NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 937

SIMPLIFIED TRUSS STABILITY CRITERIA

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Washington
July 1944

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SUMMARY

Part I covers the development of simplified criteria for the stability of planar pin-jointed trusses against buckling in the plane of the truss, based on the earlier work of Viscovich. Part II constitutes a report on tests carried out to verify the validity of the criteria developed in part I. The agreement between observed and predicted critical loads was well within the range of probable experimental error.

This investigation, conducted at the Stanford University, was sponsored by, and conducted with financial assistance from, the National Advisory Committee for Aeronautics.

I. GENERAL STABILITY OF PLANAR PIN-JOINTED TRUSSES

When designing practical trusses, an engineer seldom considers the general stability of the truss as a whole, and very rarely treats the stability of a single member as a function of the stiffnesses of those adjacent to it. Usually the conventional design procedures lead to truss designs which are stable. When, however, these procedures are used and it is found that the axial force computed for some member is zero, the calculated required area of that member is also zero. If such a member were omitted from a statically determinate truss, the structure would usually be unstable. Also when the computed axial load in a member is very small, the use of an area which has been computed by the conventional procedures may result in such a flexible member that the stability of the truss is impaired. In practice the experienced engineer will usually recognize such situations and use arbitrarily selected member sizes. If he lacks a rational method of computing the necessary stiffness and must rely on experience or intuition, he may use much larger sectional areas than are really needed. Since this would result in unnecessary structural weight, a rational method of attacking the problem is desirable. The first part of this report is devoted to the development of a simple and practical method for predicting the critical intensity of loading for a pin-connected planar truss, with a simple and practicable design procedure for the rational design of the zero or slightly loaded members of a given truss configuration.

NOTATION

A cross-sectional area

E modulus of elasticity

K spring constant

L length

P axial load in link

U work

V axial load in supporting spring

W external load on truss

α angle of rotation or of deviation from nominal position

γ deflection parallel to original direction of link axis

δ deflection normal to original direction of link axis

η ratio of lengths

The significance of subscripts and primes, and a few seldom-used symbols, is indicated where they are introduced.

VISCOVICH'S STABILITY CRITERION

The method of analysis presented here is an extension of that developed by S. Viscovich in reference 1. It will therefore be helpful to begin the development of the new method by a brief statement of Viscovich's method as it would be applied in a specific problem. For this purpose consider the truss of figure 1, for which the lengths and sectional dimensions of all members are assumed to be known. It is also to be assumed that the truss was so cambered that, when the load W is applied at joint E, the bars AB and BC form a straight line. Then, according to the usual methods of stress analysis, the design load for member BE would be zero. If member BE were left out, the truss would continue to carry the load at E so long as joint B remained on the straight line AC. Because of the pin joint at B, however, its equilibrium would be unstable. It would also be unstable if member BE were too flexible to counteract any tendency of joint B to move away from the line AC. The problem is to determine the minimum stiffness required of member BE in order to obtain positive stability, or whether any specific stiffness of that member is in excess of such minimum.

Let the axial loads on the members produced by the load W at joint E and associated reactions at D and F be called the "primary" axial loads. If member AB is subjected to a unit couple while the truss is subjected to this primary load system, each member of the truss will rotate with respect to the line joining the supports. The magnitudes of these rotations may be computed by the method of virtual work or any equivalent procedure. The unit couple should be assumed to be so small that the angles of rotation, measured in radians, may be assumed numerically equal to their sines and tangents and that the cosines of these angles of rotation may be assumed equal to unity. The rotations produced by the unit couple acting on AB will be termed the "unit rotations" and that for any member XY will be designated α_{xy}.

One effect of the unit rotations would be to change the geometry of the truss and therefore to modify the axial loads developed to resist the load W at joint E. Since, however, it is assumed that the unit couple and the resulting unit rotations are small, such changes in the primary axial loads may be neglected. Though these primary axial loads may be assumed unchanged in magnitude, they are not unchanged in direction; but their lines of action have been subjected to the unit rotations. Therefore at each joint the axial loads on the members may be resolved into components parallel and perpendicular to the original directions of the members on which they act. The components parallel to those original directions, $P_{xy} \cos \alpha_{xy}$, may be assumed equal to the primary axial loads, P_{xy}. The perpendicular components, $P_{xy} \sin \alpha_{xy}$, may similarly be assumed equal to $P_{xy}\alpha_{xy}$.

Since the parallel components are equal in magnitude and parallel to the primary axial loads found from the original truss analysis and the primary axial loads are in equilibrium at each truss joint, the parallel components must be similarly in equilibrium at each joint. Furthermore, since each truss member, XY, is designed to carry its axial load P_{xy}, these forces alone would not produce instability.

The perpendicular components, $P_{xy}\alpha_{xy}$, are induced by the unit rotations of the members and are therefore termed the "induced loads." In general, these induced loads would not be in equilibrium at each joint but would cause additional rotations of the truss members which may be termed their "induced rotations." The magnitudes of the induced rotations can be computed from the induced loads by the method of virtual work or any equivalent procedure.

Viscovich's stability criterion is that if the induced rotation of member AB is less than its rotation owing to the unit couple applied to it, that member is in stable equilibrium; while if the induced rotation exceeds its rotation due to the unit couple, the equilibrium of that member is unstable. In a statically determinate truss like that under consideration, if any member is in unstable equilibrium, the whole truss will be unstable.

In applying this criterion it is necessary to start with the assumption of a specific system of unit rotations produced by an arbitrarily located unit couple. For complete proof of the stability of a truss it would be necessary to investigate all possible locations for applying the unit couple, and the designer would have to apply it not only to each single member but also to each possible group of members. In fact, it might be necessary to assume several unit couples acting simultaneously. In practice, however, very few of the theoretically possible unit rotation systems need be investigated, and the critical ones are easily identified. Thus for the truss of figure 1 the investigation could be limited to determining the effect of using too small a cross-sectional area for member BE, and applying the unit couple to member AB, Unless some member is present for which the design axial load is much smaller than those of its neighbors, the resulting design sizes are large enough so that if each is capable of carrying its design load there will be little danger of general instability of the truss. The stability of members adjacent to an unstressed or very lightly loaded member, however, may be impaired through failure to assign sufficiently large sectional dimensions to the latter. The designer's problem is therefore to identify these "critical" members, assign sectional areas to them, and then to make sure that the adjacent members have been made stable.

If different sizes are assigned to the critical members and Viscovich's criterion is applied to each size, the engineer may thus investigate the adequacy of his design. This criterion, however, indicates only whether an assumed size for the critical member is sufficient to provide stability. To obtain the most efficient design, several trials may be needed, since Viscovich failed to develop a procedure for the direct determination of the size of lightly loaded member needed for stability. If his method were short, simple, and free from abnormal hazards of calculation error, it would be acceptable in practice. The opposite is true, however, and the method, as developed by Viscovich, is not suitable for practical design work. The desirability of a simpler method for the rational design of "unstressed" members and prediction of the stability of pin-jointed planar trusses, has led to the extension of his procedure that is developed below.

STABILITY OF SYSTEMS OF ELASTICALLY SUPPORTED BARS

The stability of a pin-connected truss may be determined by suitable application of the stability criteria of a small number of type systems of elastically supported, absolutely rigid, pin-connected links. In fact, only three such systems are needed for handling almost any statically determinate truss pattern, and the first step is to develop the stability criteria for these three systems.

The first to be considered is that shown in figure 2 where an absolutely rigid link AB is connected to a rigidly supported frictionless pin at A and is supported at B by the elastic member BC which has a spring constant K. The lower end of BC is connected to a rigidly supported pin at C. It is assumed that the load P is applied horizontally to the originally horizontal member AB. Timoshenko has shown (reference 2) that the critical value of P for this system would be

$$P_{cr} = K\,L \tag{1}$$

An equivalent statement is that the critical value for the spring constant K is

$$K_{cr} = \frac{P}{L} \tag{2}$$

The second system to be considered is that shown in figure 3. Here the rigid links AB and BC are supported at A and C. The support at A is assumed completely restrained from movement in translation. That at C is restrained against vertical motion, but is free to move horizontally. There is no restraint against rotation at either A or C. At B the two links are joined by a frictionless pin which is supported by the elastic member BD of spring constant K. By extending to this system the method used by Timoshenko to analyze that of figure 2, Viscovich showed that the critical value of the spring constant K would be

$$K_{cr} = \left(\frac{P_{ab}}{L_{ab}} + \frac{P_{bc}}{L_{bc}}\right) \tag{3}$$

and if the axial load is the same for both links, its critical value would be

$$P_{cr} = \frac{K\,L_{ab}\,L_{bc}}{L_{ab} + L_{bc}} \tag{4}$$

The third system to be analyzed is that of figure 4 where the rigid link AB is supported at its ends by the elastic members AC and BD with spring constants K_1 and K_2, respectively. Viscovich analyzed this system, assuming the axial load P in AB to be constant, and found as the criterion for stability

$$P_{cr} = \frac{K_1\,K_2\,L}{K_1 + K_2} \tag{5}$$

From this relation, if K_1 and P are specified, the critical value of K_2 is

$$K_{2\,cr} = \frac{K_1 P}{K_1 L - P} \tag{6}$$

In extending Viscovich's work it will be assumed that the axial load P, instead of being constant, varies linearly from P_1 at A to $P_1 + \mu x$ at any point X at the distance x from A. In studying this system it is convenient to measure the movements of all points along AB with respect to vertical and horizontal axes through A. In effect this is equivalent to replacing the system of figure 4 by that of figure 5, but this is allowable since the stability criteria for the two systems are identical.

