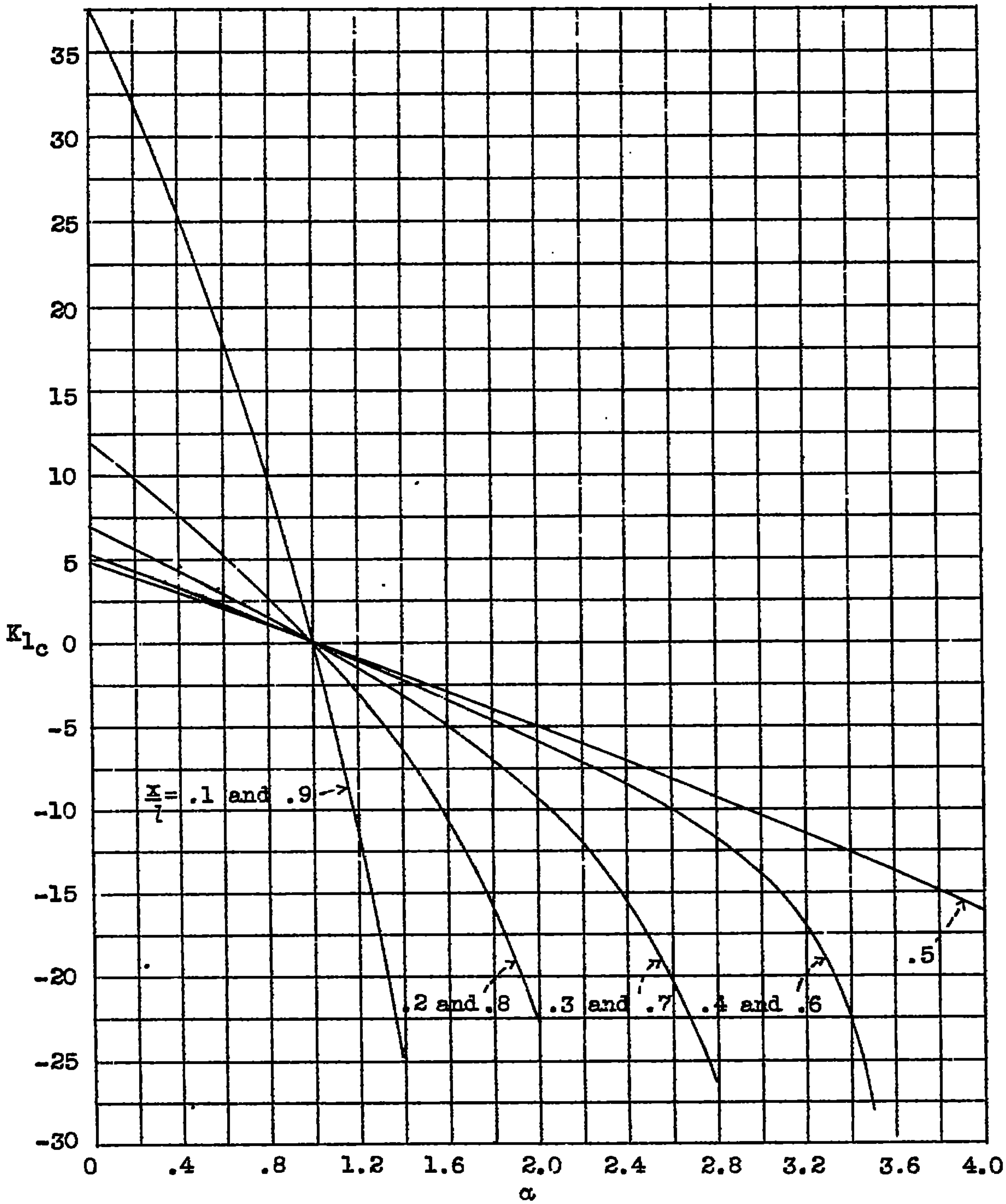


Figure 21.- Plot of K_{1_c} against α.

