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No. 1344

CRITICAL STRESS OF THIN-WALLED CYLINDERS IN TORSION

By S. B. Batdorf, Manuel Stein, and Murry Schildcrout

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Langley Field, Va.

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SUMMARY

A theoretical solution is given for the critical stress of thin-walled cylinders loaded in torsion. The results are presented in terms of a few simple formulas and curves which are applicable to a wide range of cylinder dimensions from very short cylinders of large radius to long cylinders of small radius. Theoretical results are found to be in somewhat better agreement with experimental results than previous theoretical work for the same range of cylinder dimensions.

INTRODUCTION

For most practical purposes the solution to the problem of the buckling of cylinders in torsion was given by Donnell in an important contribution to shell theory published in 1933 (reference 1). The present paper, which gives a solution to the same problem, has two main objectives: first, to present a theoretical solution of somewhat improved accuracy; second, to help complete a series of papers treating the buckling strength of curved sheet from a unified viewpoint based on a method of analysis essentially equivalent to that of Donnell but considerably simpler. (See, for example, references 2 and 3.)

The method of solution in the present paper is that developed in reference 3. The steps in the theoretical computations of the critical stress are contained in the appendix. The results are given in the form of nondimensional curves and simple approximate formulas which follow these curves closely in the usual range of cylinder dimensions.

SYMBOLS

j,m,n	integers
p	arbitrary constant
r	radius of cylinder
t	thickness of cylinder wall
u	axial component of displacement; positive in x-direction
v	circumferential component of displacement; positive in y-direction
w	radial component of displacement; positive outward
x	axial coordinate of cylinder
y	circumferential coordinate of cylinder
D	flexural stiffness of plate per unit length $\left(\frac{Et^3}{12(1-\mu^2)}\right)$
E	Young's modulus
L	length of cylinder
Q	mathematical operator defined in appendix
Z	curvature parameter $\left(\frac{L^2}{rt}\sqrt{1-\mu^2} \text{ or } \left(\frac{L}{r}\right)^2 \frac{r}{t}\sqrt{1-\mu^2}\right)$
a_n, b_n	coefficients of deflection functions
k_s	critical shear-stress coefficient appearing in formula $\tau_{cr} = k_s \frac{\pi^2 D}{L^2 t}$

$$M_n = \frac{\pi}{8\beta}\left[(n^2 + \beta^2)^2 + \frac{12Z^2n^4}{\pi^4(n^2 + \beta^2)^2}\right]$$

V_m,W_m	deflection functions defined in appendix

$\beta = \frac{L}{\lambda}$

λ half wave length of buckles in circumferential direction

μ Poisson's ratio

τ_{cr} critical shear stress

$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\, \partial y^2} + \frac{\partial^4}{\partial y^4}$

∇^{-4} inverse of ∇^4, defined by $\nabla^{-4}\nabla^4 w = w$

RESULTS AND DISCUSSION

The critical shear stresses for cylinders are obtained from the equation

$$\tau_{cr} = k_s \frac{\pi^2 D}{L^2 t}$$

The values of k_s for cylinders with either simply supported or clamped edges are given in the form of logarithmic plots in figure 1. The ordinate in this figure is the critical shear-stress coefficient k_s The abscissa is a curvature parameter Z which is given directly by the theory and involves the dimensions of the cylinder and Poisson's ratio.

For very short cylinders the value of the shear-stress coefficient approaches the values for flat plates, 5.34 when the edges are simply supported and 8.98 when the edges are clamped. As Z increases k_s also increases and the curves which defined k_s are given approximately by straight lines. For simply supported cylinders,

$$k_s = 0.85\, Z^{3/4}$$

For cylinders with clamped edges,

$$k_s = 0.93\, Z^{3/4}$$

The range of validity of these formulas is approximately
$100 < Z < 10 \frac{r^2}{t^2}$.

For the case of long cylinders the curves of figure 1 split into a series of curves depending upon the radius-thickness ratio. These curves, which correspond to buckling of the cylinder into two circumferential waves $(n = 2)$, depart from the straight lines at approximately $Z = 10\frac{r^2}{t^2}$ or approximately $\frac{L}{r} = 3\sqrt{\frac{r}{t}}$. Because the critical shear stress of a long cylinder is almost independent of end conditions, the curves for different values of r/t apply both to cylinders with simply supported edges and to cylinders with clamped edges. These curves are probably somewhat inaccurate, however, because one of the requirements for the validity of the simplified equation of equilibrium used is that $n^2 >> 1$. A calculation for long cylinders made by Schwerin and reported in reference 1 by Donnell suggests that all values corresponding to the curves given in the present paper for $n = 2$ are slightly high.

In figure 2 the results of the present paper are compared with those given by Donnell (reference 1) and Leggett (reference 4). The present solution agrees quite closely with that of Donnell except in the transition region between the horizontal part and the sloping straight-line part of the curves. In this region the present results are appreciably less than those of Donnell (maximum deviation about 17 percent) but are in close agreement with Leggett's results, which are limited to low values of Z.

In figure 3 the present solution and that of Donnell for the critical shear stress of simply supported cylinders are compared on the basis of agreement with test results obtained by a number of investigators. (See references 1, 5, 6, and 7.) The curves giving the present solution are appreciably closer to the test points. More than 80 percent of the test points are within 20 percent of the values corresponding to the theoretical curve for simply supported cylinders given in the present paper, and all points are within 35 percent of values corresponding to the curve.

In figure 4 the present solution for critical shear-stress coefficients of long cylinders which buckle into two half waves is given more fully than in figure 1 and is compared with test results of references 1 and 8.

The computed values from which the theoretical curves presented in this paper were drawn are given in tables 1 and 2.

CONCLUDING REMARKS

A theoretical solution is given for the buckling stress of thin-walled cylinders loaded in torsion. The results are applicable to a wide range of cylinder dimensions from very short cylinders of large radius to very long cylinders of small radius. The theoretical results are found to be in somewhat better agreement with experimental results than previous theoretical work for the same range of cylinder dimensions.