Assume the link AB to rotate through the small angle α, the center of rotation being any point E along its length. The resulting horizontal movement of any point X at the distance x from A would be

$$\gamma_x = x \operatorname{vers} \alpha \tag{7}$$

If the angle α is small, and it is so assumed, $\operatorname{vers} \alpha$ is approximately equal to $\alpha^2/2$, whence

$$\gamma_x = \frac{\alpha^2 x}{2} \tag{8}$$

Consider now a differential element of the link AB with its left end at X. This element and the forces acting on it are shown in figure 6, in which the upper portion represents conditions before, and the lower portion conditions after, the assumed rotation through the angle α. To satisfy the conditions of equilibrium

$$P_1 + \mu x + \mu dx - P_1 - \mu(x + dx) = 0 \tag{9}$$

As a result of the assumed rotation, the left end of the element would move horizontally through the distance γ_x and the right end through the distance $\gamma_{x + dx}$. The work done by the horizontal forces acting on the element would therefore be

$$dU_e = -(P_1 + \mu x)\,\frac{\alpha^2}{2}\,x - \mu dx\,\frac{\alpha^2}{2}\left(x + \frac{dx}{2}\right)$$

$$+ \left[P_1 + \mu(x + dx)\right]\frac{\alpha^2}{2}\,(x + dx) \tag{10}$$

Combine terms and neglect second-order differentials, and this becomes

$$dU_e = \frac{\alpha^2}{2}\,(P_1\,dx + \mu x\,dx) \tag{11}$$

Since, however, the angle α has been assumed small, it may be represented by

$$\alpha = \frac{\delta_1 + \delta_2}{L} \tag{12}$$

whence

$$dU_e = \frac{(\delta_1 + \delta_2)^2}{2L^2}\,(P_1 + \mu x)dx \tag{13}$$

The total work done by the horizontal forces on the link can therefore be found by integration to be

$$U_e = \frac{(\delta_1 + \delta_2)^2}{2L^2}\int_0^L (P_1 + \mu x)dx = (\delta_1 + \delta_2)^2\left(\frac{P_1}{2L} + \frac{\mu}{4}\right) \tag{14}$$

Since P_2, the axial load at B is equal to $P_1 + \mu L$, μ can be replaced by $(P_2 - P_1)/L$ and equation (14) becomes

$$U_e = \frac{(\delta_1 + \delta_2)^2}{4L}\,(P_1 + P_2) \tag{15}$$

The total strain energy stored in the springs as a result of their elongations δ_1 and δ_2 is

$$U_i = \frac{K_1\delta_1^2}{2} + \frac{K_2\delta_2^2}{2} \tag{16}$$

According to the energy theory used by Timoshenko, the critical loading is that at which $U_e = U_i$. Equating the expressions for those quantities and simplifying gives

$$K_1\delta_1^2 + K_2\delta_2^2 = \frac{(\delta_1 + \delta_2)^2}{2L} (P_1 + P_2) \qquad (17)$$

In order to satisfy the requirements of equilibrium, the tension in one of the elastic members supporting the link AB must be equal to the compression in the other. If this force is designated by V, $\delta_1 = V/K_1$ and $\delta_2 = V/K_2$. If these values for the deflections are inserted in equation (17), it may be simplified to

$$\frac{P_1 + P_2}{2} = \frac{K_1 K_2}{K_1 + K_2} L \qquad (18)$$

If the axial load is constant, the left side of equation (18) may be replaced by P and that equation becomes identical with equation (5), checking Viscovich's result. Thus Viscovich's stability criterion is valid for a linearly varying as well as for a constant axial load in the link if the average axial load is used for P.

In the above development of stability criteria, the links were assumed perfectly rigid - that is, inextensible. The criteria found are equally applicable, however, to extensible links, since it is assumed that any virtual rotations of the links take place after the axial loads have been imposed and that those loads, and consequently the link lengths, remain unchanged during the rotations. A slight error may be introduced as a result of the axial load being changed by the rotation, but as long as the rotations are small, such errors would be negligible.

SIMPLIFIED TRUSS STABILITY COMPUTATIONS

In applying Viscovich's procedure for determining the stability of a truss the entire structure must be dealt with simultaneously. The simplified method proposed here is to isolate and analyze small portions of the truss, each including a member which is of such light construction as to make the stability questionable. These isolated portions would be treated as if they were systems of the types analyzed above. This involves assuming rigid support for the pins at which the isolated portion is attached to the remainder of the truss. It will be convenient to illustrate the application of the procedure before attempting to demonstrate the validity of this underlying assumption.

If the section of a pin-jointed truss shown in figure 7 is loaded as shown, no axial load will be imposed on member BC. Member AB, however, will be subjected to an axial compression, P_{ab}, equal to the external load W_b. If joint C is assumed rigidly supported, members AB and BC form a system of the type shown in figure 2, the similarly lettered members are equivalent to each other. The critical, or minimum allowable, spring constant for member BC will therefore be $K_{bc} = P_{ab}/L_{ab}$. The practical design problem, however, is to determine not the critical spring constant, but the minimum allowable size for member BC. If that member is assumed to be elastic, its elongation under load is obtainable from the relation

$$\Delta L = \frac{P\ L}{A\ E} \tag{19}$$

where ΔL is the elongation, L the original length, P the axial load, A the cross-sectional area, and E the modulus of elasticity of the member. Since the spring constant or "stiffness" of a member is the ratio of its axial load to the resulting elongation, for any member

$$K = \frac{P}{\Delta L} = \frac{A\ E}{L} \tag{20}$$

The critical value of the spring constant of member BC is therefore

$$K_{bc} = \frac{P_{ab}}{L_{ab}} = \frac{A_{bc}\ E_{bc}}{L_{bc}} \tag{21}$$

from which the minimum allowable value for the sectional area of BC is

$$A_{bc} = \frac{P_{ab}}{E_{bc}}\,\eta \tag{22}$$

where η is the ratio L_{bc}/L_{ab}.

The same basic method can be used to determine the required sectional area of member BE of the truss of figure 8 for the loading shown. For this example the type system to be used consists of members AB, BC, and BE, which is equivalent to the system of figure 3. In this case, since the truss and its loading are symmetrical, equation (3) for the critical value of the spring constant when applied to member BE, becomes

$$K_{be} = \frac{2P_{ab}}{L_{ab}} \tag{23}$$

Combination of this expression with equation (20) and solving for A_{be} gives

$$A_{be} = \frac{2\ P_{ab}}{E_{be}}\eta \tag{24}$$

where η is the ratio L_{be}/L_{ab}.

ACCURACY OF THE SIMPLIFIED METHOD

As previously mentioned, this simplified method of investigating the stability of a truss is based on the assumption that the pins connecting the isolated portion to the remainder of the truss may be assumed rigidly supported. Since completely rigid support is impossible, the effective spring constants of the members assumed elastic are somewhat less than those computed from equation (20). While it would be difficult to develop a general proof that the resulting error in the computed areas required for these members would be negligible in a reasonably well designed practical truss, it is not difficult to show that this would probably be the case.

In the foregoing discussion the points at which the system under consideration were supported were assumed to be rigidly supported. An alternative is to assume that each such point is elastically supported, and to define as the "spring constant of a point" the ratio of load imposed on that point to the resulting movement of the point parallel to the line of action of the load. Each point therefore must be assumed to have two spring constants, one based on its movement under vertical and the other based on its movement under horizontal force, and these may be termed its vertical and its horizontal spring constants, respectively.

The criterion for the required area obtained for member BE of figure 8 implies vertical spring constants of infinity for points A, C, and E. It should be compared with the criterion which would be obtained if the spring constants of those points were reduced to values which would be associated with reasonable selections for the dimensions of the truss members. In the appendix the analysis of a truss like that of figure 8 is summarized. In selecting member sizes for this truss an allowable working stress of 30,000 psi was assumed for the tension members, and the Euler formula was used for the design of the compression members, which were assumed square in cross section. The sectional areas having been selected, the next step was to determine for each member its value of L/AE, termed by J. Clerk Maxwell its "extensibility" (reference 3)*. For this step E was taken as 30,000,000 psi. Once the extensibilities of the members had been determined it was a simple matter to compute, by the method of virtual work, the vertical deflections of joints A, C, and E that would be produced by unit vertical loads imposed at those joints. The vertical spring constants thus determined were $K_a = K_c = 158,591$ pounds per inch and $K_e = 106,400$ pounds per inch.

This procedure included no basis for the design of member BE. According to the simplified stability criterion described, however, the minimum allowable spring constant for that member would be $\frac{2P_{ab}}{L_{ab}} = \frac{2 \times 30000}{180} = 666$ pounds per inch. Member BE was therefore assigned the sectional area needed to produce this value.

In the development of the simplified criterion for stability the group of members converging at B was assumed equivalent to the system of figure 3. For the more accurate investigation it is assumed equivalent to that of figure 9a, where members AB, BC, and BE are elastically supported at A, C, and E by springs which have for their effective stiffnesses K_a, K_c, and K_e, respectively. Member BE is assumed to have the

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*It is to be noted that the extensibility of any member is the reciprocal of its stiffness or spring constant.

effective stiffness K_{be} = 666 pounds per inch as calculated.

This equivalent structure requires further simplification in order to develop a satisfactory criterion for its stability. From inspection of figure 9a, bearing in mind that the conditions of equilibrium must remain satisfied, it can be seen that if points A and C move upward when ABC is subjected to a horizontal load, points B and E must move downward. In the system under consideration $K_a = K_c$ and the vertical movements of points A and C would be equal. If the deflections are measured with respect to a line through A and C instead of one through the supports, the system of figure 9b may be used in place of that of figure 9a. In this substitute structure the members meeting at B are supported by a fixed pin at A, a vertically fixed pin at C and a pair of springs at D, one with a spring constant equal to K_a and the other with a spring constant equal to K_c. This pair of springs in parallel may be combined into a single spring of stiffness $K_a + K_c$, which in this structure would make its spring constant equal $2\ K_a$. This modification is represented by figure 9c.

The spring ED and the spring between D and the fixed foundation act in series, the first having as its spring constant K_e, and the second $2\ K_a$. The effective spring constant of the combination - that is, the spring constant of point E with respect to the foundation - will then be (reference 4)

$$K_1 = \frac{(2\ K_a)\ K_e}{(2\ K_a) + K_e} \tag{25}$$

and for the specific truss under study

$$K_1 = \frac{317000 \times 106000}{317000 + 106000} = 79{,}400 \text{ pounds per inch}$$

This effective system is represented in figure 9d. Similarly the effective spring constant of point B can be found from

$$K_b = \frac{K_{be} K_1}{K_{be} + K_1} \qquad (26)$$

which gives

$$K_b = \frac{666 \times 79400}{666 + 79400} = 660.47 \text{ pounds per inch}$$

Thus the effect of neglecting the elasticity of the supports of joints A, C, and E results in an error of only 666 − 660.47 = 5.53 pounds per inch or 0.837 percent.

Although the error resulting from the application of the simplified criterion to the truss of figure 8 is less than 1 percent, this is not a proof that such errors will be comparably small for all practical trusses. If, in practice, it were proposed to use an unstressed member with a spring constant little if any larger than that called for by the simplified criterion a more refined analysis similar to that made in this section would be in order. In practice, however, it will nearly always be found that the size required to satisfy other conditions, such as those of handling, will be so much larger than that called for by the simplified criterion that the inherent error due to the simplification may clearly be ignored.

EFFECT OF DEVIATIONS FROM NOMINAL TRUSS DIMENSIONS

The required stiffness of a critical member as calculated in the preceding sections is the minimum required for the stability of the truss under the loading considered. While these calculations were based on the assumption that the truss would be geometrically perfect, that would never be the case in a practical structure. A complete investigation into the problem of stability of pin-jointed planar trusses must, therefore, include a discussion of the effects of deviations of the actual from the nominal truss dimensions upon the validity of the calculations or, if more convenient, the inclusion of the effects of such deviations directly in the computations.

