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SUMMARY

The problem of determining values of structural damping for use in flutter calculations is discussed. The concept of equivalent viscous damping is reviewed and its relation to the structural damping coefficient g introduced in N.A.C.A. Technical Report No. 685 is shown. The theory of normal modes is reviewed and a number of methods are described for separating the motions associated with different modes. Equations are developed for use in evaluating the damping parameters from experimental data.

Experimental results of measurements of damping in several flutter models are presented.

INTRODUCTION

One important step in the study of the flutter properties of an airplane structure is the determination of the structural damping. In an investigation of flutter carried out in the N.A.C.A. 8-foot high-speed tunnel, it was desirable to determine the damping in the models tested. The present report is a description of the methods used, together with a review and a critical discussion of the principles and the derivations pertaining to the measurement of damping parameters.

The presence of damping in a structure can be inferred from and its amount can be measured by a number of different effects. The principal effects that depend upon damping are:

(1) The rate of decay of free vibrations.

(2) The amplitudes produced by given applied forces at a resonant frequency.

(3) The heat produced by vibration.

(4) The elastic hysteresis loop.

A complete physical theory would have to account for all the observed results in all the preceding effects. For the purpose of flutter calculations, however, a complete physical theory is unnecessary. For example, the exact shape of the hysteresis loop may be unaccounted for. It is sufficient to express the damping in terms of certain effective parameters that can be conveniently incorporated into the flutter analysis. The present report reviews some of the results of other investigators on the physical laws of damping but makes no attempt to find the true law of damping in the flutter models.

The results of damping experiments are often expressed in terms of the loss of energy per cycle during vibration. This energy can be considered as consisting of two parts: the part that is independent of frequency and the part that depends on frequency. The independent part has been termed "statical hysteresis"; the dependent part, "hereditary hysteresis." The results of numerous investigations indicate that, for most solids, the statical hysteresis accounts for practically all of the internal damping. For example, Kimball and Lovell (reference 1) found that, in various metals, rubber, glass, celluloid, and maple wood, the loss of energy per cycle was independent of frequency and could be represented with sufficient accuracy for most purposes by a constant times the square of the stress amplitude. Keulegan (reference 2) found that the loss of energy per cycle in Armco iron was the same whether determined from a static-hysteresis test or from the rate of decay of vibrations. Other investigators (references 3, 4, and 5) have likewise found the loss per cycle to be independent of the frequency.

Although the damping of materials is conveniently expressed in terms of the loss of energy per cycle, it is difficult to write an analytical equation of motion that will represent the observed damping properties. Nearly all of the published analytical treatments express the damping as a force proportional to the velocity. The constant of proportionality is thus a measure of the equivalent viscous damping. This constant may then be regraded as a function of amplitude and frequency. The use of the analysis of viscous damping for cases of other types of damping is based upon the approximation of assuming that the hysteresis loop can be replaced by an ellipse having the same amplitude and the same area as the actual loop. The damping coefficient determining this equivalent ellipse is then used to express the results of experiments. Von Schlippe (reference 6) has applied this method to the analysis of internal damping.

Another analytical method of describing internal damping has been used in reference 7. If the displacement is represented by a complex variable, a damping force proportional to amplitude but independent of frequency can be represented by a complex stiffness constant. If this complex stiffness is written in the form $k(1 + ig)$, then g is a nondimensional damping coefficient. In the following analysis, the relation of the coefficient g to the viscous damping coefficient is shown and the principal formulas for use in evaluating these coefficients from actual data are derived.

The basis of the analysis of vibrations in continuous structures is the theory of normal modes. A normal mode means a type of vibration in which each particle of a structure vibrates in simple harmonic motion with the same frequency and passes through its equilibrium position at the same time. The important property of normal modes that makes them useful in vibration is the fact that, when they exist, any possible type of vibration of a system can be represented by the superposition of vibrations in each of the normal modes and each normal mode can be treated independently as a system of one degree of freedom. It has been shown by various writers (see, for example, reference 8, pp. 107-108) that normal modes will certainly exist if there is no damping and if the potential and the kinetic energies are quadratic functions of the coordinates and the velocities of the system. In the theory of vibrations the amplitudes are usually assumed to be small enough that, in the expressions for the energies, all but the quadratic terms may be neglected. Even with damping, normal modes will still exist under certain conditions. It can also be shown that, when the damping is small, the theory of normal modes always gives a good approximation to the actual vibrations.

When an external periodic force is applied to a structure, all the normal modes are excited to a greater or a lesser extent. But when the amount of damping is small and the frequency of the applied force is near to one of the resonant frequencies of the structure, one of the normal modes becomes predominant in comparison with all of the others. Under this condition, the analysis for one degree of freedom will provide a good approximation for the response curve in that particular range of frequencies. The deviations of the actual response from the values appropriate to one degree of freedom may be called the normal-mode interference effect. Whenever the shapes of the deflection curve for the various modes are known, this interference effect can be evaluated. A number of methods are described for experimentally evaluating this effect. Simple theoretical expressions for the exact response curves have also been found for the cases of a uniform cantilever beam in bending and in torsion.

It is to be noted that measurements of the type considered in this report yield the total damping, including internal damping, air damping, losses in the supports, and whatever types of damping are present.

SYMBOLS

m, mass.

$f(x)$, restoring force.

k, spring constant of a vibrating system.

$g(x)$, deviation of restoring force from Hooke's law.

F, F_0, applied force; instantaneous and maximum value.

t, time.

x, displacement in system of one degree of freedom, or position coordinate in a continuous structure.

ΔW, loss of energy or work done per cycle.

A_n, B_n, C_n, D_n, constants.

c, damping force per unit velocity.

ω, angular frequency.

ω_0, natural angular frequency with no damping, $\sqrt{k/m}$.

ω_1, natural angular frequency with damping, $\sqrt{\omega_0^2 - \lambda^2}$.

ω_m, angular frequency of maximum response.

λ, $c/2m$.

T, period of free vibrations, $2\pi/\omega_1$.

δ, δ_n, nondimensional damping parameter, $c/m\omega_0$.

c_{cr}, value of c for critical damping, $2m\omega_0$.

a_1, $\frac{1}{\pi n} \log \frac{x_0}{x_n}$; πa_1 is the logarithmic decrement.

M, constant.

g, damping coefficient used in reference 7.

K, spring constant of a vibrator.

R, radius of crank arm.

x_0', amplitude corrected for effect of coupling with vibrator.

u(x), displacement function for a continuous structure.

q_n, generalized coordinate.

$X_n(x)$, normal function.

T, kinetic energy.

V, potential energy.

F, dissipation function.

a_n, b_n, c_n, coefficients of mass, damping, and stiffness.

Q_n, generalized force.

Δx_i, interval of distance along a beam.

D_m, denominator of mth term in equation (36).

ρ, mass density.

I_p, polar moment of inertia of section.

J, torsion modulus of section.

G, shear modulus of material.

$\theta(x)$, angular displacement.

E, Young's modulus.

I, moment of inertia of section.

A, area of section.

φ, $(2n - 1)\sqrt{\omega/\omega_n}$ for large values of n.

ANALYSIS

Equivalent Viscous Damping

The analysis of viscous damping is often applied to systems having a different physical law of damping. Fortunately, this convenient but inexact analysis can give useful information because certain approximations are justified when the amount of damping is small. The basis of this method follows. In viscous damping, the hysteresis loop is an ellipse. Corresponding to any other type of damping, an equivalent viscous damping coefficient can be defined such that the hysteresis ellipse will have the same amplitude and the same area as the actual loop. The parameter characterizing this equivalent ellipse can then be used as a measure of damping.

Consider a typical elastic hysteresis loop (fig. 1). This curve is seldom directly measured, but a typical shape can be inferred for the purpose of this discussion. Now the area of this loop is a measure of the energy dissipated per cycle, that is, a measure of the damping. If the amount of damping is small, the loop must be narrow. From Hooke's law, the mean slope should be approximately constant. The equation of motion can be written

$$m\ddot{x} + f(x) = F \sin \omega t \tag{1}$$

$$m\ddot{x} + g(x) + kx = F_o \sin \omega t$$

$$f(x) = kx + g(x)$$

where kx is the elastic force corresponding to Hooke's law and $g(x)$ is an undetermined function to take account of the damping.

The energy dissipated per cycle is

$$\Delta W = \int_0^{2\pi/\omega} [kx + g(x)] \frac{dx}{dt} dt \qquad (2)$$

and, whatever the actual law of damping may be, the amplitude of steady forced vibrations is determined by the condition that the work done per cycle by the external force is equal to the dissipation of energy by the damping.

$$\int_0^{2\pi/\omega} F \frac{dx}{dt} dt = \int_0^{2\pi/\omega} f(x) \frac{dx}{dt} dt = \Delta W \qquad (3)$$

Now the deflection will be very nearly a sine function of the time but, for generality, a Fourier series will be assumed.

