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A THEORETICAL STUDY OF THE DYNAMIC PROPERTIES OF HELICOPTER-BLADE SYSTEMS

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By H. Reissner and M. Morduchow

SUMMARY

The work herein presented on a theoretical study of the dynamic properties of lifting rotors covers:

1. The derivations of the angles of attack of the inflow, of the blade-position variables – pitch, flapping, and lagging – and of the aerodynamic and inertia forces acting on hinged helicopter (lifting-rotor) blades in the hovering and in the traveling states

2. The development and solution of the equilibrium conditions of the blade system in the hovering and in the traveling states

3. The development of the frequency, stability, and damping properties of the hinged, sufficiently rigid rotor blades during hovering and traveling

The method of solution of small oscillations about a state of simultaneous rotation and traveling has in this paper been carried through for only small speed ratios.

This method was applied to four cases of diverse constraint conditions between the three angles of pitch, flapping, and lagging. The results are significant in regard to restoring force and to mode, frequency, phase, damping, and amplitude ratios. The number of modes and of independent amplitudes is, of course, equal to the degree of freedom. Each mode in hovering corresponds to one frequency, but in traveling each mode consists of three frequencies of fixed amplitude ratios and fixed phase differences but with only one free amplitude. However, the amplitude ratios of the two additional frequencies to the original amplitude are, in all cases which have been computed numerically, smaller than the speed ratio. The results for the four cases treated show marked advantages obtainable by appropriate kinematic constraints between the three angles in regard to safety against resonance, to damping, and to automatic adjustment.

INTRODUCTION

The problems of blade-angle control, stability of motion, resonance, and flutter of helicopter-blade systems have not as yet been fully treated. This fact seems to be substantiated by the presence of disturbances in present-day rotor operations which have not been fully explained. No complete theory encompassing the effects of different methods of articulation and angle control appears to be known. These problems are closely interrelated through their dependence on the dynamic equations of blade motion.

Previous publications have dealt mainly with the performance of the helicopter or with the stability of the equilibrium of steady flight of the complete helicopter system with very special assumptions in regard to the blade and hub connections.

It is believed that the problem of smooth operation of a helicopter must be attacked in a more general way, and for this reason must be divided into at least two parts; namely,

(1) The free oscillation of the blade system about the different steady states of flight

(2) The forced oscillations of this system caused by the reaction of the fuselage, by irregularities of torque, by gusts, by transition to another state of flight, by flying in a curve, and so on.

Problem (1) again falls into several parts. The first part deals with the conclusions which can be drawn from the results of the theory of small oscillations about a steady state of motion applied to a system of sufficiently rigid blades hinged to a driving hub. The rigidity of blades is sufficient to give the real behavior of the blade system if the natural frequencies of a blade, treated as rigid, are small in comparison with the lowest natural frequency of elastic vibrations of a blade.

The second part, not treated in this paper, would have to deal with the superposition and interference of elastic vibrations, or what is the same, of elastic waves of the blades on or with the rigid-blade oscillations in those cases when the blades are appreciably flexible. The analysis of this phenomenon would require the integration of the equations of deflection - and twist - vibrations of the blades, under the action of the local aerodynamic and inertia (including centrifugal) forces and under the effect of the boundary conditions at the hinge and at the tip.

The flutter problem, meaning the determination of the critical velocity at which damping coefficients become negative, so that self-excited oscillations arise, has not been covered in this paper for two reasons. The first reason is that the critical velocity at the low average reduced ratio $\frac{V}{\Omega R}$ of the blade will, in general, not be reached, especially since the blade is stiffened by the restoring centrifugal force. The second reason is that all other sources of instability and resonance should be removed first before going into this difficult problem more deeply than previous authors, who have simply applied results of straight-moving airfoils. It is not improbable that sometimes unstable oscillations appearing in stationary flow, with no phase differences between bending and twisting, have been mistaken for flutter.

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SYMBOLS

x,y,z	right-hand Cartesian coordinates, fixed to hub of blade (see fig. 1)
x_1,y_1,z_1	right-hand Cartesian coordinates, fixed to blade system (see fig. 3)
Ω	angular velocity of hub
$\overline{V}$	velocity of flight (called traveling) in any direction, for example, forward, sideways, or backward
γ	angle between plane of rotation and velocity vector $\overline{V}$
$\psi = \Omega t$	angular position of blade center line ($\psi = 0$ when x-axis coincides with projection of $\overline{V}$ on plane of rotation)
t	time, seconds
t_b	maximum thickness of blade cross section
$\cdot \equiv \frac{d}{dt}$	
$' \equiv \frac{d}{d\psi}$	
θ	angle between plane of rotation and zero-lift line of chord (called pitch angle), positive from y to z
α	angle of attack

β	angle between plane of rotation and center line of blade (called flapping angle), positive upward, that is, from x to z
ζ	angle between x-axis and projection of center line of blade on plane of rotation (called lagging angle), positive backward, that is, from y to x
$\theta_o, \beta_o, \zeta_o$	values of θ, β, ζ in steady state of flight
$\theta_c, \beta_c, \zeta_c$	values of θ, β, ζ if steady state of flight is hovering state (subscript c ≡ Constant)
$\bar{\theta}, \bar{\beta}, \bar{\zeta}$	deviations of θ, β, ζ from values in steady state of equilibrium, that is, from θ_o, β_o, ζ_o
e	distance of blade-hinge center from axis of rotation (see fig. 1)
r	distance of point of center line of blade from hinge center (see fig. 1)
R	tip radius of blade
$\eta \equiv \frac{e}{r}$	
$\eta_e \equiv \frac{e}{R}$	
$\bar{V}$	vector of resultant velocity $\bar{v} + \overline{\Omega r}$ in steady flight
V_x, V_y, V_z	rectangular components of $\bar{V}$
R_e	real part of complex oscillation frequency p
r_i	inner radius of blade length (see fig. 1)

$\mu \equiv \frac{v}{\Omega r}$; also a geometric constraint constant in case C, OSCILLATIONS OF BLADE SYSTEM IN HOVERING, and IN LOW-SPEED TRAVELING

$\mu_e \equiv \frac{v}{\Omega R}$; that is, speed ratio

$s \equiv \frac{r}{R}$

$s_i \equiv \frac{r_i}{R}$

S_i	imaginary part of complex oscillation frequency p
k	defined by equation (13), $k \simeq 1$

c	chord of cross section of blade
c_i	chord of innermost cross section ($r = r_i$)
$\bar{c}_1/c_1$	unit vector of zero-lift line of cross section of blade
V_{nc_1}	component of relative inflow velocity in plane of cross section of blade
V_{c_1}	component of V_{nc_1} in direction of $\bar{c}_1$
V_n	component of V_{nc_1} normal to $\bar{c}_1$
Γ	circulation around cross section of blade
$f \equiv \frac{l_{cg} - l_{ac}}{c}$	distance ratio between center of gravity and aerodynamic center of cross section of blade (see fig. 4)
M_{x1}, M_{y1}, M_{z1}	moment components about x_1-, y_1-, z_1-axes, respectively (see fig. 3)
$\bar{B}$	moment of momentum vector of blade
A	area of blade cross section
I_p, I_1, I_2	moments of inertia of cross section of blade
$\bar{I}_{12}$, $\bar{I}_x$, $\bar{I}_y$	volume integrals of moments of inertia (see equation (61))
$\bar{I}_H \equiv \int_{r_i}^{R} Ar^2\, dr$	
$\bar{S} \equiv \int_{r_i}^{R} Ar\, dr$	
ρ	density of air
σ	density of blade material (average density in case of framed structure)
$q_{complex}$	complex circular frequency of oscillation
$p_n = \frac{q_{complex}}{\Omega}$	complex frequency ratio $\left(R_{e,n} + i\, S_{i,n}\right)$
q_n	real frequency, cycles per second $\left(\frac{\Omega}{2\pi} S_{i,n}\right)$

$\log \frac{A_n}{A_{n+1}}$ logarithmic decrement $\left(2\pi \frac{R_{e,n}}{S_{1,n}}\right)$

B, F, D complex amplitudes of oscillation of θ, β, ζ, respectively, about steady state of flight

κ, λ, μ constants in kinematic constraint conditions

L_m Lagrange multiplier

$$E = \pi \frac{\rho}{\sigma} \Omega^2 R^4 c_1$$

$$T = \pi \frac{\rho}{\sigma} \frac{R^4 c_1}{\overline{I}_H} \frac{32}{315} \left(1 + \frac{1}{2} s_1 + \frac{3}{2} \eta_e\right)$$

$$C_2 = \frac{4}{315} \left(1 + \frac{1}{2} s_1\right) \pi \frac{\rho}{\sigma} \frac{R^4 c_1}{\overline{I}_H}$$

$$C_2' = \frac{C_2}{1 + 2\eta_e}$$

k_1, k_2, k_3 values of integrals given by equation (21)

g acceleration due to gravity

AERODYNAMIC AND CENTRIFUGAL FORCES AND MOMENTS IN STEADY HORIZONTAL FLIGHT

The aerodynamic forces acting on a blade when it is rotating in a conical path and also moving horizontally will be determined basically by means of the Kutta-Joukowski lift theorem. This principle requires that the components of the total relative inflow velocity $\overline{V}$ and also of the circulation $\overline{\Gamma}$ first be obtained.

