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STRESS ANALYSIS BY RECURRENCE FORMULA OF REINFORCED CIRCULAR CYLINDERS UNDER LATERAL LOADS

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SUMMARY

A recurrence formula is developed for the stress analysis of reinforced circular cylinders loaded in the planes of their rings. In contrast to the elementary engineering analysis, deformations of rings and sheet are considered. The recurrence formula in conjunction with appropriate boundary equations can be used to obtain sets of simultaneous linear algebraic equations. The solutions of these equations enable the stress analyst to find the shear flows and direct stresses in the sheet, as well as the shear forces, axial forces, and bending moments in the rings.

In order to reduce the amount of computation involved in the stress analysis of relatively long reinforced cylinders, an approximate method of analysis is presented. In this method the cylinder under consideration is assumed to be infinitely long, and the recurrence formula is treated as a fourth-order finite-difference equation. It is recommended that the approximate solution be utilized for the stress analysis of cylinders loaded at rings located two or more bays from external restraints.

INTRODUCTION

Experimental data on stresses in reinforced circular cylinders indicate the inadequacy of the elementary theory of bending and torsion when applied to the relatively flexible shell structures used in airframe construction. Several investigators have presented methods for the stress analysis of cylinders laterally loaded at the reinforcing rings (references 1 to 3). The theory of reference 1, developed only for the case of a one bay cylinder, involves the assumption that stringer strains can be entirely neglected and, consequently, leads to inaccurate results. The more precise theory of references 2 and 3, developed for cantilevered cylinders having identical bays, becomes tedious and unwieldy when extended to nonuniform cylinders.

The present paper contains the development of a general recurrence formula suitable for the stress analysis of cylinders that may be nonuniform in construction, arbitrarily supported at the boundaries, and arbitrarily loaded in the planes of the reinforcing rings. The development is based upon the maintenance of continuity of deformation between the rings and shell. In any particular problem the recurrence formula together with appropriate boundary equations are used to obtain sets of simultaneous linear equations for the corrections to the stresses given by the elementary theory. (For a cantilevered uniform cylinder the results obtained in this manner are identical with those obtained by the method of reference 2 or 3.)

If a cylinder is composed of many bays, as in conventional fuselage construction, the number of simultaneous equations requiring consideration may be prohibitive. For a uniform cylinder, however, good approximations to the correction stresses can be obtained if the cylinder is assumed to be infinitely long. The recurrence formula for this case is solved as a homogeneous finite difference equation of the fourth order and yields a relatively simple solution. For practical purposes this solution can be applied to arbitrarily supported cylinders provided the loads are located a few bays from external restraints. When the recurrence formula, together with the boundary equations presented, is applied to a cantilevered uniform cylinder discussed in reference 3, good agreement is obtained among the recurrence-formula solution, difference-equation solution, and experimental stresses.

SYMBOLS

$A = \dfrac{R^6 t'}{IL^3}$

$B = \dfrac{Et'R^2}{GtL^2}$

C function of ring loading

$D_n = \dfrac{2(\beta_n - 1)}{\gamma_n^2}$

E Young's modulus

G	shear modulus
H	axial force in ring
I	moment of inertia of cross section
L	length of bay
M	bending moment
M_c	concentrated ring bending moment
P	radial load
Q	static moment about neutral axis of cross-sectional area lying between extreme fiber and plane under consideration
R	radius of cylinder and ring
T	tangential load on ring
V	shear force
a, b	Fourier coefficients in Fourier expansions of q
c	distance from neutral axis
i, k	general numbers of bay or ring
m	designation of root bay
n	general number of Fourier coefficient
q	shear flow in skin
t	thickness of skin
t'	effective sheet thickness, that is, thickness of all material carrying bending stresses in cylinder if uniformly distributed around perimeter
u, v, w	axial, tangential, and radial displacements of points on cylinder
x, y, z	axial, tangential, and radial coordinates of cylinder

α arbitrary constant of integration

$$\beta_n = 3 + \frac{n^2 + 3B}{3A\gamma}$$

$$\gamma_n = -2 + \frac{n^2 - 6B}{12A\gamma}$$

$$\gamma = \frac{1}{n^2\left(n^2 - 1\right)^2}$$

λ, μ, υ constants dependent upon bay lengths

$$\rho_n = \frac{1}{2}\cosh^{-1}\left[\frac{\beta_n - 1}{2} - \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^{\,2}}\right]$$

σ longitudinal direct stress in skin

φ angular coordinate of point on cylinder

$$\psi_n = \frac{1}{2}\cosh^{-1}\left[\frac{\beta_n - 1}{2} + \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^{\,2}}\right]$$

$$\chi_n = \frac{1}{2}\cos^{-1}\left[\frac{\beta_n - 1}{2} - \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^{\,2}}\right]$$

Subscripts:

R rigid

m moment

r radial

t tangential

STRESS ANALYSIS OF REINFORCED CYLINDERS

Inadequacy of Elementary Theory

The elementary engineering theory for bending and torsion of reinforced cylinders loaded at the ring reinforcements yields the well-known formulas Mc/I for direct bending stress, VQ/It for shear stress due to bending, and $T/2At$ for shear stress due to torsion (where T and A are the torque on cross section and the area inclosed by perimeter of cross section, respectively). This simple theory is based upon the assumption that radial displacements of both rings and sheet can be neglected. Since the dimensions of most monocoque structures are such that radial displacements of the structural components cannot be ignored without appreciable inaccuracies in the results of analysis, the elementary theory must be modified so as not only to satisfy the laws of statics but also to maintain continuity between rings and sheet. The present development, consequently, is directed towards finding self-equilibrating stress distributions that, when superimposed upon the elementary stress distributions, yield results which, in addition to satisfying the laws of statics, preserve the continuity of the structure. These correction stresses are found from the recurrence formula that is developed herein.

Basic Assumptions of Present Theory

In the development of the recurrence formula that can be used to obtain the desired stress corrections, several simplifying assumptions are made. That part of the sheet area which is considered to resist normal stresses is added to the stringer area and the combination is uniformly distributed about the periphery of the cylinder. This resulting combination is an effective sheet thickness t' that resists normal stresses. The actual sheet area is considered capable of supporting only shear stresses. It then follows that within a bay the shear stresses vary in the circumferential direction but are constant in the longitudinal direction. Inextensional deformation of rings and sheet is also assumed, and Poisson's ratio is considered to be zero.

Development of Recurrence Formula

Procedure.- For the skin of any bay i of the structure (see figs. 1 and 2), the corrections to the elementary shear flow, direct stress, axial displacement, and radial displacement are each expressed as Fourier series with undetermined Fourier coefficients. Through static, elastic, and geometric considerations of rings and sheet, a recurrence formula is obtained relating the Fourier coefficient of the shear flow of any bay i with the coefficients of the two bays on each side of bay i, that is, bays i+1 and i+2 and bays i-1 and i-2. From the recurrence formula, simultaneous equations may be obtained from which the values of the shear-flow coefficients are determined. With these values the loads and stresses in the rings and sheet can be found.

Sheet stresses and deformations.- The system of coordinate axes to be used is shown in figures 1 and 2. Positive displacements in x-, y-, and z-directions are designated u, v, and w, respectively. For convenience, the external loading on the reinforcing rings of a cylinder is considered to be either symmetrical or antisymmetrical about $\varphi = 0^{\circ}$. (See figs. 1 and 2.) In accordance with the basic assumptions the corrections to the elementary shear flow, direct stress, axial displacement, and radial displacement at any point $\left(x_i, \varphi\right)$ in bay i can be expressed for symmetrical loading as the Fourier expansions

$$q_i(\varphi) = \sum_{n=2}^{\infty} a_{in} \sin n\varphi \tag{1a}$$

$$\sigma_i\left(x_i, \varphi\right) = \sum_{n=2}^{\infty} \sigma_{in}\left(x_i\right) \cos n\varphi \tag{1b}$$

$$u_i\left(x_i, \varphi\right) = \sum_{n=2}^{\infty} u_{in}\left(x_i\right) \cos n\varphi \tag{1c}$$

$$w_i\left(x_i, \varphi\right) = \sum_{n=2}^{\infty} w_{in}\left(x_i\right) \cos n\varphi \tag{1d}$$

respectively, in which a_{in}, $\sigma_{in}(x_i)$, $u_{in}(x_i)$, and $w_{in}(x_i)$ are Fourier coefficients. Inasmuch as only corrections to the elementary stresses and displacements are desired, Fourier terms corresponding to $n = 0$ and $n = 1$ are omitted since they correspond to the elementary stress and displacement distributions.

If antisymmetrical loading is considered, the harmonic functions in equations (1) are replaced by their cofunctions. It is then convenient to designate the Fourier coefficient of the shear flow by b_{in}.

Relationships among sheet stresses and deformations.- Within any bay i the following relationships exist (fig. 2): by the equilibrium equation

$$t'_i \frac{\partial \sigma_i(x_i, \varphi)}{\partial x_i} + \frac{1}{R} \frac{\partial q_i(\varphi)}{\partial \varphi} = 0 \tag{2a}$$

by Hooke's law for direct stress

$$\sigma_i(x_i, \varphi) = E \frac{\partial u_i(x_i, \varphi)}{\partial x_i} \tag{2b}$$

by Hooke's law for shear stress

$$\frac{q_i(\varphi)}{Gt_i} = \frac{1}{R} \frac{\partial u_i(x_i, \varphi)}{\partial \varphi} + \frac{\partial v_i(x_i, \varphi)}{\partial x_i} \tag{2c}$$

and by the inextensional deformation equation (p. 208 of reference 4)

$$\frac{\partial v_i(x_i, \varphi)}{\partial \varphi} - w_i(x_i, \varphi) = 0 \tag{2d}$$

where

t_i	actual skin thickness of bay i
t'_i	effective skin thickness of bay i
R	radius of cylinder
E	Young's modulus
G	shear modulus
$v_i(x_i, \varphi)$	circumferential displacement of any point in bay i

If equations (1a) and (1b) are substituted into equation (2a) and if coefficients of like cosine terms are equated, the following expression for the Fourier coefficient $\sigma_{in}(x_i)$ is obtained:

$$\frac{\partial \sigma_{in}(x_i)}{\partial x_i} = -\frac{n}{Rt'_i}a_{in}$$

Integration of this equation yields

$$\sigma_{in}(x_i) = -\frac{nx_i}{Rt'_i}a_{in} + \sigma_{in}(0) \tag{3}$$

in which $\sigma_{in}(0)$ is the direct-stress Fourier coefficient at $x_i = 0$.