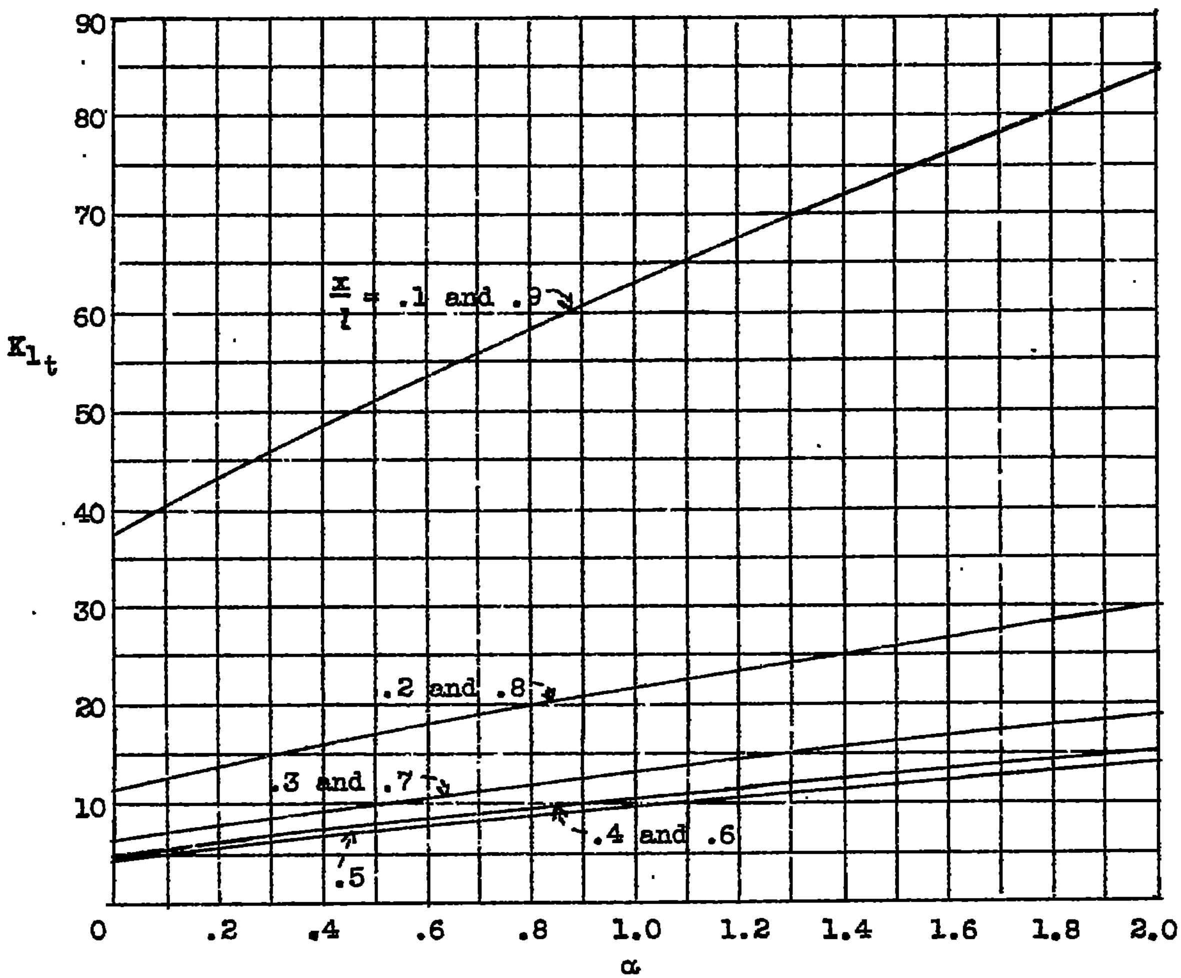


Figure 22.- Plot of K_{1_t} against α.