A right-hand Cartesian coordinate system will be used, in which the z-axis coincides with the axis of rotation, or axis of the cone, and is directed upward, though not necessarily exactly vertical. The axis of the conical path of a blade is supposed to coincide with the z-axis, which, from an accurate standpoint, implies that the connection of the hub with the driving shaft is such that the blade system tilts with the hub. The x-axis, moreover, is in the direction of the arm (of length e) of the hub. (See fig. 1.) Instead of using the direction of flight, which is assumed as horizontal in this paper, as the axis of reference, it is equivalent and more convenient for the kinematic analysis to consider the longitudinal axis of a blade as fixed and the line of action of the velocity vector as rotating. The velocity vector will, in general, not lie in the plane of rotation xy but will make an angle γ with it. (See fig. 2.) Thus, while the direction of travel, that is, of $\bar{v}$, is assumed to be horizontal (or nearly horizontal), the plane of rotation xy will be tilted at an angle γ to the direction of flight (travel), so that the normal z of this plane may be tilted forward in the direction of flight and possibly also sideways. It is possible in this way to treat the motion of all blades by analyzing only one as a sample.

As seen from figure 1, a concentric arrangement of the flapping and lagging hinges at distance e from the axis of rotation is assumed. Such an arrangement appears to the authors to have the advantage of transmitting the large centrifugal force on one (spherical) bearing surface with lower (unit) pressure than in two smaller surfaces of the sleeve bearings. Moreover, it is shown in this analysis that the actual value of the small distance between the hinges has little influence on the stability characteristics of the rotor system.

Relative Velocity Components

Let v_x, v_y, v_z be the components of the relative (travel) velocity of flight, that is, relative to the x, y, z coordinate system, which is for this purely geometric discussion considered fixed in space. Moreover, let $v_{\Omega x}$, $v_{\Omega y}$, $v_{\Omega z}$ be the components of the relative rotational velocity of a blade. Then the components of the resultant inflow velocity (excluding the induced velocity) will be

$$\left.\begin{aligned} V_x &= v_x + v_{\Omega x} \\ V_y &= v_y + v_{\Omega y} \\ V_z &= v_z + v_{\Omega z} \end{aligned}\right\} \qquad (1)$$

As in ordinary wing theory, the induced flow across a helicopter blade will be taken into account by the changed (induced) direction of the total inflow; whereas the induced change of magnitude of the velocity is negligible.

The following notation is used in figures 1 and 2:

Ω angular velocity of hub due to torque of engine

r distance along blade measured outward from hinge point H

$\bar{r}$ centroid axis of blade

β flapping angle, that is, angle between line r and xy-plane

ζ lagging angle, that is, angle between x-axis and projection of $\bar{r}$ on xy-plane (It may be noted that the positive direction of ζ is opposite to that of Ω.)

ψ angle periodically traversed by velocity vector, that is, angle between x-axis and projection of relative velocity of flight $\bar{v}$ on xy-plane (in accordance with the kinematic inversion)

γ angle between relative velocity of flight $\bar{v}$ and plane of rotation, that is, xy-plane

From figure 2, the components of $\bar{v}$ are:

$$\left.\begin{aligned} v_x &= -v \cos\gamma \cos\psi \\ v_y &= -v \cos\gamma \sin\psi \\ v_z &= -v \sin\gamma \end{aligned}\right\} \qquad (2)$$

From figure 1 and the fact that the rotational velocity $\bar{v}_\Omega$ is perpendicular to the line r_1 and to the z-axis, the components of $\bar{v}_\Omega$ are:

$$\left.\begin{aligned} v_{\Omega x} &= -\Omega r \cos\beta \sin\zeta \\ v_{\Omega y} &= -\Omega(e + r \cos\beta \cos\zeta) \\ v_{\Omega z} &= 0 \end{aligned}\right\} \qquad (3)$$

In practical cases β, ζ, and γ will be small quantities, in the sense that powers of these quantities above the second can be neglected. Moreover, when it is considered that the appreciable contribution to lift and torque will be made only by the blade farther out from the hinge, it is permissible to treat e/r also as a small quantity for all values of r contributing to the forces. That is,

$$\beta \ll 1, \quad \zeta \ll 1, \quad \gamma \ll 1, \quad \eta \ll 1 \tag{4}$$

where

$$\eta \equiv \frac{e}{r}$$

Thus, by adding equations (2) and (3), making use of equation (4), and introducing the dimensionless variable

$$\mu \equiv \frac{v}{\Omega r}$$

the components of the resultant relative velocity, to second powers of small quantities, are seen to be:

$$\left.\begin{aligned} V_x &= -\Omega r\left[\zeta + \mu\left(1 - \frac{\gamma^2}{2}\right)\cos\psi\right] \\ V_y &= -\Omega r\left[1 + \mu\left(1 - \frac{\gamma^2}{2}\right)\sin\psi + \eta - \frac{1}{2}\left(\zeta^2 + \beta^2\right)\right] \\ V_z &= -\Omega r\,\mu\gamma \end{aligned}\right\} \tag{5}$$

Lift Components

According to the Kutta-Joukowski theorem the lift per unit length of a blade is given by the vector product

$$\bar{L}' = \rho\,\bar{V} \times \bar{\Gamma} \tag{6}$$

where ρ is the density of the fluid medium, $\bar{V}$ is the vector of the resultant relative velocity (see equation (5)), and $\bar{\Gamma}$ is the circulation vector. The direction of $\bar{\Gamma}$ coincides with that of the bound vortex line representing the blade; that is, the direction of $\bar{\Gamma}$ is the same as that of the vector $\bar{r}$. The magnitude of the circulation $\bar{\Gamma}$ is given by the condition of finite velocity at the trailing edge of the zero-lift chord $\bar{c}_l$ of a blade. The radial component, that is, the component of the velocity parallel to the blade axis r, will have no influence on the value of Γ, because the circulation integral following this radial component, which is the same above and below the blade, is zero. Thus Γ itself is determined by the velocity component in the plane perpendicular to the blade axis; whereas the lift (see equation (6)) is determined by the total velocity vector (including the radial component). A well-known formula gives the magnitude of Γ by

$$\Gamma = \pi c V_{nc_1} \alpha \tag{7}$$

where

c chord of cross section

V_{nc_1} velocity component of V in plane perpendicular to r, that is, in plane of cross section of blade

α angle of attack, that is, angle between $\overline{V}_{nc_1}$ and $\overline{c}_1$, line of of zero lift of cross section

The velocity component V_{nc_1} can be determined as resultant of V_n and and V_{c_1}, where

V_n component of V in plane of blade cross section (at any r) perpendicular to zero-lift line c_1

V_{c_1} component of V in plane of cross section parallel to c_1

If θ denotes the pitch angle (that is, the angle between the zero-lift line of an airfoil section and the plane of rotation), which like β, ζ, γ, and η may be considered a first-order small quantity, then, with the direction cosines of the vectors $\overline{c}_1$ and $\overline{r}$ (see appendix A), the expressions for V_{c_1} and V_n, to second-order small quantities, are found to be:

$$V_{c_1} = \Omega r \left[1 + \mu\left(1 - \frac{\zeta^2 + \theta^2 + \gamma^2}{2}\right)\sin\psi + \eta + \mu(\zeta - \theta\beta)\cos\psi + \mu\theta\gamma - \frac{\beta^2 + \theta^2}{2}\right] \tag{8a}$$

$$V_n = \Omega r\left\{\theta(1 + \eta) + \mu\left[(\beta + \zeta\theta)\cos\psi + (\theta - \beta\zeta)\sin\psi - \gamma\right]\right\} \tag{8b}$$

The angle α can be obtained from the relation

$$\alpha \approx \tan \alpha = \frac{V_n}{V_{c_1}} \tag{8c}$$

whereas V_{nc_1} can be obtained from

$$V_{nc_1} = \sqrt{{V_{c_1}}^2 + {V_n}^2}$$

From equations (8a) and (8b) it can be seen that V_n is first order small, whereas V_{c_1} is finite; it follows from equation (7) therefore that to second-order small quantities

$$\Gamma = \pi c V_n \tag{9a}$$

or, from equation (8b),

$$\Gamma = \pi c \Omega r \left\{\theta(1 + \eta) + \mu \left[(\beta + \zeta\theta) \cos \psi + (\theta - \beta\zeta) \sin \psi - \gamma\right]\right\} \tag{9b}$$

Therefore, by expanding the vector product of equation (6), using the velocity components (equation (5)) and the direction cosines of $\bar{r}$, the lift components per unit length of a blade, to second orders, are found to be (see appendix B):

$$\left.\begin{aligned}
L_x' &= -\pi\rho c\Omega^2 r^2 \beta(1 + \mu \sin \psi)\left[\theta + \mu(\beta \cos \psi + \theta \sin \psi - \gamma)\right] \\
L_y' &= -\pi\rho c\Omega^2 r^2 \mu(\gamma - \beta \cos \psi)\left[\theta + \mu(\cos \psi + \theta \sin \psi - \gamma)\right] \\
L_z' &= \pi c\Omega^2 r^2 \rho \left\{\theta(1 + 2\eta) + \mu\left[(\beta + 2\zeta\theta) \cos \psi + (2\theta - \beta\zeta) \sin \psi - \gamma\right]\right. \\
&\quad + \eta\mu\left[\beta \cos \psi + 2\theta \sin \psi - \gamma\right] + \mu^2(\sin \psi + \zeta \cos \psi) \\
&\quad \left. \times \left[\theta(\sin \psi + \zeta \cos \psi) + \beta(\cos \psi - \zeta \sin \psi) - \gamma\right]\right\}
\end{aligned}\right\} \tag{10a}$$

It will be observed that, as might be expected, L_x' and L_y' are of a higher (second) order small than L_z' (which is first order small).