Similarly, elimination of $\sigma_{in}(x_i)$ from equations (1b), (1c), (2b), and (3) and subsequent integration gives

$$u_{in}(x_i) = -\frac{nx_i^2}{2ERt'_i}a_{in} + \frac{x_i}{E}\sigma_{in}(0) + u_{in}(0) \tag{4}$$

in which $u_{in}(0)$ is the axial displacement coefficient at $x_i = 0$.

Elimination of $v_i(x_i, \varphi)$ from equations (2c) and (2d) yields

$$\frac{\partial w_i(x_i, \varphi)}{\partial x_i} = \frac{1}{Gt_i}\frac{\partial q_i(\varphi)}{\partial \varphi} - \frac{1}{R}\frac{\partial^2 u_i(x_i, \varphi)}{\partial \varphi^2}$$

Substitution of equations (1a), (1c), (1d), and (4) into this relationship and integration yields the following expression for the radial displacement coefficient:

$$w_{in}(x_i) = \frac{nx_i}{Gt_i}a_{in} - \frac{n^3x_i^3}{6ER^2t'_i}a_{in} + \frac{n^2x_i^2}{2ER}\sigma_{in}(0) + \frac{n^2x_i}{R}u_{in}(0) + w_{in}(0) \qquad (5)$$

in which $w_{in}(0)$ is the radial displacement coefficient of the sheet of bay i at $x_i = 0$.

Appropriate changes of the subscripts i in equations (3) to (5) permit the application of the equations to each bay of the structure.

Ring deformations.- The radial displacement at any point φ of a symmetrically loaded circular ring can be expressed as the Fourier expansion (see pp. 208 and 209 of reference 4)

$$\left[w_i(\varphi)\right]_{ring} = \sum_{n=2}^{\infty}\left(w_{in}\right)_{ring} \cos n\varphi$$

It can be shown by the method of virtual work (pp. 209 and 210 of reference 4) that for inextensional deformation the radial displacement coefficient $\left(w_{in}\right)_{ring}$ for a ring of radius R and constant moment of inertia I_i that is loaded by the shear flows in bays i and i-1 and by an arbitrary set of symmetrically applied external forces is (fig. 1)

$$\left(w_{in}\right)_{ring} = \frac{R^4}{EI_i}\frac{a_{in} - a_{i-1,n}}{n\left(n^2 - 1\right)^2} + C_{in} \tag{6}$$

In equation (6) the first expression on the right-hand side represents the part of the radial displacement coefficient due to the correction shears only, whereas the second expression represents the part of the displacement coefficient due to the external loading and the elementary shears. Values of C_{in} are given later for particular loadings. (See equations (23).)

Continuity relationships.- The following expressions can be obtained from continuity considerations of the rings and sheet of bays i-1, i, and i+1 (fig. 1):

$$\sigma_{i-1,n}\left(L_{i-1}\right) = \sigma_{in}(0) \tag{7}$$

$$\sigma_{in}\left(L_i\right) = \sigma_{i+1,n}(0) \tag{8}$$

$$u_{i-1,n}\left(L_{i-1}\right) = u_{in}(0) \tag{9}$$

$$u_{in}\left(L_i\right) = u_{i+1,n}(0) \tag{10}$$

$$\left.\begin{aligned} w_{i-1,n}(0) &= \left(w_{i-1,n}\right)_{ring} \\ w_{i-1,n}\left(L_{i-1}\right) &= \left(w_{in}\right)_{ring} \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} w_{in}(0) &= \left(w_{in}\right)_{ring} \\ w_{in}\left(L_{i}\right) &= \left(w_{i+1,n}\right)_{ring} \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} w_{i+1,n}(0) &= \left(w_{i+1,n}\right)_{ring} \\ w_{i+1,n}\left(L_{i+1}\right) &= \left(w_{i+2,n}\right)_{ring} \end{aligned}\right\} \tag{13}$$

Equations (7) to (10) are conditions of continuity of σ and u across the boundaries between bays i-1 and i and between bays i and i+1. Equations (11) to (13) state that the radial deformations of the rings bounding bays i-1, i, and i+1 are equal to the sheet deformations of these bays at the rings. Implicit in equations (11) to (13) is a statement of the continuity of w of the cylinder across the boundaries between bays.

Recurrence formula.- Substitution of the expressions for $\sigma_{in}\left(x_i\right)$, $u_{in}\left(x_i\right)$, $w_{in}\left(x_i\right)$, and $\left(w_{in}\right)_{ring}$ (equations (3) to (6), respectively) in the continuity relationships (equations (7) to (13)) yields the following seven simultaneous equations in which $n = 2, 3, 4, \ldots$

$$\left.\begin{aligned}
&\sigma_{in}(0) - \sigma_{i-1,n}(0) + \frac{nL_{i-1}}{Rt'_{i-1}}a_{i-1,n} = 0 \\
&\sigma_{i+1,n}(0) - \sigma_{in}(0) + \frac{nL_i}{Rt'_i}a_{in} = 0 \\
&u_{in}(0) - u_{i-1,n}(0) + \frac{nL_{i-1}{}^2}{2ERt'_{i-1}}a_{i-1,n} - \frac{L_{i-1}}{E}\sigma_{i-1,n}(0) = 0 \\
&u_{i+1,n}(0) - u_{in}(0) + \frac{nL_i{}^2}{2ERt'_i}a_{in} - \frac{L_i}{E}\sigma_{in}(0) = 0 \\
&\frac{R^4}{EI_i}\,\frac{a_{in} - a_{i-1,n}}{n\left(n^2 - 1\right)^2} + C_{in} = \frac{nL_{i-1}}{Gt_{i-1}}a_{i-1,n} - \frac{n^3L_{i-1}{}^3}{6ER^2t'_{i-1}}a_{i-1,n} \\
&\qquad + \frac{n^2L_{i-1}{}^2}{2ER}\sigma_{i-1,n}(0) + \frac{n^2L_{i-1}}{R}u_{i-1,n}(0) \\
&\qquad + \frac{R^4}{EI_{i-1}}\,\frac{a_{i-1,n} - a_{i-2,n}}{n\left(n^2 - 1\right)^2} + C_{i-1,n} \\
&\frac{R^4}{EI_{i+1}}\,\frac{a_{i+1,n} - a_{in}}{n\left(n^2 - 1\right)^2} + C_{i+1,n} = \frac{nL_i}{Gt_i}a_{in} - \frac{n^3L_i{}^3}{6ER^2t'_i}a_{in} \\
&\qquad + \frac{n^2L_i{}^2}{2ER}\sigma_{in}(0) + \frac{n^2L_i}{R}u_{in}(0) + \frac{R^4}{EI_i}\,\frac{a_{in} - a_{i-1,n}}{n\left(n^2 - 1\right)^2} + C_{in} \\
&\frac{R^4}{EI_{i+2}}\,\frac{a_{i+2,n} - a_{i+1,n}}{n\left(n^2 - 1\right)^2} + C_{i+2,n} = \frac{nL_{i+1}}{Gt_{i+1}}a_{i+1,n} - \frac{n^3L_{i+1}{}^3}{6ER^2t'_{i+1}}a_{i+1,n} \\
&\qquad + \frac{n^2L_{i+1}{}^2}{2ER}\sigma_{i+1,n}(0) + \frac{n^2L_{i+1}}{R}u_{i+1,n}(0) \\
&\qquad + \frac{R^4}{EI_{i+1}}\,\frac{a_{i+1,n} - a_{in}}{n\left(n^2 - 1\right)^2} + C_{i+1,n}
\end{aligned}\right\} \quad (14)$$

If the six quantities $\sigma_{i-1,n}(0)$, $\sigma_{in}(0)$, $\sigma_{i+1,n}(0)$, $u_{i-1,n}(0)$, $u_{in}(0)$, and $u_{i+1,n}(0)$ are eliminated from the seven expressions of equations (14), the following recurrence formula relating the Fourier coefficients a of five successive bays is obtained:

$$
\begin{aligned}
&- a_{i-2,n}\left(\frac{\upsilon_1}{I_{i-1}}\right) + a_{i-1,n}\left[\frac{\upsilon_1}{I_{i-1}}\left(1 + \frac{I_{i-1}}{I_i} + \frac{6B_{i-1} - n^2}{6A_{i-1}\gamma}\right) + \frac{\upsilon_2}{I_i}\right] \\
&- a_{in}\left[\frac{\upsilon_1}{I_i} + \frac{\upsilon_2}{I_i}\left(1 + \frac{I_i}{I_{i+1}} + \frac{6B_i - n^2}{6A_i\gamma}\right) + \frac{\upsilon_3}{I_{i+1}} + \frac{\upsilon_4 n^2}{2I_iA_i\gamma}\right] \\
&+ a_{i+1,n}\left[\frac{\upsilon_2}{I_{i+1}} + \frac{\upsilon_3}{I_{i+1}}\left(1 + \frac{I_{i+1}}{I_{i+2}} + \frac{6B_{i+1} - n^2}{6A_{i+1}\gamma}\right)\right] - a_{i+2,n}\left(\frac{\upsilon_3}{I_{i+2}}\right) \\
&= - \left[\upsilon_1 C_{i-1,n} - \left(\upsilon_1 + \upsilon_2\right)C_{in} + \left(\upsilon_2 + \upsilon_3\right)C_{i+1,n} - \upsilon_3 C_{i+2,n}\right]\frac{E}{R^4 n\gamma} \qquad (15)
\end{aligned}
$$

in which

$$\upsilon_1 = \frac{1}{L_{i-1}{}^2}\left(\frac{L_{i-1}}{L_i} + \frac{L_{i-1}}{L_{i+1}}\right)$$

$$\upsilon_2 = \frac{1}{L_i{}^2}\left(2\frac{L_i}{L_{i+1}} + \frac{L_{i-1}}{L_{i+1}} + 1\right)$$

$$\upsilon_3 = \frac{1}{L_{i+1}{}^2}\left(\frac{L_{i-1}}{L_i} + 1\right)$$

$$v_4 = \frac{1}{L_i^{\,2}}\left(\frac{L_i}{L_{i+1}} + \frac{L_{i-1}}{L_{i+1}} + \frac{L_{i-1}}{L_i} + 1\right)$$

$$A_i = \frac{R^6 t'_i}{I_i L_i^{\,3}}$$

$$B_i = \frac{Et'_i R^2}{Gt_i L_i^{\,2}}$$

$$\gamma = \frac{1}{n^2\left(n^2 - 1\right)^2}$$

If the cylinder is of uniform construction, equation (15) can be considerably simplified and reduces to

$$a_{i-2,n} + 2\gamma_n a_{i-1,n} + 2\beta_n a_{in} + 2\gamma_n a_{i+1,n} + a_{i+2,n}$$

$$= \left(C_{i-1,n} - 3C_{in} + 3C_{i+1,n} - C_{i+2,n}\right)\frac{EI}{R^4 n\gamma} \qquad (16)$$

in which

$$\gamma_n = -2 + \frac{n^2 - 6B}{12A\gamma}$$

$$\beta_n = 3 + \frac{n^2 + 3B}{3A\gamma}$$

The recurrence formulas (15) and (16) relate the nth shear-flow coefficient of bay i with the corresponding coefficients of the two bays on each side of bay i. One equation similar to equation (15) or equation (16) can consequently be written for each bay of a cylinder, provided that at least two bays exist on each side of this bay. For antisymmetrical loading, equations (15) and (16) can be applied if the Fourier coefficients a are replaced by the coefficients b.