Langley Memorial Aeronautical Laboratory
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Langley Field, Va., March 20, 1947

APPENDIX

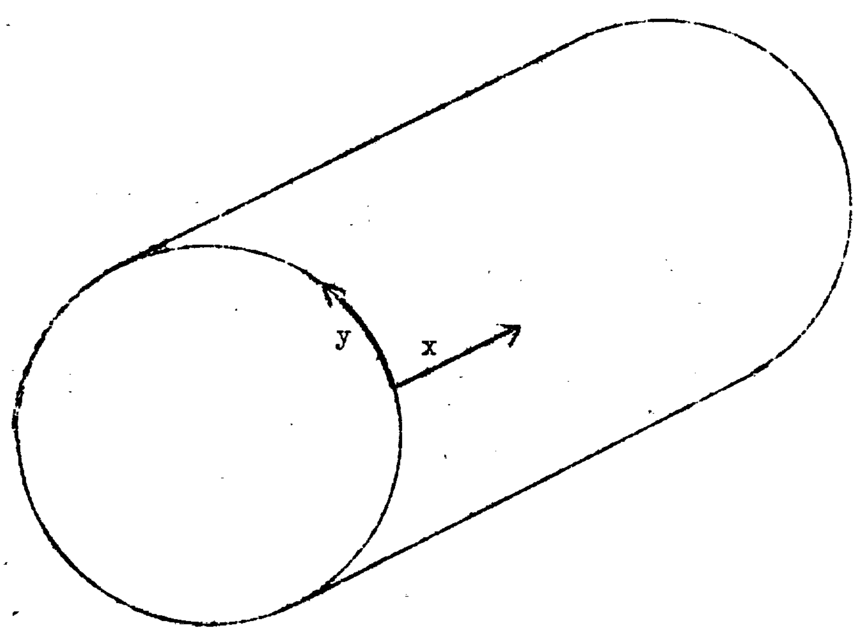
THEORETICAL SOLUTION

The critical shear stress at which buckling occurs in a cylindrical shell may be obtained by solving the equation of equilibrium.

Equation of equilibrium.- The equation of equilibrium for a slightly buckled cylindrical shell under shear is (reference 3)

$$D\nabla^4 w + \frac{Et}{r^2}\nabla^{-4}\frac{\partial^4 w}{\partial x^4} + 2\tau_{cr}t\frac{\partial^2 w}{\partial x\,\partial y} = 0 \tag{1}$$

where x is the axial direction and y the circumferential direction. The following figure shows the coordinate system used in the analysis:



Dividing through equation (1) by D gives

$$\nabla^4 w + \frac{12Z^2}{L^4}\nabla^{-4}\frac{\partial^4 w}{\partial x^4} + 2k_s\frac{\pi^2}{L^2}\frac{\partial^2 w}{\partial x\,\partial y} = 0 \tag{2}$$

where the dimensionless parameters Z and k_s are defined by

$$Z = \frac{L^2}{rt}\sqrt{1 - \mu^2}$$

and

$$k_s = \frac{\tau_{cr} t L^2}{\pi^2 D}$$

The equation of equilibrium may be represented by

$$Qw = 0 \tag{3}$$

where Q is defined by

$$Q = \nabla^4 + \frac{12Z^2}{L^4}\nabla^{-4}\frac{\partial^4}{\partial x^4} + 2k_s\frac{\pi^2}{L^2}\frac{\partial^2}{\partial x\,\partial y}$$

Method of solution.- The equation of equilibrium may be solved by using the Galerkin method as outlined in reference 9. In applying this method, equation (3) is solved by expressing w in terms of an arbitrary number of functions (V_0, V_1, . . . V_j, W_0, W_1, . . ., W_j) that need not satisfy the equation but do satisfy the boundary conditions on w; thus let

$$w = \sum_{m=0}^{j} a_m V_m + \sum_{m=0}^{j} b_m W_m \tag{4}$$

The coefficient a_m and b_m are then determined by the equations

$$\left.\begin{aligned} \int_0^{2\lambda}\int_0^L V_n Q w \, dx \, dy = 0 \\ \int_0^{2\lambda}\int_0^L W_n Q w \, dx \, dy = 0 \end{aligned}\right\} \tag{5}$$

where

$$n = 0, 1, 2, \ldots, j$$

The solutions given in the present paper satisfy the following conditions at the ends of the cylinder:

For cylinders of short and medium length with simply supported edges $w = \frac{\partial^2 w}{\partial x^2} = v = 0$ and u is unrestrained. For cylinders of short and medium length with clamped edges $w = \frac{\partial w}{\partial x} = u = 0$ and v is unrestrained. For long cylinders $w = 0$. (See references 2 and 3.)

Solution for Cylinders of Short and Medium Length

Simply supported edges.- A deflection function for simply supported edges may be taken as the infinite series

$$w = \sin\frac{\pi y}{\lambda}\sum_{m=1}^{\infty} a_m \sin\frac{m\pi x}{L} + \cos\frac{\pi y}{\lambda}\sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{L} \tag{6}$$

where λ is the half wave length of the buckles in the circumferential direction. Equation (6) is equivalent to equation (4) if

$$\left.\begin{aligned} V_n &= \sin\frac{\pi y}{\lambda}\sin\frac{n\pi x}{L} \\ W_n &= \cos\frac{\pi y}{\lambda}\sin\frac{n\pi x}{L} \end{aligned}\right\} \tag{7}$$

Substitution of expressions (6) and (7) into equations (5) and integration over the limits indicated give

$$\left.\begin{aligned} a_n\left[(n^2+\beta^2)^2+\frac{12Z^2n^4}{\pi^4(n^2+\beta^2)^2}\right]-\frac{8k_s\beta}{\pi}\sum_{m=1}^{\infty}b_m\frac{mn}{n^2-m^2} &= 0 \\ b_n\left[(n^2+\beta^2)^2+\frac{12Z^2n^4}{\pi^4(n^2+\beta^2)^2}\right]+\frac{8k_s\beta}{\pi}\sum_{m=1}^{\infty}a_m\frac{mn}{n^2-m^2} &= 0 \end{aligned}\right\} \tag{8}$$

where

$$\beta = \frac{L}{\lambda}$$

$$n = 1, 2, 3, \ldots$$

and $m \pm n$ is odd. Equations (8) have a solution if the following determinant vanishes:

$$
\begin{array}{c|cccccccccccccc|}
 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \cdots & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdots \\
n=1 & \frac{1}{k_s}M_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{2}{3} & 0 & \frac{4}{15} & 0 & \frac{6}{35} & \cdots \\
n=2 & 0 & \frac{1}{k_s}M_2 & 0 & 0 & 0 & 0 & \cdots & -\frac{2}{3} & 0 & \frac{6}{5} & 0 & \frac{10}{21} & 0 & \cdots \\
n=3 & 0 & 0 & \frac{1}{k_s}M_3 & 0 & 0 & 0 & \cdots & 0 & -\frac{6}{5} & 0 & \frac{12}{7} & 0 & \frac{2}{3} & \cdots \\
n=4 & 0 & 0 & 0 & \frac{1}{k_s}M_4 & 0 & 0 & \cdots & -\frac{4}{15} & 0 & -\frac{12}{7} & 0 & \frac{20}{9} & 0 & \cdots \\
n=5 & 0 & 0 & 0 & 0 & \frac{1}{k_s}M_5 & 0 & \cdots & 0 & -\frac{10}{21} & 0 & -\frac{20}{9} & 0 & \frac{30}{11} & \cdots \\
n=6 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_s}M_6 & \cdots & -\frac{6}{35} & 0 & -\frac{2}{3} & 0 & -\frac{30}{11} & 0 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
n=1 & 0 & -\frac{2}{3} & 0 & -\frac{4}{15} & 0 & -\frac{6}{35} & \cdots & \frac{1}{k_s}M_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
n=2 & \frac{2}{3} & 0 & -\frac{6}{5} & 0 & -\frac{10}{21} & 0 & \cdots & 0 & \frac{1}{k_s}M_2 & 0 & 0 & 0 & 0 & \cdots \\
n=3 & 0 & \frac{6}{5} & 0 & -\frac{12}{7} & 0 & -\frac{2}{3} & \cdots & 0 & 0 & \frac{1}{k_s}M_3 & 0 & 0 & 0 & \cdots \\
n=4 & \frac{4}{15} & 0 & \frac{12}{7} & 0 & -\frac{20}{9} & 0 & \cdots & 0 & 0 & 0 & \frac{1}{k_s}M_4 & 0 & 0 & \cdots \\
n=5 & 0 & \frac{10}{21} & 0 & \frac{20}{9} & 0 & -\frac{30}{11} & \cdots & 0 & 0 & 0 & 0 & \frac{1}{k_s}M_5 & 0 & \cdots \\
n=6 & \frac{6}{35} & 0 & \frac{2}{3} & 0 & \frac{30}{11} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_s}M_6 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot &
\end{array}
\qquad (9)
$$

where

$$
M_n = \frac{\pi}{8\beta}\left[(n^2 + \beta^2)^2 + \frac{12Z^2n^4}{\pi^4(n^2 + \beta^2)^2}\right]
$$

By rearranging rows and columns, the infinite determinant can be factored into the product of two infinite subdeterminants which are equivalent to each other. The critical stress may then be obtained from the following equation:

$$
\begin{array}{c|ccccccccccccccc|l}
 & a_1 & b_2 & a_3 & b_4 & a_5 & b_6 & \cdots & b_1 & a_2 & b_3 & a_4 & b_5 & a_6 & \cdots & \\
n=1 & \frac{1}{k_s}M_1 & \frac{2}{3} & 0 & \frac{4}{15} & 0 & \frac{6}{35} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
n=2 & \frac{2}{3} & \frac{1}{k_s}M_2 & -\frac{6}{5} & 0 & -\frac{10}{21} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
n=3 & 0 & -\frac{6}{5} & \frac{1}{k_s}M_3 & \frac{12}{7} & 0 & \frac{2}{3} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
n=4 & \frac{4}{15} & 0 & \frac{12}{7} & \frac{1}{k_s}M_4 & -\frac{20}{9} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
n=5 & 0 & -\frac{10}{21} & 0 & -\frac{20}{9} & \frac{1}{k_s}M_5 & \frac{30}{11} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
n=6 & \frac{6}{35} & 0 & \frac{2}{3} & 0 & \frac{30}{11} & \frac{1}{k_s}M_6 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & = 0 \\
n=1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{k_s}M_1 & -\frac{2}{3} & 0 & -\frac{4}{15} & 0 & -\frac{6}{35} & \cdots & \\
n=2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{2}{3} & \frac{1}{k_s}M_2 & \frac{6}{5} & 0 & \frac{10}{21} & 0 & \cdots & \\
n=3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{6}{5} & \frac{1}{k_s}M_3 & -\frac{12}{7} & 0 & -\frac{2}{3} & \cdots & \\
n=4 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{4}{15} & 0 & -\frac{12}{7} & \frac{1}{k_s}M_4 & \frac{20}{9} & 0 & \cdots & \\
n=5 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{10}{21} & 0 & \frac{20}{9} & \frac{1}{k_s}M_5 & -\frac{30}{11} & \cdots & \\
n=6 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{6}{35} & 0 & -\frac{2}{3} & 0 & -\frac{30}{11} & \frac{1}{k_s}M_6 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & &
\end{array}
\qquad (10)
$$

The first approximation, obtained from the second-order determinant, is given by

$$
k_s{}^2 = \left(\frac{3}{2}\right)^2 M_1 M_2 \qquad (11)
$$

The second approximation, obtained from the third-order determinant, is given by

$$k_s^2 = \frac{M_1M_2M_3}{\left(\frac{6}{5}\right)^2 M_1 + \left(\frac{2}{3}\right)^2 M_3} \tag{12}$$

The third approximation, obtained from the fourth-order determinant, is given by

$$k_s^4\left(\frac{8}{7} + \frac{8}{25}\right)^2 - k_s^2\left[\left(\frac{12}{7}\right)^2 M_1M_2 + \left(\frac{6}{5}\right)^2 M_1M_4 + \left(\frac{4}{15}\right)^2 M_2M_3 + \left(\frac{2}{3}\right)^2 M_3M_4\right]$$
$$+ M_1M_2M_3M_4 = 0 \tag{13}$$

Each of these equations shows that for a selected value of the curvature parameter Z the critical buckling stress of a cylinder depends on the wave length. Since a structure buckles at the lowest stress at which instability can occur, k_s is minimized with respect to the wave length by substituting values of β into the equation until the minimum value of k_s can be obtained from a plot of k_s against β. This procedure is permissible when $\beta > \frac{2L}{\pi r}$, that is, when the cylinder buckles into more than two circumferential waves. For the limiting case of a cylinder buckling into two waves, see the section of the present appendix entitled "Solution for a Long Cylinder" which follows.

Figure 5(a) shows the convergence of the determinant for cylinders with simply supported edges.