Let

$$x = \sum_{n=1}^{\infty} A_n \sin n\omega t + B_n \cos n\omega t$$

The phase can be adjusted arbitrarily to make $B_1 = 0$. The velocity is

$$\frac{dx}{dt} = \sum_{n=1}^{\infty} A_n n\omega \cos n\omega t - B_n n\omega \sin n\omega t$$

All of the coefficients except A_1 will be small when the amount of damping is small. Similarly, for a given amplitude and a given frequency, the function $g(x)$ can be expressed as a Fourier series.

$$g(x) = \sum_{n=1}^{\infty} C_n \sin n\omega t + D_n \cos n\omega t$$

The elastic force kx does not contribute to the damping and may be omitted from the expression for ΔW.

Then

$$\Delta W = \int_0^{2\pi/\omega} g(x) \frac{dx}{dt} dt$$

$$= \int_0^{2\pi/\omega} \sum_{n=1}^{\infty} (C_n \sin n\omega t + D_n \cos n\omega t)$$

$$\times \sum_{n=1}^{\infty} (n\omega A_n \cos n\omega t - n\omega B_n \sin n\omega t) dt$$

$$= \sum_{n=1}^{\infty} n\pi (D_n A_n - C_n B_n)$$

$$= \pi (D_1 A_1 + 2D_2 A_2 - 2C_2 B_2 + \ldots) \qquad (4)$$

All the cross products of sines and cosines and the products involving two different values of n vanish in the integration. The only remaining terms are products of each Fourier component of $g(x)$ multiplied by the corresponding Fourier component of the velocity. Thus, if the second harmonic and the higher terms in the Fourier series for x and g are small quantities of the first order, their contribution to the value of ΔW will involve only small quantities of the second order. For the definition of the equivalent viscous damping, these higher order terms are neglected in comparison with the term in A_1.

Then

$$\Delta W = \pi D_1 A_1$$

$$D_1 = \frac{\Delta W}{\pi A_1} \qquad (5)$$

Under this assumption, the loss of energy per cycle depends only upon the Fourier component of the restoring force in phase with the velocity. But it is just this component that corresponds to the term for viscous damping. The equivalent restoring force is

$$f(x) = kA_1 \sin \omega t + D_1 \cos \omega t$$

$$= kA_1 \sin \omega t + \frac{\Delta W}{\pi A_1} \cos \omega t \tag{6}$$

This function gives a hysteresis loop in the form of an ellipse. The coefficient of viscous damping, c, which gives the same energy loss per cycle is given by

$$c \frac{dx}{dt} = c\omega A_1 \cos \omega t$$

$$= \frac{\Delta W}{\pi A_1} \cos \omega t$$

Hence

$$c = \frac{\Delta W}{\pi \omega A_1{}^2} \tag{7}$$

Derivations of Formulas for One Degree of Freedom

In the following analysis, it has been recognized that the quantity $\omega_o = \sqrt{k/m}$, the natural frequency without damping, is not an observable but a conceptual quantity. Actual measurements can yield only the natural frequency with damping and the frequency of maximum response. Hence, the equations for determining damping parameters should contain only the observable frequencies and not the frequency ω_o.

Free vibration.- The equation of motion of a vibrating system with viscous damping will be taken in the form

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{8}$$

The solution is

$$x = x_o e^{-\lambda t} \sin \omega_1 t$$

where

$$\lambda = \frac{c}{2m}$$

$$\omega_1 = \sqrt{\omega_o^2 - \lambda^2}$$

The damping will be expressed in terms of a nondimensional parameter defined by

$$\delta = \frac{c}{m\omega_o} = \frac{2\lambda}{\omega_o} \tag{9}$$

This parameter is simply related to the fraction of critical damping. The condition for critical damping is $\omega_1 = 0$; hence

$$\omega_o = \lambda = \frac{c_{cr}}{2m}$$

$$\frac{c}{c_{cr}} = \frac{c}{2m\omega_o} = \frac{\delta}{2}$$

The amplitude after n complete cycles of free vibration is

$$x_n = x_o e^{-n\lambda T} = x_o e^{-\frac{nc\pi}{m\omega_1}} \tag{10}$$

$$\frac{c}{m\omega_1} = \frac{1}{\pi n} \log_e \frac{x_o}{x_n}$$

and since

$$\omega_1 = \omega_o \sqrt{1 - \frac{\lambda^2}{\omega_o^2}} = \omega_o \sqrt{1 - \frac{\delta^2}{4}} \tag{11}$$

$$\frac{\delta}{\sqrt{1 - \frac{\delta^2}{4}}} = \frac{1}{\pi n} \log_e \frac{x_o}{x_n} \tag{12}$$

Define

$$a_1 = \frac{1}{\pi n} \log_e \frac{x_o}{x_n}$$

Then

$$\delta = \frac{a_1}{\sqrt{1 + \frac{a_1^2}{4}}}$$

From this equation it is apparent that, when δ is small,

$$\delta \approx a_1$$

The quantity usually called the logarithmic decrement is equal to πa_1 in the present notation.

Forced vibration produced by a periodic force of constant amplitude.- In terms of the notation of complex variables, the equation of motion of forced vibration produced by a periodic force of constant amplitude is

$$m\ddot{x} + c\dot{x} + kx = F_o e^{i\omega t} \tag{13}$$

The solution for a steady state is obtained by making the substitution

$$x = x_o e^{i\omega t}$$

Then

$$x_o = \frac{\frac{F_o}{m}}{(\omega_o^2 - \omega^2) + i\frac{c\omega}{m}} = \frac{\frac{F_o}{m}}{(\omega_o^2 - \omega^2) + i\omega\omega_o\delta} \tag{14}$$

or, for the absolute magnitude of x_o,

$$x_o = \frac{\frac{F_o}{m}}{\omega_o^2\sqrt{\left(1 - \frac{\omega^2}{\omega_o^2}\right)^2 + \delta^2\frac{\omega^2}{\omega_o^2}}} \tag{15}$$

The frequency ω_m of the impressed force for which x_o is a maximum is given by

$$\frac{d}{d\omega^2}\left[\left(1 - \frac{\omega^2}{\omega_o^2}\right)^2 + \delta^2\frac{\omega^2}{\omega_o^2}\right] = -\frac{2}{\omega_o^2}\left(1 - \frac{\omega^2}{\omega_o^2}\right) + \frac{\delta^2}{\omega_o^2} = 0$$

$$1 - \frac{\omega_m^2}{\omega_o^2} = \frac{\delta^2}{2}$$

$$\omega_m^2 = \omega_o^2\left(1 - \frac{\delta^2}{2}\right) \tag{16}$$

The amplitude at resonance is

$$x_m = \frac{\frac{F_o}{m}}{\omega_o^2 \sqrt{\left(1 - \frac{\omega_m^2}{\omega_o^2}\right)^2 + \frac{\omega_m^2}{\omega_o^2}\,\delta^2}} \qquad (17)$$

$$= \frac{\frac{F_o}{k}}{\delta \sqrt{1 - \frac{\delta^2}{4}}}$$

By use of equations (15) and (16), there is obtained

$$\delta^2 = 2\left(1 - \frac{1}{\sqrt{a_2^2 + 1}}\right) \qquad (18)$$

where

$$a_2^2 = \frac{\left(1 - \frac{\omega^2}{\omega_m^2}\right)^2}{\left(\frac{x_m}{x_o}\right)^2 - 1}$$

<u>Forced vibration produced by a force proportional to ω^2</u>.- The equation of motion of forced vibration produced by a force proportional to ω^2 is

$$m\ddot{x} + c\dot{x} + kx = M\omega^2 e^{i\omega t} \qquad (19)$$

where M is a constant. The amplitude of steady vibration is given by

$$x_o = \frac{\frac{M}{m}\,\omega^2}{\sqrt{(\omega_o^2 - \omega^2)^2 + \frac{c^2\omega^2}{m^2}}} = \frac{\frac{M}{m}\,\omega^2}{\sqrt{(\omega_o^2 - \omega^2)^2 + \delta^2\omega^2\omega_o^2}} \qquad (20)$$

The frequency ω_m of the impressed force for which x_o is a maximum is given by

$$\frac{dx_o}{d\omega^2} = \frac{\frac{M}{m}}{\sqrt{(\omega_o^2 - \omega^2)^2 + \delta^2\omega^2\omega_o^2}} -$$

$$- \frac{\frac{1}{2}\frac{M}{m}\omega^2\left[-2(\omega_o^2 - \omega^2) + \delta^2\omega_o^2\right]}{\left[(\omega_o^2 - \omega^2)^2 + \delta^2\omega_o^2\omega^2\right]^{3/2}} = 0$$

from which is obtained

$$\omega_m^2 = \frac{\omega_o^2}{1 - \frac{\delta^2}{2}} \tag{21}$$

The amplitude at resonance is

$$x_m = \frac{\frac{M}{m}\omega_m^2}{\sqrt{(\omega_o^2 - \omega_m^2)^2 + \delta^2\omega_o^2\omega_m^2}} = \frac{\frac{M}{m}}{\delta\sqrt{1 - \frac{\delta^2}{4}}} \tag{22}$$

By use of equations (20) and (22), there is obtained

$$\delta^2 = 2\left(1 - \frac{1}{\sqrt{a_3^2 + 1}}\right) \tag{23}$$

where

$$a_3^2 = \frac{\left(\frac{\omega_m^2}{\omega^2} - 1\right)^2}{\left(\frac{x_m^2}{x_o^2} - 1\right)}$$

Properties of Damping Coefficient, g

Theodorsen and Garrick (reference 7) have used a complex stiffness function to describe the phenomena of internal damping. The properties of this function are herein developed.