Boundary Equations

Since the recurrence formula applies only to a bay having two bays on each side, incomplete or boundary equations must be found for each of the two bays at each boundary. Boundary equations, consequently, are presented for bays m and m-1 for a cylinder fixed at the right of bay m and for bays 0 and 1 for a cylinder free at the left end of bay 0. (See fig. 3.) By suitable combinations of the boundary equations and by proper manipulation of the subscripts, these equations can be used for the analysis of cylinders fixed at both ends, unrestrained at both ends, or unrestrained at one end and fixed at the other end.

Procedure for deriving boundary equations.- The general recurrence formula was derived by combining the equations for $\sigma_{in}(x_i)$, $u_{in}(x_i)$, $w_{in}(x_i)$, and $(w_{in})_{ring}$ (equations (3) to (6)) with the general continuity conditions (equations (7) to (13)) and then eliminating all Fourier coefficients except the a's. In the derivation of the boundary equations, the defining equations (3) to (6) are combined in a similar fashion with (1) all of the continuity conditions (equations (7) to (13)) that do not include quantities in nonexistent bays or rings and (2) the boundary conditions.

Thus, for clamped edges (see fig. 3) the boundary equation for bay m is obtained by combining equations (3) to (6) with the continuity conditions

$$\sigma_{m-1,n}(L_{m-1}) = \sigma_{mn}(0)$$

$$u_{m-1,n}(L_{m-1}) = u_{mn}(0)$$

$$w_{m-1,n}(0) = (w_{m-1,n})_{ring}$$

$$w_{m-1,n}\left(L_{m-1}\right) = \left(w_{mn}\right)_{ring}$$

$$w_{mn}(0) = \left(w_{mn}\right)_{ring}$$

and the boundary conditions

$$w_{mn}\left(L_m\right) = 0$$

$$u_{mn}\left(L_m\right) = 0$$

and then eliminating all the Fourier coefficients except the a's.

Boundary equations for fixed end.- If the foregoing procedure is followed, the boundary equation for bay m (fig. 3) is found to be

$$- a_{m-2,n}\left(\frac{\mu_1}{I_{m-1}}\right) + a_{m-1,n}\left[\frac{\mu_1}{I_{m-1}}\left(1 + \frac{I_{m-1}}{I_m} + \frac{6B_{m-1} - n^2}{6A_{m-1}\gamma}\right) + \frac{\mu_2}{I_m}\right]$$

$$- a_{mn}\left[\frac{\mu_1}{I_m} + \frac{\mu_2}{I_m}\left(1 + \frac{6B_m - n^2}{6A_m\gamma}\right) + \frac{\mu_3 n^2}{2I_m A_m \gamma}\right]$$

$$= - \left[\mu_1 C_{m-1,n} - \left(\mu_1 + \mu_2\right)C_{mn}\right]\frac{E}{R^4 n\gamma} \tag{17}$$

in which

$$\mu_1 = \frac{1}{L_{m-1}^{\;2}}$$

$$\mu_2 = \frac{1}{L_m^{\;2}}\left(2\frac{L_m}{L_{m-1}} + 1\right)$$

$$\mu_3 = \frac{1}{L_m^{\;2}}\left(\frac{L_m}{L_{m-1}} + 1\right)$$

and similarly for bay m-1

$$- a_{m-3,n}\left(\frac{\mu_4}{I_{m-2}}\right) + a_{m-2,n}\left[\frac{\mu_4}{I_{m-2}}\left(1 + \frac{I_{m-2}}{I_{m-1}} + \frac{6B_{m-2} - n^2}{6A_{m-2}\gamma}\right) + \frac{\mu_5}{I_{m-1}}\right]$$

$$- a_{m-1,n}\left[\frac{\mu_4}{I_{m-1}} + \frac{\mu_5}{I_{m-1}}\left(1 + \frac{I_{m-1}}{I_m} + \frac{6B_{m-1} - n^2}{6A_{m-1}\gamma}\right) + \frac{\mu_6}{I_m} + \frac{\mu_7 n^2}{2I_{m-1}A_{m-1}\gamma}\right]$$

$$+ a_{mn}\left[\frac{\mu_5}{I_m} + \frac{\mu_6}{I_m}\left(1 + \frac{6B_m - n^2}{6A_m\gamma}\right)\right] = -\Big[\mu_4 C_{m-2,n} - \left(\mu_4 + \mu_5\right)C_{m-1,n}$$

$$+ \left(\mu_5 + \mu_6\right)C_{mn}\Big]\frac{E}{R^4 n\gamma} \qquad (18)$$

where

$$\mu_4 = \frac{1}{L_{m-2}^{\,2}}\left(\frac{L_{m-2}}{L_{m-1}} + \frac{L_{m-2}}{L_m}\right)$$

$$\mu_5 = \frac{1}{L_{m-1}^{\,2}}\left(2\frac{L_{m-1}}{L_m} + \frac{L_{m-2}}{L_m} + 1\right)$$

$$\mu_6 = \frac{1}{L_m^{\,2}}\left(\frac{L_{m-2}}{L_{m-1}} + 1\right)$$

$$\mu_7 = \frac{1}{L_{m-1}^{\,2}}\left(\frac{L_{m-1}}{L_m} + \frac{L_{m-2}}{L_m} + \frac{L_{m-2}}{L_{m-1}} + 1\right)$$

For cylinders of uniform construction the fixed-end boundary equations (17) and (18) for bays m and m-1, respectively, reduce to

$$\left.\begin{aligned} &a_{m-2,n} + \left(2\gamma_n - 1\right)a_{m-1,n} + \left(2\beta_n - 2\gamma_n - 6\right)a_{mn} \\ &\qquad = \left(C_{m-1,n} - 4C_{mn}\right)\frac{EI}{R^4 n\gamma} \\ &a_{m-3,n} + 2\gamma_n a_{m-2,n} + 2\beta_n a_{m-1,n} + \left(2\gamma_n + 1\right)a_{mn} \\ &\qquad = \left(C_{m-2,n} - 3C_{m-1,n} + 3C_{mn}\right)\frac{EI}{R^4 n\gamma} \end{aligned}\right\} \qquad (19)$$

For antisymmetrical loading the Fourier coefficients a in equations (17) to (19) are replaced by the corresponding coefficients b.

In order to apply equations (17) to (19) to the left end of a cylinder, the signs of the shear-flow coefficients must be changed and the subscripts of the various terms altered. If the cylinder of figure 3 is fixed at the left of bay 0, subscripts m, m-1, . . . are replaced by 0,1, . . ., respectively, for those terms pertaining to the sheet of the bays and by 1,2, . . ., respectively, for those terms pertaining to the rings.

Boundary equations for unrestrained end.- The boundary equations for the unrestrained end of the cylinder shown in figure 3 are also found by following the procedure outlined. The boundary condition at the free edge is

$$\sigma_{0n}(0) = 0$$

The boundary equation for bay 0 is found to be

$$a_{0n}\left[\frac{\lambda_1}{I_0}\left(1 + \frac{I_0}{I_1} + \frac{6B_0 - n^2}{6A_0\gamma}\right) + \frac{\lambda_2}{I_1} + \frac{\lambda_3 n^2}{2I_0A_0\gamma}\right]$$

$$- a_{1n}\left[\frac{\lambda_1}{I_1} + \frac{\lambda_2}{I_1}\left(1 + \frac{I_1}{I_2} + \frac{6B_1 - n^2}{6A_1\gamma}\right)\right] + a_{2n}\left(\frac{\lambda_2}{I_2}\right)$$

$$= - \left[\lambda_1 c_{0n} - \left(\lambda_1 + \lambda_2\right)c_{1n} + \lambda_2 c_{2n}\right]\frac{E}{R^4 n\gamma} \qquad (20)$$

in which

$$\lambda_1 = \frac{1}{L_0 L_1}$$

$$\lambda_2 = \frac{1}{L_1^2}$$

$$\lambda_3 = \frac{1}{L_0^2}\left(\frac{L_0}{L_1} + 1\right)$$

Similarly, the boundary equation for bay 1 is

$$- a_{0n}\left(\frac{\lambda_2}{I_1} - \frac{\lambda_5 n^2}{2I_0A_0\gamma}\right) + a_{1n}\left[\frac{\lambda_2}{I_1}\left(1 + \frac{I_1}{I_2} + \frac{6B_1 - n^2}{6A_1\gamma}\right) + \frac{\lambda_4}{I_2} + \frac{\lambda_6 n^2}{2I_1A_1\gamma}\right]$$

$$- a_{2n}\left[\frac{\lambda_2}{I_2} + \frac{\lambda_4}{I_2}\left(1 + \frac{I_2}{I_3} + \frac{6B_2 - n^2}{6A_2\gamma}\right)\right] + a_{3n}\left(\frac{\lambda_4}{I_3}\right)$$

$$= - \left[\lambda_2 C_{1n} - \left(\lambda_2 + \lambda_4\right)C_{2n} + \lambda_4 C_{3n}\right]\frac{E}{R^4 n\gamma} \qquad (21)$$

where

$$\lambda_4 = \frac{1}{L_1 L_2}$$

$$\lambda_5 = \frac{1}{L_0^2}\left(\frac{L_2}{L_1} + 1\right)$$

$$\lambda_6 = \frac{1}{L_1^2}\left(\frac{L_2}{L_1} + 1\right)$$

For cylinders of uniform construction the unrestrained-end boundary equations (20) and (21) for bays 0 and 1, respectively, are

$$\left.\begin{aligned}
\left(2\gamma_n + 2\beta_n + 1\right)a_{0n} + \left(2\gamma_n + 1\right)a_{1n} + a_{2n} &= -\left(C_{0n} - 2C_{1n} + C_{2n}\right)\frac{EI}{R^4 n\gamma} \\
2\gamma_n a_{0n} + 2\beta_n a_{1n} + 2\gamma_n a_{2n} + a_{3n} &= \left(C_{0n} - 3C_{1n} + 3C_{2n} - C_{3n}\right)\frac{EI}{R^4 n\gamma}
\end{aligned}\right\} \quad (22)$$

For antisymmetrical loading the coefficients a are replaced by b in equations (20) to (22). In order to apply equations (20) to (22) to the right end of a cylinder, the signs of the shear-flow coefficients must be changed and the subscripts of the various terms suitably altered.