If the truss of figure 10 is assumed to be manufactured to a given degree of accuracy, the effects of deviations from the nominal dimensions would be to make the angles ABE and CBE differ slightly from their nominal value of 90°. It can also be seen that if the truss were originally designed without camber, the loads coming on to the structure would cause an additional deviation from 90° of the angles ABE and CBE. Furthermore the use of initial camber could eliminate this added angle of deviation for but one loading condition. As the load at E is increased, the angles ABE and CBE will change according to the increase in the load. Thus the total deviation of the practical truss from the ideal truss previously considered may be represented by the total deviations, α_o, of the angles ABE and CBE from 90°, caused by rotations due to the elongations of the members under load and deviations in manufacture from the nominal dimensions. The angle α_o will be termed the initial angle.

For any given value for the initial angle, it is possible to design the entire truss, including member BE, by the usual methods of truss design. The relation for finding the axial load on member BE is

$$P_{be} = 2 P_{ab} \alpha_o \tag{27}$$

ABE and CBE are not the only angles that would be affected by the deviations of the actual from the nominal dimensions, but it should be obvious that members like BE, which would be subjected to no load if it were not for such deviations, are the only ones where the percentage change in axial load due to the deviations would be appreciable.

After the magnitudes of the initial angles have been decided upon and the axial load on BE has been computed by the usual methods of truss analysis, the sectional dimensions of that member may be obtained in the usual manner. In the problem at hand, if the allowable working stress for BE is σ_w, the required sectional area for that member will be

$$A_{be} = \frac{2 P_{ab} \alpha_o}{\sigma_w} \tag{28}$$

Essentially, the calculation of the required stiffness for BE in the ideal truss by the simplified stability criterion may be interpreted as a computation of the minimum area for BE consistent with stability. Equation (24) may be rewritten,

$$A_{be} = \frac{2\ P_{ab}}{E}\ \eta \qquad (24)$$

Thus two separate criteria are obtained for the design of member BE. In equation (28) it can be seen that the required area is directly proportional to the load in AB and the angle α_0, and inversely proportional to the working stress σ_w. In equation (24) the required area of member BE is directly proportional to the load in AB and to the ratio η, and is inversely proportional to the modulus of elasticity of the material. For different trusses and different materials it is obvious that first one, and then the other criterion might yield the larger value for the minimum allowable sectional area for member BE. Naturally, the criterion calling for the larger area is that which should be used in design. It is therefore desirable to develop a convenient method for choosing the criterion to be used in any specific design problem.

If the areas from equations (24) and (28) are set equal to each other and α_0 is plotted against σ_w/E, the relation may be represented by a family of straight lines through the origin, one for each value of η. A diagram of this type is given in figure 11. If the point $(\alpha_0, \sigma_w/E)$ lies on the line for the associated value of η, the areas computed by the two criteria will be the same. If that point should lie above the line for the associated value of η, the initial angle criterion will yield the larger area; while if it falls below that line, the simplified stability criterion is the more severe. Figure 11 can therefore be used to determine the criterion to be employed in the design of a truss like that of figure 10.

If the effects of an initial angle upon the design of a member such as BC in figure 12 are to be investigated, it can be seen that the support at A may be taken as fixed. The total effect of the deviations of actual from nominal dimensions may be reduced to a single small acute angle between member AB and the vertical.

If this angle is designated α_o, the usual method of truss analysis yields for the minimum allowable sectional area of member BC

$$A_{bc} = \frac{P_{ab}\,\alpha_o}{\sigma_w} \tag{29}$$

where A_{bc} is the required sectional area of member BC, P is the axial load in AB, and σ_w is the allowable working stress for member BC.

The simplified stability criterion for an ideal truss of this type is given by equation (21) and the area computed from it is

$$A_{bc} = \frac{P_{ab}}{E}\,\eta \tag{22}$$

where η is the ratio L_{be}/L_{ab}.

If equations (29) and (22) are set equal to each other, the relations between α_o, σ_w/E, and η will be represented by figure 11, though that figure was originally drawn up for a different truss pattern. Figure 11 can therefore be used in the design of a truss like that of figure 12 in the same manner as in that of a truss like the one shown in figure 10.

The method of investigating truss stability used in developing the criteria of this report differs considerably from that proposed by Von Mises and Ratzersdorfer in reference 5. It would be of interest to compare the results of applying these alternative methods to some specific truss designs. Limitation of time and personnel, however, prevented the inclusion of such a comparison in this report.

II. EXPERIMENTAL INVESTIGATION OF TRUSS STABILITY CRITERIA

Since no theoretical formula should be relied upon until its validity has been established by tests, the second part of the investigation covered by this report was devoted to the construction and the testing of a small pin-jointed truss to determine its actual critical load. The truss used was of the pattern shown in figures 1 and 8, the unstressed vertical BE being so designed that its stiffness could be varied over a considerable range.

Originally it was intended to compare the observed critical loads for this truss with those computed by Viscovich's criterion. While the computations to determine suitable sizes for the truss members were in progress it became evident that Viscovich's procedure was too complicated and tedious for practical design. It was also noticed that the differences in calculated extensibilities between those for unstressed verticals likely to produce instability and those of the other members were very great. Study of the formula for the effective spring constant of two springs in series (equation (25)) indicated that for practical purposes it would be reasonable to treat these differences as if they were between finite and infinite quantities. Thus if K is the effective spring constant of a pair of springs in series, one with a small spring constant K_1 and the other with a very large spring constant K_2, and K_1/K_2 is assumed negligible in comparison with unity,

$$K = \frac{K_1 K_2}{K_1 + K_2} = \frac{K_1}{1 + \frac{K_1}{K_2}} = K_1 \tag{30}$$

The simplified criterion was therefore developed as described in part I of this report.

Had Viscovich's criterion been used for determining the theoretical critical load for the test truss, it would have been necessary to determine the extensibility of each member. The development of the simplified criterion made this superfluous for all except the unstressed vertical, but it was decided to determine the extensibilities of all the members in order to have as complete information as possible on the properties of the test truss. Tests were therefore made to obtain three types of data: extensibilities of members subject to finite primary stress, stiffnesses of the member used for the unstressed vertical, and critical loads for the truss. This part of the report is the record of those tests.

TEST MATERIAL

The test specimen was a truss, of the pattern shown in figures 1 and 8, which was specially designed for the purpose. The tension members were 1/16- by 3/16-inch annealed tool steel and the compression members were 7/32- by 7/32-inch square polished drill rod, or 7/32-by 7/32-inch square cold rolled steel. Thus all the materials were comparatively soft, and easily machined to within 0.001 inch of nominal dimensions.

If the truss joints are lettered as in figure 8, joints A and D are located symmetrically to joints C and F about the midplane of the truss. The joints were so constructed that the resultant loads on the individual truss members were within about 0.001 inch of being coplanar and acting along the centroidal axes of the members. This was true although the members were not actually connected to single pins at joints A, C, and E. Figure 13 shows the truss assembled. Figures 14 and 15 show the construction of joint C in detail. Figure 16 shows joint D, and figure 17 shows joint B. Figure 18 shows joint E assembled and figure 19 shows the same joint with one of the plates removed.

The unstressed vertical, or critical member BE of figure 8, was so constructed that the axial stiffness could be varied. This member had two main elements, as can be seen from figure 20. The principal element was a steel rod bent 90° in two places to form a letter U. The other, called the spacing bar, could be set to produce a stiffness for the combination of almost any value from 2 pounds per inch up to about 70,000 pounds per inch. The U-shape element was made of 1/16-by 3/16-inch annealed tool steel. The spacing bar was made of 7/32-by 7/32-inch square polished drill rod. A loop of steel was provided over each end of the spacing bar, so that the legs of the U could be clamped against the ends of the spacing bar by set screws which were located in these loops. The outer surfaces of the legs of the U were center-punched at equal intervals so that the conical ends of the set screws could fit snugly into the conical center punch marks. After a stiffness had been determined for a certain set of corresponding center punch marks, it was always possible to regain that same stiffness by fitting the set screws into the same two marks. Thus it was possible to repeat experiments without remeasuring the stiffness of the member after each setting.

TEST APPARATUS

Apparatus for Measuring Extensibilities

Figure 21 shows the apparatus used for obtaining the extensibilities of the tension members. A 3- by 8-inch steel I-beam was erected with the outer face of one flange vertical, and a trussed cantilever bracket was bolted to its upper end. A fitting which was drilled and slotted to accommodate the 1/16- by 3/16-inch members of the truss was bolted to the free end of this cantilever bracket. The tension members were hung directly from this fitting and a similarly drilled and slotted fitting was provided at the bottom end of each such member. From a milled knife edge in this lower fitting there was hung a U-shape link of 1/4-inch steel rod which tended to reduce the flexural rigidity of the system. A 1/2- by 1/2-inch square piece of steel 2 inches long was drilled along a diagonal and held on the U member with a nut on each leg of the U. This acted as a sort of knife edge which supported a steel wire of about 3/64 inch diameter that was strung over it. The lower end of this wire supported the weight pan. Thus there were a total of four joints, which tended to eliminate almost all flexural rigidity of the load-applying system. Load was applied directly to the weight pan. Thus, except for an extremely minute amount of flexural rigidity, the tension load was applied vertically and axially to each member.

Two optical micrometers or microscopes, graduated to read to 1/28000 inch, were clamped to a piece of $1\frac{1}{4}$- by $1\frac{1}{4}$- by 1/4-inch steel angle and were set at a distance equal to the axial distance between the pins at the ends of the member being tested. This angle was supported by a structure independent of the rest of the apparatus so the loads on the specimen would not affect the distance between the microscopes. Under the usual increment of the load, about 25 pounds, the specimen as a whole moved measurably, due to the flexibility of the cantilever bracket. Thus if readings were taken from both micrometers at zero load, the total elongation for a given load would be the movement read at the bottom microscope minus the movement read at the top microscope.

When the square section compression members were tested this apparatus was modified, as shown in figure 22. The member was hung from the cantilever bracket end fitting by a short piece of 1/16- by 7/32-inch stock which fit into the milled slots in both the fitting and the member tested. The lower slotted fitting was connected to the lower end of the member by another short piece of 1/16- by 7/32-inch steel which fit into the slots of the fitting and the member tested. The other parts of the apparatus were the same as in the tension member tests. Tension loads were applied to these members, Young's modulus for tension and compression being assumed equal.

Apparatus for Measuring Stiffness of the Unstressed Vertical Member

The apparatus for measuring the spring constant of the U-shape member is shown in figure 23. The end of one leg of the U was rigidly supported against horizontal movement by a thick cast iron block and was supported vertically by a small steel block. The center of the bottom of the U was set on a small hard steel roller which rested on a hard steel block. The end of the other leg of the U also rested on a hard steel roller. The rollers eliminated almost all friction. The U was supported at these three points so that it lay in a horizontal plane.

The load was applied vertically to a weight pan, and was transmitted through a flexible string over an aluminum ball bearing V sheave to the horizontal direction. The top of the V sheave was set in the same horizontal plane as the U spring.

An optical micrometer, calibrated to read 0.000267 inch per division, was set over the end of the free leg of the U spring. Since it was assumed, for the small loads applied, that the end of the leg which bore directly on the cast iron block did not move at all, the measurement of the movement of the end of the free leg was taken as the change in distance between the two ends of the legs. The ratio of the load applied to the deflection observed was taken as the effective stiffness of the member BE.