Drag Components

The drag components per unit length of span of a blade can be determined as follows: Inasmuch as the drag D' will be parallel to the resultant relative velocity V, it follows that

$$D_x' = D'\frac{V_x}{V}, \quad D_y' = D'\frac{V_y}{V}, \quad D_z' = D'\frac{V_z}{V} \tag{11}$$

The total drag per unit length can be expressed by the equation

$$D' = L_z'\left(\alpha_p + \alpha_i\right) \tag{12}$$

where α_p and α_i are the parasite and induced changes of the angle of attack, respectively. It may be remarked that equation (12), although not exact, is correct to quantities of second order, since, by observing the second-order smallness of L_x' and L_y', the total lift to quantities of second order may be given only by the z-component L_z'.

From equations (11), (12), and (5) it is seen, by considering α_p and α_i as first order small, that D_z' will be a third-order small quantity. Hence to second orders,

$$D_z' = 0$$

As can be seen from equation (12), it is sufficient, in order to determine D_x' and D_y' to second orders, to consider only the finite terms of V_x, V_y, and V_z. Thus, from equation (5),

$$V \approx \sqrt{{V_x}^2 + {V_y}^2} = \Omega r(1 + \mu \sin \psi)\sqrt{1 + \left(\frac{\mu \cos \psi}{1 + \mu \sin \psi}\right)^2}$$

For purposes of investigating stability it will suffice, in order to avoid needlessly complicated integrations, to replace the radical factor in the foregoing expression for V by an average value. Thus, taking $\mu \sim \frac{2}{3}$ as the highest expected speed ratio,

$$\cos^2\psi \sim \frac{1}{2}, \quad \sin^2\psi \sim \frac{1}{2}$$

it is seen that the maximum of $\sqrt{1 + \left(\frac{\mu \cos \psi}{1 + \mu \sin \psi}\right)^2} \sim \sqrt{1 + 0.11} \sim 1.087$

for the highest expected velocity $v(\psi = 2/3)$. Therefore, there can be written

$$V \approx \Omega r(1 + \mu \sin \psi)k \tag{13}$$

where k varies from 1 (for hovering, or $v = 0$) to 1.087 $\left(\text{for high-speed ratio } \frac{v}{\Omega R} \equiv \mu_e\right)$. Thus, by use of equations (10a), (11), (12), and (13), the expressions for the x- and y-components of the drag per unit length, to second-order small terms, are found to be:

$$\left.\begin{aligned} D_x' &= -\frac{\pi}{k}\left(\alpha_p + \alpha_i\right)c\Omega^2 r^2 \rho\mu \cos \psi\left[\mu(\beta \cos \psi + \theta \sin \psi - \gamma) + \theta\right] \\ D_y' &= -\frac{\pi}{k}\left(\alpha_p + \alpha_i\right)c\Omega^2 r^2 \rho(1 + \mu \sin \psi)\left[\mu(\beta \cos \psi + \theta \sin \psi - \gamma) + \theta\right] \end{aligned}\right\} \tag{14}$$

For purposes of investigating stability, when the assumptions or approximations need not be as accurate as for performance, the induced angle of attack α_i may be approximated by the average along the radius, which can be taken from the well-known formula for elliptical wings of aspect ratio AR in rectilinear flight:

$$\alpha_i = \frac{C_L}{\pi\ AR} = \frac{\frac{2\pi}{1 + \frac{2}{AR}}\alpha}{\pi\ AR} = \frac{2}{AR + 2}\alpha$$

The angle of attack α can be determined from

$$\alpha \approx \frac{V_n}{V_{c_1}}$$

Hence, by using equations (8a) and (8b) and supposing $AR = 8$, the expression for the induced angle, to first orders, becomes

$$\alpha_i = \frac{1}{5}\,\frac{\theta + \mu\left(\beta \cos \psi + \theta \sin \psi - \gamma\right)}{1 + \mu \sin \psi} \tag{15}$$

When, for simplicity, the case of hovering $(\mu_e = 0)$ is considered, it will be noted from equation (15) that if the blade angle θ is assumed as constant along the blade (see next section), then the induced angle of attack has implicitly been assumed to be constant. However, because the chord will be taken as a parabolic function along the blade decreasing toward the tip (see equation (18)), the drag force per unit length will also decrease parabolically. According to the theory of minimum drag of rotating airfoils, the induced angle will increase slightly near the root and then decrease, also slightly, toward the tip, according to a function of the form α_i proportional to $\dfrac{\Omega r/w}{1 + \left(\dfrac{\Omega r}{w}\right)^2}$, where w is the axial component of inflow velocity, which is equal to the induced velocity in hovering. When the multiplication by the chord is considered, a more exact determination of the drag force according to the aforementioned induced inflow distribution cannot cause much difference from the results of the assumptions made in this paper, particularly because for the blade treated as a rigid body oscillating about its hinge, only the resultant moment of drag enters the dynamic equations.

Pitch Angle

The blade angle is determined by the weight which the blade has to carry by means of its lift force. For this purpose it will be sufficient to consider only quantities up to first order small. Thus, from equation (10a),

$$L_x' = 0$$

$$L_y' = 0$$

$$L_z' = \pi\rho c\Omega^2 r^2(1 + \mu_1 \sin\psi)\left[\theta + \mu(\beta\cos\psi + \theta\sin\psi - \gamma)\right] - \Omega^2\sigma A r \frac{\partial^2\beta_o}{\partial\psi^2} \quad (10b)$$

The last term of equation (10b), where A is the area of a cross section, represents the inertia force for the case in which, for $\mu_e \neq 0$, the flapping angle β_o in steady flight varies periodically with $\psi(= \Omega t)$, that is, with time. The ideal requirement would be that the lift of a blade element be constant during a cycle. The pitch angle θ would then, according to equation (10b), have to vary (with ψ) along the circumference as follows:

$$\theta = \frac{\frac{L_z'}{\rho} + \Omega^2 \frac{\sigma}{\rho} Ar \frac{\partial^2 \beta_o}{\partial \psi^2}}{(\Omega r)^2 \pi c(1 + \mu \sin \psi)^2} - \frac{\mu(\beta \cos \psi - \gamma)}{1 + \mu \sin \psi} \tag{16}$$

In equation (16) L_z', Ωr, μ, β, and γ must be considered as constant on the circle at radius r while ψ is changing from 0 to 2π. Equation (16) would require a freely twisting blade, which, of course, would be difficult to realize. Therefore, it may henceforth be required that only the total force component $\int L_z' \, dr$ be constant (that is, be independent of ψ). Then θ, which will be dependent on ψ, can be determined as follows: Assume

$$\theta = \theta_c + \Delta\theta \tag{17}$$

where $\theta_c = f(r)$ and $\Delta\theta = g(\psi, \beta, \gamma, v)$ but is independent of r. This means that the blade is designed sufficiently rigid so that, for change of pitch, it can practically be rigidly rotated only about its radial axis.

Let r_1 be the radius of the innermost section of the blade, or the value of r at which the blade begins to become effective in lift. The following dimensionless quantities, moreover, in which R is the tip radius of a blade, will henceforth be used:

$$s \equiv \frac{r}{R}$$

$$\eta_e \equiv \frac{e}{R}$$

$$s_1 \equiv \frac{r_1}{R}$$

$$\mu_e \equiv \frac{v}{\Omega R}$$

As an example, it will be assumed that the variation of the blade chord c with the radius r is parabolic. This provides a decrease of the lift to zero at the tip of the blade and represents in a way the decrease of circulation toward the tip, without making it necessary to enter into the theory of the trailing distorted helical vortex sheet. Thus, it is supposed that

$$c = c_1\left(\frac{1 - s}{1 - s_1}\right)^{1/2} \tag{18}$$

It seems appropriate, moreover, to choose for an example the basic pitch angle θ_c for the hovering state $(\mu = 0)$ as constant (with r). It should be remarked, however, that this assumption, as well as that of equation (18), is suggested only by way of an example to fix the ideas and the order of magnitude and approximate form of the functions appearing. Whether these or other exemplifying reasonable assumptions are chosen will have only a negligible influence on the frequencies and on the stability conditions.