Clamped edges.– A procedure similar to that used for cylinders with simply supported edges may be followed for cylinders with clamped edges. The deflection function used is the following series:

$$w = \sin\frac{\pi y}{\lambda}\sum_{m=0}^{\infty} a_m\left[\cos\frac{m\pi x}{L} - \cos\frac{(m+2)\pi x}{L}\right]$$
$$+ \cos\frac{\pi y}{\lambda}\sum_{m=0}^{\infty} b_m\left[\cos\frac{m\pi x}{L} - \cos\frac{(m+2)\pi x}{L}\right] \tag{14}$$

Each term of this series satisfies the condition on w at the edges. The functions V_n and W_n are now defined as follows:

$$\left.\begin{aligned} V_n &= \sin \frac{\pi y}{\lambda}\left[\cos \frac{n\pi x}{L} - \cos \frac{(n+2)\pi x}{L}\right] \\ W_n &= \cos \frac{\pi y}{\lambda}\left[\cos \frac{n\pi x}{L} - \cos \frac{(n+2)\pi x}{L}\right] \end{aligned}\right\} \tag{15}$$

where

$$n = 0, 1, 2, \ldots$$

When the same operations as those carried out for the case of simply supported edges are performed, the following simultaneous equations result:

For $n = 0$,

$$a_0(2M_0 + M_2) - a_2M_2 + k_s \sum_{m=1,3,5}^{\infty} b_m\left[-\frac{m^2}{m^2 - 4} + \frac{(m+2)^2}{(m+2)^2 - 4}\right] = 0$$

For $n = 1$,

$$a_1(M_1 + M_3) - a_3M_3 + k_s \sum_{m=0,2,4}^{\infty} b_m\left[\frac{m^2}{m^2 - 1} - \frac{m^2}{m^2 - 9} - \frac{(m+2)^2}{(m+2)^2 - 1} + \frac{(m+2)^2}{(m+2)^2 - 9}\right] = 0$$

For $n = 2, 3, 4 \ldots,$

$$a_n(M_n + M_{n+2}) - a_{n-2}M_n - a_{n+2}M_{n+2} + k_s \sum_{m=0}^{\infty} b_m\left[\frac{m^2}{m^2 - n^2} - \frac{m^2}{m^2 - (n+2)^2} - \frac{(m+2)^2}{(m+2)^2 - n^2} + \frac{(m+2)^2}{(m+2)^2 - (n+2)^2}\right] = 0$$

where $m \pm n$ is odd.

For $n = 0$,

$$b_0(2M_0 + M_2) - b_2M_2 - k_s \sum_{m=1,3,5}^{\infty} a_m \left[- \frac{m^2}{m^2 - 4} + \frac{(m + 2)^2}{(m + 2)^2 - 4} \right] = 0$$

For $n = 1$,

$$b_1(M_1 + M_3) - b_3M_3 - k_s \sum_{m=0,2,4}^{\infty} a_m \left[\frac{m^2}{m^2 - 1} - \frac{m^2}{m^2 - 9} - \frac{(m + 2)^2}{(m + 2)^2 - 1} \right.$$

$$\left. + \frac{(m + 2)^2}{(m + 2)^2 - 9} \right] = 0$$

For $n = 2, 3, 4, \ldots,$

$$b_n(M_n + M_{n+2}) - b_{n-2}M_n - b_{n+2}M_{n+2} - k_s \sum_{n=0}^{\infty} a_m \left[\frac{m^2}{m^2 - n^2} - \frac{m^2}{m^2 - (n + 2)^2} \right.$$

$$\left. = \frac{(m + 2)^2}{(m + 2)^2 - n^2} + \frac{(m + 2)^2}{(m + 2)^2 - (n + 2)^2} \right] = 0 \qquad (16)$$

where $m \pm n$ is odd and

$$M_n = \frac{\pi}{8\beta} \left[(n^2 + \beta^2)^2 + \frac{12Z^2n^4}{\pi^4(n^2 + \beta^2)^2} \right]$$

The infinite determinant formed by these equations can be rearranged so as to factor into the product of two determinants which are equivalent to each other. The vanishing of one of these determinants leads to the following equation (limited for convenience to the sixth order):

$$
\begin{array}{c|cccccc|l}
 & a_0 & b_1 & a_2 & b_3 & a_4 & b_5 & \\
n=0 & \frac{1}{k_s}(2M_0+M_2) & \frac{32}{15} & -\frac{1}{k_s}M_2 & -\frac{64}{105} & 0 & -\frac{32}{315} & \\
n=1 & \frac{32}{15} & \frac{1}{k_s}(M_1+M_3) & -\frac{352}{105} & -\frac{1}{k_s}M_3 & \frac{32}{35} & 0 & \\
n=2 & -\frac{1}{k_s}M_2 & -\frac{352}{105} & \frac{1}{k_s}(M_2+M_4) & \frac{1472}{315} & -\frac{1}{k_s}M_4 & -\frac{1376}{1155} & \\
n=3 & -\frac{64}{105} & -\frac{1}{k_s}M_3 & \frac{1472}{315} & \frac{1}{k_s}(M_3+M_5) & -\frac{4160}{693} & -\frac{1}{k_s}M_5 & = 0 \quad (17) \\
n=4 & 0 & \frac{32}{35} & -\frac{1}{k_s}M_4 & -\frac{4160}{693} & \frac{1}{k_s}(M_4+M_6) & \frac{9440}{1287} & \\
n=5 & -\frac{32}{315} & 0 & -\frac{1376}{1155} & -\frac{1}{k_s}M_5 & \frac{9440}{1287} & \frac{1}{k_s}(M_5+M_7) &
\end{array}
$$

The first approximation, obtained from the second-order determinant, is given by

$$k_s^2 = \left(\frac{15}{32}\right)^2 (2M_0 + M_2)(M_1 + M_3) \tag{18}$$

The second approximation, obtained from the third-order determinant, is given by

$$k_s^2 = \frac{(M_1 + M_3)\left[(2M_0 + M_2)(M_2 + M_4) - M_2^2\right]}{\left(\frac{32}{15}\right)^2 (M_2 + M_4) - \frac{64}{15}\,\frac{352}{105}M_2 + \left(\frac{352}{105}\right)^2 (2M_0 + M_2)} \tag{19}$$

The third approximation, obtained from the fourth-order determinant, is given by

$$k_s^4\left(\frac{32}{15}\frac{1472}{315} - \frac{352}{105}\frac{64}{105}\right)^2 - k_s^2\left[\left(\frac{1472}{315}\right)^2(2M_0 + M_2)(M_1 + M_3) + \left(\frac{352}{105}\right)^2(2M_0 + M_2)(M_3 + M_5)\right.$$

$$+ \left(\frac{64}{105}\right)^2(M_1 + M_3)(M_2 + M_4) + \left(\frac{32}{15}\right)^2(M_2 + M_4)(M_3 + M_5) - \frac{128}{105}\frac{1472}{315}M_2(M_1 + M_3)$$

$$- \frac{64}{15}\frac{352}{105}M_2(M_3 + M_5) - \frac{704}{105}\frac{1472}{315}M_3(2M_0 + M_2) - \frac{64}{15}\frac{64}{105}M_3(M_2 + M_4)$$

$$\left. + 2\left(\frac{64}{105}\frac{352}{105} + \frac{32}{15}\frac{1472}{315}\right)M_2M_3\right] + \left[2M_0(M_2 + M_4) + M_2M_4\right]\left[M_1(M_3 + M_5) + M_3M_5\right] = 0 \quad (20)$$

These equations may be solved in the same way as in the previous problem of simply supported edges, by substituting values of β into the equation until the minimum value of k_s is obtained from a plot of β and corresponding values of k_s. The restriction that $\beta > \frac{2L}{\pi r}$ applies for cylinders with clamped edges as well as for cylinders with simply supported edges. Figure 5(b) shows the convergence of this determinant.