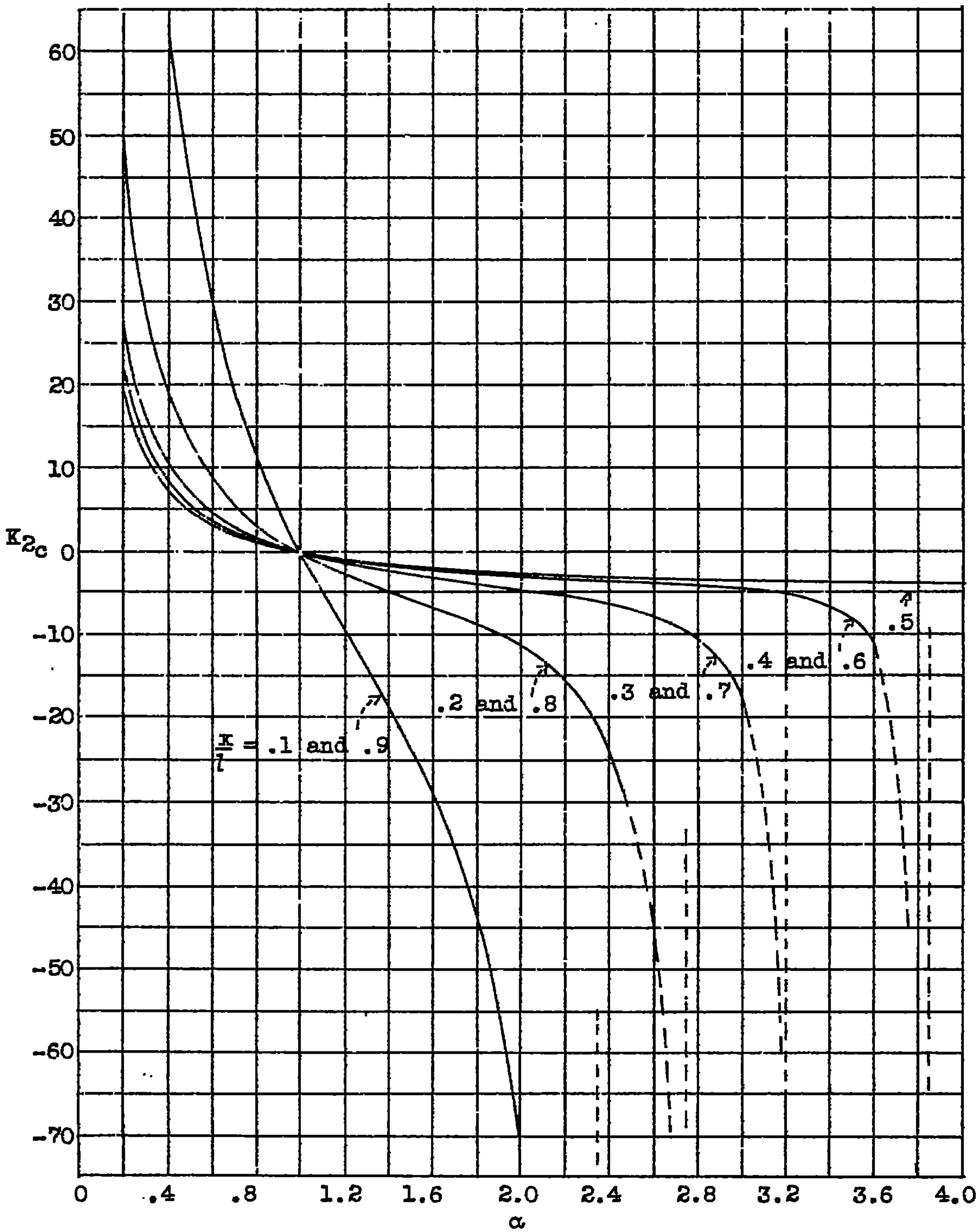


Figure 23.- Plot of K_{2_c} against α.