Truss-Testing Apparatus

As the truss had rather small dimensions in a lateral direction, it was evident that it might become laterally unstable before becoming unstable in its plane. Since the objective was to determine its stability in its plane, it was necessary to provide lateral support. As shown in figure 13, a rectangular steel plate 1/2 inch thick was supported at each end of the bottom edge so that the longer edge was horizontal and the face of the plate was in a vertical plane. Two 120° V grooves were cut in the top edge of the plate, 24 ± 0.001 inch apart. Two machined and case-hardened knife edges were set into these grooves. A 10-inch long bar of 3/4- by 3/4-inch cold rolled steel hung from each knife edge and lay against the face of the plate. These bars hung vertically and gave the effect of having one end of the truss simply supported and the other on rollers, since they could rotate slightly as the truss deformed under load. Slots 7/32 + 0.010 inch wide were milled in the lower ends of these bars to accommodate the ends of the truss and to allow sufficient clearance for free movement of the 7/32- by 7/32-inch truss members. Holes were drilled in the bars the same size as the pins in the ends of the truss members. Each end of the truss was set in the milled slots and held there by the pins. A piece of 0.005-inch shim stock 7/32 inch in diameter was placed on each side of the truss member and drilled so that the pin held each shim in place. This assured clearance between the 7/32-inch truss members and the material on either side of the 7/32-+ 0.010-inch slot.

A brass lateral support was provided near each of the upper ends of the outside diagonal members. A single lateral support was provided just to the left of the center pin joint in the upper chord. Since this support had to have as low a coefficient of friction as possible, two knife edges of tool steel were made "dead hard" by heating and quenching without subsequent drawing. They were then polished and supported by brass fittings. The knife edges were spaced to give 0.001 inch-clearance for the upper chord member which moved between them. Thus the entire truss was laterally supported at five positions, this being the minimum number for a truss having such configuration and loading conditions. The friction forces caused by these lateral supports was very small compared to the loads in the truss members, and was ignored.

Two adjustable stops, clamped to the vertical plate which laterally supported the truss model, were provided above and below the upper chord members to prevent them from rotating through too great an angle while the truss was under load. The upper stop consisted of the spindle and thimble of a micrometer; the lower stop was the rounded end of an extra-fine-thread screw. The total movement of the portion of the upper chord between these stops could be measured to ±0.001 inch by reading the micrometer and then turning the thimble until the spindle pushed the member into contact with the lower stop. Light from a small flashlight was reflected from the supporting plate through the gap between the contact points of the stops and the horizontal member which moved vertically between them. In this way the first 0.0005 inch of movement of the member away from the stop could be observed.

TEST PROCEDURES AND RESULTS

Determination of Extensibilities

Each member was tested in direct tension while supported in the apparatus described. The weight pan and fittings between it and the lower end of the member tested were the only tare loads on the members. A reading of each optical micrometer was taken at zero load and at all subsequent loads. The usual load increment was 25 pounds and the usual maximum load was 150 pounds. There were, therefore, six points at which load and elongation were observed. The elongation was plotted against the load for each member and the slope of the straight line drawn through these points was then taken as the extensibility of the member. Four of these load-elongation curves, one from each pair of symmetrically located members, are shown in figures 24 and 25.

The extensibility of each member was measured in this manner at least twice. For each member the agreement between the measured extensibilities was within 2 percent. With each pair the difference between the average measured extensibilities for the individual members was less than the spread of the measured extensibilities for each of the pair. The average of all the measured extensibilities obtained from tests on both members of a pair was therefore taken as the extensibility of both of those members. The extensibilities thus obtained were: for members AB and BC, 8.9×10^{-6} inch per pound; for AD and CF, 9.30×10^{-6} inch per pound; for AE and CE, 36.43×10^{-6} inch per pound; and for DE and EF, 39.82×10^{-6} inch per pound.

Since the measured elongations were the changes in distance between the pins through the ends of the members and were affected by the sudden changes in section near those ends, no attempt was made to obtain a close comparison between the observed values and values computed from the dimensions of the members and an assumed value for Young's modulus. Approximate calculations, however, showed that the observed elongations were of reasonable magnitude. The spread of 2 percent between separate tests of a single member is assumed to be an effect of imperfections in the test apparatus rather than a measure of deviations from some assumed ideal nominal dimensions. It is of interest to note that this spread between the measured extensibilities of individual members is considerably greater than the computed error in the critical load for the truss investigated in part I of this report. This is one of the factors which justifies the use of the simplified criterion instead of the more precise but more tedious criteria for which it is offered as a substitute.

Determination of the Stiffness of the U Member BE

The stiffness of the U member was measured with the spacing bar at each one of the five sets of center punch marks in the legs of the principal element. The set screws at each end of the spacing bar were tightened into corresponding center punch marks and the entire member was placed in the apparatus for measuring the deflection of one leg with respect to the other with each increment of load. For each increment of load, usually an ounce, the increment of deflection of the free leg was observed through the optical micrometer and readings of load and deflection were recorded. The set screws were then loosened, the spacing bar was moved to the next set of corresponding center punch marks, and a new set of deflections and loads was recorded. This procedure was repeated five times tnd the results of the observations are plotted with load versus deflection. The experimental spring constant then is the slope of the loading-deflection curve. The five curves obtained are shown in figure 26.

After having gone through a series of tests to determine the spring constants, the procedure was completely repeated to determine whether the assumption that the spring constant would be the same after changing the spacing bar's position actually was justifiable. It was found by these check tests that the observed values of the spring constants changed negligibly when the spacing bar was taken from one set of center punch marks and then returned to the same marks after other tests upon the U member had been completed.

Determination of the Critical Load for the Truss

For determining the critical load for the truss it was first assembled and placed in the testing apparatus. The effects of friction in the joints were investigated and it was found that, when the truss was carefully loaded, if the upper chord members were carefully alined after each increment of load and there were no vibrations or other disturbing forces, the truss without the center vertical could be loaded up to the ultimate for the material used. The ability to carry such load in this condition was due to the very accurate alinement of the three pins in the upper chord members and the small amount of friction in the pin joints. Although it was possible for the truss to be in equilibrium without the center vertical, the structure was not stable in this condition.

When a structure like a truss is subjected to load and is deformed to a configuration in which all forces are in equilibrium, that configuration may be termed the equilibrium configuration for the given loading. The stability status under these conditions depends on what happens when the configuration is slightly modified. If there is a tendency to return to the equilibrium configuration, the equilibrium is stable; if there is a tendency to remain in the new configuration, the truss is in neutral equilibrium; and if the change in configuration tends to become more pronounced, the equilibrium is unstable. The tendency regarding return to the equilibrium position is a function of the load in actual structures and, as the load on a structure increases, the tendency to return decreases until it changes into a tendency to deflect further. The load at which the tendency to return to the equilibrium position disappears is taken as the critical load. In these tests only this tendency to return to the equilibrium could be investigated.

In each test the U member was adjusted to a given stiffness and then was placed in the truss. The truss was next loaded to within a few pounds of the critical load calculated for the structure by the simplified stability criterion. The upper chord members were then so rotated by hand that the center pin joint connecting these members was moved vertically 0.030 inch in each direction from the mean equilibrium position, the top and bottom stops having been adjusted so that the rotations could not exceed these values. This involved a rotation of the members of about ±0.005 radian. When the members were rotated while the load on the truss was below the calculated critical load, a perceptible amount of spring back from the stops was observed. As the load was increased, the amount of spring back decreased; and when the spring back completely disappeared, the total magnitude of the load was recorded as an observed critical load.

This procedure was repeated until the complete series of five stiffnesses of the U member had been used in the truss, and critical loads corresponding to these stiffnesses had been observed. The results of the tests are summarized in the table below.

Stiffness of U member, K (lb/in.)	Calculated critical load (lb)	Observed critical load (lb)	
		(max.)	(min.)
6.09	24.36	24.8	23.8
8.36	33.44	34.2	31.8
11.54	46.16	47.25	45.0
16.39	65.56	67.0	64.0
26.00	104.00	106.0	102.0

It is interesting to note that the absolute difference between the observed critical load and that calculated by the simplified criterion from the given stiffnesses of the critical member increased directly as the load; the percentage difference remained approximately constant. This may be explained as mostly due to the effect of the friction in the joints A, B, and C of the members AB and BC. Since the forces transmitted through the pins to the holes at these joints are proportional to the load on the truss, if the static coefficient of friction is assumed to be constant, the friction forces vary directly with the load on the truss. It was found that rotating the upper chord members by hand and observing the tendency to spring back tended to eliminate most of the frictional effects, but the spread of the test results indicates that the elimination was not complete.

From the simplified stability criterion it is evident that the critical load on the truss should be directly proportional to the stiffness of the U member. Therefore, if the stiffness of the critical member is plotted against the observed critical load, the resulting curve should be a straight line. Figure 27 shows the agreement of the observed critical loads, for the various stiffnesses of the unstressed vertical, with the calculated critical loads computed from the same stiffnesses. The short horizontal lines intersecting the diagonal indicate the range of observed critical loads for each stiffness of the vertical.

CONCLUSIONS

1. The simplified stability criteria developed in part I provide a convenient tool for investigating the stability of pin-jointed trusses against buckling in the truss plane. They also provide a rational method for designing those members of a truss for which the axial loads computed by the standard methods of analysis are very small.

2. The simplified criteria are applicable when the loads in the truss members are due primarily to deviations of actual from nominal dimensions.

3. The tests of part II indicate that the simplified criteria of part I are valid.

Stanford University,
Stanford University, Calif., March 30, 1944.

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APPENDIX

COMPUTATION OF THE VERTICAL STIFFNESS OF JOINTS A, C, AND E OF THE TRUSS OF FIGURE 8

Figure A-1 is a line diagram of the truss of figure 8. Alongside each member are listed in order: the computed extensibility of the member in inches per pound multiplied by 10^9, the axial load due to 80 kips at joint E, the axial load due to a unit load at joint A, and the axial load due to a unit load at joint E. In the design of the tension members the allowable stress was taken as 30,000 psi. The compression members were assumed square in cross section and were designed by the Euler formula as pin-end columns, with E = 30,000,000. Formulas for the sectional area required were developed as follows:

Tension members: $A_{xy} = \frac{P_{xy}}{\sigma_w} = \frac{P_{xy}}{30000} = 33.33 \times 10^{-6} P_{xy} \text{ in.}^2$

Compression members: $P = \frac{\pi^2 EI}{L^2}$; $I = 33.77 \times 10^{-10} PL^2 \text{ in.}^4$

but $I_{xy} = \frac{b^4}{12}$ for a square cross section

$$b^4 = 12I = 12 \times 33.77 \times 10^{-10} PL^2$$

$$b^4 = 405.2 \times 10^{-10} PL^2$$

$E = 30 \times 10^6 \text{ lb/in.}^2$

With these formulas the extensibilities were computed as indicated in table A-1.