Special boundary equations.- The boundary equations developed are suitable for cylinders having four or more bays. For the special case of the center bay of a three bay cylinder, the boundary equation, which depends upon the conditions at both boundaries, can also be found by means of the general procedure previously outlined. The boundary equations for cylinders of one or two bays can be similarly derived.

Application of Recurrence Formula and Boundary Equations

Specific loadings.- As mentioned previously, the stress analysis of a reinforced cylinder arbitrarily loaded in the planes of the rings can be carried out conveniently if the stresses caused by the symmetric and antisymmetric components of the external forces are suitably combined. Further simplification of the analysis is obtained if the loadings are resolved into concentrated radial forces, concentrated tangential forces, and concentrated bending moments. For each ring loaded at $\varphi = 0^\circ$ (see fig. 4) the load function C_{in}, obtained in the derivation of equation (6) for $\left(w_{in}\right)_{ring}$, for a concentrated radial force, a concentrated tangential force, and a concentrated bending momont are, respectively,

$$\left.\begin{aligned} C_{r_{in}} &= \frac{P_i n}{\pi R} \frac{R^4 n\gamma}{EI_i} \\ C_{t_{in}} &= \frac{T_i}{\pi R} \frac{R^4 n\gamma}{EI_i} \\ C_{m_{in}} &= -\frac{M_{c_i}\left(n^2 - 1\right)}{\pi R^2} \frac{R^4 n\gamma}{EI_i} \end{aligned}\right\} \tag{23}$$

where P, T, and M_c are the symmetrical radial load, the antisymmetrical tangential load, and the antisymmetrical bending-moment load, respectively, acting on any ring i at $\varphi = 0^\circ$.

Simultaneous equations.- A typical set of equations applicable to a cantilevered uniform cylinder with six bays ($m = 5$ in fig. 3) is presented in table 1. The first two and last two rows were obtained from the unrestrained-end and fixed-end boundary relationships of equations (22) and (19), respectively, and the intermediate rows were obtained from the recurrence formula of equation (16). For a nonuniform cylinder these expressions are replaced by those of equations (20), (21), (18), (17), and (15). It is to be noted that the coefficients of the unknown a's and b's are independent of C_{in} (load term of equations (23));

consequently, numerical solution of the equations (reference 5) for various loadings is greatly facilitated. A set of simultaneous linear equations similar to that of table 1 must be solved for each n-value chosen. The number of n-values required depends upon the desired accuracy. The Fourier coefficients obtained for a given load P, T, or M_c at $\varphi = 0^\circ$ can be used to determine the coefficients for similar loads at any other value of φ since the z-axis (fig. 2) can be chosen to coincide with any radius.

Stresses and loads in cylinder.- After the coefficients a and b are computed, substitution in the formulas (A1) to (A4) presented in appendix A enables the stress analyst to compute the shear flow in the actual sheet, the direct stress in the fictitious sheet, and the moments, shears, and axial forces in the rings. The stresses due to loads acting at several rings and at various values of φ can be superimposed to give the stresses caused by these loads acting simultaneously.

APPROXIMATE METHOD OF ANALYSIS BY SOLUTION OF FINITE DIFFERENCE EQUATION

Difference-Equation Solution for Infinitely Long Cylinders

Equation (15) referred to previously as a general recurrence formula is also a fourth-order finite difference equation with variable coefficients. Since the variable coefficients prohibit the solution of this equation in closed form, only the solution of the equation that pertains to a uniform cylinder is discussed herein. A general procedure for solving the fourth-order finite difference equation with constant coefficients (see equation (16)) is presented in reference 6. When the right-hand side of equation (16) is set equal to zero, the following homogeneous equation is obtained:

$$a_{i-2,n} + 2\gamma_n a_{i-1,n} + 2\beta_n a_{in} + 2\gamma_n a_{i+1,n} + a_{i+2,n} = 0 \qquad (24)$$

From reference 6, the general solution of this homogeneous equation consists of the following six independent solutions: for

$$D_n = 2\frac{(\beta_n - 1)}{\gamma_n^{\ 2}} > 1 \quad \text{and} \quad \gamma_n < 0$$

$$a_{in} = e^{-\psi_n k}\left(\alpha_{1n} \cos k\chi_n + \alpha_{2n} \sin k\chi_n\right)$$

$$+ e^{\psi_n k}\left(\alpha_{3n} \cos k\chi_n + \alpha_{4n} \sin k\chi_n\right) \tag{25a}$$

for $D_n > 1$ and $\gamma_n > 0$

$$a_{in} = (-1)^k e^{-\psi_n k}\left(\alpha_{1n} \cos k\chi_n + \alpha_{2n} \sin k\chi_n\right)$$

$$+ (-1)^k e^{\psi_n k}\left(\alpha_{3n} \cos k\chi_n + \alpha_{4n} \sin k\chi_n\right) \tag{25b}$$

for $D_n < 1$ and $\gamma_n < 0$

$$a_{in} = e^{-\psi_n k}\left(\alpha_{1n} \cosh k\rho_n + \alpha_{2n} \sinh k\rho_n\right)$$

$$+ e^{\psi_n k}\left(\alpha_{3n} \cosh k\rho_n + \alpha_{4n} \sinh k\rho_n\right) \tag{25c}$$

for $D_n < 1$ and $\gamma_n > 0$

$$a_{in} = (-1)^k e^{-\psi_n k}\left(\alpha_{1n} \cosh k\rho_n + \alpha_{2n} \sinh k\rho_n\right)$$

$$+ (-1)^k e^{\psi_n k}\left(\alpha_{3n} \cosh k\rho_n + \alpha_{4n} \sinh k\rho_n\right) \tag{25d}$$

for $D_n = 1$ and $\gamma_n < 0$

$$a_{in} = e^{-\psi_n k}\left(\alpha_{1n} + \alpha_{2n} k\right) + e^{\psi_n k}\left(\alpha_{3n} + \alpha_{4n} k\right) \tag{25e}$$

for $D_n = 1$ and $\gamma_n > 0$

$$a_{in} = (-1)^k e^{-\psi_n k}\left(\alpha_{1n} + \alpha_{2n} k\right) + (-1)^k e^{\psi_n k}\left(\alpha_{3n} + \alpha_{4n} k\right) \tag{25f}$$

in which

$$\psi_n = \frac{1}{2}\cosh^{-1}\left[\frac{\beta_n - 1}{2} + \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^{\,2}}\right]$$

$$\chi_n = \frac{1}{2}\cos^{-1}\left[\frac{\beta_n - 1}{2} - \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^{\,2}}\right]$$

$$\rho_n = \frac{1}{2}\cosh^{-1}\left[\frac{\beta_n - 1}{2} - \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^{\,2}}\right]$$

$$k = i = 0, 1, 2 \ldots$$

and α_{1n}, α_{2n}, α_{3n}, and α_{4n} are arbitrary constants.

The analysis of a uniform cylinder that extends longitudinally to infinity in both directions from a loaded ring is readily carried out with the aid of equations (16), (24), and (25). If the loaded ring is considered to be a boundary between the two halves of the beam and if no load other than that at the boundary is assumed to act, the difference equation (16) with the right-hand side set equal to zero applies equally well to both parts of the cylinder (see fig. 5); consequently, only one-half of the cylinder need be considered in the analysis. Since the difference equation applicable is the homogeneous equation (24), equations (25) together with the appropriate boundary conditions are solutions of the present problem.

The distortions caused by the concentrated load have no effect on the stress distribution in the cylinder at $k = \infty$; therefore, $a_{\infty n} = 0$. The first term on the right-hand side of each of equations (25) satisfies this condition; however, the second term does not satisfy this requirement and, hence, must vanish. The solutions then that are compatible with the boundary condition at infinity are from equations (25): for $D_n > 1$ and $\gamma_n < 0$

$$a_{in} = e^{-\psi_n k}\left(\alpha_{1n} \cos k\chi_n + \alpha_{2n} \sin k\chi_n\right) \tag{26a}$$

for $D_n > 1$ and $\gamma_n > 0$

$$a_{in} = (-1)^k e^{-\psi_n k}\left(\alpha_{1n} \cos k\chi_n + \alpha_{2n} \sin k\chi_n\right) \tag{26b}$$

for $D_n < 1$ and $\gamma_n < 0$

$$a_{in} = e^{-\psi_n k}\left(\alpha_{1n} \cosh k\rho_n + \alpha_{2n} \sinh k\rho_n\right) \tag{26c}$$

for $D_n < 1$ and $\gamma_n > 0$

$$a_{in} = (-1)^k e^{-\psi_n k}\left(\alpha_{1n} \cosh k\rho_n + \alpha_{2n} \sinh k\rho_n\right) \tag{26d}$$

for $D_n = 1$ and $\gamma_n < 0$

$$a_{in} = e^{-\psi_n k}\left(\alpha_{1n} + \alpha_{2n} k\right) \tag{26e}$$

for $D_n = 1$ and $\gamma_n > 0$

$$a_{in} = (-1)^k e^{-\psi_n k}\left(\alpha_{1n} + \alpha_{2n} k\right) \tag{26f}$$

From the conditions of symmetry about the loaded ring, modification of equation (16) leads to the determination of two boundary equations applicable to the present problem. If the load function at the loaded ring is designated C_{0n} (see equations (23)) and equation (16) is written for bay 0, the first boundary equation is

$$\left(2\beta_n - 2\gamma_n\right)a_{0n} + \left(2\gamma_n - 1\right)a_{1n} + a_{2n} = -3C_{0n} \frac{EI}{R^4 n\gamma} \tag{27}$$

since $a_{0n} = -a_{-1n}$ and $a_{1n} = -a_{-2n}$. If equation (16) is written for bay 1, the second boundary equation is seen to be

$$\left(2\gamma_n - 1\right)a_{0n} + 2\beta_n a_{1n} + 2\gamma_n a_{2n} + a_{3n} = C_{0n} \frac{EI}{R^4 n\gamma} \tag{28}$$

since $a_{0n} = -a_{-1n}$.