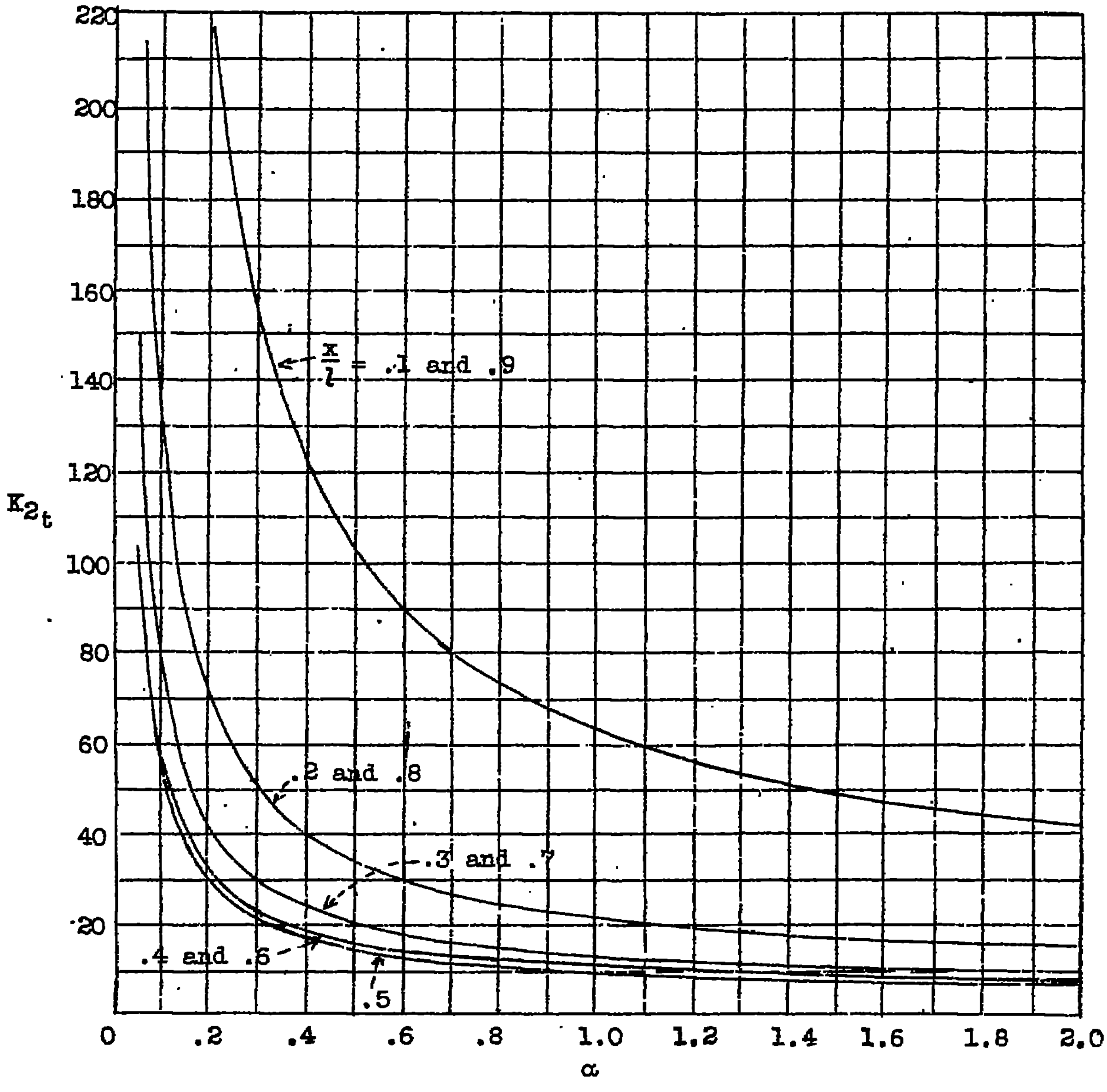


Figure 24.- Plot of K_{2_t} against α.