Put $\theta = \theta_c$, $\mu = 0$, and equation (18) into equation (10b), and set

$$\int_{r_1}^{R} L_z' \, dr = \frac{W}{\cos\gamma}\frac{1}{n} \approx \frac{W}{n} \tag{19}$$

where W is the weight of the helicopter, and n is the number of blades. Then by evaluation of the integral in equation (19), with the substitution of equation (10b), the expression obtained for θ_c, to first powers of s_1, is found to be

$$\theta_c = \frac{W}{\rho R^3\,\Omega^2 c_1}\,\frac{105}{8n\pi(2 + s_1)} \tag{20}$$

In order to find $\Delta\theta$, put equation (17) into equation (10b) and again require the condition of equation (19), substituting for W in terms of θ_c by means of equation (20). Then the expression for $\Delta\theta$, if powers of s_1 higher than the first are neglected, is found to be

$$\Delta\theta = \frac{-\mu_e\left[\theta_o(\sin\psi)\left(2k_2 + \mu_e k_1\sin\psi\right) + (\beta\cos\psi - \gamma)\left(k_2 + \mu_e k_1\sin\psi\right)\right] + \frac{\sigma}{\rho}\,\frac{S}{\pi k_3}\,\frac{\partial^2\beta_o}{\partial\psi^2}}{1 + 2\mu_e k_2\sin\psi + \mu_e^{\,2}k_1\sin^2\psi} \tag{21}$$

where

$$k_1 \equiv \frac{R^2\int_{r_1}^{R} c\,dr}{\int_{r_1}^{R} r^2 c\,dr},\quad k_2 \equiv \frac{R\int_{r_1}^{R} rc\,dr}{\int_{r_1}^{R} r^2 c\,dr},\quad k_3 \equiv R^3\int_{s_1}^{1} ds\,s^2 c$$

With the assumption (18), the values of k_1, k_2, and k_3, to first powers of s_1, are:

$$k_1 = \frac{35}{8}\left(1 - \frac{3}{2} s_1\right), \quad k_2 = \frac{7}{4}, \quad k_3 = \frac{2c_1R^3}{105}\left(8 + 4s_1\right)$$

Only the total lift force L_z has been calculated for the purpose of determining the blade angle. The other force resultants L_x, L_y, D_x, and D_y follow, by integration, from their unit values (equations (10a) and (14)), and determine, together with L_z, the reaction of the hub and the fuselage. They are not needed in this paper, which confines itself to the blade system.

Aerodynamic and Centrifugal Moments

In order to determine the steady-state flapping and lagging angles β_o and ζ_o and to investigate the stability of the helicopter rotor, it is necessary to obtain the moments acting on the blades. The moments about the following axes must be determined (see fig. 3):

The x_1-axis, giving the twisting moment which in some way must be held in balance by the pitch-changing mechanism of the hinge

The y_1-axis (perpendicular to the x_1-axis, and, like x_1, parallel to the xy-plane), giving a condition for the flapping angle β_o

The z-axis, determining the torque per blade

The z_1-axis parallel to z and passing through the hinge point, giving a condition for the lagging angle ζ_o.

Twisting moment about x_1-axis.- By referring to figures 3 and 4, it is found that:

$$M_{x1} = M_{ac} - \sigma\int_{r_1}^{R} dr(I_1 - I_2)\theta\Omega^2 + \int_{r_1}^{R} \left[\left(l_{cg} - l_{ac}\right) L_z' \cos\theta \cos\beta - \left(l_{cg} - l_{ac}\right)\left(L_{x1}' + D_{x1}'\right)\cos\theta \sin\beta\right] dr - \sigma\Omega^2 \frac{\partial^2\theta}{\partial\psi^2}\int_{r_1}^{R} I_p \, dr \quad (22a)$$

where M_{ac} is the (aerodynamic) moment about the aerodynamic center and can be expressed as

$$M_{ac} = \int_{r_1}^{R} C_{M_{ac}} \frac{\rho}{2} V_{c_1}{}^2 c^2 \, dr$$

The density of the blade material is σ; I_1 and I_2 are the principal moments of inertia of a blade cross section $\left(I_1 > I_2\right)$. As is well known, $\frac{1}{2}\left(I_1 - I_2\right) \sin 2\theta$ is the product of inertia about the x_s- and y_s-axes (referred to in equation (60)) of a cross section. In the following text the abbreviation $I_{12} \equiv \int dr \left(I_1 - I_2\right)$ will be used; I_p denotes the polar moment of inertia of a cross section about the y_s- and z_s-axes (see equation (61)); l_{cg} and l_{ac} (see fig. 3) are the distances of the centroid and of the aerodynamic center, respectively, of a blade cross section from the leading edge of the airfoil section. The abbreviation $f \equiv \frac{l_{cg} - l_{ac}}{c}$ will henceforth be used. The last term of equation (22) represents the twisting moment of the inertia force for the case in which, for $\mu_e \neq 0$, the pitch angle θ, in steady flight, varies periodically with $\psi(= \Omega t)$, that is, with time. Analogous inertia terms will also appear in the expression for the moments M_{y1} and M_{z1}, equations (23) and (29), respectively. These inertia terms are all derived in detail under INERTIA FORCES AND MOMENTS AND EQUATIONS OF OSCILLATION and are given in advance here only to make the expressions for the moments M_{x1}, M_{y1}, and M_{z1} complete.

The second term of equation (22a) represents the moment contribution of the distribution of the centrifugal forces in the cross sections. The third term of equation (22a) is due to the fact that for rigid-body oscillations the centroid axis (see fig. 3) (which will generally not pass through the aerodynamic center) must be taken as the reference axis. For elastic vibrations with the twist angle equal to zero at the root the shear center (which for airfoil sections, however, lies very near the centroid) must be substituted for the centroid.

By keeping in mind the orders of magnitude of the terms involved and rejecting all terms smaller than the second order, remembering that $C_{M_{ac}}$ will, in general, be negative, and assuming (as appears reasonable) that $\left(\frac{l_{cg} - l_{ac}}{c}\right) \equiv f$ is constant (≈ 0.15) along the blade length r, the expression (equation (22a)) for M_{x1} may be written as follows:

$$M_{x1} = -\frac{\rho}{2}\int_{r_1}^{R} \left|C_{M_{ac}}\right| V_{c_1}{}^2 c^2 \, dr + f\int_{r_1}^{R} L_z{}'c \, dr$$

$$-\sigma\Omega^2\theta\int_{r_1}^{R}\left(I_1 - I_2\right) dr - \Omega^2\sigma \frac{\partial^2\theta}{\partial\psi^2}\int_{r_1}^{R} I_p \, dr \qquad (22b)$$

Moment about y_1-axis.- It will be seen from figure 3 that this moment is:

$$M_{y1} = -\int_{r_1}^{R}\left[L_z{}'r \cos\beta - \left(L_{x1}{}' + D_{x1}{}'\right) r \sin\beta\right] dr$$

$$+\int_{r_1}^{R} r \beta \cos\left(\zeta - \zeta_1\right) dC + \sigma\Omega^2 \frac{\partial^2\beta}{\partial\psi^2}\bar{I}_H \qquad (23)$$

where $\bar{I}_H \equiv \int_{r_1}^{R} Ar^2 \, dr$.

In equation (23), the differential dC of the centrifugal force is given by

$$dC = \Omega^2 r_1 \, dm = \Omega^2 r_1 A\sigma \, dr \qquad (24)$$

where A is the cross-sectional area of a blade element at the distance r. The value of r_1 can be obtained from

$$r_1{}^2 = e^2 + (r \cos\beta)^2 + 2e(r \cos\beta) \cos\zeta$$

or

$$r_1 \approx r\left(1 + \eta - \frac{1}{2}\beta^2\right) \qquad (25)$$

to second-order small quantities.

The value of $\left(\zeta - \zeta_1\right)$ can be determined (see fig. 5) from the approximate relation

$$\left(\zeta - \zeta_1\right) r \approx (\zeta r) \frac{e}{e + r}$$

or

$$\left(\zeta - \zeta_1\right) \approx \zeta\eta \tag{26}$$

to second-order small quantities.

When equations (24), (25), and (26) are put into equation (23a) and terms smaller than the second order are rejected, the expression for M_{y1} becomes

$$M_{y1} = -\int_{r_1}^{R} L_z' \, r \, dr + \sigma\Omega^2\beta\left(\bar{I}_H + \bar{S}e\right) + \sigma\Omega^2 \frac{\partial^2 \beta_o}{\partial\psi^2} \bar{I}_H + \sigma g \bar{S} \tag{27}$$

where

$$\bar{S} \equiv \int_{r_1}^{R} Ar \, dr$$

The last term of equation (27) represents the effect of the weight of a blade.

Moment about z_1-axis.- From figure 3, this moment is seen to be given by

$$M_{z1} = \int_{r_1}^{R} \left(L_{y1}' + D_{y1}'\right) r \, dr + \int_{r_1}^{R} r \sin\left(\zeta - \zeta_1\right) dC + \sigma\Omega^2 \frac{\partial^2 \zeta_o}{\partial\psi^2} \bar{I}_H \tag{28a}$$

From equations (24) and (26), M_{z1} to second-order small terms can be written as

$$M_{z1} = \int_{r_1}^{R} \left(L_{y1}' + D_{y1}'\right) r \, dr + \sigma\Omega^2\zeta e \, \bar{S} + \sigma\Omega^2 \frac{\partial^2 \zeta_o}{\partial\psi^2} \bar{I}_H \tag{28b}$$

Finally, the moment about the z-axis will be

$$M_z = \int_{r_i}^{R} \left(L_{y1}' + D_{y1}'\right) r_1 \cos\left(\zeta - \zeta_1\right) dr - \int_{r_i}^{R} \left(L_{x1}' + D_{x1}'\right) r_1 \sin\left(\zeta - \zeta_1\right) dr + \sigma\Omega^2 \frac{\partial^2 \zeta_o}{\partial \psi^2} \bar{I}_H \qquad (29a)$$

To second-order small terms, this may be written simply as

$$M_z = \int_{r_i}^{R} \left(L_{y1}' + D_{y1}'\right) r \, dr = M_{z1} - \sigma\Omega^2 \zeta e \, \bar{S} \qquad (29b)$$

It may be remarked that the center of pressure on a blade at any speed ratio μ_e moves radially with the angle of rotation ψ. This motion of the center of pressure is, however, taken into account by the moment-equilibrium relations between the aerodynamic, centrifugal, weight, and inertia forces. This radial displacement is compensated for by the change of pitch angle θ with the azimuth angle ψ, which can be accomplished by either a swash plate (sufficient near hovering for the first harmonic term) or a cam plate (for higher speed ratios). At higher speed ratios (not treated in this report) the problem of the blackout of lift toward the root of the retreating blade would have to be considered.