The boundary equations (27) and (28) permit the determination of the arbitrary constants α_{1n} and α_{2n}. For a given value of n, substitution of the appropriate value for a_{in} from equations (26) into equations (27) and (28) yields a set of two simultaneous equations; for example, if $D_n > 1$ and $\gamma_n < 0$

$$\begin{aligned}
&\alpha_{1n}\left[\left(2\beta_n - 2\gamma_n\right) + \left(2\gamma_n - 1\right)e^{-\psi_n} \cos \chi_n + e^{-2\psi_n} \cos 2\chi_n\right] \\
&\quad + \alpha_{2n}\left[\left(2\gamma_n - 1\right)e^{-\psi_n} \sin \chi_n + e^{-2\psi_n} \sin 2\chi_n\right] = -3C_{0n} \frac{EI}{R^4 n\gamma} \\
&\alpha_{1n}\left[\left(2\gamma_n - 1\right) + 2\beta_n e^{-\psi_n} \cos \chi_n + 2\gamma_n e^{-2\psi_n} \cos 2\chi_n + e^{-3\psi_n} \cos 3\chi_n\right] \\
&\quad + \alpha_{2n}\left(2\beta_n e^{-\psi_n} \sin \chi_n + 2\gamma_n e^{-2\psi_n} \sin 2\chi_n + e^{-3\psi_n} \sin 3\chi_n\right) = C_{0n} \frac{EI}{R^4 n\gamma}
\end{aligned} \tag{29}$$

The constants α_{1n} and α_{2n} are obtained from the solution of these equations. To each value of n there corresponds one value each for α_{1n} and α_{2n}. Since D_n and γ_n are functions of n as well as the elastic properties of the cylinder, for a particular cylinder more than one of equations (26) may be required for the determination of all the values of α_{1n} and α_{2n}. With the values of these constants determined for each value of n, corresponding values of a_{in} for each bay are obtained from equations (26).

As in the application of the recurrence formula, a's corresponding to several harmonics, that is, n varying from 2 to the value that yields the desired accuracy, must be found. For antisymmetrical loading, a is replaced by b in equations (24) to (29). The values of the coefficients a and b obtained are substituted in equations (A1), (A2), and (A4) for the desired load values. Since the expression for the direct stress in the sheet now involves an infinite summation along the cylinder of the shear-flow coefficients, simplified formulas for the direct stresses at any ring k are presented in appendix B.

If equations (27) and (28) are replaced by the unrestrained-end boundary equations (22), with all values of C_{in} except C_{0n} set equal to zero, a tip loaded cylinder extending to infinity in one direction can be analyzed with a procedure similar to that developed herein.

Application to Finite Cylinders

Whereas a concentrated load causes distortion in the region in the immediate vicinity of the load, for most practical purposes the part of the cylinder a few bays away from the load can be assumed undisturbed. Consequently, if the load is located a sufficient distance from external restraints, the distortions of the cylinder in the region of the load are independent of those restraints. If then a uniform cylinder of finite length is to be analyzed and this cylinder is loaded in a manner such that the load is not in the proximity of an external restraint, the elementary stresses and loads are found as usual by considering the cylinder to be finite, whereas the corrections may be found by use of the difference-equation method by considering the cylinder to be infinite. Because the effect of the concentrated load dissipates quite rapidly, values of a and b are usually of interest only for those bays in the vicinity of the load. The desired forces and moments in this region can then be determined as before from the equations given in appendixes A and B.

Adequacy of Difference-Equation Solution

Although the solution in closed form of the problem of a uniform reinforced circular cylinder is exact only for infinitely long cylinders symmetrical about a loaded ring, comparisons of the finite-difference-equation solution, the recurrence-formula solution, the standard solution (reference 7), and experimental data for cylinder 2 of reference 3 were made for a cylinder fixed at one end, unrestrained at the other, and having only four bays. The cylinder was loaded with a concentrated radial force at a ring located two bays from each end. (See fig. 6.) In figures 7 and 8 curves are given for bending moments in the loaded ring and adjacent rings as well as for the shear flows in the two bays adjacent to the loaded ring. Inasmuch as the cylinder contains relatively few bays, an extreme case is represented that is unlikely to be met in practice. The more bays a cylinder has the more closely it approximates an infinite cylinder for which the finite-difference-equation solution is exact; consequently, the good agreement shown in figures 7 and 8 among the finite-difference-equation solution, the recurrence-formula solution and experimental data indicates that the simplified solution is quite adequate.

Advantages of Difference-Equation Solution

Since airplane fuselages approximating circular cylinders are composed of a relatively large number of bays for most practical cases, the simplified solution should be a good approximation to that obtained by the use of the recurrence formula. As mentioned previously, when the recurrence formula is applied, sets of simultaneous equations containing as many unknowns as there are bays in the structure must be solved for each n-value required. For structures having many bays the amount of computations involved may be prohibitive; however, no such computations are involved when use is made of the infinite-cylinder solution. In addition, this solution is adaptable to the construction of design charts similar to Wise's charts of reference 7. The analysis of any long uniform cylinder is dependent only on the values of the structural parameters A and A/B. For various representative values of these parameters, charts can be constructed from which the analyst can determine desired stress coefficients. For extreme cases, such as a cylinder loaded only one to two bays away from a restraint, the recurrence-formula method is recommended for accurate solutions.

CONCLUDING REMARKS

The recurrence formula developed in the present paper facilitates the stress analysis of circular cylinders loaded in the planes of the reinforcing rings. The cylinders can be composed of bays of different cross sections and lengths and can be supported by rings having different moments of inertia. The boundary equations presented are applicable to cylinders fixed at both ends, unrestrained at both ends, or unrestrained at one end and fixed at the other end.

For the analysis of cylinders composed of relatively few bays, it is recommended that the recurrence formula be used to obtain sets of simultaneous linear algebraic equations. The solutions of these equations lead to an accurate determination of the stresses in the rings and sheet of the cylinders. The analysis of cylinders composed of many bays, as are semimonocoque fuselages, can more conveniently be accomplished by the solution of the recurrence formula as a finite difference equation. Although the stresses obtained with this solution are approximations to the more accurate stresses found with the simultaneous equations, for long cylinders the computations involved are considerably shorter. In addition, since for the three basic loads the stresses determined by this method are dependent only upon the structural parameters of the cylinder, charts facilitating the rapid determination of the stresses in reinforced cylinders can be readily constructed.

Langley Memorial Aeronautical Laboratory
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APPENDIX A

FORMULAS FOR LOADS AND STRESSES IN CYLINDERS

After the coefficients a and b are computed, the shear flows in the actual sheet, the direct stresses in the fictitious sheet, and the bending moments, shears, and axial forces in the rings can be found with the aid of the equations given in the appendix of reference 3. For the sake of completeness these equations, with some additions, are presented herein.

Shear Flow

The total shear flow $q_i(\varphi)$ in any bay i for any ring loading on a cylinder can be expressed as

$$q_i(\varphi) = q_R + \sum_{n=2}^{\infty} a_{in} \sin n\varphi + \sum_{n=2}^{\infty} b_{in} \cos n\varphi \qquad \text{(A1)}$$

in which q_R represents the elementary shear flow calculated on the basis of rigid rings. For a cantilevered cylinder, q_R is zero for those bays located between the tip and a loaded ring. For those bays between a loaded ring and the root, the values of q_R for a radial load P, a tangential load T, and a concentrated ring bending moment M_c, each applied to ring i at $\varphi = 0^\circ$, are given in table 2. Positive forces and bending moments are indicated in figure 4. If more than one ring is loaded or if the cylinder is not of cantilever construction, P_i, T_i, and M_{c_i} are replaced by the resultant radial, tangential, and moment load, respectively, acting on a cross section of bay i.

Direct Stress in Skin

For a cantilevered cylinder such as that shown in figure 3, if the longitudinal skin stress at ring 0 is assumed to be zero, the direct stress at ring i is (see equations (1b) and (3))

$$\sigma_i(0, \varphi) = \sigma_R - \frac{1}{R}\sum_{n=2}^{\infty}\left(a_{0n}\frac{L_0}{t'_0} + a_{1n}\frac{L_1}{t'_1} + \cdot\cdot\cdot + a_{i-1,n}\frac{L_{i-1}}{t'_{i-1}}\right)n\cos n\varphi$$

$$+ \frac{1}{R}\sum_{n=2}^{\infty}\left(b_{0n}\frac{L_0}{t'_0} + b_{1n}\frac{L_1}{t'_1} + \cdot\cdot\cdot + b_{i-1,n}\frac{L_{i-1}}{t'_{i-1}}\right)n\sin n\varphi \qquad \text{(A2)}$$

in which σ_R is the stress given by the simple engineering theory of bending. Since the shear stress is constant in the longitudinal direction within a bay, σ varies linearly between rings.

If the cylinder is rigidly fixed at ring 0 as well as at ring m+1, the initial boundary stress $\sum_{n=2}^{\infty}\sigma_{0n}(0)\cos n\varphi$ (for symmetrical loads) must be added to the direct stress obtained with equation (A2). The value of the Fourier coefficient $\sigma_{0n}(0)$ is determined for a cylinder having at least three bays from the continuity condition

$$w_{0n}(L_0) = \left(w_{1n}\right)_{ring}$$

and the boundary conditions

$$u_{0n}(0) = 0$$

$$w_{0n}(0) = 0$$

together with the defining equations (5) and (6). The relationship obtained is

$$\sigma_{0n}(0) = \frac{2L_0A'_0\gamma}{Rt'_0n}\left[-a_{0n}\left(1 + \frac{6B_0 - n^2}{6A'_0\gamma}\right) + a_{1n} + \frac{EI_1}{R^4n\gamma}C_{1n}\right] \qquad \text{(A3)}$$

in which

$$A'_0 = \frac{R^6 t'_0}{I_1 L_0{}^3}$$

For antisymmetrical loading, a_{0n} and a_{1n} are replaced by b_{0n} and b_{1n}, respectively.

