NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1073

NOTE ON THE THEOREMS OF BJERKNES AND CROCCO

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SUMMARY

The theorems of Bjerknes and Crocco are of great interest in the theory of flow around airfoils at Mach numbers near and above unity. A brief note shows how both theorems are developed by short vector transformations.

INTRODUCTION

The flow field at supercritical velocities may be obtained by inserting the required shock wave or waves and adjusting the fields to satisfy all boundary conditions. If the shock waves are inserted by proper procedure, there will exist a potential field up to the first shock wave and a field with rotation extending between and behind the shock waves. The theorems of Bjerknes and Crocco (reference 1) refer to such rotational fields.

SYMBOLS

$\bar{q}$	fluid velocity vector
ρ	density
p	pressure
R	gas constant
T	absolute temperature
T_o	stagnation temperature

k adiabatic constant

S entropy

r distance from axis of symmetry

c_p specific heat at constant pressure

THEOREM OF BJERKNES

The equation of motion of a fluid for the stationary case is

$$\overline{q} \times \text{curl } \overline{q} = \frac{1}{\rho}\nabla p + \frac{1}{2}\nabla q^2 \tag{1}$$

For a perfect gas

$$p = \rho RT$$

and

$$\frac{1}{2}q^2 = c_p(T_o - T)$$

where T_o is the constant stagnation temperature and c_p is the specific heat at constant pressure. Equation (1) hence transforms to

$$\frac{\overline{q} \times \text{curl } \overline{q}}{T} = R\frac{\nabla p}{p} + c_p\frac{\nabla(T_o - T)}{T} \tag{2}$$

The right-hand side of equation (2) is seen to be

$$\nabla\left(\log p^R - \log \nabla T^{c_p}\right) = \nabla \log\left(\frac{p^R}{T^{c_p}}\right) = R\nabla \log \frac{p}{T^{\frac{k}{k-1}}} = \nabla S$$

where S is the entropy. Thus

$$\frac{\overline{q} \times \text{curl } \overline{q}}{T} = \nabla S \tag{3}$$

which is the theorem of Bjerknes.

THEOREM OF CROCCO

The theorem of Crocco may be obtained from the theorem of Bjerknes by a vector transformation. Since $T = \frac{p}{\rho R}$, the left-hand side of equation (3) may be written

$$\rho\bar{q} \times \frac{\text{curl } \bar{q}}{p} R$$

Take the curl of this expression, omitting the constant R:

$$\text{curl}\left(\rho\bar{q} \times \frac{\text{curl } \bar{q}}{p}\right) = \rho\bar{q} \text{ div}\left(\frac{\text{curl } \bar{q}}{p}\right) - \frac{\text{curl } \bar{q}}{p} \text{ div } \rho\bar{q} + \left(\frac{\text{curl } \bar{q}}{p} \cdot \nabla\right)\rho\bar{q} - (\rho\bar{q} \cdot \nabla)\frac{\text{curl } \bar{q}}{p}$$

For two-dimensional flow the three first terms on the right of this equation are identically zero since the vector curl $\bar{q}$ is perpendicular to the vector $\bar{q}$ and since div $\rho\bar{q} = 0$. With reference to equation (3) it is therefore seen that

$$(\rho\bar{q} \cdot \nabla)\frac{\text{curl } \bar{q}}{p} = 0$$

or that $\frac{\text{curl } \bar{q}}{p}$ is constant along the streamlines. The useful result obtained then is that the rotation remains proportional to the absolute pressure p along each and all streamlines. This is the theorem of Crocco.

Similarly, for flow with rotation symmetry, the following expression may be written:

$$\rho\bar{q}r \times \frac{\text{curl } \bar{q}}{pr} R$$

where r is the radius from the center of symmetry. By the same reasoning $\frac{\text{curl } \bar{q}}{pr}$ is shown to be constant along each streamline. This expression, when translated into words, means that the rotation along a streamline is proportional to the pressure times the distance from the axis of symmetry.

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REFERENCE

1. Crocco, L.: Eine neue Stromfunktion für die Erforschung der Bewegung der Gase mit Rotation. Z.f.a.M.M., Bd. 17, Heft 1, Feb. 1937, pp. 1-7.