Solution for a Long Cylinder

A long slender cylinder $\left(Z > 10\,\frac{r^2}{t^2}\right)$ will buckle into two waves in the circumferential direction. If, in the previous cases of cylinders with simply supported or clamped edges, the half wave length in the circumferential direction λ is taken as $\pi r/2$, it is possible to find the critical stress of a long slender cylinder with the corresponding edge conditions. This method of solution is laborious, however, because determinants of high order must be employed to obtain solutions of reasonable accuracy. The labor is greatly reduced by the use of the following deflection function:

$$w = a_1\left\{\cos\left(\frac{p\pi x}{L} + \frac{2y}{r}\right) - \cos\left[\frac{(p+2)\pi x}{L} + \frac{2y}{r}\right]\right\} \tag{21}$$

where $p + 1$ is the phase difference of the circumferential waves at the two ends of the cylinder measured in quarter-revolutions. This equation satisfies the single boundary condition $w = 0$. With this deflection function, the functions V and W all vanish except

$$V_1 = \cos\left(\frac{p\pi x}{L} + \frac{2y}{r}\right) - \cos\left[\frac{(p+2)\pi x}{L} + \frac{2y}{r}\right] \tag{22}$$

Use of equations (5), (21), and (22) and the relation $2\lambda = \pi r$ results in the following equation:

$$k_s = \frac{\pi}{8\frac{L}{r}(p+1)}\left\{\left[p^2 + \frac{4}{\pi^2}\left(\frac{L}{r}\right)^2\right]^2 + \frac{12Z^2p^4}{\pi^4\left[p^2 + \frac{4}{\pi^2}\left(\frac{L}{r}\right)^2\right]^2} + \left[(p+2)^2 + \frac{4}{\pi^2}\left(\frac{L}{r}\right)^2\right]^2 + \frac{12Z^2(p+2)^4}{\pi^4\left[(p+2)^2 + \frac{4}{\pi^2}\left(\frac{L}{r}\right)^2\right]^2}\right\} \tag{23}$$

This equation may be written

$$k_s = \frac{\pi\sqrt{\frac{r}{Zt}\sqrt{1-\mu^2}}}{8(p+1)}\left\{\left(p^2 + \frac{4}{\pi^2}\frac{Zt}{r\sqrt{1-\mu^2}}\right)^2 + \frac{12Z^2p^4}{\pi^4\left(p^2 + \frac{4}{\pi^2}\frac{Zt}{r\sqrt{1-\mu^2}}\right)^2}\right.$$

$$\left. + \left[(p+2)^2 + \frac{4}{\pi^2}\frac{Zt}{r\sqrt{1-\mu^2}}\right]^2 + \frac{12Z^2(p+2)^4}{\pi^4\left[(p+2)^2 + \frac{4}{\pi^2}\frac{Zt}{r\sqrt{1-\mu^2}}\right]^2}\right\} \tag{24}$$

For given values of Z and $\frac{r}{t}\sqrt{1-\mu^2}$, p is varied until a minimum value of k_s is obtained from a plot of p and corresponding values of k_s. The critical stress of a long slender cylinder is very insensitive to edge restraint; therefore, the solution applies with sufficient accuracy to cylinders with either simply supported or clamped edges. The shear-stress coefficient for long slender cylinders is plotted against the curvature parameter in figure 4, and parts of these curves also appear in figure 1.

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TABLE 1

THEORETICAL SHEAR-STRESS COEFFICIENTS AND WAVE LENGTHS

OF BUCKLES FOR SHORT- AND MEDIUM-LENGTH CYLINDERS

Z	First approximation		Second approximation		Third approximation	
	k_s	β	k_s	β	k_s	β
Cylinders with simply supported edges						
0	5.60	0.770	5.34	0.790	-----	-----
1	5.69	.805	5.42	.860	5.41	0.865
5	6.68	1.00	6.22	1.015	-----	-----
10	8.36	1.24	7.55	1.265	7.545	1.27
30	14.93	1.82	12.69	1.875	-----	-----
100	34.09	2.74	27.86	2.91	-----	-----
300	76.80	3.86	62.47	4.18	61.47	4.32
1,000	189.5	5.40	153.0	5.95	-----	-----
10,000	1072	10.0	871.2	11.2	851.9	11.8
100,000	6050	17.9	4920	20.1	4800	23.0
Cylinders with clamped edges						
0	9.55	1.175	9.31	1.205	9.09	1.205
1	9.57	1.18	9.32	1.21	-----	-----
5	9.90	1.23	9.62	1.27	-----	-----
10	10.79	1.35	10.42	1.38	10.19	1.38
30	16.13	1.89	14.99	1.97	-----	-----
100	35.40	2.95	30.68	3.14	30.65	3.12
1,000	206.3	6.12	167.5	6.70	165.7	7.00
10,000	6860	20.85	5449	23.2	5310	24.8

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TABLE 2

THEORETICAL SHEAR-STRESS COEFFICIENTS

FOR LONG CYLINDERS

$\frac{r}{t}\sqrt{1 - \mu^2}$	Z	k_s
20	4×10^3	428
	3×10^4	2,450
	10^5	7,780
	10^6	76,500
50	2.5×10^4	1,680
	10^5	5,380
	10^6	47,900
	10^7	476,000
100	10^5	4,800
	10^6	35,200
	10^7	334,500