Bending Moments and Forces in Ring

The bending moments, shear forces, and axial forces in the reinforcing rings of a cylinder arbitrarily supported at its ends are, respectively,

$$\left.\begin{aligned}
M_i &= M_R + R^2 \sum_{n=2}^{\infty} \frac{a_{in} - a_{i-1,n}}{n\left(n^2 - 1\right)} \cos n\varphi - R^2 \sum_{n=2}^{\infty} \frac{b_{in} - b_{i-1,n}}{n\left(n^2 - 1\right)} \sin n\varphi \\
V_i &= V_R - R \sum_{n=2}^{\infty} \frac{a_{in} - a_{i-1,n}}{\left(n^2 - 1\right)} \sin n\varphi - R \sum_{n=2}^{\infty} \frac{b_{in} - b_{i-1,n}}{\left(n^2 - 1\right)} \cos n\varphi \\
H_i &= H_R + R \sum_{n=2}^{\infty} \frac{n\left(a_{in} - a_{i-1,n}\right)}{\left(n^2 - 1\right)} \cos n\varphi - R \sum_{n=2}^{\infty} \frac{n\left(b_{in} - b_{i-1,n}\right)}{\left(n^2 - 1\right)} \sin n\varphi
\end{aligned}\right\} \quad \text{(A4)}$$

in which M_R, V_R, and H_R are the bending moment, shear force, and axial force in the rings, respectively, determined on the basis of elementary shear flow in the skin. Positive values of the bending moments and loads in a cylinder are indicated in figure 4. Formulas for M_R, V_R, and H_R corresponding to a radial load P, a tangential load T, and a concentrated ring bending moment M_c, each applied to a ring i at $\varphi = 0^\circ$, are given in table 2. For rings not loaded externally, only the series expression in equations (A4) are required.

APPENDIX B

DIRECT STRESSES IN INFINITELY LONG CYLINDERS

For the determination of the direct stress in the skin of a uniform infinitely long cylinder, equation (A2) can be replaced by

$$\sigma_i(0,\ \varphi) = \sigma_R + \frac{L}{Rt'}\sum_{n=2}^{\infty}\sum_{i=\infty}^{k} a_{in}\ n \cos n\varphi \tag{B1}$$

or

$$\sigma_i(0,\ \varphi) = \sigma_R + \frac{L}{Rt'}\sum_{n=2}^{\infty}\left(\sum_{i=0}^{\infty} a_{in} - \sum_{i=0}^{k-1} a_{in}\right) n \cos n\varphi \tag{B2}$$

in which only the coefficients a_{in} are considered. In these equations, $\sigma_i(0,\ \varphi)$ is the direct stress at ring i=k. Corresponding to the six values of a_{in} from equations (26), six solutions for $\sigma_i(0,\ \varphi)$ can be determined by summation along the cylinder. As an illustration of the procedure involved, equation (26a) is used herein for the value of a_{in}. Consequently, equation (B2) becomes

$$\sigma_i(0,\ \varphi) = \sigma_R + \frac{L}{Rt'}\sum_{n=2}^{\infty}\left[\sum_{i=0}^{\infty} e^{-\psi_n k}\left(\alpha_{1n} \cos k\chi_n + \alpha_{2n} \sin k\chi_n\right)\right.$$
$$\left. - \sum_{i=0}^{k-1} e^{-\psi_n k}\left(\alpha_{1n} \cos k\chi_n + \alpha_{2n} \sin k\chi_n\right)\right] n \cos n\varphi \tag{B3}$$

The summations from $i = 0$ to $i = \infty$ and $i = 0$ to $i = k - 1$ are readily accomplished with the aid of formulas 6.830, 6.833, and 3.61, numbers 12 and 13, of reference 8. The resulting formula for the direct stress at any ring k is for $D_n > 1$, $\gamma_n < 0$, and $i = k = 0,1,2, \ldots$

$$\sigma_i(0,\ \varphi) = \sigma_R + \frac{L}{2Rt'} \sum_{n=2}^{\infty} e^{-\psi_n k} \frac{\alpha_{1n}\left[e^{\psi_n} \cos k\chi_n - \cos (k-1)\chi_n\right] + \alpha_{2n}\left[e^{\psi_n} \sin k\chi_n - \sin (k-1)\chi_n\right]}{\cosh \psi_n - \cos \chi_n} n \cos n\varphi \tag{B4}$$

With a procedure analogous to that used for the determination of this equation, the following solutions are obtained for the direct stresses corresponding to the remaining five values of a_{in}:

for $D_n > 1$ and $\gamma_n > 0$

$$\sigma_i(0,\varphi) = \sigma_R + (-1)^k \frac{L}{2Rt'} \sum_{n=2}^{\infty} e^{-\psi_n k} \frac{\alpha_{1n}\left[e^{\psi_n}\cos k\chi_n + \cos(k-1)\chi_n\right] + \alpha_{2n}\left[e^{\psi_n}\sin k\chi_n + \sin(k-1)\chi_n\right]}{\cosh \psi_n + \cos \chi_n} n \cos n\varphi \quad \text{(B5a)}$$

for $D_n < 1$ and $\gamma_n < 0$

$$\sigma_i(0,\varphi) = \sigma_R + \frac{L}{2Rt'} \sum_{n=2}^{\infty} e^{-\psi_n k} \frac{\alpha_{1n}\left[e^{\psi_n}\cosh k\rho_n - \cosh(k-1)\rho_n\right] + \alpha_{2n}\left[e^{\psi_n}\sinh k\rho_n - \sinh(k-1)\rho_n\right]}{\cosh \psi_n - \cosh \rho_n} n \cos n\varphi \quad \text{(B5b)}$$

for $D_n < 1$ and $\gamma_n > 0$

$$\sigma_i(0,\varphi) = \sigma_R + (-1)^k \frac{L}{2Rt'} \sum_{n=2}^{\infty} e^{-\psi_n k} \frac{\alpha_{1n}\left[e^{\psi_n}\cosh k\rho_n + \cosh(k-1)\rho_n\right] + \alpha_{2n}\left[e^{\psi_n}\sinh k\rho_n + \sinh(k-1)\rho_n\right]}{\cosh \psi_n + \cosh \rho_n} n \cos n\varphi \quad \text{(B5c)}$$

for $D_n = 1$ and $\gamma_n < 0$

$$\sigma_i(0,\varphi) = \sigma_R + \frac{L}{2Rt'} \sum_{n=2}^{\infty} e^{-\psi_n k} \frac{\alpha_{1n}\left(e^{\psi_n} - 1\right) + \alpha_{2n}\left[ke^{\psi_n} - (k - 1)\right]}{\cosh \psi_n - 1} n \cos n\varphi \quad \text{(B5d)}$$

for $D_n = 1$ and $\gamma_n > 0$

$$\sigma_i(0,\varphi) = \sigma_R + (-1)^k \frac{L}{2Rt'} \sum_{n=2}^{\infty} e^{-\psi_n k} \frac{\alpha_{1n}\left(e^{\psi_n} + 1\right) + \alpha_{2n}\left[ke^{\psi_n} + (k - 1)\right]}{\cosh \psi_n + 1} n \cos n\varphi \quad \text{(B5e)}$$

If the coefficients b_{in} are considered, $\cos n\varphi$ is replaced by $-\sin n\varphi$ in equations (B1) to (B5).

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TABLE 1.- GENERAL SCHEME OF EQUATIONS FOR UNIFORM CYLINDER OF SIX BAYS

	Left-hand side (2)						Right-hand side (3)		
Coefficient (1) / Equation number	a_{0n}	a_{1n}	a_{2n}	a_{3n}	a_{4n}	a_{5n}	(Load term)		
	b_{0n}	b_{1n}	b_{2n}	b_{3n}	b_{4n}	b_{5n}	Radial	Tangential	Bending moment
0	$2\gamma_n + 2\beta_n + 1$	$2\gamma_n + 1$	1	---	---	---	$-\left(P_0 - 2P_1 + P_2\right)\frac{n}{\pi R}$	$-\left(T_0 - 2T_1 + T_2\right)\frac{1}{\pi R}$	$\left(M_0 - 2M_1 + M_2\right)\frac{n^2 - 1}{\pi R^2}$
1	$2\gamma_n$	$2\beta_n$	$2\gamma_n$	1	---	---	$\left(P_0 - 3P_1 + 3P_2 - P_3\right)\frac{n}{\pi R}$	$\left(T_0 - 3T_1 + 3T_2 - T_3\right)\frac{1}{\pi R}$	$-\left(M_0 - 3M_1 + 3M_2 - M_3\right)\frac{n^2 - 1}{\pi R^2}$
2	1	$2\gamma_n$	$2\beta_n$	$2\gamma_n$	1	---	$\left(P_1 - 3P_2 + 3P_3 - P_4\right)\frac{n}{\pi R}$	$\left(T_1 - 3T_2 + 3T_3 - T_4\right)\frac{1}{\pi R}$	$-\left(M_1 - 3M_2 + 3M_3 - M_4\right)\frac{n^2 - 1}{\pi R^2}$
3	---	1	$2\gamma_n$	$2\beta_n$	$2\gamma_n$	1	$\left(P_2 - 3P_3 + 3P_4 - P_5\right)\frac{n}{\pi R}$	$\left(T_2 - 3T_3 + 3T_4 - T_5\right)\frac{1}{\pi R}$	$-\left(M_2 - 3M_3 + 3M_4 - M_5\right)\frac{n^2 - 1}{\pi R^2}$
4	---	---	1	$2\gamma_n$	$2\beta_n$	$2\gamma_n + 1$	$\left(P_3 - 3P_4 + 3P_5\right)\frac{n}{\pi R}$	$\left(T_3 - 3T_4 + 3T_5\right)\frac{1}{\pi R}$	$-\left(M_3 - 3M_4 + 3M_5\right)\frac{n^2 - 1}{\pi R^2}$
5	---	---	---	1	$2\gamma_n - 1$	$2\beta_n - 2\gamma_n - 6$	$\left(P_4 - 4P_5\right)\frac{n}{\pi R}$	$\left(T_4 - 4T_5\right)\frac{1}{\pi R}$	$-\left(M_4 - 4M_5\right)\frac{n^2 - 1}{\pi R^2}$

[1]Coefficients a apply to radial loads; coefficients b to tangential or bending-moment loads.

[2]Symbols are defined in equation (16).

[3]Symbols are defined in equation (23).