Figure 3 shows that any force components in the x_1- and y_1-directions are related to those in the x- and y-directions by

$$F_{x1}' = F_x' \cos \zeta - F_y' \sin \zeta, \quad F_{y1}' = F_y' \cos \zeta + F_x' \sin \zeta$$

When it is remembered that F_x' and F_y' are in this case the second-order small quantities L_{x1}', L_{y1}', D_x', and D_y' and that ζ is first order small, it follows that to second orders

$$F_{x1}' \approx F_x', \quad F_{y1}' \approx F_y'$$

This simplification will be used in the following section.

Explicit expressions for moments.— The four moments, as given by equations (22b), (23b), (28b), and (29b) can be evaluated by using equations (10a) and (14) for the lift and drag components and the assumption (18) for the values of the zero-lift chord c in terms of r(or s), that is, along the blade axis. In the evaluation of these moments, for the analysis given here, all terms smaller than the second order have been rejected, and for further simplification, even second-order terms have been neglected when they appear as additions to first-order or finite terms (for example, as in L_z'). Moreover, powers of s_i higher than the first have for simplicity been neglected. The results obtained for the moments are then (see appendix C):

$$M_{x1} = -\rho\Omega^2R^3c_i^2\left\{\frac{\left|C_{M_{ac}}\right|}{24}\left[1 + s_i + 4\mu_e(1 + s_i)\sin\psi + 3\mu_e^2(1 - s_i)(1 - \cos 2\psi)\right]\right.$$

$$-\pi\left(\frac{l_{cg} - l_{ac}}{c}\right)(1 + s_i)\left[\frac{\theta}{12} + \mu_e\left(\frac{1}{3}\theta\sin\psi - \frac{\gamma}{6} + \frac{\beta}{6}\cos\psi\right)\right.$$

$$\left.\left. + \frac{\mu_e^2}{4}(1 - 2s_i)(\beta\sin 2\psi + \theta(1 - \cos 2\psi) - 2\gamma\sin\psi)\right]\right\}$$

$$-\sigma\Omega^2\theta\overline{I}_{12} - \sigma\Omega^2\frac{\partial^2\theta_o}{\partial\psi^2}\int I_p\,dr \qquad (30a)$$

For hovering $(\mu_e = 0,\ \theta = \theta_c)$,

$$\left(M_{x1}\right)_c = -\frac{\rho\Omega^2R^3c_i^2}{24}(1 + s_i)\left[\left|C_{M_{ac}}\right| - 2\pi\theta_c\left(\frac{l_{cg} - l_{ac}}{c}\right)\right]$$

$$-\sigma\Omega^2\,\theta_c\,\overline{I}_{12} \qquad (30b)$$

where, as before:

$$\overline{I}_{12} \equiv \int_{r_i}^{R}\left(I_1 - I_2\right)dr$$

$$M_{y1} = -\frac{4}{315}\left(1 + \frac{1}{2}s_i\right)\pi\rho\Omega^2R^4c_i\left[8\theta + 12\mu_e\left(\beta\cos\psi + 2\theta\sin\psi - \gamma\right)\right.$$

$$\left. + 21\mu_e^2(\beta\cos\psi + \theta\sin\psi - \gamma)\sin\psi\right]$$

$$+ \sigma\Omega^2\beta\left(\overline{I}_H + \overline{S}e\right) + \sigma\Omega^2\frac{\partial^2\beta_o}{\partial\psi^2}\overline{I}_H + \bar{S}\sigma g \qquad (31a)$$

For hovering,

$$\left(M_{y1}\right)_c = -\frac{32}{315}\left(1 + \frac{1}{2}e_1\right)\pi\rho\Omega^2R^4c_1\ \theta_c + \sigma\Omega^2\beta\left(\bar{I}_H + \bar{S}e\right) + \bar{S}\sigma g \qquad (31b)$$

$$M_{z1} = -4\pi\rho\Omega^2R^4c_1\left\{\frac{8\theta}{315k}\left(\alpha_p + \frac{1}{5}\theta\right) + \mu_e\frac{12}{315}\left[\theta\left(\gamma - \beta\cos\psi + \frac{\alpha_p}{k}\sin\psi\right)\right.\right.$$

$$\left. + \frac{1}{k}\left(\alpha_p + \frac{2}{5}\theta\right)(\beta\cos\psi + \theta\sin\psi - \gamma)\right] + \mu_e{}^2\frac{1}{15}(\beta\cos\psi$$

$$\left.\left. + \theta\sin\psi - \gamma\right)\left[\gamma - \beta\cos\psi + \frac{\alpha_p}{k}\sin\psi + \frac{1}{5k}(\beta\cos\psi + \theta\sin\psi - \gamma)\right]\right\}$$

$$+ \sigma e\zeta\Omega^2\bar{S} + \sigma\Omega^2\frac{\partial^2\zeta}{\partial\psi^2}\bar{I}_H \qquad (32a)$$

For hovering,

$$\left(M_{z1}\right)_c = -\frac{32}{315}\pi\rho\Omega^2R^4c_1\ \theta_c\left(\alpha_p + \frac{1}{5}\theta_c\right) + \sigma e\Omega^2\bar{S}\ \zeta \qquad (32b)$$

$$M_z = M_{z1} - \sigma\Omega^2\zeta e\ \bar{S} \qquad (33)$$

Expressions (31a) and (32a) for the moments M_{y1} and M_{z1}, respectively, will be used in the section STEADY STATE IN HOVERING AND IN LOW-SPEED TRAVELING to determine the steady-state values of β and ζ, which will be constant in hovering, and functions of ψ and μ_e in traveling. These expressions for the moments will also be used in the following section to determine the "quasi-elastic" moments (that is, moments depending on the deviation from the steady-state equilibrium position) during an oscillation.

FORCES AND MOMENTS DUE TO SMALL OSCILLATORY DISPLACEMENTS AND TO VELOCITIES

In order to treat fully the questions of frequencies, amplitudes, and stability, the forces and moments due to oscillatory displacements

$\Delta\beta \equiv \overline{\beta}$, $\Delta\zeta \equiv \overline{\zeta}$, $\Delta\theta \equiv \overline{\theta}$ and velocities $\dot{\beta} \equiv \frac{d\beta}{dt}$, $\dot{\zeta} \equiv \frac{d\zeta}{dt}$, $\dot{\theta} \equiv \frac{d\theta}{dt}$ must be determined. These will be the quasi-elastic and the aerodynamic-damping terms in the final dynamic equations of oscillations about a state of steady motion. The oscillatory displacements and velocities which must be considered herein are the angular displacements $\overline{\beta}$, $\overline{\zeta}$, $\overline{\theta}$ and velocities $\dot{\beta}$, $\dot{\zeta}$, $\dot{\theta}$. Because only natural (or free) oscillations about a state of steady flight are treated in this analysis, the fluctuations (in magnitude or direction) of the rotational speed as caused by change of engine torque and of the translational inflow velocity v_o as caused by gusts or change of angular position of the driving shaft will be left for a later research. These latter fluctuations would give rise to forced oscillations, which could be treated by the same general method as given herein for the free oscillations.

In the following analysis, squares of oscillatory displacements and velocities will in all cases be neglected.

Quasi-Elastic Moments

The quasi-elastic terms, which must be used in the dynamic equations of oscillation, can be determined from expressions (30a), (31a), and (32a) for the moments by putting

$$\theta = \theta_o + \overline{\theta}, \quad \beta = \beta_o + \overline{\beta}, \quad \zeta = \zeta_o + \overline{\zeta} \tag{34}$$

in these expressions. In equation 34, the subscript o denotes the value for the steady state, which is constant for hovering $(\mu_e = 0)$ but a function of ψ for traveling $(\psi_e \neq 0)$, whereas the bar denotes the small oscillatory changes (varying with time) about the steady state. The quasi-elastic terms are the moments due to the changes $\overline{\theta}$, $\overline{\beta}$, and $\overline{\zeta}$. By putting, therefore, equation (34) into equations (30a) to (32a), subtracting in each case the moment for the steady state $(\theta = \theta_o;\ \beta = \beta_o;$ and $\zeta = \zeta_o)$, and neglecting powers of the oscillatory changes higher than the first, the following expressions are obtained for the (aerodynamic and centrifugal) quasi-elastic terms:

$$\left(M_{x1}\right)_{qe} = \frac{\pi\rho\Omega^2R^3c_1{}^2\left(1 + s_1\right)}{12} f\left\{\overline{\theta} + 2\mu_e(2\overline{\theta}\sin\psi + \overline{\beta}\cos\psi)\right.$$
$$\left. + 3\mu_e{}^2(1 - 2s_1)\left[\overline{\beta}\sin 2\psi + \overline{\theta}(1 - \cos 2\psi)\right]\right\} - \sigma\Omega^2\,\overline{\theta}\,I_{12} \tag{35}$$