Table A-1

Member	P_{xy}	L	L^2	b^4	$b^2=A_{xy}$	$\frac{L}{AE}\times 10^9$
AB	-60,000	180	32,400	78.78	8.876	676.0
BC	-60,000	180	32,400	78.78	8.876	676.0
AD	-50,000	300	90,000	182.36	13.504	740.5
CF	-50,000	300	90,000	182.36	13.504	740.5
AE	50,000	300	90,000	------	1.667	5,999.0
CE	50,000	300	90,000	------	1.667	5,999.0
DE	30,000	360	129,600	------	1.000	12,000.0
EF	30,000	360	129,600	------	1.000	12,000.0
BE	0	240	129,600	------	------	--------

Computations of the vertical stiffnesses of joints A, C, and E by the method of virtual work are outlined in tables A-2 and A-3.

Table A-2

Joint A or C

Member	p_a	p_a^2	$\frac{p_a^2 L}{AE}\times 10^9$	
AB	0.3750	0.1406	95.0	$\delta_x = 10^{-9}\ \frac{p_x^2 L}{AE}\ 10^9$ in.
BC	.3750	.1406	95.0	
AD	.9375	.8789	650.8	
CF	.3125	.0977	73.3	$K_x = \frac{1}{\delta_x}$ lb/in.
AE	.3125	.0977	586.1	
CE	-.3125	.0977	586.1	
DE	-.5625	.3164	3,796.8	$\delta_a = 6305.5 \times 10^{-9}$ in/1 lb
EF	-.1875	.0352	422.4	load.
BE	0	0	0	
			--------	$K_a = K_c = 158,591$ lb/in.

$$\sum \frac{p_a^2 L}{AE} \times 10^9 = 6,305.5$$

Table A-3

Joint E

Member	p_a	p_a^2	$\frac{p_a^2 L}{AE} \times 10^9$	
AB	-0.7500	0.5625	380.3	δ_e = 9399.6 × 10^{-9} lb/1 lb load.
BC	-.7500	.5625	380.3	
AD	-.6250	.3906	289.2	K_e = 106,400 lb/in.
CF	-.6250	.3906	289.2	
AE	.6250	.3906	2,343.1	
CE	.6250	.3906	2,343.1	
DE	.3750	.1406	1,687.2	
EF	.3750	.1406	1,687.2	
BE			--------	

$$\sum \frac{p_e^2 L}{AE} \times 10^9 = 9,399.6$$

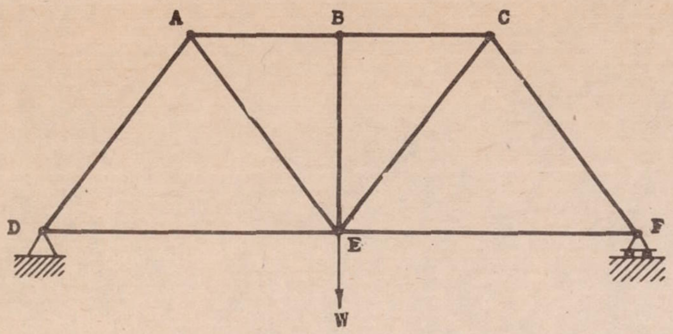

Figure 1.- Typical truss.

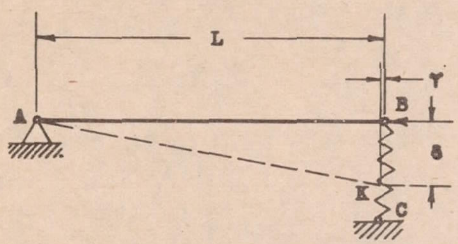

Figure 2.- Link with one elastically supported end.

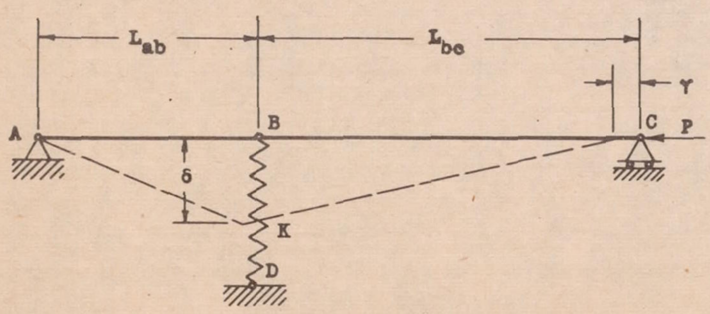

Figure 3.- Two links with single elastic support.

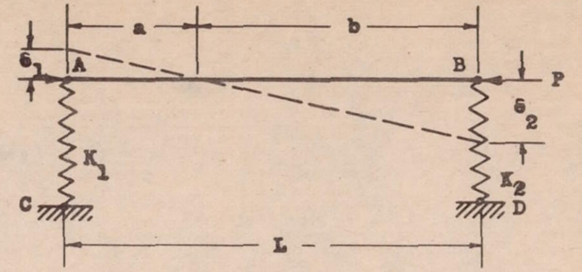

Figure 4.- Link elastically supported at both ends.

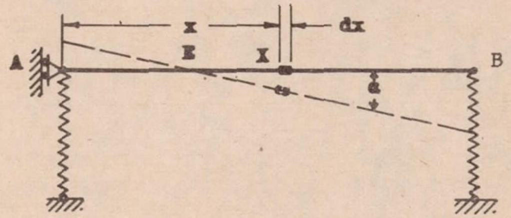

Figure 5.- Link of figure 4 with horizontal restraint.

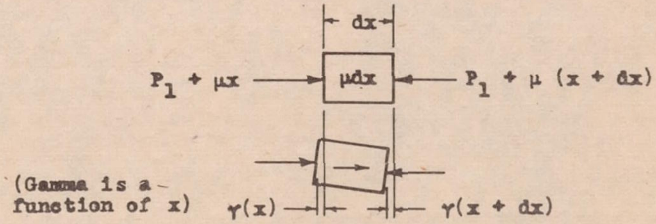

Figure 6.- Forces on element of link of figure 5.

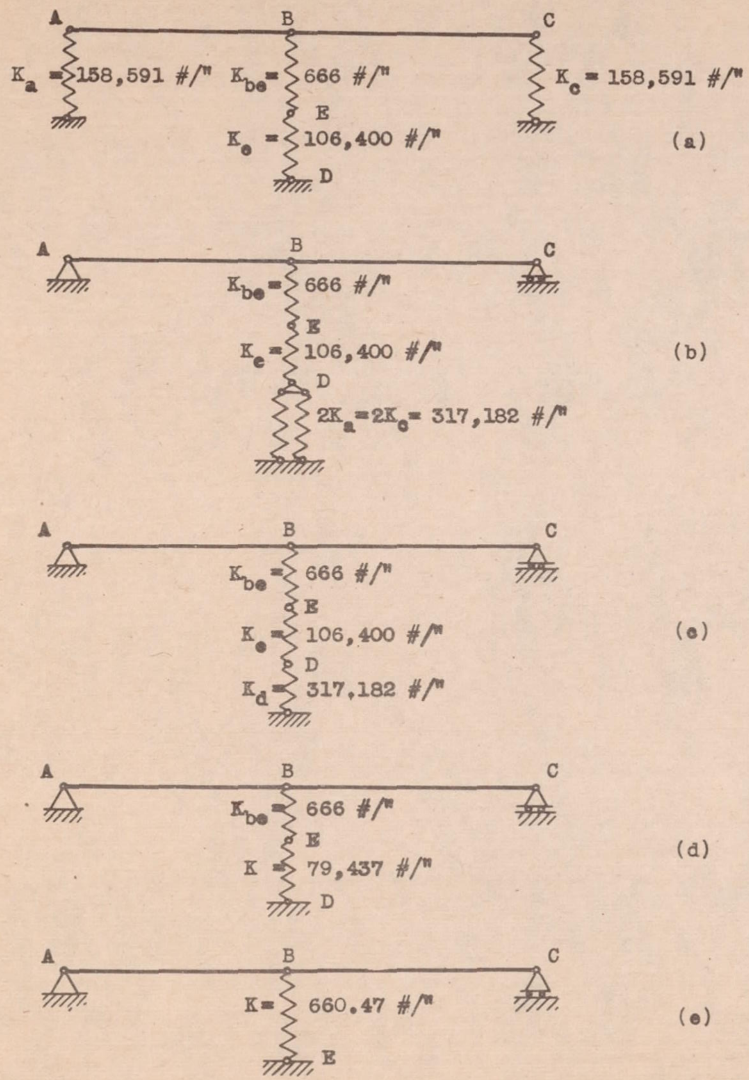

Figure 9.- Equivalent spring systems.

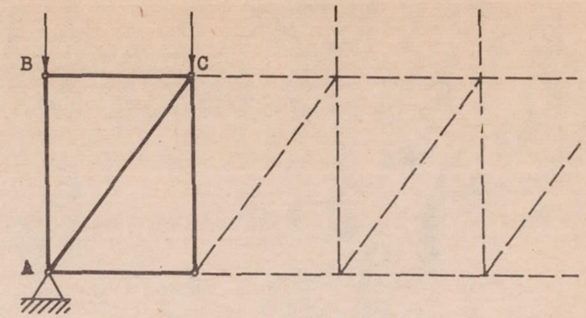

Figure 7.- Portion of N truss.

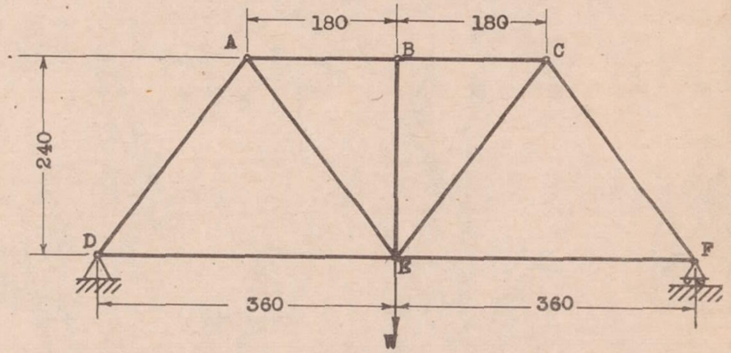

Figure 8.- Truss investigated.

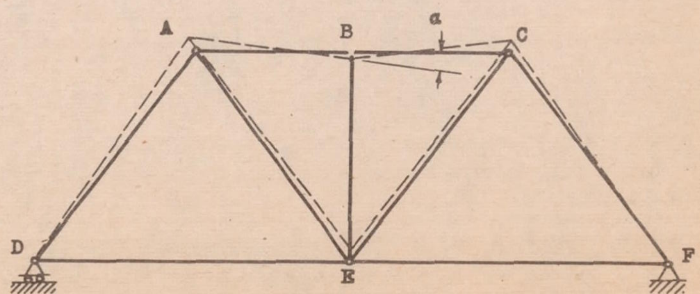

Figure 10.- Truss with initial deformation.