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TABLE 2.- ELEMENTARY SHEARS, BENDING MOMENTS, AND LOADS IN CYLINDER

CORRESPONDING TO BASIC RING LOADINGS

[Sign convention shown in fig. 4]

External ring loading at $\varphi = 0^\circ$	q_R	M_R	V_R	H_R
P_i	$-\frac{P_i}{\pi R}\sin\varphi$	$\frac{P_iR}{2\pi}\left[1+\frac{\cos\varphi}{2}-(\pi-\varphi)\sin\varphi\right]$	$\frac{P_i}{2\pi}\left[\frac{\sin\varphi}{2}-(\pi-\varphi)\cos\varphi\right]$	$-\frac{P_i}{2\pi}\left[(\pi-\varphi)\sin\varphi+\frac{3}{2}\cos\varphi\right]$
T_i	$-\frac{T_i}{\pi R}\left(\frac{1}{2}+\cos\varphi\right)$	$\frac{T_iR}{2\pi}\left[(\pi-\varphi)(1-\cos\varphi)-\frac{3}{2}\sin\varphi\right]$	$\frac{T_i}{2\pi}\left[(\pi-\varphi)\sin\varphi-\frac{\cos\varphi}{2}-1\right]$	$\frac{T_i}{2\pi}\left[\frac{\sin\varphi}{2}-(\pi-\varphi)\cos\varphi\right]$
M_{c_i}	$-\frac{M_{c_i}}{2\pi R^2}$	$\frac{M_{c_i}}{2\pi}\left[(\pi-\varphi)-2\sin\varphi\right]$	$-\frac{M_{c_i}}{2\pi R}(1+2\cos\varphi)$	$-\frac{M_{c_i}}{\pi R}\sin\varphi$

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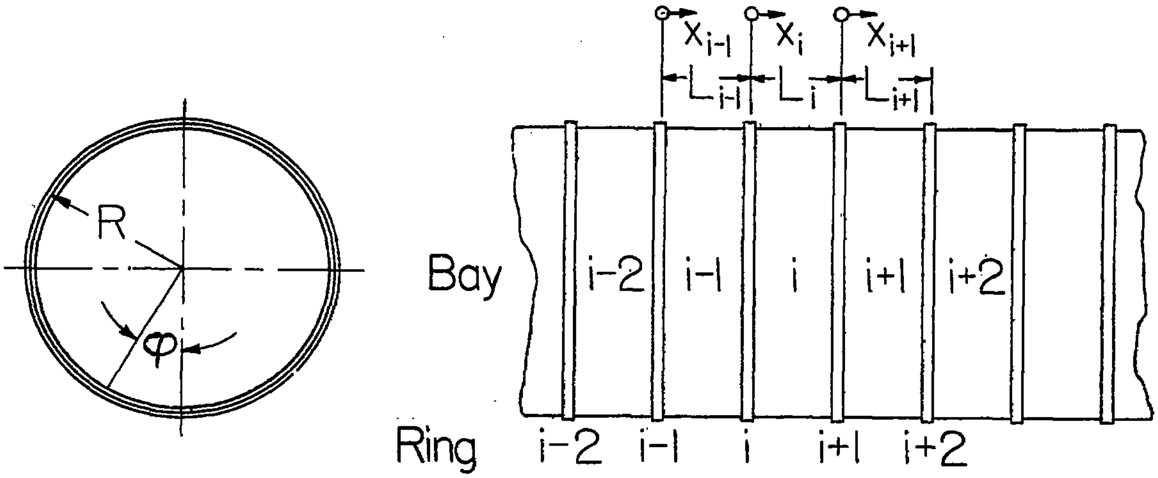

Figure 1.- Part of typical cylinder.

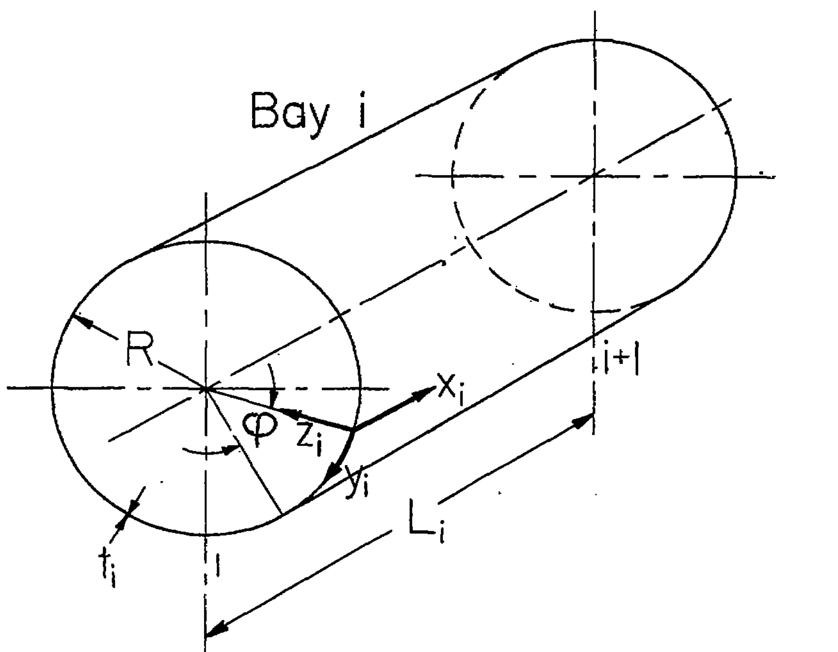

Figure 2.- Coordinate system for typical bay.

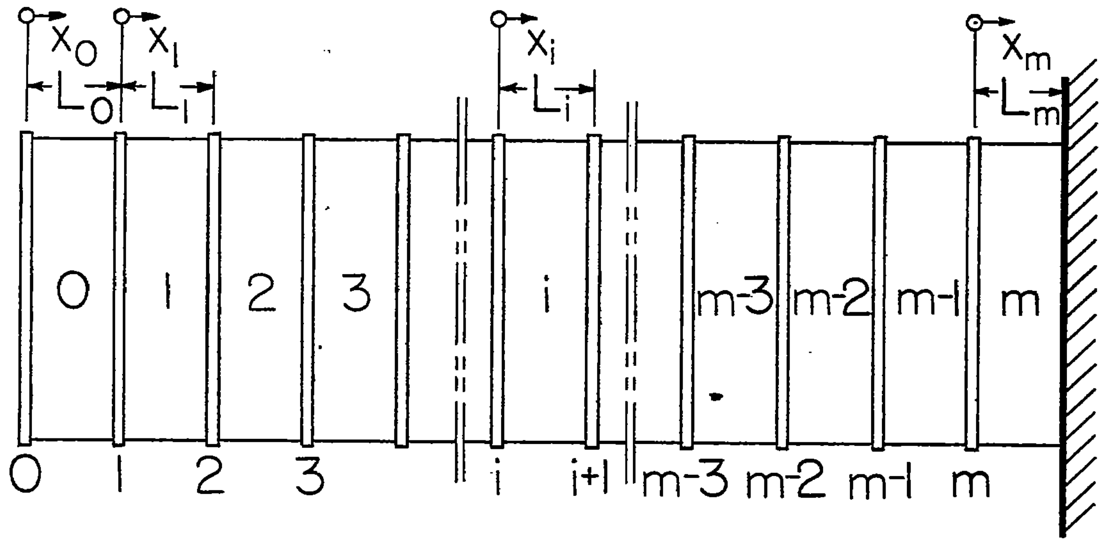

Figure 3.- Side view of cantilevered cylinder.

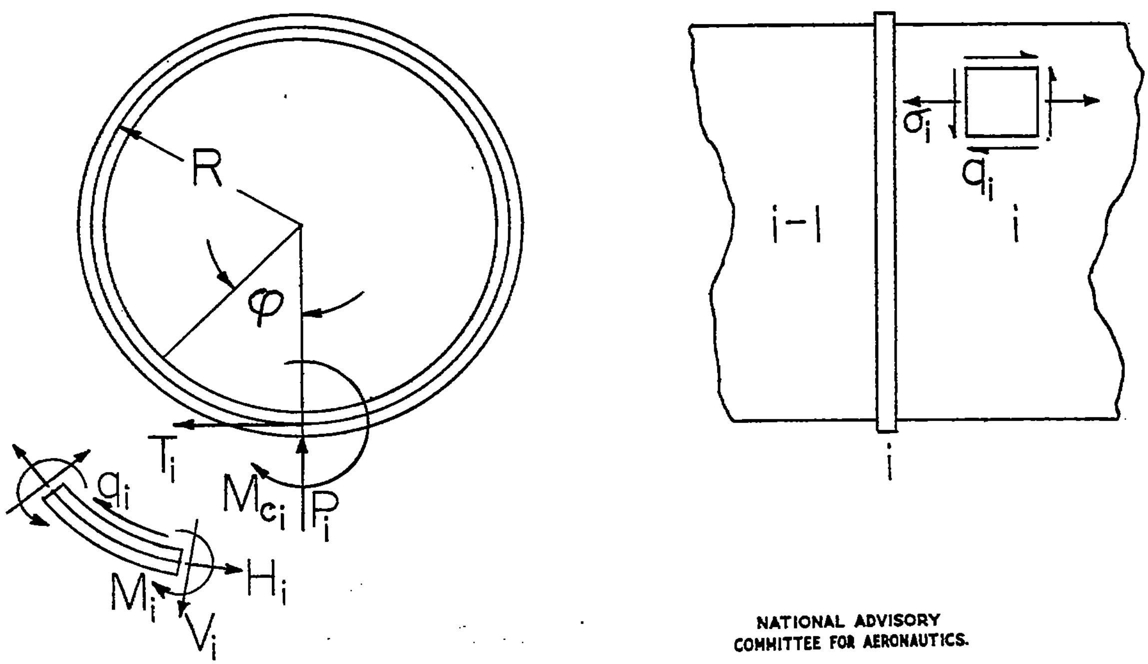

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Figure 4.- Sign convention used in analysis.

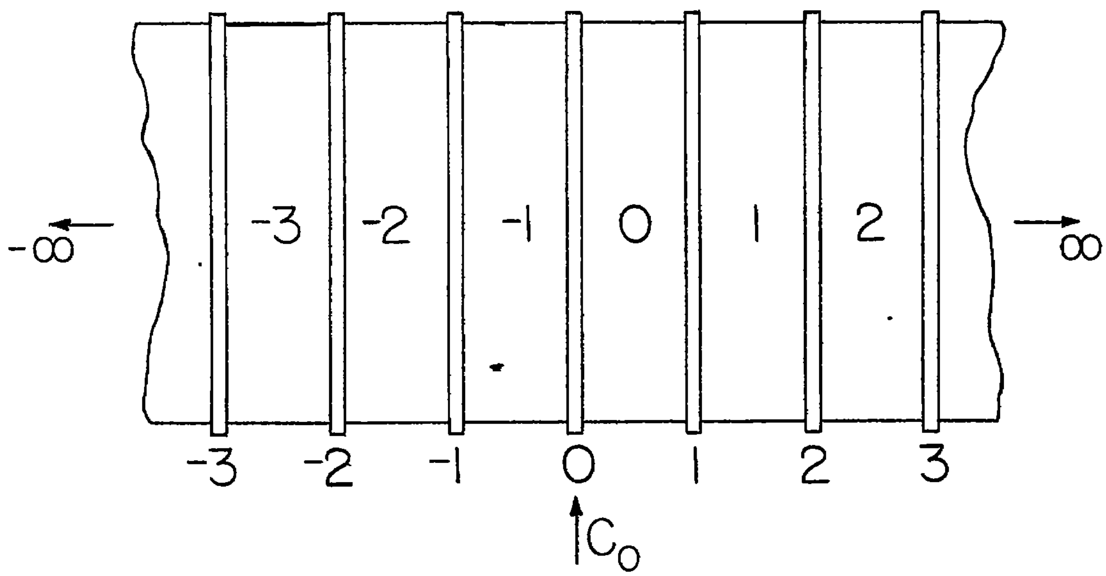

Figure 5.- Loaded part of infinitely long cylinder.