$$\left(M_{y1}\right)_{qe} = -\frac{4}{315}\left(1 + \frac{1}{2}s_1\right)\pi\rho\Omega^2R^4c_1\left[8\overline{\theta} + 12\mu_e(\overline{\beta}\cos\psi + 2\overline{\theta}\sin\psi)\right.$$

$$\left. + 21\mu_e{}^2(\overline{\beta}\cos\psi + \overline{\theta}\sin\psi)\sin\psi\right] + \sigma\Omega^2\overline{\beta}\left(\overline{I}_H + \overline{S}\,e\right) \qquad (36)$$

$$\left(M_{z1}\right)_{qe} = -4\pi\rho\Omega^2R^4c_1\left(\frac{8}{315k}\overline{\theta}\left(\alpha_p + \frac{2}{5}\theta_o\right) + \mu_e\frac{4}{105}\left[\overline{\theta}\left(\gamma - \beta_o\cos\psi\right.\right.\right.$$

$$\left. + \frac{\alpha_p}{k}\sin\psi\right) - \overline{\beta}\,\theta_o + \frac{1}{k}(\overline{\beta}\cos\psi + \overline{\theta}\sin\psi)\left(\alpha_p + \frac{2}{5}\theta_o\right) + \frac{2}{5}\overline{\theta}\left(\beta_o\cos\psi\right.$$

$$\left.\left. + \theta_o\sin\psi - \gamma\right)\right] + \mu_e{}^2\frac{1}{15}\left\{\left(\overline{\beta}\cos\psi + \overline{\theta}\sin\psi\right)\left[\frac{2}{5k}\left(\beta_o\cos\psi\right.\right.\right.$$

$$\left.\left. + \theta_o\sin\psi - \gamma\right) + \gamma - \beta_o\cos\psi + \frac{\alpha_p}{k}\sin\psi\right] - (\overline{\beta}\cos\psi)(\beta_o\cos\psi$$

$$\left.\left. + \theta_o\sin\psi - \gamma)\right\}\right) + \sigma e\Omega^2\,\overline{\zeta}\,\overline{S} \qquad (37)$$

Components of Total Relative Inflow Velocity of Oscillation

By referring to figure 1 and using the fact that, at any distance r along a blade from the hinge, the linear velocity $r\dot{\beta}$ due to angular oscillations of the flapping angle will be perpendicular to $\overline{r}$ and will lie in the plane which contains r and is perpendicular to the xy-plane, it is seen that the velocity components due to $r\dot{\beta}$ will be:

$$\left.\begin{aligned} V_{x\dot{\beta}} &= r\dot{\beta}\sin\beta\cos\zeta \approx r\,\dot{\beta}\beta \\ V_{y\dot{\beta}} &= r\dot{\beta}\sin\beta\sin\zeta \approx 0 \\ V_{z\dot{\beta}} &= r\dot{\beta}\cos\beta \approx r\,\dot{\beta} \end{aligned}\right\} \qquad (38)$$

Similarly, use of the fact that the linear velocity $r\dot{\zeta}$, due to oscillations of the lagging angle, will be parallel to the xy-plane and perpendicular to the projection (of length $r\cos\beta$) of the blade axis r on the xy-plane yields for the velocity components of $r\dot{\zeta}$:

$$\left.\begin{aligned} V_{x\dot{\zeta}} &= r \cos \beta \sin \zeta \approx r\dot{\zeta}\zeta \\ V_{y\dot{\zeta}} &= r \cos \beta \cos \zeta \approx r\dot{\zeta} \\ V_{z\dot{\zeta}} &= 0 \end{aligned}\right\} \tag{39}$$

All approximations made in this entire section are up to second-order small terms.

Damping Forces

The contributions to the aerodynamic forces following from the changes of the velocities calculated in equations (38) and (39) can be determined in the following general manner. The velocity increments must be added to the corresponding expressions (equations (5)) for the velocity components. The forces and moments are then determined by the same procedure as in the section AERODYNAMIC AND CENTRIFUGAL FORCES AND MOMENTS IN STEADY HORIZONTAL FLIGHT and the damping terms will be those containing the quantities $\dot{\zeta}$ and $\dot{\beta}$ in the expressions for the forces and moments. In this manner (see appendix D), the expression for the total circulation to second orders is found to be:

$$\Gamma = \pi c \Omega r\left[\theta(1 + \eta) + \mu(\beta + \zeta\theta) \cos \psi + (\theta - \beta\zeta) \sin \psi - \gamma\right] - \theta \frac{\dot{\zeta}}{\Omega} - \frac{\dot{\beta}}{\Omega} \tag{40}$$

Then, by the use of the Kutta-Joukowski relation given in equation (6) (see appendix D), the increments due to $\dot{\beta}$ and $\dot{\zeta}$, in the lift components per unit length, are found to be

$$\Delta L_x' = \pi\rho\Omega^2r^2c\beta(1 + \mu \sin \psi) \frac{\dot{\beta}}{\Omega} \tag{41a}$$

$$\Delta L_y' = -\pi\rho\Omega^2r^2c \frac{\dot{\beta}}{\Omega} \left[\theta + \mu(2\beta \cos \psi + \theta \sin \psi - 2\gamma)\right] \tag{41b}$$

$$\Delta L_z' = -\pi\rho\Omega^2r^2c \left\{(1 + \eta + \mu\zeta \cos \psi + \mu \sin \psi) \frac{\dot{\beta}}{\Omega} + \frac{\dot{\zeta}}{\Omega} \left[2\theta + \mu(\beta \cos \psi + \theta \sin \psi - \gamma)\right]\right\} \tag{41c}$$

The additional drag components per unit of blade span, to second orders, can in accordance with equation (11) be determined from

$$\Delta D_x' = \frac{V_x}{V} \Delta D', \quad \Delta D_y' = \frac{V_y}{V} \Delta D', \quad \Delta D_z' = \frac{V_z}{V} \Delta D' \tag{42}$$

where $\Delta D'$ is the increment in the resultant drag and may be given, according to equation (12), by

$$\Delta D' = \left(\alpha_p + \alpha_i\right) \Delta L_z' \tag{43}$$

Therefore, by putting equation (41c) into equation (43) and then putting equations (43), (5), and (13) into equations (42), the expressions for the additional drag components are found to be:

$$\left.\begin{aligned} \Delta D_x' &= \frac{\left(\alpha_p + \alpha_i\right)}{k} \pi c \Omega^2 r^2 \mu (\cos \psi) \frac{\dot{\beta}}{\Omega} \\ \Delta D_y' &= \frac{\left(\alpha_p + \alpha_i\right)}{k} \pi c \ \rho \Omega^2 r^2 (1 + \mu \sin \psi) \frac{\dot{\beta}}{\Omega} \\ \Delta D_z' &= 0 \end{aligned}\right\} \tag{44}$$

As in the section AERODYNAMIC AND CENTRIFUGAL FORCES AND MOMENTS IN STEADY HORIZONTAL FLIGHT, α_i may be obtained from equation (15).

Damping Moments

Damping moments about the hinge point are caused by damping forces distributed over the length of the span of a blade and acting with their radial levers. Such moments appear furthermore as the effect of the pitch-angle oscillation. This latter effect, for the moderate velocities appearing in helicopter flight, can be calculated by means of an apparent change of local angle of attack under the assumption of quasi-stationary flow. This calculation (see appendix E and fig. 6) leads to the relation

$$\left(M_{xl}\right)_{\dot{\theta}} = -\frac{\pi}{12} \rho \dot{\theta} \int_{r_i}^{R} dr V_{c_l} c^3$$

where, from equations (8a) and (18),

$$\int_{r_i}^{R} dr V_{c_l} c^3 = \Omega R^2 c_i^3 \frac{2}{35} \left(1 + s_i\right) \left[2 + s_i + 7\eta_e + 7\mu_e \left(\sin \psi + \zeta \cos \psi\right)\right] \tag{45}$$

This simplified approach is made for the damping moments, although it is well known that for high values of oscillatory and inflow velocities an effective change of camber of the airfoil, as well as the reaction due to the trailing vortices of the flow around the airfoil, would have to be considered. As has been made plausible from the theory of straight-moving airfoils, however, these effects are small for velocity changes occurring in helicopters. A sufficiently exact theory, moreover, including radial velocities and helical distorted vortex sheets has not yet been developed.

Another problem which must be considered in a further development of the dynamic theory of flexible blades, with or without fixed roots, is the influence of phase differences between flapping and twisting due to the distance between the center-of-gravity axis and the elastic axis. This problem does not appear with hinged and sufficiently rigid blades, which are assumed in this paper.