The equation of motion of a vibrating system can be written in the form

$$m\ddot{x} + xk\,(1 + ig) = F_o e^{i\omega t} \tag{24}$$

The use of the imaginary term ig implies that the displacement is a sine function of time and that there is a component of force proportional to the amplitude and independent of frequency but in phase with the velocity. The solution of equation (24) is

$$x = \frac{\frac{F}{k}\, e^{i\omega t}}{1 - \frac{\omega^2}{\omega_o^2} + ig} \tag{25}$$

This equation agrees with Schlippe's analysis (reference 6) and is similar to the result just given for viscous damping except that the damping term is ig instead of $i(\omega/\omega_o)\delta$. Thus, the numerical value of g is nearly equal to the value of δ obtained by measurements close to a resonant peak.

The frequency of maximum response is obtained as before by differentiating x with respect to ω^2. The result for this case is

$$\omega_m = \omega_o$$

Hence

$$x_o = \frac{\frac{F}{k}}{1 - \frac{\omega^2}{\omega_m^2} + ig}$$

$$x_m = \frac{\frac{F}{k}}{ig}$$

$$g^2 = \frac{\left(1 - \frac{\omega^2}{\omega_m^2}\right)^2}{\left(\frac{x_m}{x_o}\right)^2 - 1} \tag{26}$$

This equation shows that g is given by the same expression as the quantity a_2 of equation (18).

Further Details of Practical Importance

In the elementary analysis, the applied force is usually assumed to be independent of the amplitudes produced in the vibrating system. In actual apparatus, however, the motion of the point where the external force is applied frequently affects the value of the force, so that the actual force transmitted to the structure changes with the amplitude of vibration. When the force is produced by a spring of stiffness K fastened to a crank of radius R (see fig. 2(a)), the force on the structure is $-K(x - R \sin \omega t)$ and the equation of motion is:

$$m\ddot{x} + kx = -K(x - R \sin \omega t) \tag{27}$$

Then

$$\frac{1}{x} = \frac{-\omega^2 m + k}{KR \sin \omega t} + \frac{1}{R \sin \omega t}$$

$$\frac{1}{x_o} = \frac{1}{x_o'} + \frac{1}{R}$$

$$\frac{1}{x_o'} = \frac{1}{x_o} - \frac{1}{R} \tag{28}$$

Similarly, for a rotating mass fastened to the structure (see fig. 2(b)), the equation of motion is:

$$m\ddot{x} + kx = -M \frac{d^2}{dt^2} (x + R \sin \omega t) \tag{29}$$

The solution is:

$$\frac{1}{x} = \frac{-m\omega^2 + k}{M\omega^2 R \sin \omega t} - \frac{1}{R \sin \omega t}$$

$$\frac{1}{x_o} = \frac{1}{x_o'} - \frac{1}{R}$$

$$\frac{1}{x_o'} = \frac{1}{x_o} + \frac{1}{R} \tag{30}$$

The corrected amplitude, x_o', corresponds to the amplitudes given by theoretical equations in other sections of this report. Equations (28) and (30) thus provide a method of correcting the measured amplitudes for the effect of a vibrator having a small stroke. The natural frequencies are changed to

$$\sqrt{\frac{k + K}{m}}$$

and

$$\sqrt{\frac{k}{m + M}}$$

for the two types of vibrators shown in figure 2(a) and 2(b), respectively.

Derivation of Formulas for Continuous Structures

General case.- The general theory of vibration has been presented by a number of authors. (See, for example, reference 8, chs. IV and V.) This theory deals with the problem of finding the normal coordinates of a system, that is, the coordinates in terms of which the equations of motion have only one coordinate occurring in each equation. The existence of such coordinates for systems with damping has been discussed by Lord Rayleigh, who shows that these coordinates exist for certain distributions of damping and that, in all cases of small damping, the errors introduced by assuming normal modes are of the second order.

A brief outline of the method of normal coordinates is given. The application of this method to the determination of damping in each mode of a cantilever beam is then discussed.

Suppose that the functions $X_n(x)$, giving the shape of the deflection curve for each mode, are known. Then the displacements corresponding to any motion of the system can be expressed as a summation of the displacements in each normal mode

$$u(x) = \sum_n q_n X_n(x) \tag{31}$$

According to Lord Rayleigh (reference 8, pp. 130-131), in damped systems for which normal modes exist, the kinetic and the potential energy functions, T and V, and the dissipation function, F, can be expressed as sums of squares of generalized coordinates or velocities.

Then

$$2T = a_1 \dot{q}_1^2 + a_2 \dot{q}_2^2 + \ldots + a_n \dot{q}_n^2 + \ldots$$

$$2F = b_1 \dot{q}_1^2 + b_2 \dot{q}_2^2 + \ldots + b_n \dot{q}_n^2 + \ldots$$

$$2V = c_1 q_1^2 + c_2 q_2^2 + \ldots + c_n q_n^2 + \ldots$$

The equations of motion will be obtained by the use of Lagrange's equations.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) + \frac{\partial F}{\partial \dot{q}_n} + \frac{\partial V}{\partial q_n} = Q_n \tag{32}$$

$$a_n \ddot{q}_n + b_n \dot{q}_n + c_n q_n = Q_n \tag{33}$$

where Q_n is the generalized force corresponding to q_n. The form of Q_n is found from the relation

$$Q_n = \frac{\delta W}{\delta q_n}$$

where δW is the work done on the system by the external force during a displacement δq_n.

For example, in the case of a force $F = F_o e^{i\omega t}$ applied at the point $x = x_1$,

$$\delta W = Q_n \ \delta q_n = F \delta u = F_o e^{i\omega t} X_n(x_1) \ \delta q_n$$

Hence

$$Q_n = X_n(x_1) \ F_o e^{i\omega t} \tag{34}$$

The solution of the equation of motion is

$$q_n = \frac{X_n(x_1) \ F_o e^{i\omega t}}{-\omega^2 a_n + i\omega b_n + c_n}$$

$$= \frac{X_n(x_1) \ F_o e^{i\omega t}}{a_n \omega_{on}^2 \left(1 - \frac{\omega^2}{\omega_{on}^2} + i \frac{\omega}{\omega_{on}} \delta_n\right)} \tag{35}$$

where

$$\omega_{on}^2 = \frac{c_n}{a_n}$$

$$\delta_n = \frac{b_n}{a_n\ \omega_{on}}$$

The displacements are then given by

$$u(x) = \sum_n q_n X_n = \sum_n \frac{X_n(x)\ X_n(x_1)\ F_o e^{i\omega t}}{a_n\ \omega_{on}^2 \left(1 - \frac{\omega^2}{\omega_{on}^2} + i\ \frac{\omega}{\omega_{on}}\ \delta_n\right)} \qquad (36)$$

In the summation, it is necessary to sum the real and the imaginary parts separately. If only real quantities are used, the equations become

$$Q_n = X_n(x_1)\ F_o \sin \omega t$$

$$q_n = A_n \sin \omega t + B_n \cos \omega t$$

$$A_n = \frac{X_n(x_1)\ F_o \left(1 - \frac{\omega^2}{\omega_o^2}\right)}{a_n\ \omega_{on}^2 \left[\left(1 - \frac{\omega^2}{\omega_{on}^2}\right)^2 + \left(\frac{\omega}{\omega_{on}}\ \delta_n\right)^2\right]}$$

$$B_n = \frac{-X_n(x_1)\ F_o \left(\frac{\omega}{\omega_{on}}\ \delta_n\right)}{a_n\ \omega_{on}^2 \left[\left(1 - \frac{\omega^2}{\omega_{on}^2}\right)^2 + \left(\frac{\omega}{\omega_{on}}\ \delta_n\right)^2\right]}$$

$$u(x) = \sum_n A_n X_n \sin \omega t + B_n X_n \cos \omega t \qquad (37)$$

Equation (36) is the connecting link between the analysis of vibration in one degree of freedom and in continuous structures. Each term of this equation has the same form as the solution for a system having one degree of freedom. When a force is applied to a structure, all the modes are excited to a greater or a lesser extent and the problem is to find the amounts of damping associated with each mode. There are several ways of solving this problem.