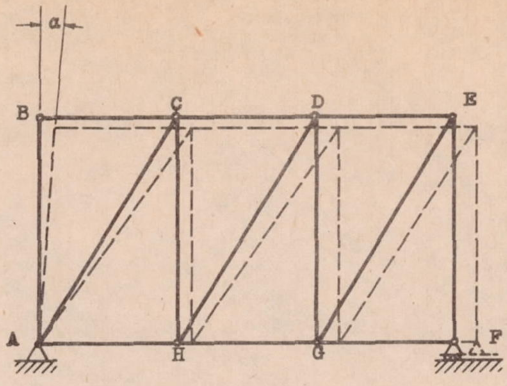

Figure 12.- Initially deformed N truss.

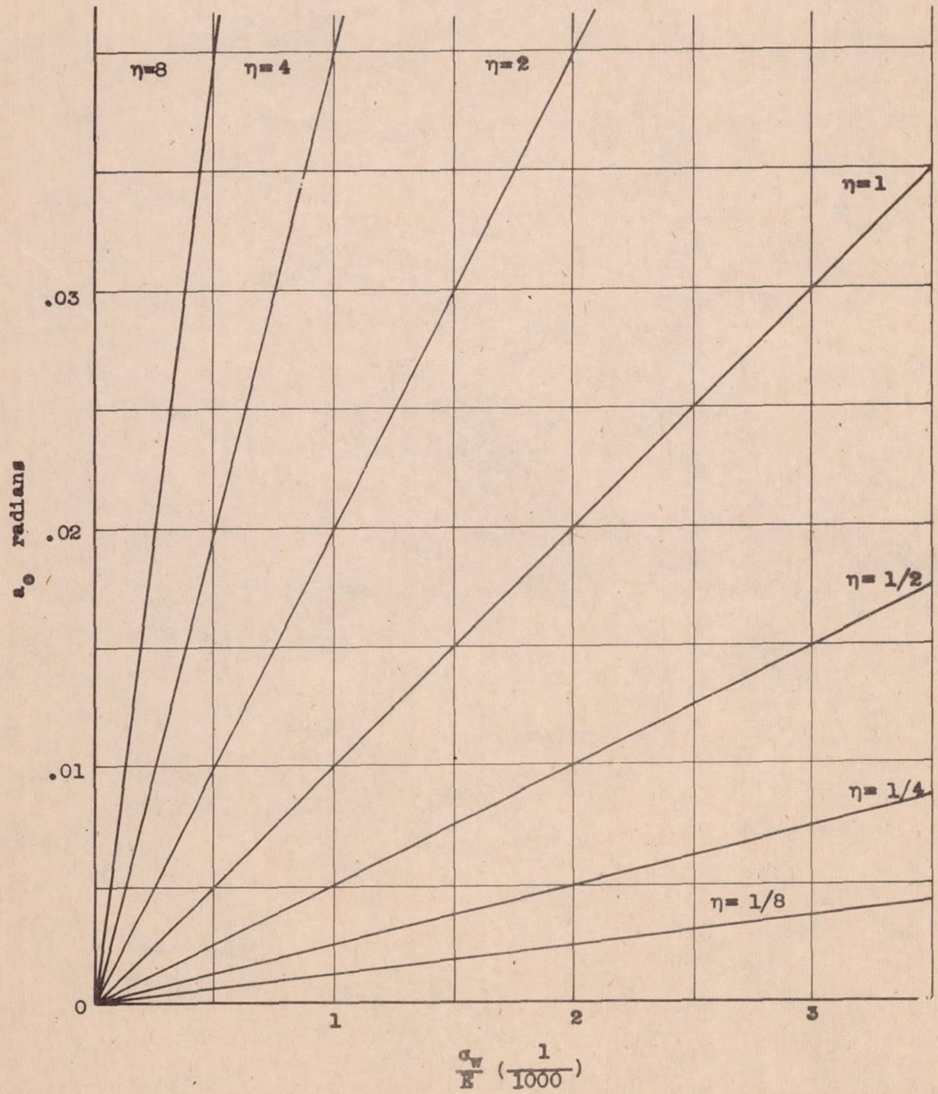

Figure 11.- Curves for determining applicable criterion.

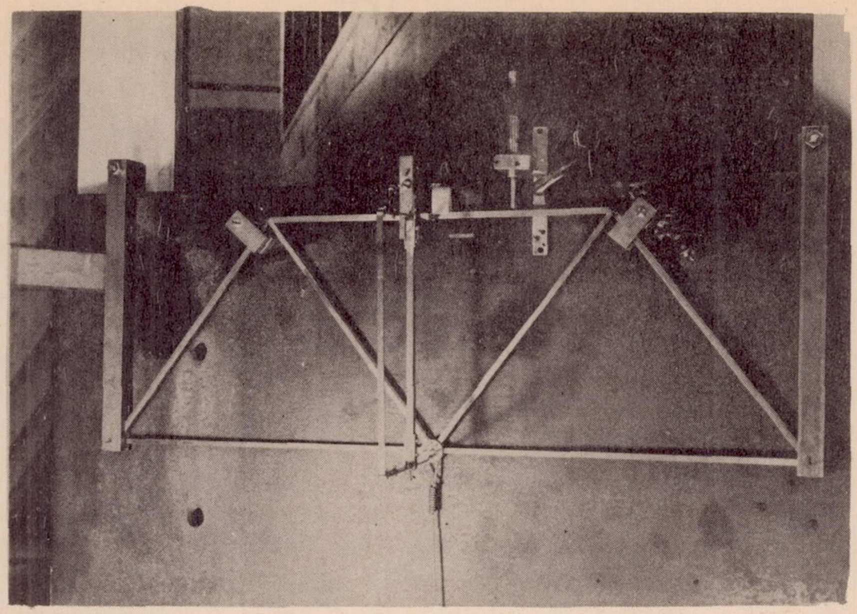

Figure 13.- Experimental truss.

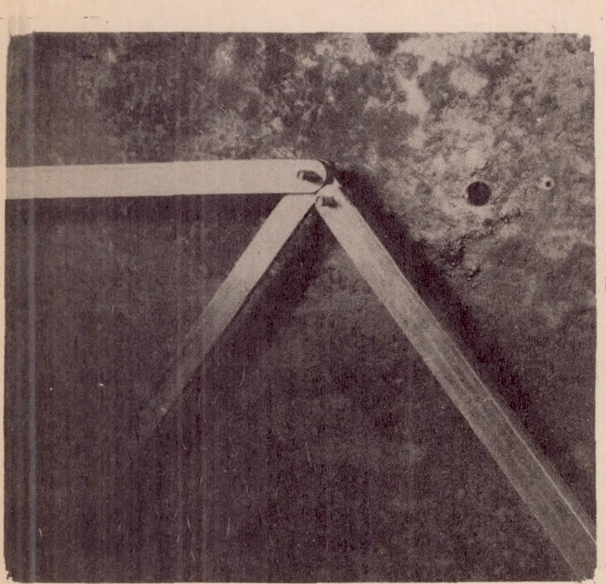

Figure 14 Joint C, assembled

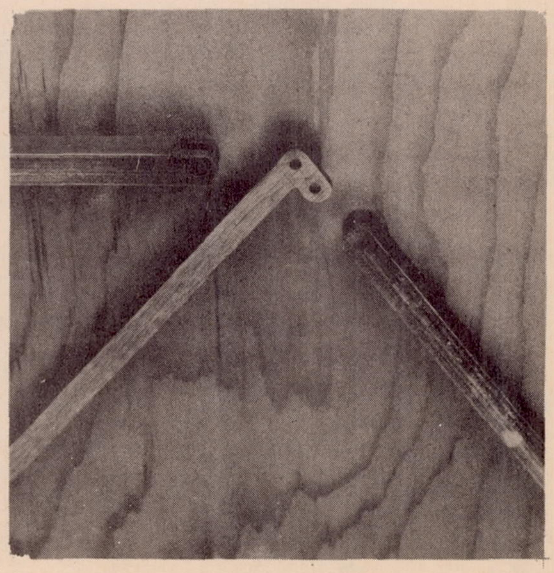

Figure 15.- Joint C, exploded.

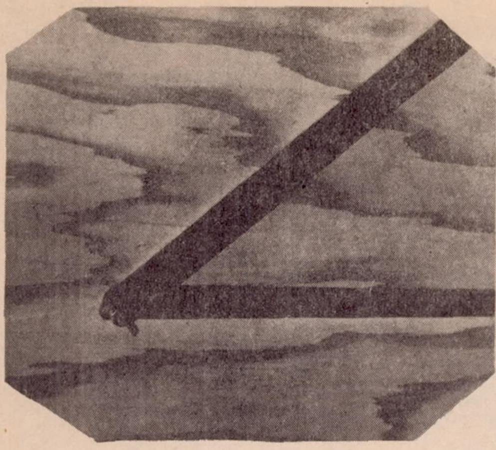

Figure 16.- Joint D.

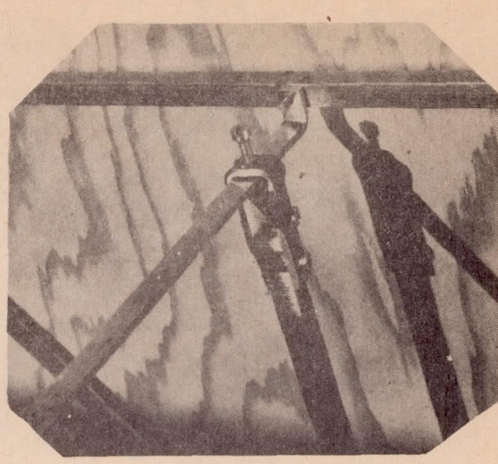

Figure 17.- Joint B.

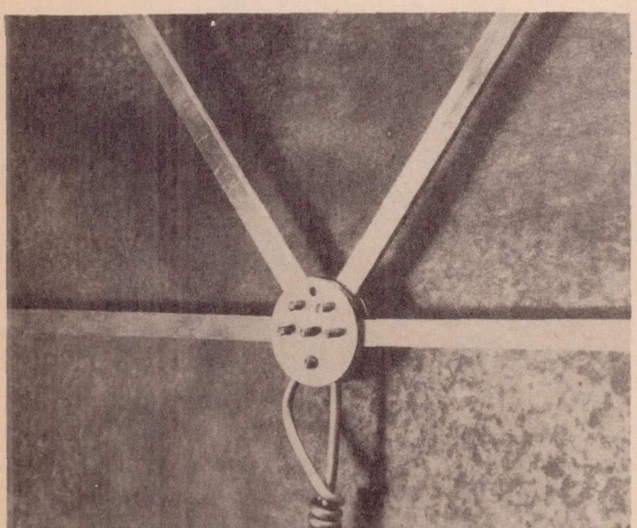

Figure 18.- Joint E, complete.