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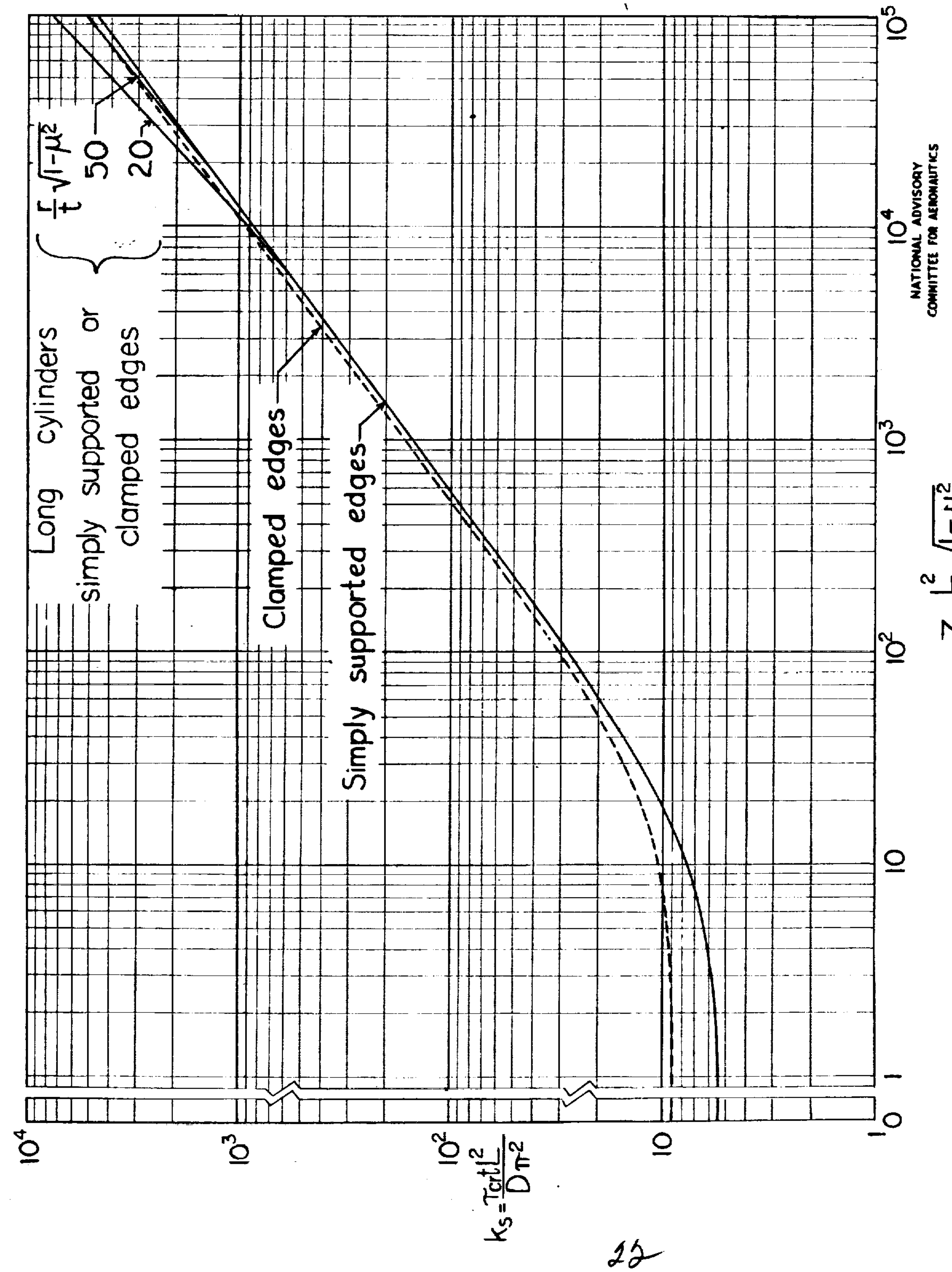

Figure 1.- Critical shear-stress coefficients for thin-walled cylinders in torsion.