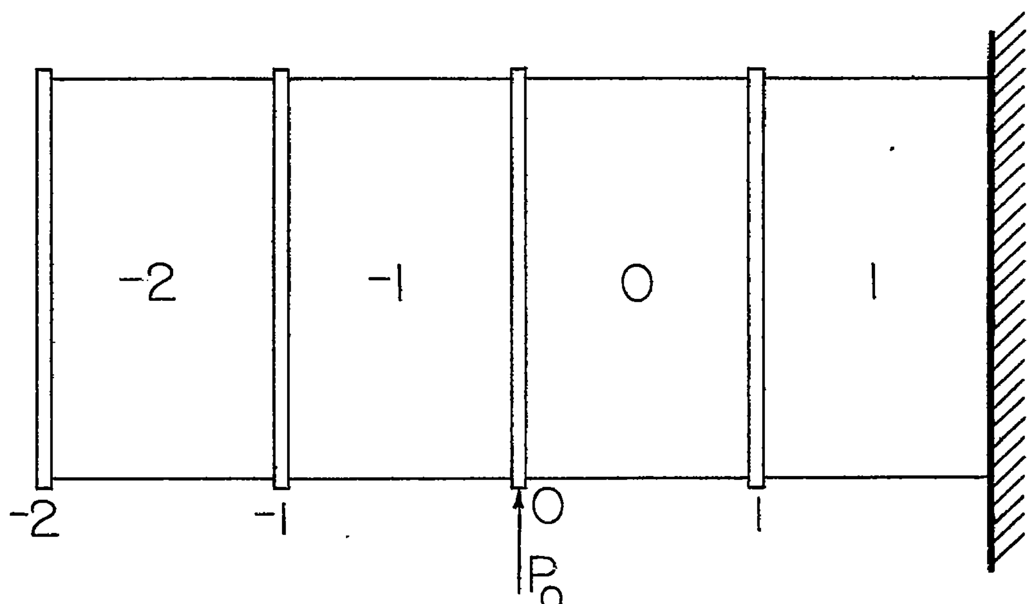

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Figure 6.- Side view of cylinder 2 analyzed in reference 3.

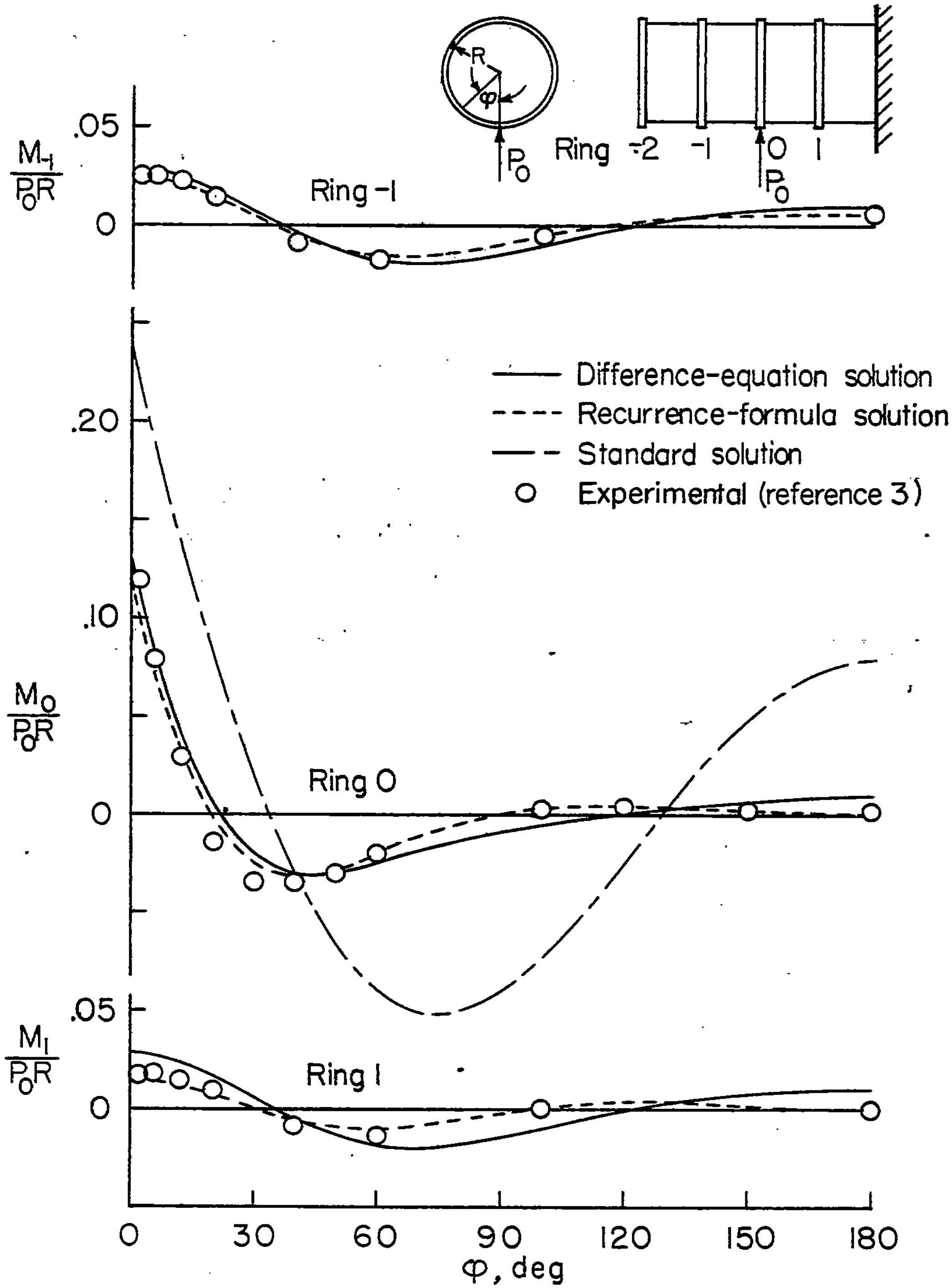

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Figure 7.- Comparison between calculated and experimental ring-bending moments for cantilevered cylinder.

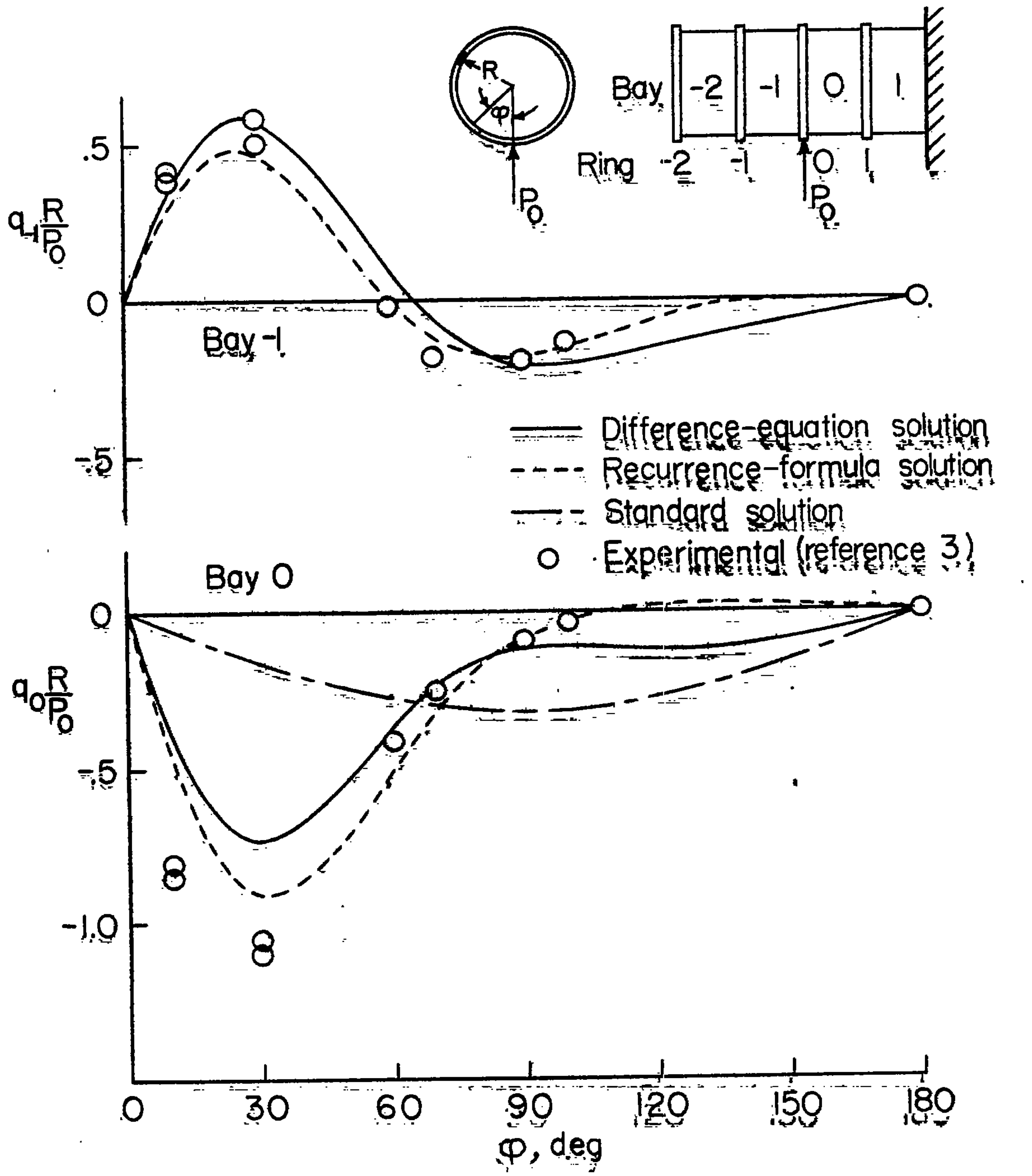

Figure 8.– Comparison between calculated and experimental skin-shear flows for cantilevered cylinder.