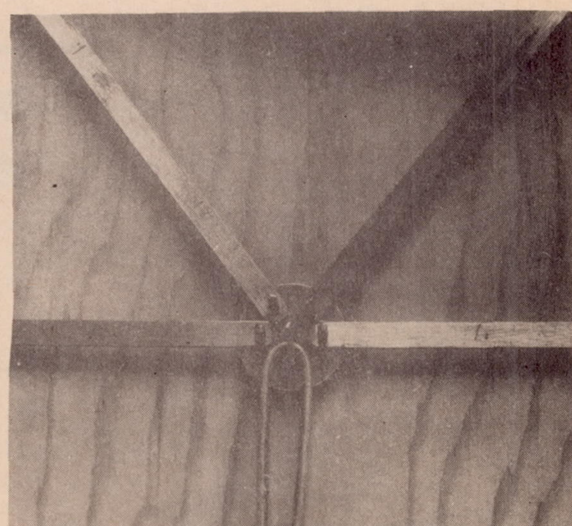

Figure 19.- Joint E, gusset plate removed.

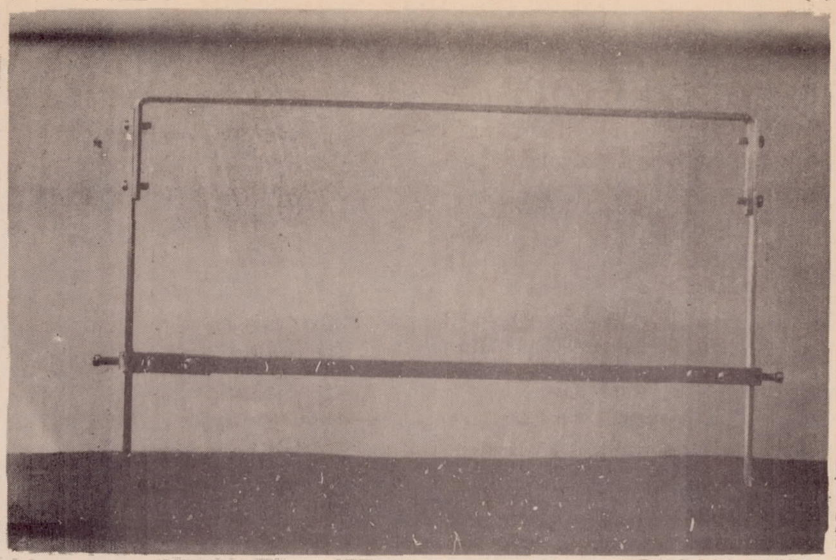

Figure 20.- Variable stiffness member BE.

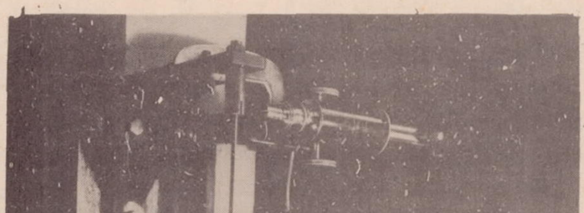

Figure 21.- Extensibility test, tension member.

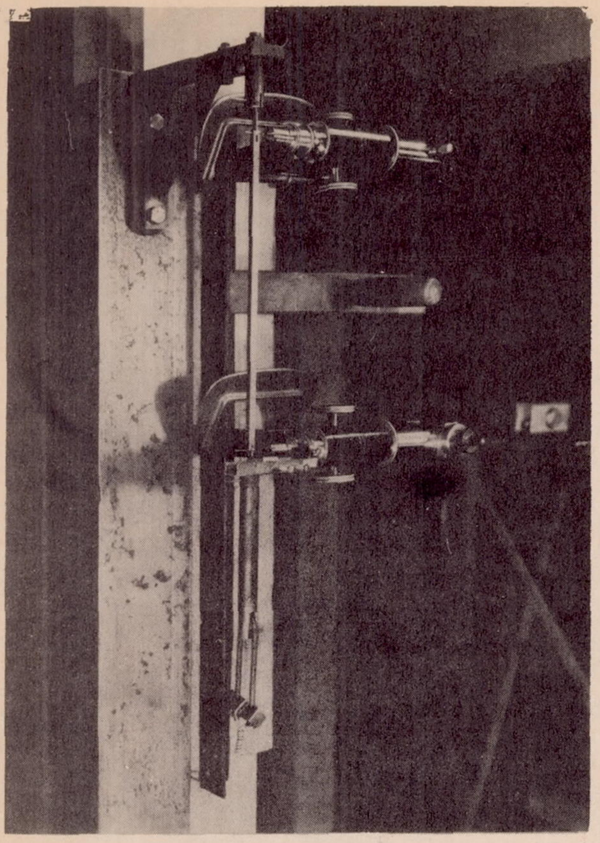

Figure 22.- Extensibility test, compression member.

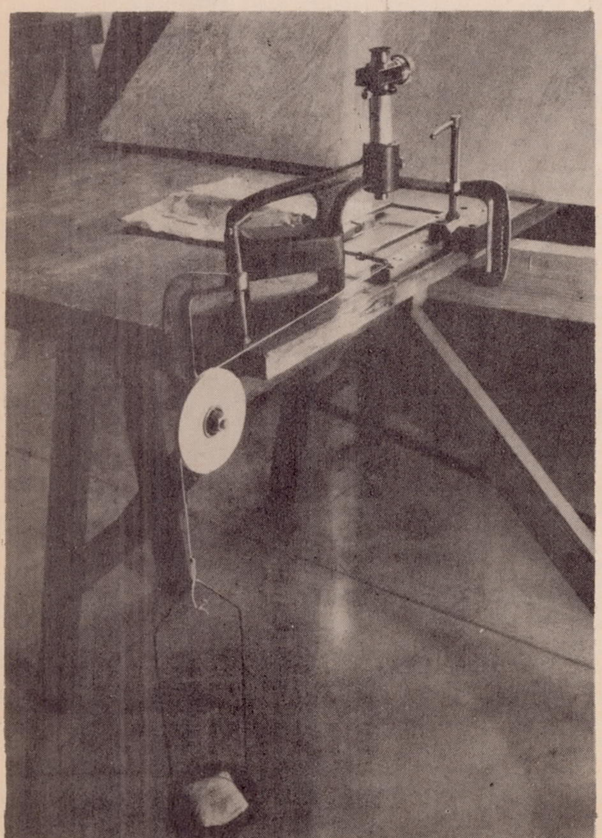

Figure 23.- Stiffness test, member BE,

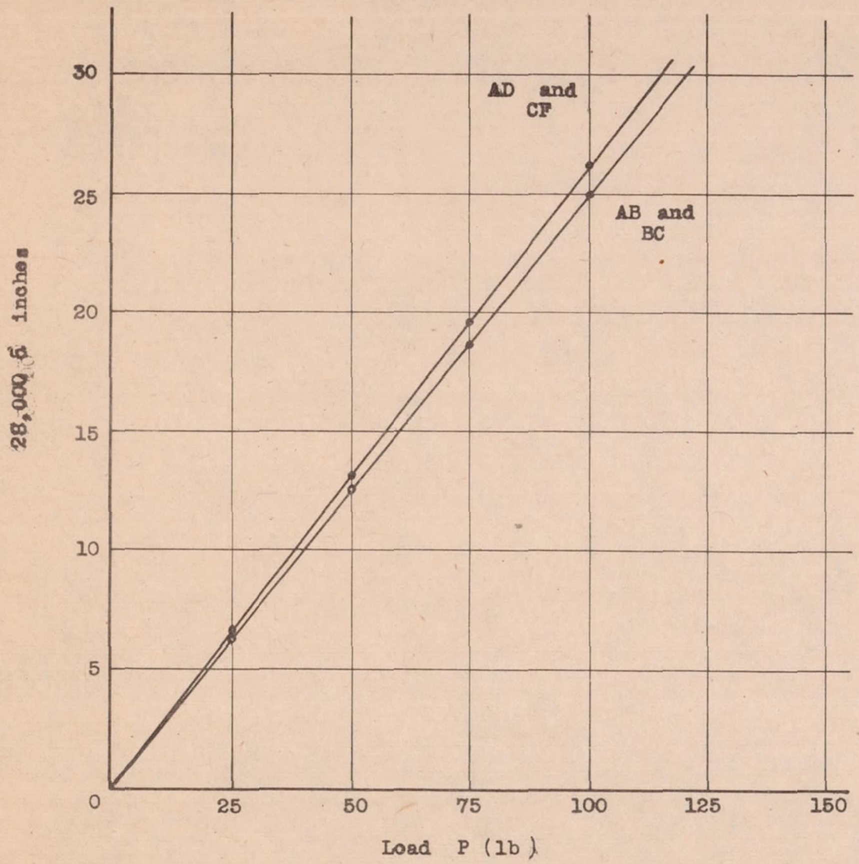

Figure 24.- Load-elongation curves, compression members.

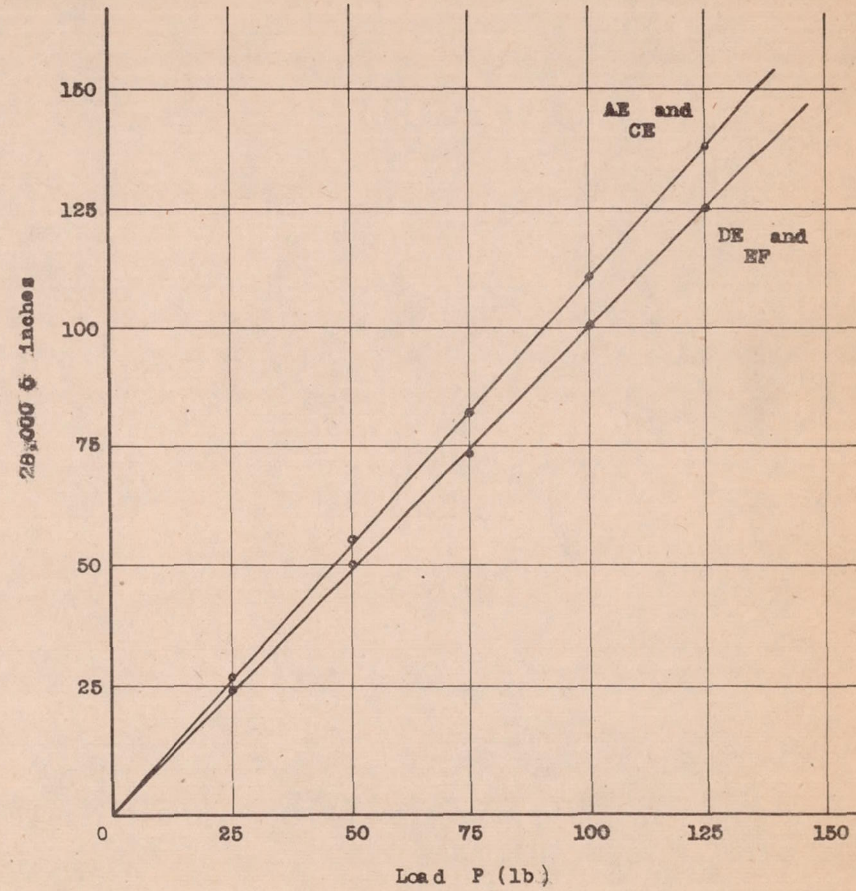

Figure 25.- Load-elongation curves, tension members.