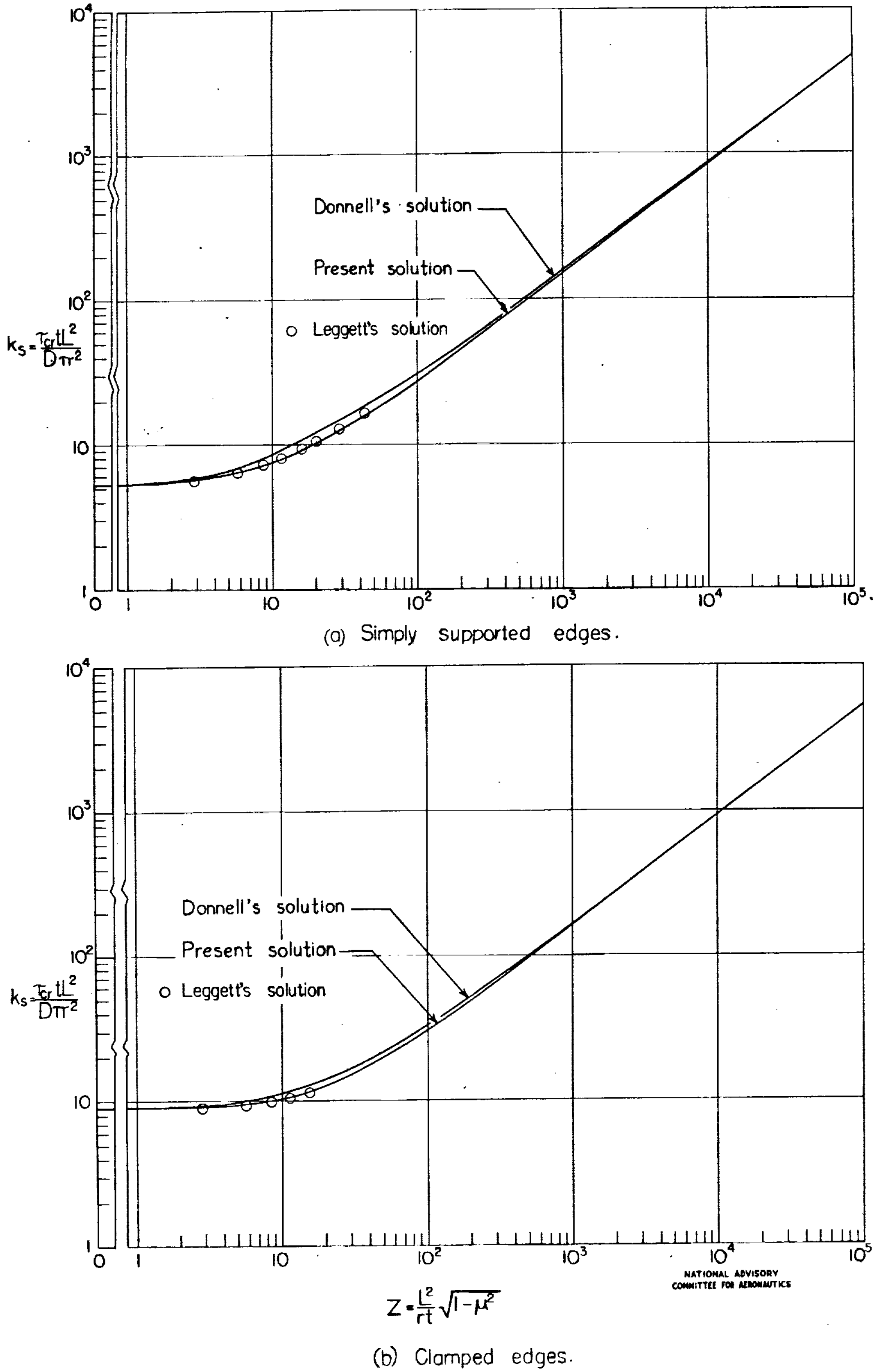

Figure 2.- Comparison of theoretical curves for critical stress of thin-walled cylinders in torsion.

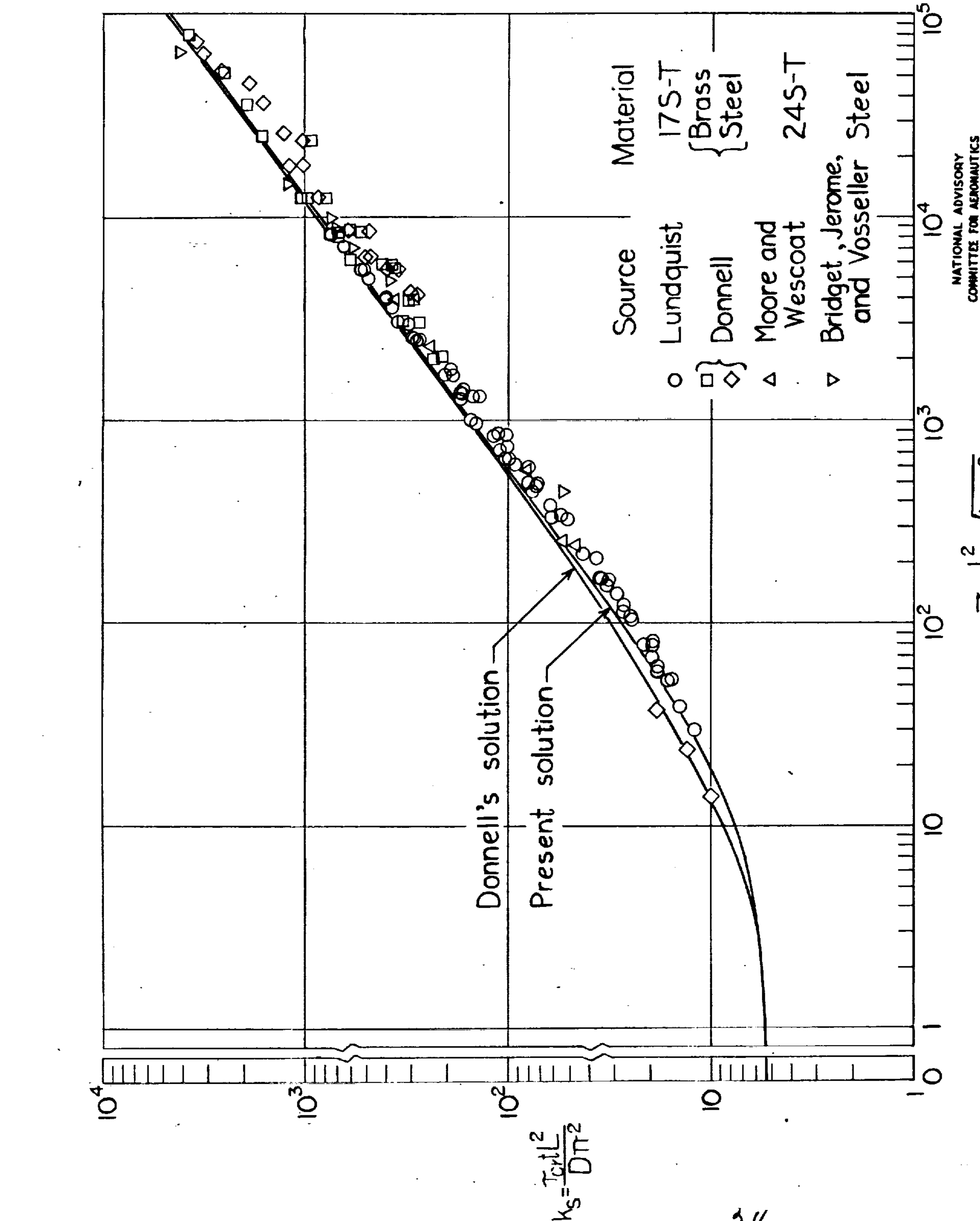

Figure 3.- Comparison of theoretical solutions for critical stress of simply supported cylinders in torsion with test data.