The damping effect of flapping and lagging can be obtained from the expressions for the damping forces per unit of blade length in equations (41a), (41b), (41c), and (44). If the moments of these unit forces are integrated over the length of a blade analogously to the method of equations (22b), (23b), and (28b) for steady flight, the following damping moments, including also the effect of pitch-angle oscillation, are obtained. The increment in M_{x1} due to damping will be

$$\Delta M_{x1} = -\rho\left|C_{M_{ac}}\right|\int_{r_1}^{R} V_{c_1}\,\Delta V_{c_1}c^2\,dr + f\int_{r_1}^{R}\Delta L_z'\,c\,dr - \frac{\pi}{12}\rho\dot{\theta}\int_{r_1}^{R} V_{c_1}c^3\,dr \tag{46}$$

where V_{c_1} (see appendix F) is to be determined from the components ΔV_x, ΔV_y, and ΔV_z of equations (38) and (39) and from the direction cosines l_{c_1}, m_{c_1}, and n_{c_1} (see appendix A). Similarly, from equation (23), the increment in M_{y1} is seen to be

$$\Delta M_{y1} = -\int_{r_1}^{R}\Delta L_z'\,r\,dr \tag{47}$$

Finally, in accordance with equation (28b), the aerodynamic and centrifugal damping moment about the z_1-axis will be

$$\Delta M_{z1} = \int_{r_1}^{R}\left(\Delta L_{y1}' + \Delta D_{y1}'\right) r\,dr \tag{48a}$$

As shown before,

$$L_{y1}' \left(\text{or } D_{y1}'\right) \approx L_y' \left(\text{or } D_y'\right)$$

to second-order small quantities. Therefore equation (44) may, to second orders, be written thus:

$$\Delta M_{z1} = \int_{r_i}^{R} \left(\Delta L_y' + \Delta D_y'\right) r\, dr \tag{48b}$$

Putting equations (41a), (41b), (41c) and (44) into equations (46), (47), and (48b), neglecting terms smaller than the second order (taking $C_{M_{ac}}$ and all oscillatory velocities as first order small), and rejecting powers of s_i above the first results in the following explicit expressions for the aerodynamic damping moments (see appendix F):

$$\Delta M_{x1} = -\frac{\rho\Omega^2 R^3 c_i{}^2}{12}\left(1 + s_i\right)\left(\frac{\dot{\theta}}{\Omega}\pi\frac{c_i}{R}\frac{2}{35}\left[2 + s_i + 7\left(1 - 2s_i\right)\eta_e + 7\left(1 - 2s_i\right)\mu_e(\sin\psi + \zeta\cos\psi)\right] + \frac{\dot{\zeta}}{\Omega}\left\{f\pi\left[2\theta + 2\mu_e(\beta\cos\psi + \sin\psi - \gamma)\right] - \left|C_{M_{ac}}\right|\left(1 + 2\mu_e\sin\psi\right)\right\} + \frac{\dot{\beta}}{\Omega}f\pi\left[1 + 2\eta_e + 2\mu_e(\sin\psi + \zeta\cos\psi)\right]\right) \tag{49}$$

$$\Delta M_{y1} = \pi\rho\Omega^2 R^4 c_i\frac{16}{315}\left(1 + \frac{1}{2}s_i\right)\left\{\frac{\dot{\zeta}}{\Omega}\left[4\theta + 3\mu_e(\beta\cos\psi + \theta\sin\psi - \gamma)\right] + \frac{\dot{\beta}}{\Omega}\left[2 + 3\eta_e + 3\mu_e(\sin\psi + \zeta\cos\psi)\right]\right\} \tag{50}$$

$$\Delta M_{z1} = -\pi\rho\Omega^2 R^4 c_i\frac{16}{315}\left(1 + \frac{1}{2}s_i\right)\frac{\dot{\beta}}{\Omega}\left\{2\left(1 - \frac{1}{5k}\right)\theta - \frac{2}{k}\alpha_p + 3\mu_e\left[\left(2 - \frac{1}{5k}\right)\beta\cos\psi + \left(1 - \frac{1}{5k}\right)\theta\sin\psi - \frac{\alpha_p}{k}\sin\psi - 2\left(1 - \frac{1}{5k}\right)\gamma\right]\right\} \tag{51}$$

In addition to the aerodynamic damping moments given in equations (49) to (51), there may, in general, be "pseudo-damping" moments due to a change in the angular velocity Ω caused by $\dot{\zeta}$. This change can be determined by substituting $(\Omega - \dot{\zeta})$ for Ω in expressions (30a) to (32a) for the aerodynamic and centrifugal moments, subtracting in each case the moment when $\dot{\zeta} = 0$, and neglecting squares of $\dot{\zeta}$. It will be observed from equations (30a) to (32a) that each of the moments in the steady state is proportional to Ω^2, that is

$$M = G\Omega^2 \tag{52}$$

where G is a function of θ, β, ζ, ψ, and so forth, but not of Ω. Therefore the pseudo-damping moments $\Delta_{pd} M$ will be of the form

$$\Delta_{pd} M = -2G \Omega \dot{\zeta} \tag{53}$$

In steady flight, however, the angles ζ_o and β_o will so adjust themselves that the moments M_{y1} and M_{z1} are both zero. Moreover, it may be assumed that by some means or other, for example, counterweights, the moment M_{x1} will also be balanced. Therefore the value of G as defined by equation (52) will be zero. It follows then from equation (53) that (because G is obviously the same function in equations (52) and (53))

$$\Delta_{pd} M_{x1} = \Delta_{pd} M_{y1} = \Delta_{pd} M_{z1} = 0 \tag{54}$$

Hence, the effect of $\dot{\zeta}$ on Ω need not be considered.

INERTIA FORCES AND MOMENTS AND EQUATIONS OF OSCILLATION

The derivation of the inertia forces and moments may be based on the time rate of change of the moment of momentum vector $\overline{B}$ referred to a coordinate system rotating about the driving shaft of the blade system with angular velocity Ω. It is well known that with such a coordinate system the centrifugal forces, the centrifugal moments about the blade axis, and the Coriolis forces must be added to the other impressed forces (in this case, aerodynamic and gravity forces).

The blade system is assumed to consist of blades, each hinged to the hub driven by the engine shaft, and the hinge system for each blade is assumed to have a common hinge center, which is then the natural moment center for the moment of momentum vector.

It is true that for reasons of detail design it is often found that the flapping hinge and the lagging hinge are not concentric; as only the eccentricity of the lagging hinge is of importance, however, it is a very small loss of generality but a great advantage in simplicity to assume concentric arrangement of the hinges. It may even, for the development of larger systems, be convenient to support the entire centrifugal forces by one spherical bearing surface instead of by several necessarily smaller cylindrical bearings, each of which must support the entire centrifugal force.

Moment of Momentum Vector

The basic dynamic theorem is expressed in vector notation by the equation

$$\dot{\overline{B}} = \overline{M} \tag{55a}$$

where $\overline{B}$ is the moment of momentum vector,

$$\dot{\overline{B}} \equiv \frac{d\overline{B}}{dt}$$

and $\overline{M}$ is the moment vector of centrifugal, relative acceleration, Coriolis, aerodynamic, and gravity forces.

In Cartesian coordinates, equation (55a) can be expressed by

$$\dot{B}_x = M_x, \quad \dot{B}_y = M_y, \quad \dot{B}_z = M_z \tag{55b}$$

The vector $\overline{B}$ and its rectangular components must now be developed in terms of the positions and velocities of the mass particles of the blades, whereby it is sufficient to consider one sample blade.

The general relation in vector notation is given by

$$\overline{B} = \int_{r_1}^{R} dm\left(\overline{V}_b \times \overline{r}\right) \tag{56a}$$

where $\overline{V}_b$ denotes the resultant velocity of a blade element, and $\overline{r}$ the radius vector from the moment center (hinge center) to the particle.

With u, v, and w as Cartesian components of $\overline{V}_b$ and x, y, and z as those of $\overline{r}$, equation (56a) is equivalent to the following three component equations:

$$\left.\begin{aligned} B_x &= \int dm(wy - vz) \\ B_y &= \int dm(uz - wx) \\ B_z &= \int dm(vx - uy) \end{aligned}\right\} \tag{56b}$$

The integrals in equations (56a) and (56b) must be taken over all mass elements of a blade.

The velocity vector $\overline{V}_b$, or its rectangular components, must now be expressed by the condition that the blade is kinematically constrained to move about the hinge center. This fact, in vector notation, is expressed by

$$\overline{V}_b = \overline{r} \times \overline{\omega} \tag{57a}$$

where $\overline{\omega}$ is the vector, with components ω_x, ω_y, and ω_z of the angular velocity of the blade. In Cartesian notation equation (57a) is expressed by

$$\left.\begin{aligned} u &= z\omega_y - y\omega_z \\ v &= x\omega_z - z\omega_x \\ w &= y\omega_x - x\omega_y \end{aligned}\right\} \tag{57b}$$