For any case in which the normal functions are known, the damping in different modes can be separated by an integration. If both sides of equation (36) are multiplied by $X_m(x)$ and integrated over the length, then all but one term in the summation will vanish.

$$u(x) = \sum_n q_n X_n$$

$$\int_0^l X_m(x)\, u(x)\, dx = \int_0^l X_m(x) \sum_n q_n X_n(x)\, dx$$

$$= q_m \int_0^l X_m^2(x)\, dx$$

$$q_m = \frac{\int_0^l X_m(x)\, u(x)\, dx}{\int_0^l X_m^2(x)\, dx} = \frac{X_m(x_1)\, F_o e^{i\omega t}}{a_m\, \omega_{om}^2 \left(1 - \frac{\omega^2}{\omega_{om}^2} + i \frac{\omega}{\omega_{om}} \delta_m\right)} \quad (38)$$

The quantity q_m obtained by graphical or numerical integration of the measured amplitudes $u(x)$ can then be used as though it were the amplitude in a system having only one degree of freedom. The process of evaluating this integral in practical cases consists in measuring the amplitude at a number of points and evaluating the sum:

$$q_m = \frac{\sum_i X_m(x_i)\, u(x_i)\, \Delta x_i}{\sum_i X_m^2(x_i)\, \Delta x_i} \quad (39)$$

In the present tests, the procedure was simplified still further. The disturbance from two of the modes can be eliminated by applying the force at a node of one of them and measuring amplitudes at a node of the other. The justification of this method follows immediately from equation (36). For measurements in the range of frequencies near to ω_{om}, this equation can be written:

$$u(x_2) = \frac{X_{m-1}(x_2)\ X_{m-1}(x_1)\ F_0 e^{i\omega t}}{D_{m-1}} + \frac{X_m(x_2)\ X_m(x_1)\ F_0 e^{i\omega t}}{D_m}$$

$$+ \frac{X_{m+1}(x_2)\ X_{m+1}(x_1)\ F_0 e^{i\omega t}}{D_{m+1}} + \text{small terms} \qquad (40)$$

where D_m is written for the denominator in the mth term. If x_1 and x_2 are chosen such that

$$X_{m-1}(x_1) = X_{m+1}(x_2) = 0$$

then the amplitude is given by

$$u(x_2) = 0 + \frac{X_m(x_2)\ X_m(x_1)\ F_0 e^{i\omega t}}{a_m\ \omega_{om}^2 \left(1 - \frac{\omega^2}{\omega_{om}^2} + i\ \frac{\omega}{\omega_{om}}\ \delta_m\right)} + 0 + \text{small terms} \qquad (41)$$

Another method of finding the damping is by evaluating the infinite series given by equation (36) to find the resultant response curve. A comparison of this exact curve with the response curve for one degree of freedom shows how to correct the analysis for one degree of freedom to take account of all the disturbing modes. Expressions for the sum of the infinite series (equation (36)) have been found for the cases of torsion and bending of a cantilever beam with the vibrator at the tip.

Torsion of a uniform beam.- Consider a cantilever beam excited in torsional vibrations by an oscillating torque applied at the free end.

$$\left.\begin{aligned} T &= \int_0^l \tfrac{1}{2}\ \rho\ I_p\ \dot{\theta}^2\ dx = \sum_n \dot{q}_n^2 \int_0^l \tfrac{1}{2}\ \rho\ I_p\ X_n^2(x)\ dx \\ V &= \int_0^l \tfrac{1}{2}\ GJ \left(\frac{\partial\theta}{\partial x}\right)^2 dx = \sum_n q_n^2 \int_0^l \tfrac{1}{2}\ GJ \left(\frac{\partial X_n}{\partial x}\right)^2 dx \end{aligned}\right\} \qquad (42)$$

Take $F = 0$

Hence

$$a_n = \int_0^l \tfrac{1}{2}\, \rho\, I_p\, X_n^2(x)\, dx$$

$$b_n = 0$$

$$c_n = \int_0^l \tfrac{1}{2}\, GJ \left(\frac{\partial X_n}{\partial x}\right)^2 dx$$

Suppose that the applied torque is supplied by a rotating unbalanced mass at the tip of the beam. Then the torque is

$$P = M\omega^2 \sin \omega t$$

The generalized force is

$$Q_n = X_n(l)\, M\omega^2 \sin \omega t \tag{43}$$

Then

$$\theta(x) = \sum_n q_n X_n = \sum_n \frac{X_n(x)\, X_n(l)\, M\omega^2 \sin \omega t}{\omega_{on}^2 \int_0^l \frac{1}{2} \rho\, I_p\, X_n^2(x)\, dx \left(1 - \frac{\omega^2}{\omega_{on}^2}\right)} \tag{44}$$

The normal functions are known for the case of constant I_p.

$$X_n(x) = \sin\, (2n - 1)\, \frac{\pi}{2}\, \frac{x}{l};\ n = 1, 2, 3, \ldots \tag{45}$$

$$\theta(l) = \frac{M \sin \omega t}{\frac{1}{4}\, \rho\, I_p\, l} \sum_n \frac{1}{\left(\frac{\omega_{on}^2}{\omega^2} - 1\right)}$$

Put

$$\frac{\omega_{on}}{\omega_1} = 2n - 1$$

$$\theta(l) = \frac{M \sin \omega t}{\frac{1}{4}\, \rho\, l\, I_p} \sum_n \frac{1}{\left[\frac{(2n - 1)^2\, \omega_1^2}{\omega^2} - 1\right]} \tag{46}$$

The sum of this series can be evaluated by the following method. Using the relation (reference 9)

$$\tan x = -\sum_{-\infty}^{\infty} \frac{1}{x - \left(n + \frac{1}{2}\right)\pi} \qquad (47)$$

there is obtained

$$x \tan x = \sum_{1}^{\infty} \left(\frac{1}{\left(n - \frac{1}{2}\right)\frac{\pi}{x} - 1} - \frac{1}{\left(n - \frac{1}{2}\right)\frac{\pi}{x} + 1} \right)$$

$$= \sum_{1}^{\infty} \frac{2}{\left(n - \frac{1}{2}\right)^2 \frac{\pi^2}{x^2} - 1} \qquad (48)$$

Put

$$\frac{2x}{\pi} = \frac{\omega}{\omega_1}$$

Then

$$\frac{\pi\omega}{4\omega_1} \tan \frac{\pi\omega}{2\omega_1} = \sum_{1}^{\infty} \frac{1}{\frac{(2n - 1)^2 \ \omega_1^2}{\omega^2} - 1}$$

$$\theta_o(l) = \frac{M}{\frac{1}{2} \rho \, l \, I_p} \frac{\pi\omega}{4\omega_1} \tan \frac{\omega\pi}{\omega_1 2} \qquad (49)$$

In figure 3 the value of

$$\frac{\theta_o(l) \frac{1}{2} \rho \, l \, I_p}{M}$$

has been plotted against ω/ω_1 for the first two peaks. In figure 4, the exact response curve is compared with the curve for one degree of freedom.

Bending of a uniform beam.- By a method similar to that used for torsion, the following equations are obtained.

$$\left.\begin{aligned} T &= \int_0^l \tfrac{1}{2}\,\rho\,A\,\dot{u}^2\,dx = \sum_n \dot{q}_n^2 \int_0^l \tfrac{1}{2}\,\rho\,A\,X_n^2\,dx \\ V &= \int_0^l \tfrac{1}{2}\,EI\left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx = \sum_n q^2 \int_0^l \tfrac{1}{2}\,EI\left(\frac{\partial^2 X}{\partial x^2}\right)^2 dx \end{aligned}\right\} \tag{50}$$

$$a = \int_0^l \tfrac{1}{2}\,\rho\,AX_n^2\,dx$$

$$b = 0$$

$$c = \int_0^l \tfrac{1}{2}\,EI\left(\frac{\partial^2 X}{\partial x^2}\right)^2 dx$$

Suppose that the applied force is produced by a rotating unbalanced mass at the tip of the beam. Then

$$Q_n = X_n(l)\;M\omega^2\,\sin\,\omega t$$

$$u(x) = \sum_n q_n X_n = \sum_n \frac{X_n(x)\;X_n(l)\;M\omega^2\,\sin\,\omega t}{\omega_{on}^2 \int \tfrac{1}{2}\,\rho\,A\,X_n^2(x)\,dx\left(1 - \dfrac{\omega^2}{\omega_{on}^2}\right)} \tag{51}$$

$$u(x) = \sum_n \frac{M\;X_n(x)\,\sin\,\omega t}{\left(\dfrac{\rho A l}{8}\right) X_n(l)\left(\dfrac{\omega_{on}^2}{\omega^2} - 1\right)} \tag{52}$$

Put

$$y = \sum_n \frac{1}{\dfrac{\omega_n^2}{\omega^2} - 1} \tag{53}$$

and define a new function

$$y' = \sum_{n=1}^{\infty} \frac{1}{\dfrac{(2n-1)^4}{\varphi^4} - 1} \tag{54}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\left[\frac{(2n-1)^2}{\varphi^2} - 1\right]\left[\frac{(2n-1)^2}{\varphi^2} + 1\right]}$$

$$= \frac{1}{2} \sum \left[\frac{1}{\frac{(2n-1)^2}{\varphi^2} - 1} - \frac{1}{\frac{(2n-1)^2}{\varphi^2} + 1}\right]$$

$$y' = \frac{\varphi\pi}{8}\left(\tan \frac{\varphi\pi}{2} - \tanh \frac{\varphi\pi}{2}\right) \quad (55)$$

If φ is defined by $(2n - 1)^2/\varphi^2 = \omega_n/\omega$ for large values of n, the function y' is practically identical with the function y, except for the first two terms of the series. By use of this function y' and suitable corrections for the first two terms of the series, the response curve has been calculated for the first three bending modes of a beam and is shown in figure 5. Figure 6 shows a comparison of these exact resonance peaks with the corresponding curve for one degree of freedom.