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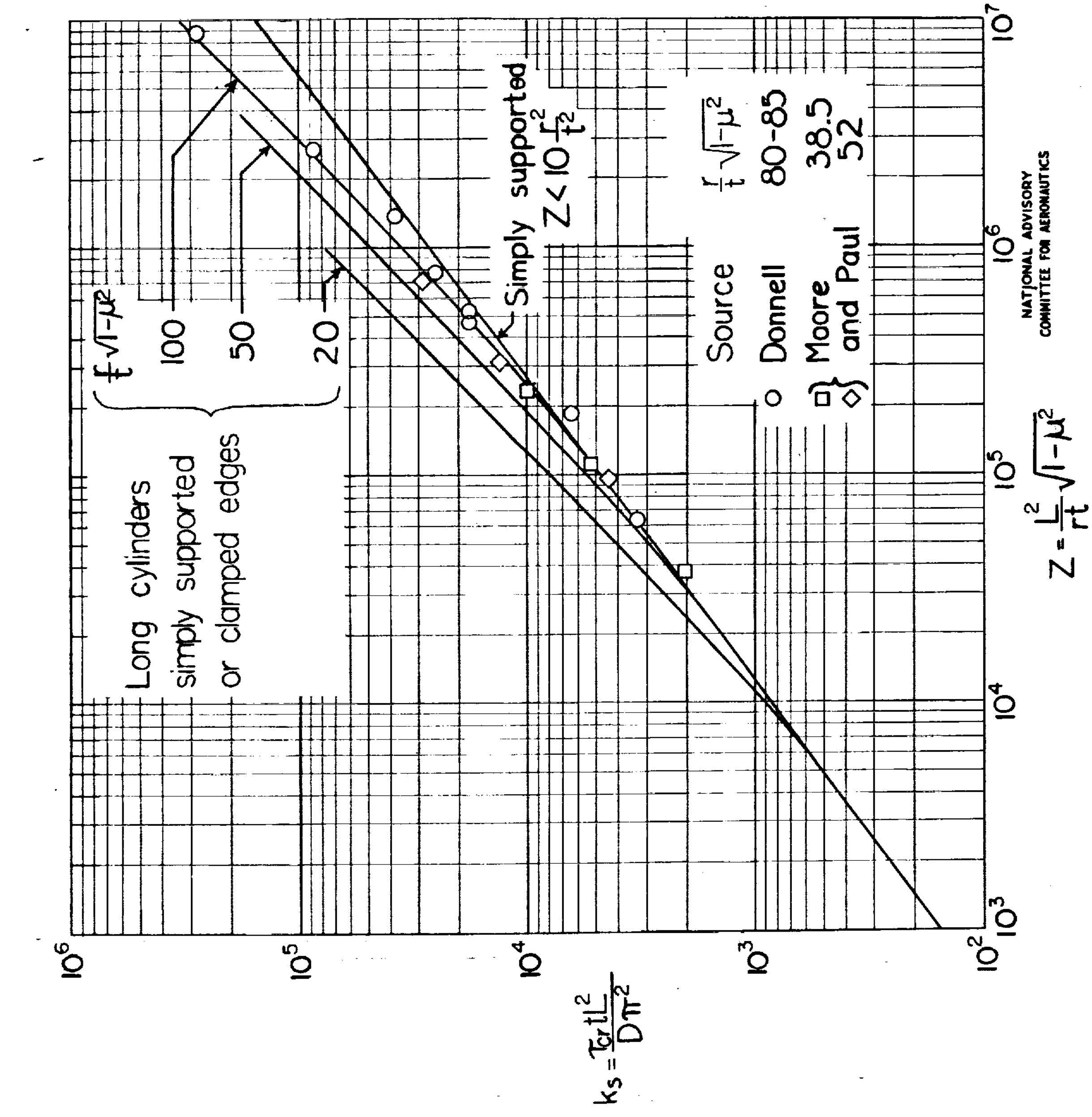

Figure 4.- Comparison of test data with theoretical buckling stress for long cylinders in torsion.

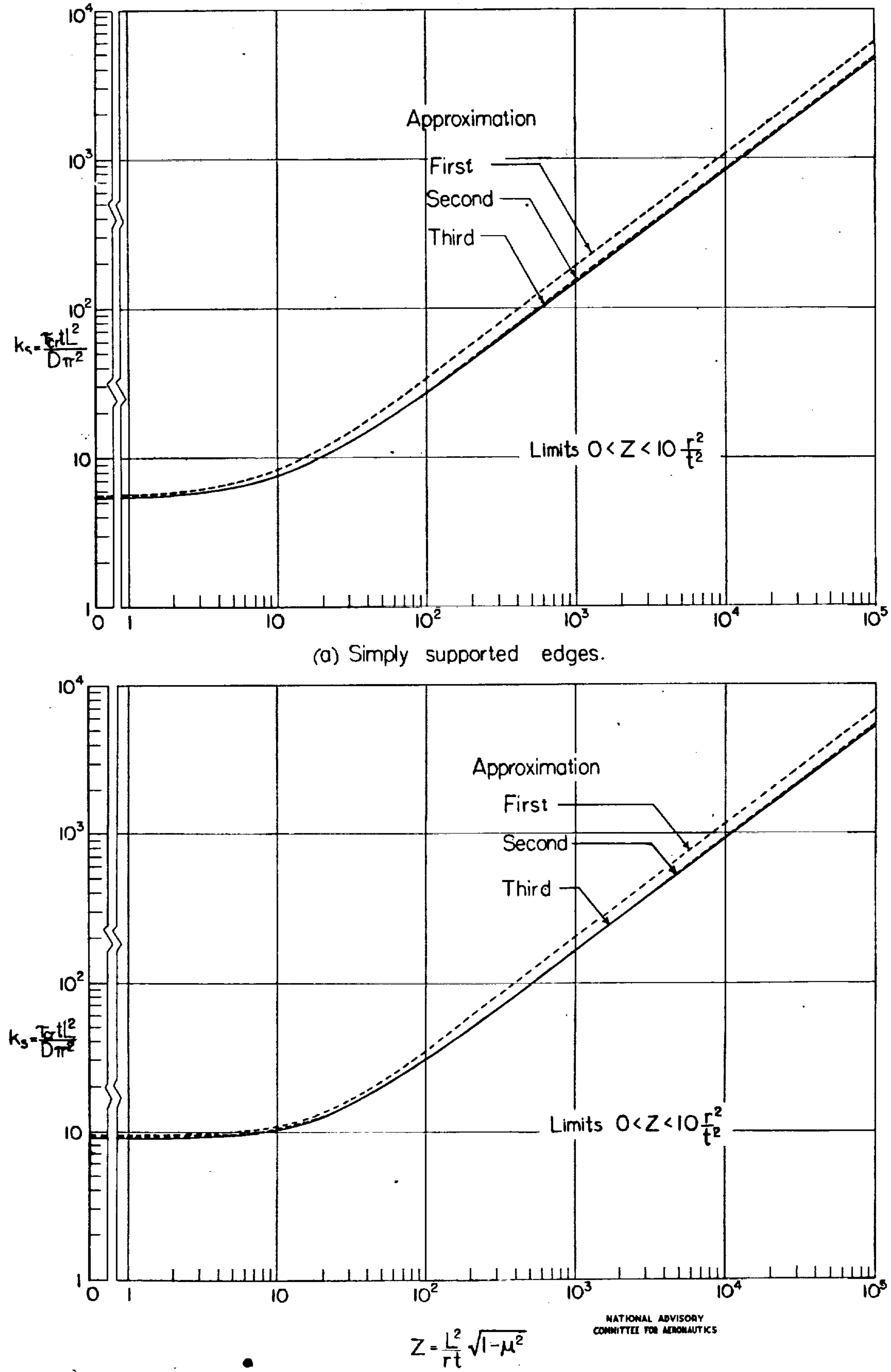

Figure 5.- Successive approximations of critical shear-stress coefficients for thin-walled cylinders in torsion.