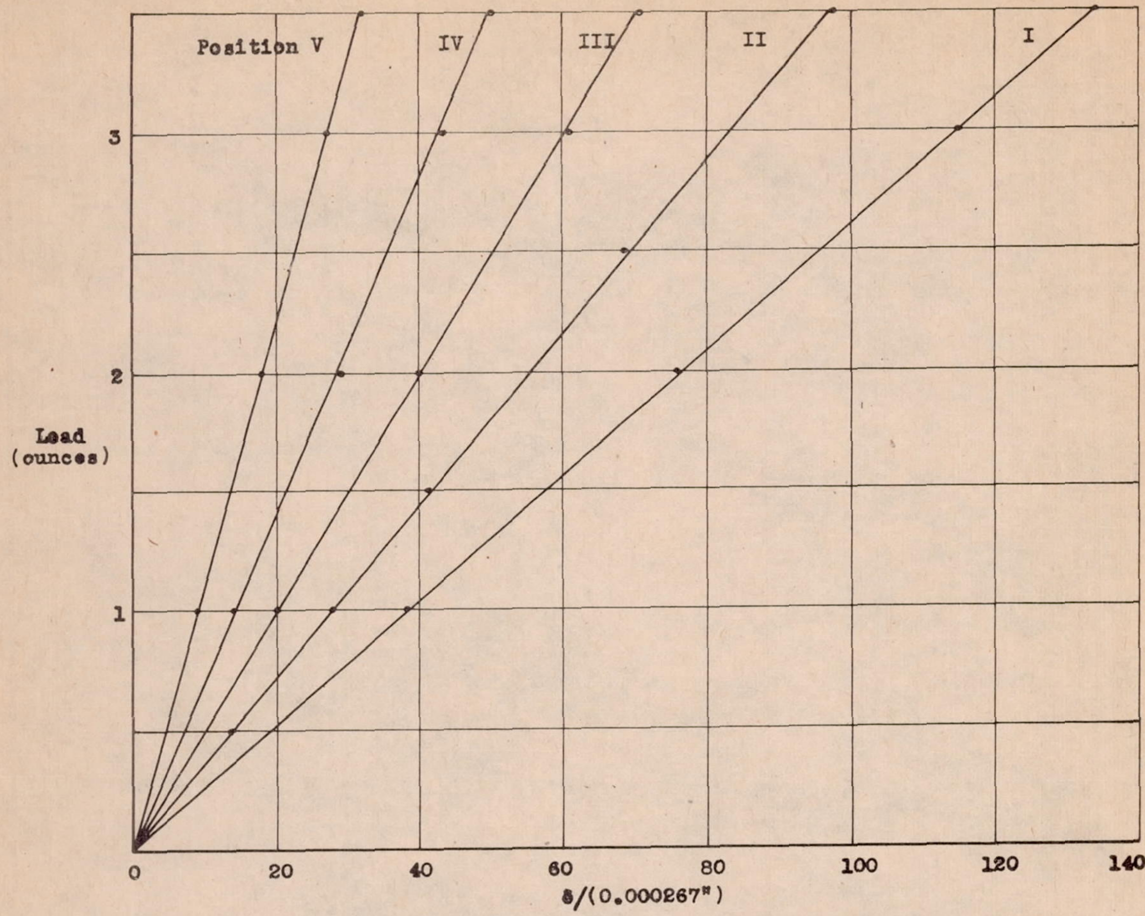

Figure 26.- Load-elongation curves, member BE.

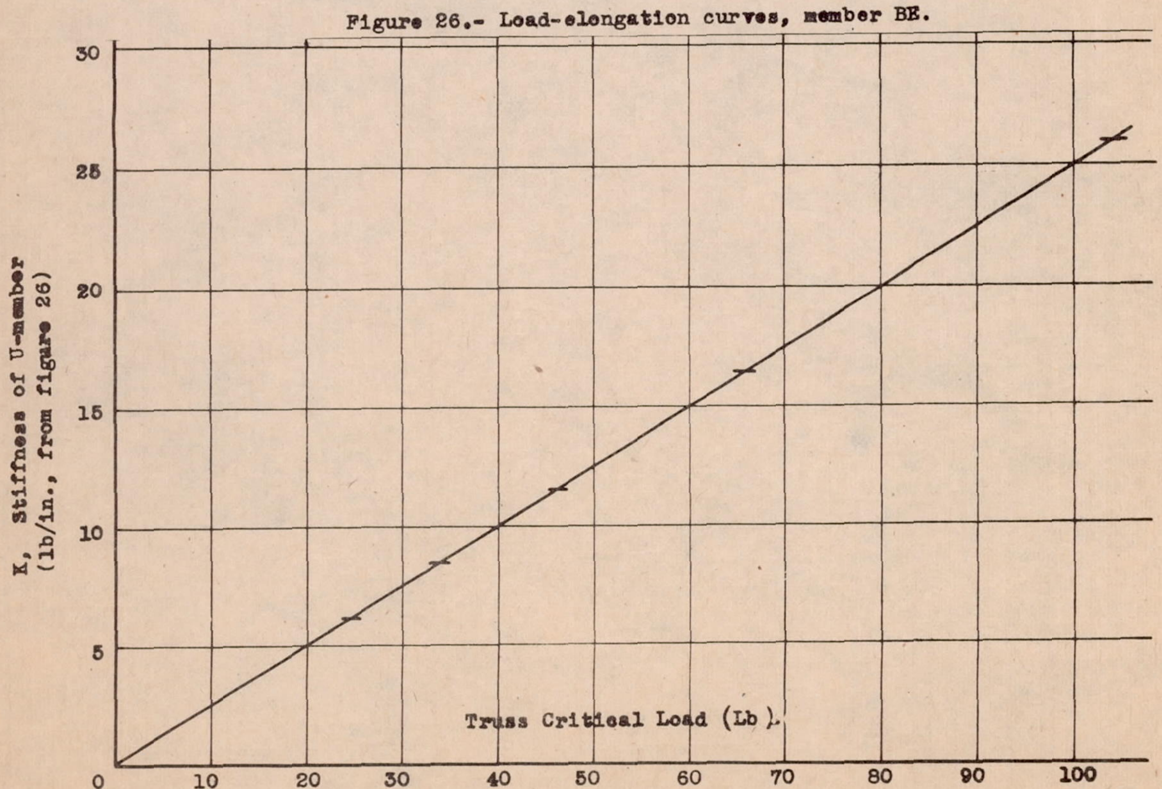

Figure 27.- Truss critical load versus stiffness of member BE.

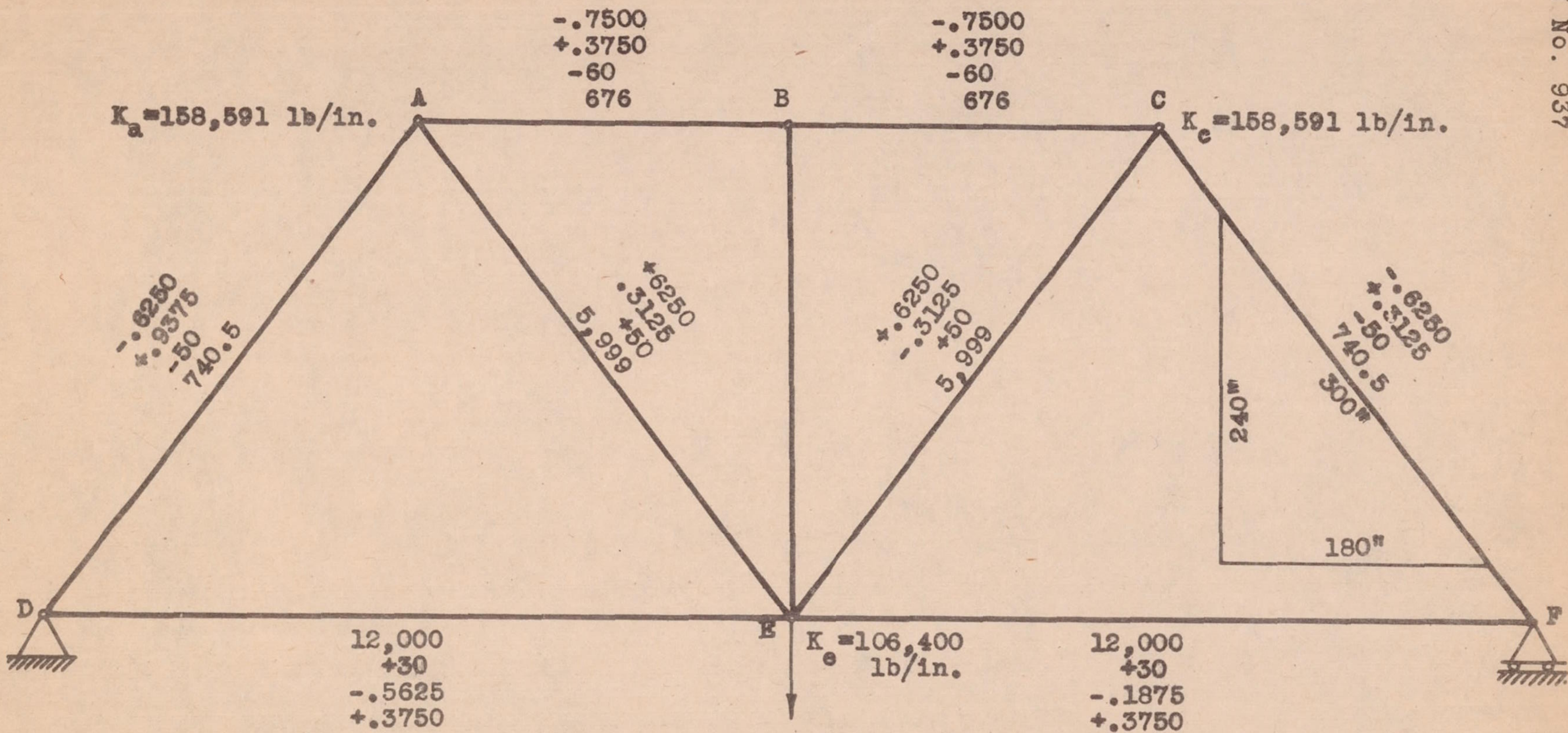

Figure A-1.- Extensibilities and unit load systems for truss of figure 8.

Extensibilities x10^9 are noted nearest each member, then forces due to 80-kip load at center, then forces due to unit load at A and finally forces due to unit load at E.