APPARATUS

The apparatus used for these tests was very simple. The applied force was produced in part of the tests by a small rotating unbalanced mass made by tapping a screw into the side of a rotating shaft, fitted in light bearings and clamped to the structure that was being tested. The shaft was turned by a small electric motor fitted with a flexible coupling and suitable gears; the frequencies were measured by an electric tachometer. For the rest of the tests, the rotating mass was replaced by a small crank coupled to the structure by a rubber band.

The amplitudes were measured by observing the magnified image of the filament of a small bulb produced by a lens of 1/4-inch diameter and 2-3/4-inch focal length held on the structure by a small brass mounting.

The models tested were the wings used in the flutter investigation (reference 7) and were in the form of cantilever beams 6 feet 9 inches long with symmetrical-airfoil cross sections. Model 1 was rectangular with 12-inch chord and 1/2-inch maximum thickness. It was made of duralumin with closely drilled 1/2-inch holes and was covered by a 0.006-inch sheet of duralumin. Models 2A, 2C, 3, and 4 were made of solid duralumin with two rows of chordwise slits to decrease the torsional stiffness. Models 6 and 7 were made by covering a balsa structure with 1/16-inch mahogany.

EXPERIMENTAL RESULTS

Three methods of measuring damping were used in these tests. The damping in the first bending mode of each model was determined by measuring the rate of decay of free vibration. For the higher modes, some of the tests were made with the rotating-mass type and some with the crank-and-rubber-band type of vibrator. In order to minimize the influence of normal modes other than the one being investigated, the force was applied and the amplitude was measured at the most suitable positions. For example, in the measurement of the damping in the third bending mode, the nearest disturbing frequencies are the torsion and the fourth bending. In this case, the force was applied to the tip of the beam at the position of the node in torsion and the amplitudes were measured at the edge of the beam at the position of a node of the fourth bending mode.

Figure 7 shows plots of the response curves of the models tested. In these tests, it was thought unnecessary to try to obtain the complete functional dependence of damping on amplitude and other variables. Consequently, the response curves were analyzed only to the extent of finding a representative value of the damping parameter δ. It will be noted that the results for all models of a given type of construction gave values of δ within a range of about a factor of 2. In table I are given the numerical results for δ. These values were computed from the data of figure 7 by use of equations (12), (18), and (23). The value of the amplitude at a frequency ratio of 1.1 was taken for determining the best single representative value of δ.

CONCLUSIONS

1. In the determination of the damping of structures by means of the shape of the response curve obtained by applying an alternating load at one point of the structure, the use of the analysis for one degree of freedom is justified when the following conditions are met:

(a) The damping is small.

(b) The points of applying the force and measuring the amplitudes are appropriate from considerations of disturbing normal modes.

(c) Only amplitudes close to a resonant peak are used to determine the nondimensional damping parameter, δ.

2. When the normal functions for a structure are known, the damping in the different modes can be separately determined from the measured amplitudes at several points along the structure.

3. The measured values of δ for a homogeneous structure such as a cantilever beam of duralumin are approximately equal in the different modes of vibration.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., December 21, 1939.

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TABLE I

Values of the Damping Parameter, δ

Model		Mode				
		Bending				Torsion
		1	2	3	4	1
1	Duralumin skin	0.062	0.083	0.1130	--	0.087
2A	Solid duralumin	.0050	.014	--	--	.0087
2C	do.	.0050	.0056	.0073	--	.015
3	do.	.0045	.0075	.0040	0.0047	--
4	do.	.0089	.0080	.0050	.0053	.0034
6	Wood	.020	--	.0174	.030	.028
7	do.	.021	--	--	--	.020

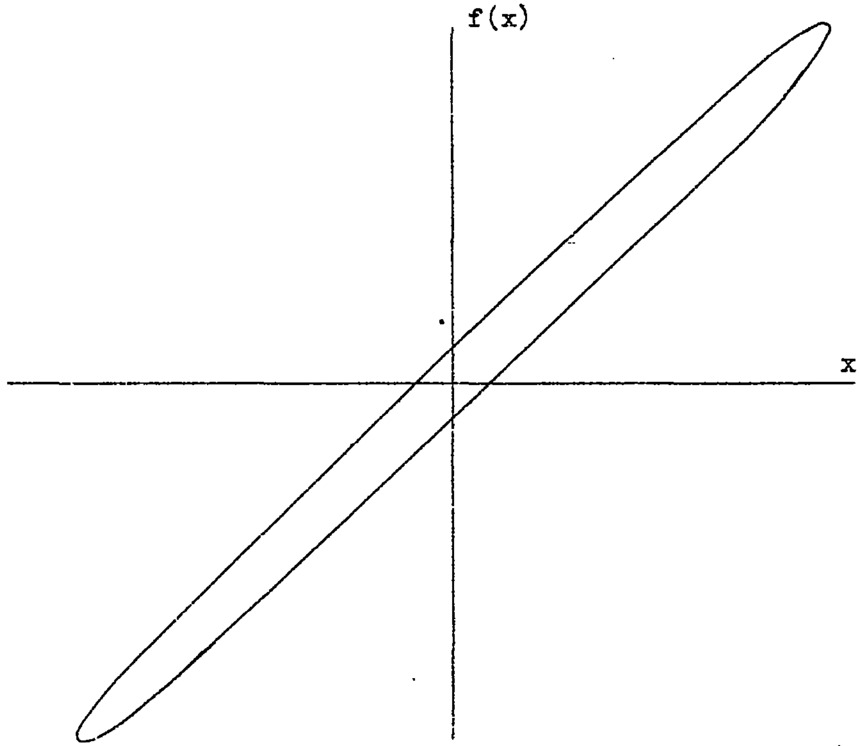


Figure 1.- A typical elastic hysteresis loop.

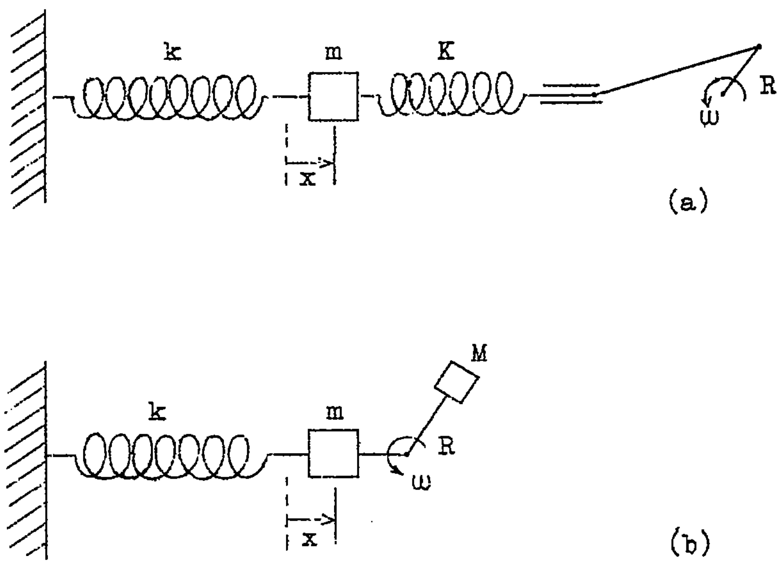


Figure 2.- Simple mechanical circuits to illustrate the effect of a vibrator.

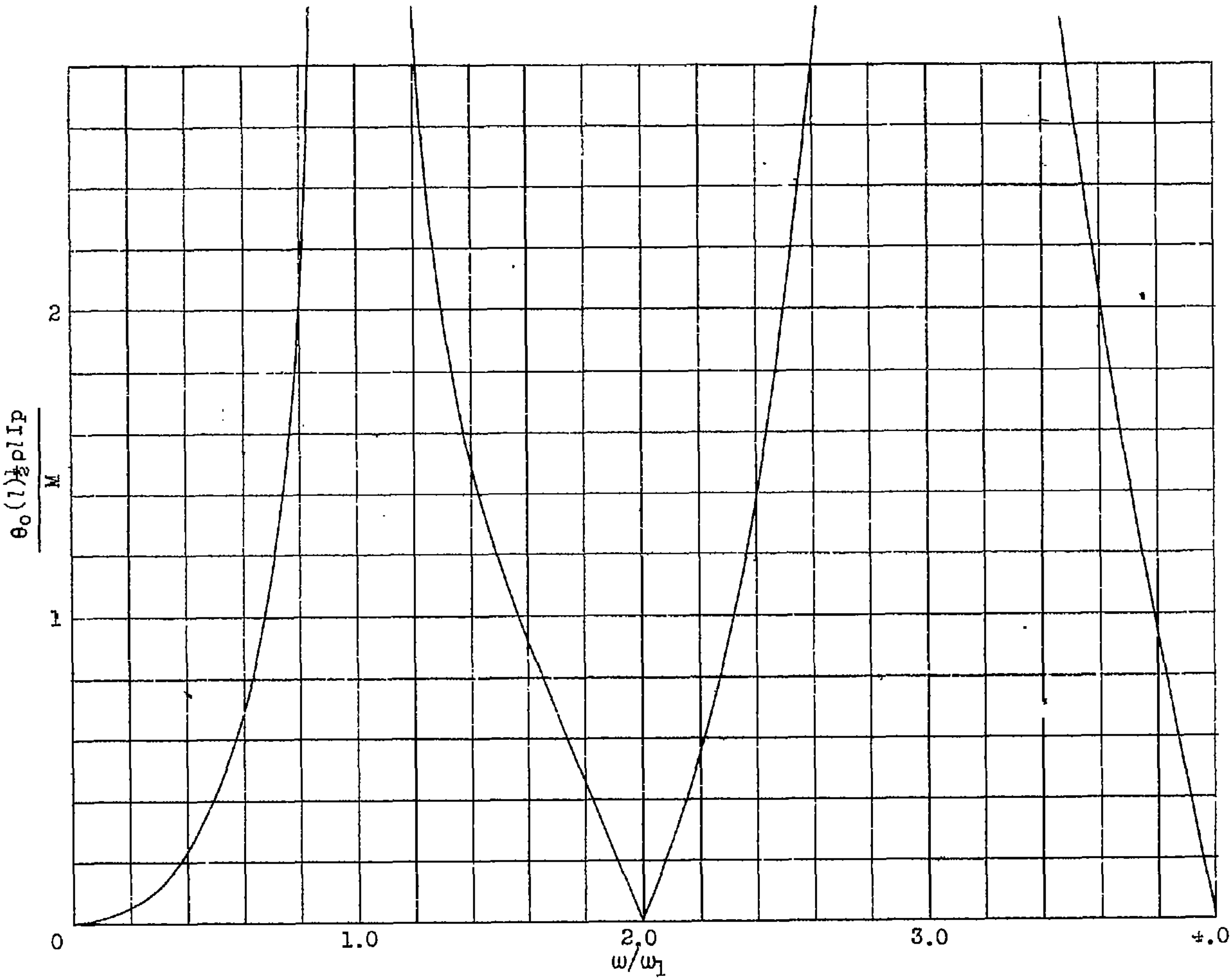


Figure 3.- Response curve for torsional vibration of a uniform beam.

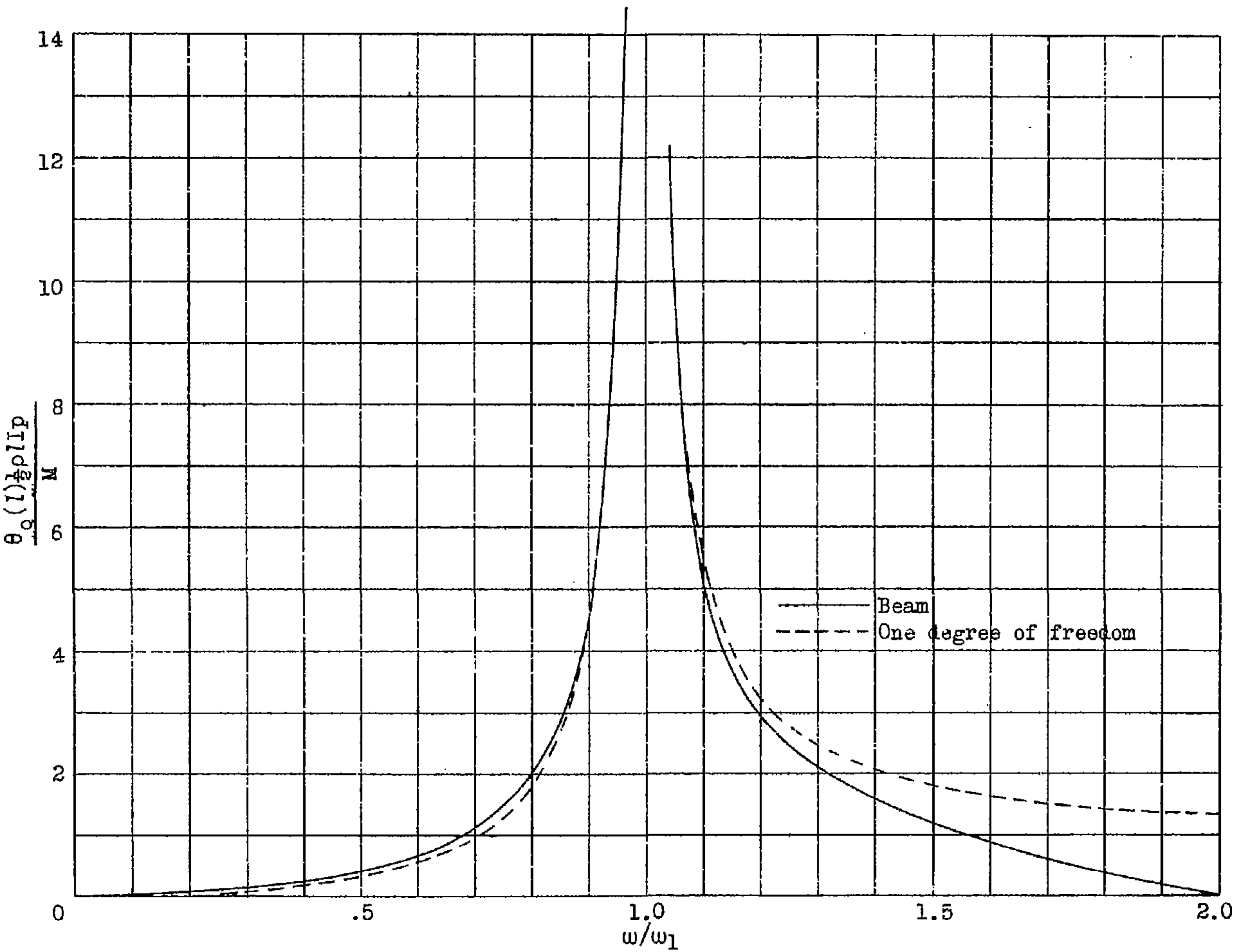


Figure 4.- Comparison of exact torsional response curve with curve for one degree of freedom.

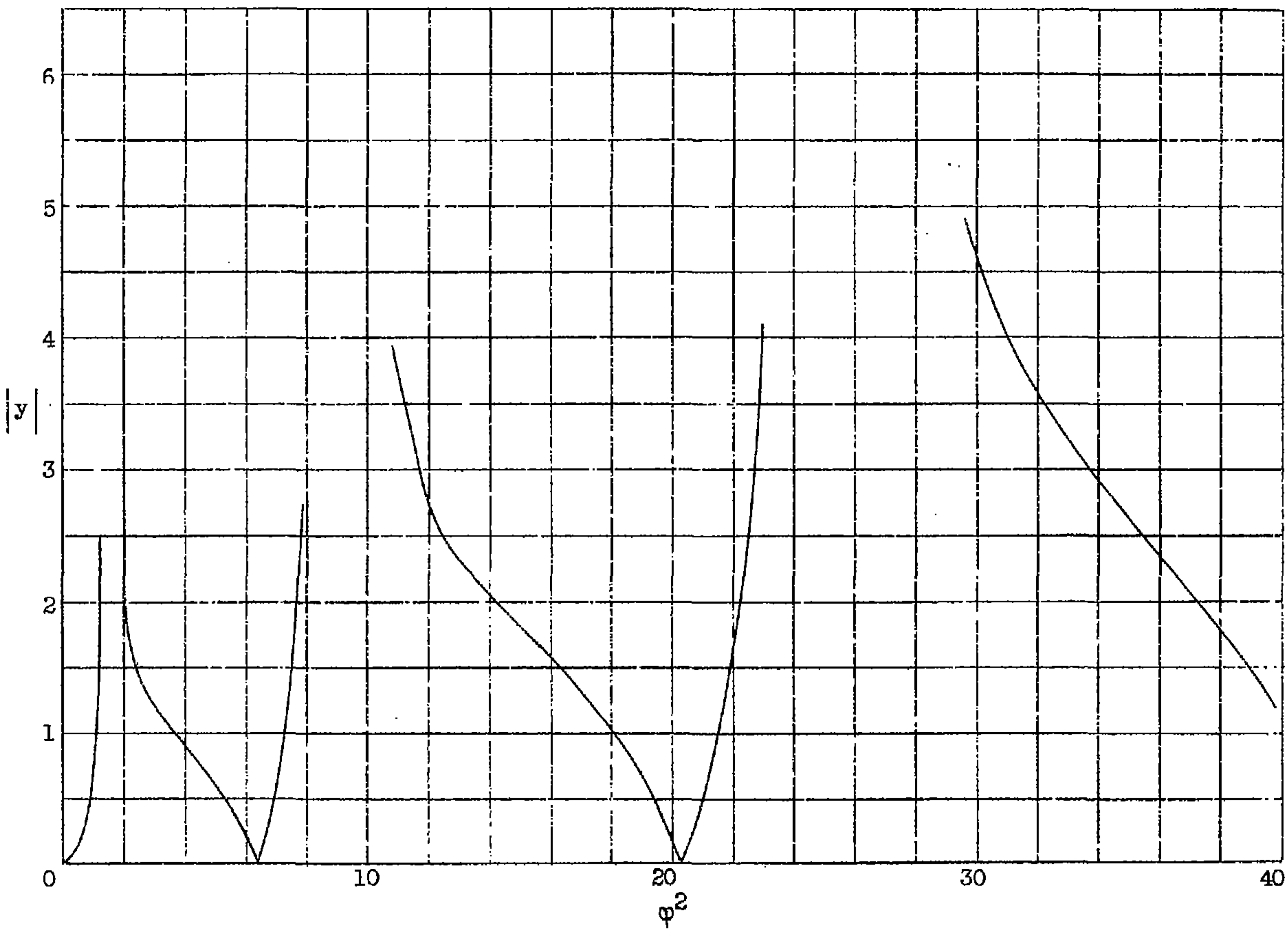


Figure 5.- Response curve for bending of a uniform beam.

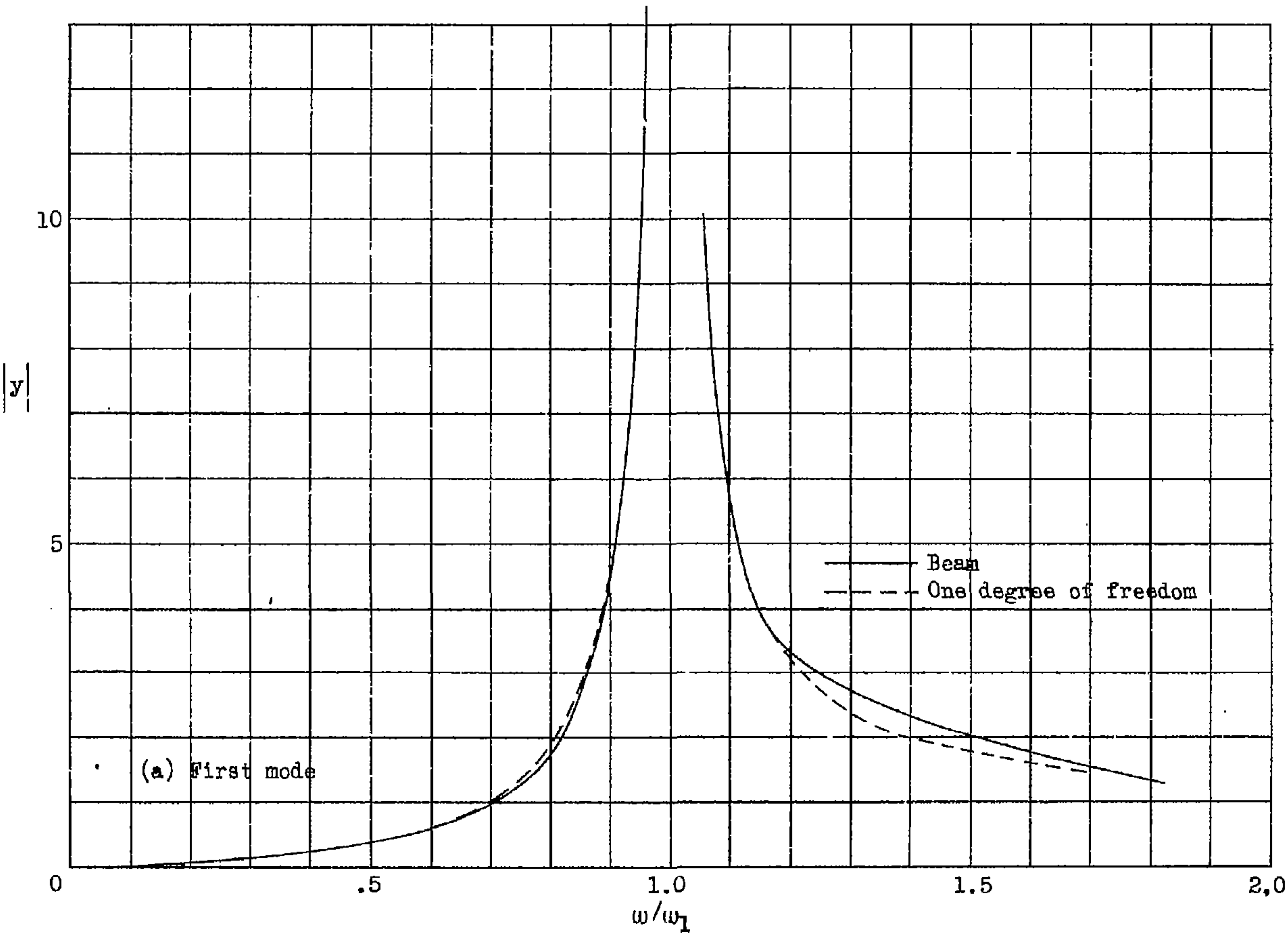


Figure 6a to c.- Comparison of response curve in bending with curve for one degree of freedom.

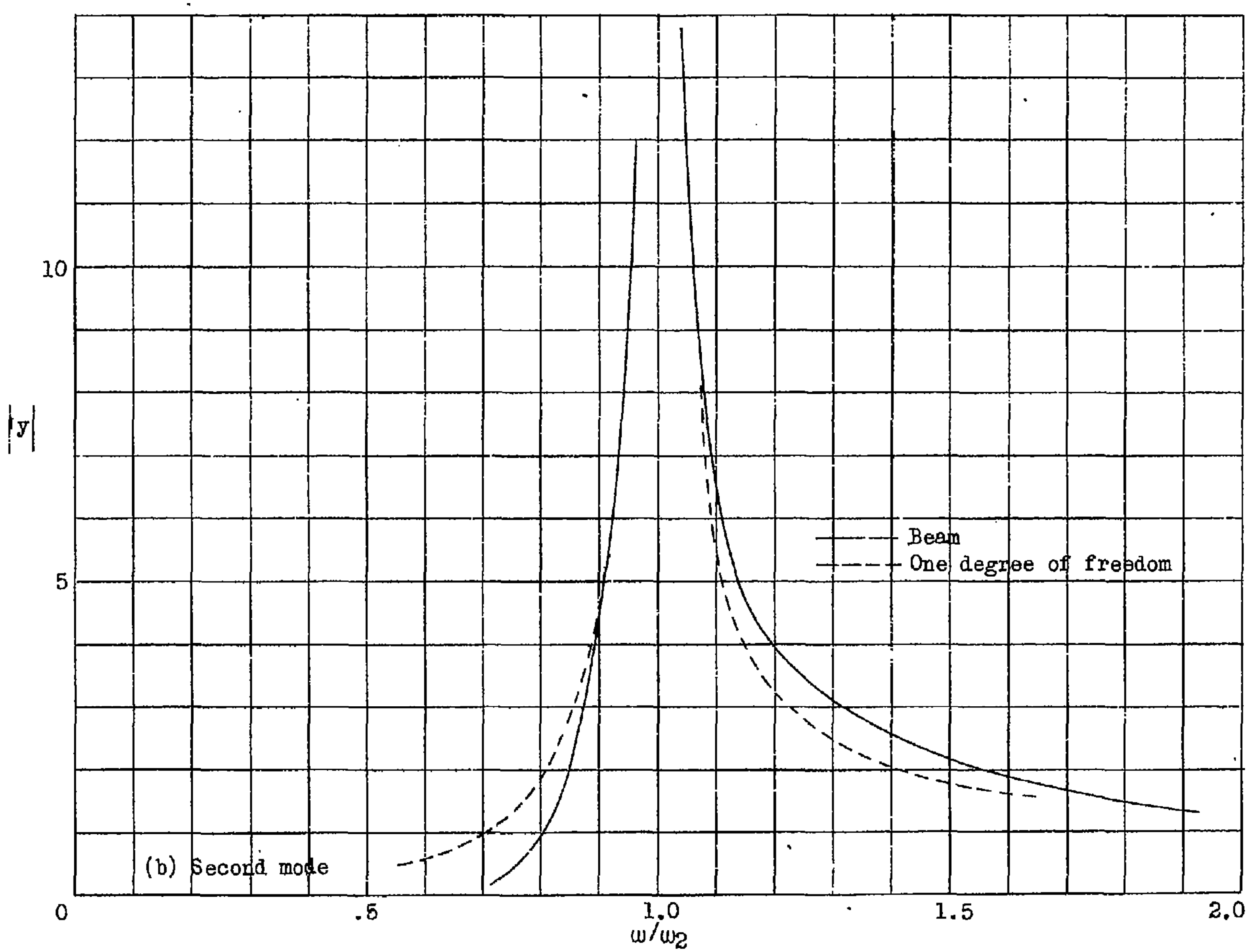


Figure 6.- Continued.

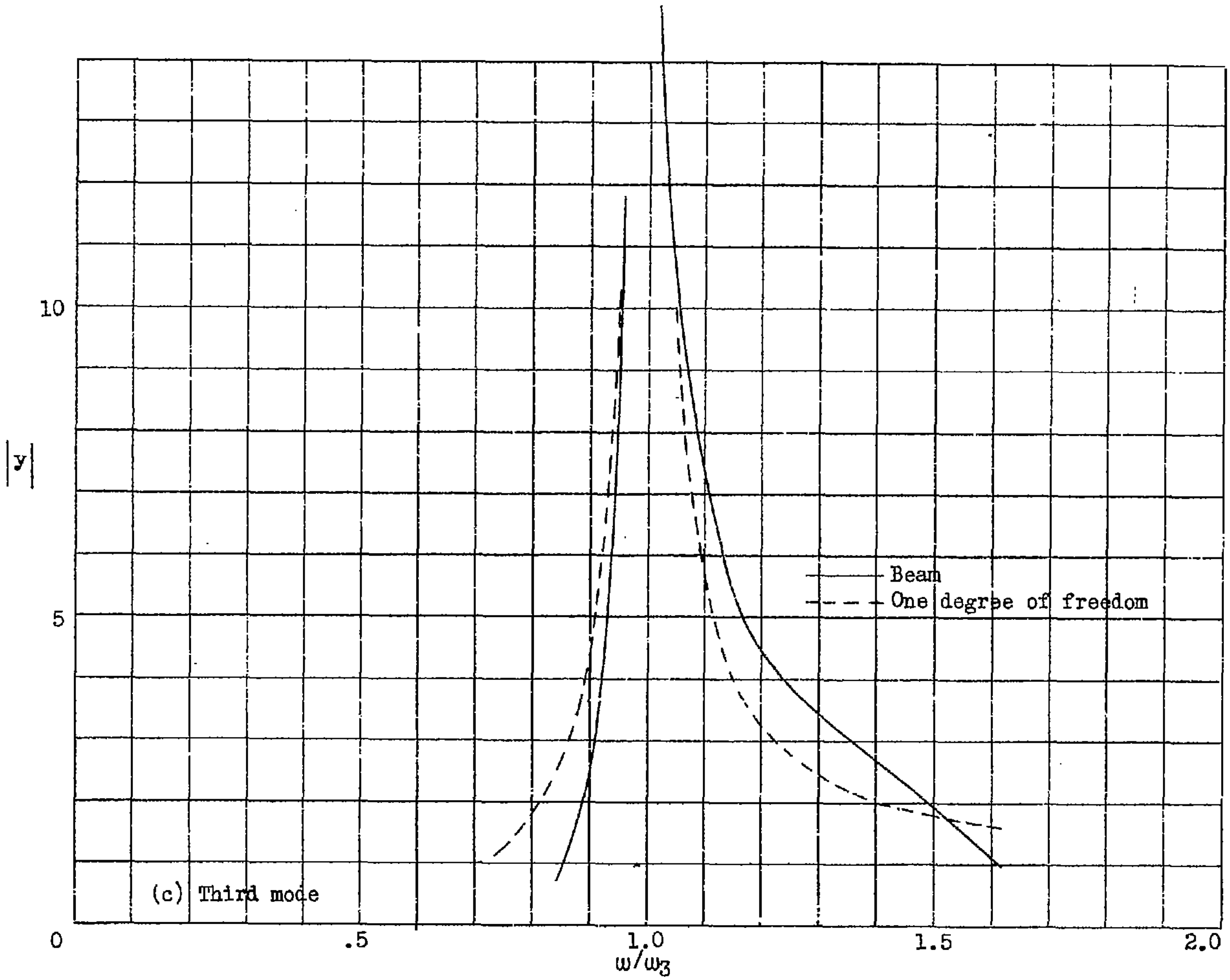
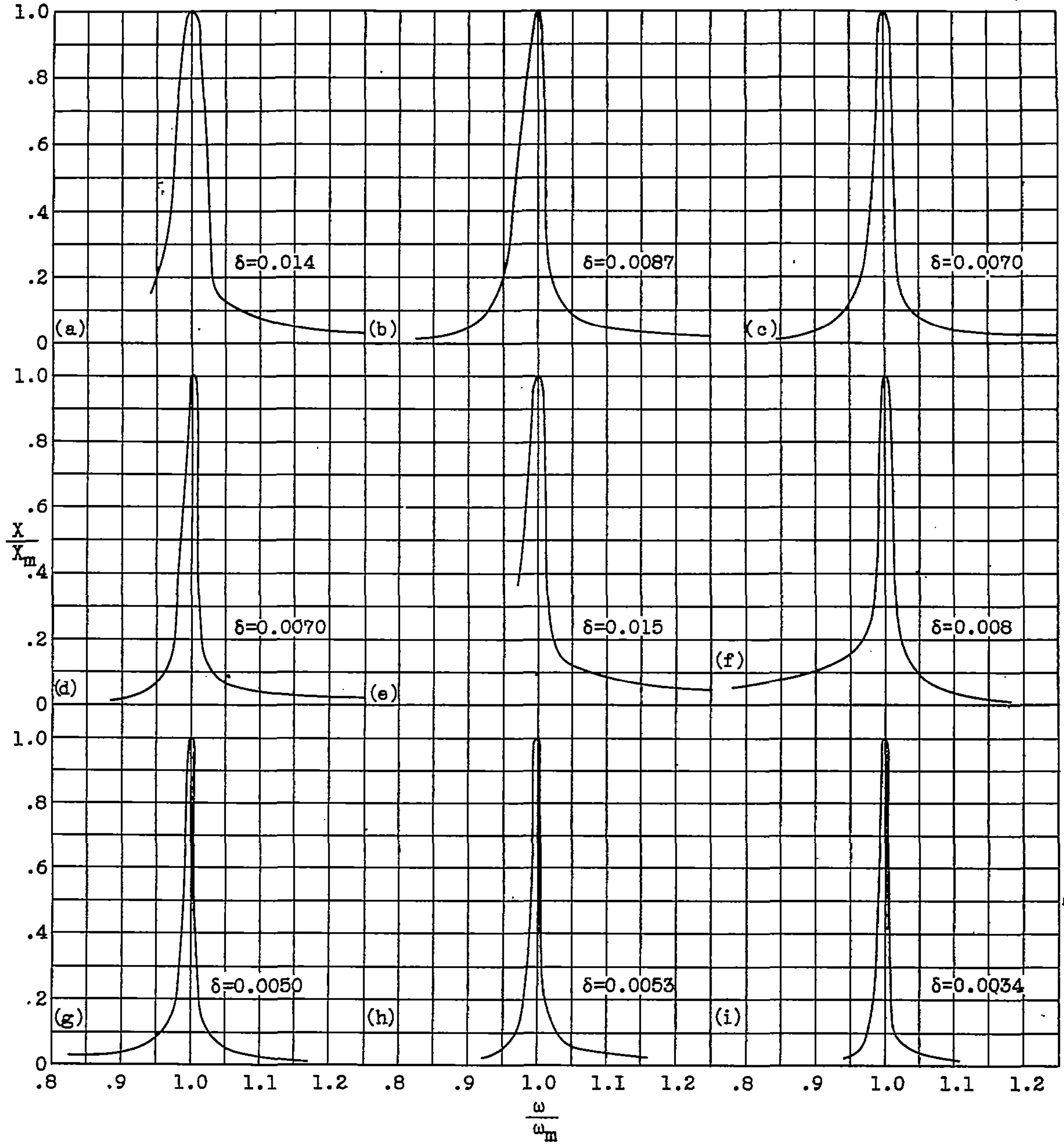


Figure 6.- Concluded.



(a) Model 2A; second bending mode; rotating-weight method.

(b) Model 2A; torsion mode; rotating-weight method.

(c) Model 3; second bending mode; rotating-weight method.

(d) Model 3; third bending mode; rotating-weight method.

(e) Model 3; fourth bending mode; rotating-weight method.

(f) Model 4; second bending mode; rubber-band method.

(g) Model 4; third bending mode; rubber-band method.

(h) Model 4; fourth bending mode; rubber-band method.

(i) Model 4; torsion mode; rubber-band method.

Figure 7,a-i.-Typical experimental response curves.