Inserting equation (57b) into equation (56b) and factoring out the density σ of the material of the blade gives

$$\left.\begin{aligned} B_x &= \left(\overline{I}_x\omega_x - \overline{I}_{xy}\omega_y - \overline{I}_{xz}\omega_z\right)\sigma \\ B_y &= \left(\overline{I}_y\omega_y - \overline{I}_{yz}\omega_z - \overline{I}_{yx}\omega_x\right)\sigma \\ B_z &= \left(\overline{I}_z\omega_z - \overline{I}_{zx}\omega_x - \overline{I}_{zy}\omega_y\right)\sigma \end{aligned}\right\} \tag{58}$$

where the moments and products of inertia are defined (in the usual way) by

$$\left.\begin{aligned} &\overline{I}_x = \iint dA\ dr\left(y^2 + z^2\right), \quad \overline{I}_y = \iint dA\ dr\left(z^2 + x^2\right), \quad \overline{I}_z = \iint dA\ dr\left(x^2 + y^2\right) \\ &\overline{I}_{yz} = \iint dA\ dr\ yz, \quad \overline{I}_{zx} = \iint dA\ dr\ zx, \quad \overline{I}_{xy} = \iint dA\ dr\ xy \end{aligned}\right\} \tag{59}$$

and where the bar over I is intended to emphasize the volume integrals as opposed to area integrals appearing later (cf. equation (61)). The coordinate system x, y, z (see fig. 1) in general does not coincide with the centroid coordinate system fixed in the blade, except when the parameters β and ζ are zero. It is therefore advisable to transform the moments and products of inertia to the coordinate system fixed in the blade. This transformation is simplified by the fact that x_s coincides with r, y_s with c, and z_s is normal to c and that only the first powers of the angles β and ζ need be considered. Thus

$$x = r, \quad y = y_s - \zeta r, \quad z = z_s + \beta r \tag{60}$$

The expressions (59) then take the following forms:

$$\left.\begin{aligned}
\bar{I}_x &= \bar{I}_r = \int dr\ I_p, \quad \bar{I}_{xy} = -\zeta\bar{I}_H \equiv -\zeta\int dr\ r^2 A, \quad \bar{I}_{xz} = \beta\bar{I}_H \\
\bar{I}_y &= \bar{I}_H + \int dr\ I_{ys}, \quad \bar{I}_{yz} = \theta\int dr\left(\bar{I}_1 - I_2\right) \equiv \theta\bar{I}_{12}, \quad \bar{I}_{yx} = -\zeta\bar{I}_H \\
\bar{I}_z &= \bar{I}_H + \int dr\ I_{zs}, \quad \bar{I}_{zx} = \beta\bar{I}_H, \quad \bar{I}_{zy} = \theta\bar{I}_{12}
\end{aligned}\right\} \tag{61}$$

where

$$I_p \equiv \int\left(y_s^2 + z_s^2\right) dA, \quad I_{ys} \equiv \int z_s^2\ dA, \quad I_{zs} \equiv \int y_s^2\ dA$$

and where A denotes the cross section (varying with r) of a blade. In equation (61) it has been observed that the static moments about the centroid axes are zero.

Time Rate of Change of Moment of Momentum Vector

The time rate of change of the moment of momentum vector $\bar{B}$, with the condition that squares and products of angular accelerations and angular velocities shall be neglected, whereas the products of accelerations and the stationary angles θ_o, β_o, and ζ_o shall be retained, can now from equations (58) and (61) be expressed as follows:

$$\left.\begin{aligned}
\dot{B}_x &= \left(\dot{\omega}_x\bar{I}_r + \dot{\omega}_y\ \zeta_o\ \bar{I}_H - \dot{\omega}_z\beta_o\bar{I}_H\right)\sigma \\
\dot{B}_y &= \left[\dot{\omega}_y\left(\bar{I}_H + \int dr\ I_{ys}\right) - \dot{\omega}_z\theta_o\bar{I}_{12} + \dot{\omega}_x\ \zeta_o\bar{I}_H\right]\sigma \\
\dot{B}_z &= \left[\dot{\omega}_z\left(\bar{I}_H + \int dr\ I_{zs}\right) - \dot{\omega}_x\beta_o\bar{I}_H - \dot{\omega}_y\theta_o\bar{I}_{12}\right]\sigma
\end{aligned}\right\} \tag{62}$$

The blade angles θ, β, and ζ in this equation are provided with the subscript o in order to emphasize that they are parameters of the steady state, which are constant in hovering and functions of $\psi(=\Omega t)$

in traveling. In order to express the moment of momentum vector by the angular accelerations $\ddot{\theta}$, $\ddot{\beta}$, and $\ddot{\zeta}$ instead of, as before, by the angular accelerations $\dot{\omega}_z$, $\dot{\omega}_x$, and $\dot{\omega}_y$, the following transformation may be applied by the projection of the components $\omega_r = \dot{\theta}$ and $\omega_{y1} = -\dot{\beta}$ on the axes x and y.

$$\left.\begin{aligned} \omega_x &= \omega_r \cos\beta \cos\zeta + \omega_{y1} \sin\zeta \approx \dot{\theta} - \dot{\beta}\zeta \\ \omega_y &= -\omega_r \cos\beta \sin\zeta + \omega_{y1} \cos\zeta \approx -\dot{\theta}\zeta - \dot{\beta} \\ \omega_z &= -\dot{\zeta} \end{aligned}\right\} \tag{63}$$

The corresponding components of the vector $\dot{\bar{B}}$ can be derived similarly by the relations

$$\left.\begin{aligned} \dot{B}_r &= \dot{B}_x - \zeta_o \dot{B}_y \\ \dot{B}_{y1} &= \dot{B}_x \zeta_o + \dot{B}_y \end{aligned}\right\} \tag{64}$$

Applying equations (62) and (63) yields

$$\left.\begin{aligned} \dot{B}_r &= \sigma\left(\ddot{\theta}\bar{I}_r - \ddot{\beta}\zeta_o \int dr\, I_{zs} + \ddot{\zeta}\beta_o \bar{I}_H\right) \\ \dot{B}_{y1} &= \sigma\left[-\ddot{\beta}\left(\bar{I}_H + \int dr\, I_{ys}\right) + \ddot{\zeta}\theta_o \bar{I}_{12} + \ddot{\theta}\zeta_o \int dr\, I_{zs}\right] \\ \dot{B}_z &= \sigma\left[-\ddot{\zeta}\left(\bar{I}_H + \int dr\, I_{zs}\right) - \ddot{\theta}\beta_o \bar{I}_H + \ddot{\beta}\theta_o \bar{I}_{12}\right] \end{aligned}\right\} \tag{65}$$

The centrifugal (inertia) forces and moments are taken into account in the sections giving the conditions of equilibrium of aerodynamic and centrifugal forces, especially in steady flight (see STEADY STATE IN HOVERING AND IN LOW-SPEED TRAVELING).

Coriolis Moment Vector

The vector $\bar{a}_c$ of the Coriolis acceleration is given by

$$\bar{a}_c = 2\bar{\Omega} \times \bar{V}_{rel} \tag{66a}$$

In the case considered herein the vector of rotational velocity coincides with the z-axis, that is,

$$\bar{\Omega} = \Omega_z .$$

The vector product of equation (66a) therefore consists only of the two components:

$$\left.\begin{aligned} a_{cx} &= -2\Omega v \\ a_{cy} &= 2\Omega u \end{aligned}\right\} \tag{66b}$$

By virtue of the relations (equation (57b)) between the velocities u and v of a mass particle and the angular velocities ω_x, ω_y, and ω_z of oscillation about the hinge, the Coriolis forces of a mass element become:

$$\left.\begin{aligned} dF_{cz} &= 0 \\ dF_{cx} &= dm\ 2\Omega\left(\omega_z x - \omega_x z\right) \\ dF_{cy} &= -dm\ 2\Omega\left(\omega_y z - \omega_z y\right) \end{aligned}\right\} \tag{66c}$$

The total moments of this vector distribution up to small quantities (cross products of ω_x, ω_y, ω_z, β, and ζ) of second order, if $\eta \left(\equiv \frac{e}{r}\right)$ as compared with 1, is neglected, are thus:

$$\left.\begin{aligned}
M_{cx} &= -\int_{r_1}^{R} dF_{cy}\, r\beta = 2\Omega\beta\left(\omega_y\int_{r_1}^{R} dm\, rz - \omega_z\int_{r_1}^{R} dm\, ry\right)\\
M_{cy} &= \int_{r_1}^{R} dF_{cx}\, r\beta = 2\Omega\beta\left(\omega_z\int dm\, rx - \omega_x\int dm\, rz\right)\\
M_{cz} &= \int_{r_1}^{R}\left(dF_{cx}\, r\zeta + dF_{cy}\, r\right) = 2\Omega\left[\zeta\left(\omega_z\int dm\, rx - \omega_x\int dm\, rz\right)\right.\\
&\qquad \left. - \left(\omega_y\int dm\, rz - \omega_z\int dm\, ry\right)\right]
\end{aligned}\right\} \qquad (67a)$$

By use of the transformations (equations (60)) of the moments of inertia to the blade axes and the notations (equations (61)) of the moments of inertia of the blade, it is found that

$$\left.\begin{aligned}
M_{cx} &= 0\\
M_{cy} &= 2\Omega\sigma\bar{I}_H\beta\omega_z = -2\Omega\sigma\bar{I}_H\beta\dot{\zeta}\\
M_{cz} &= -2\Omega\sigma\bar{I}_H\beta\omega_y = 2\Omega\sigma\bar{I}_H\beta\dot{\beta}
\end{aligned}\right\} \qquad (67b)$$

where, in the last terms on the right-hand sides, the relations (63) have been inserted.

The transformation to the moments about the hinge axis y_1 of the flapping angle β and about the axis r of the pitch angle θ is again derived as in equation (64) and yields, to second-order small quantities, the results

$$\left.\begin{aligned}
M_{cr} &= 0\\
M_{cy_1} &= -2\Omega\sigma\bar{I}_H\beta\dot{\zeta}\\
M_{cz} &= 2\Omega\sigma\bar{I}_H\beta\dot{\beta}
\end{aligned}\right\} \qquad (67c)$$

The entire moment of momentum vector is now given by equations (65) and (67c), that is, by:

$$\left.\begin{aligned} &\dot{B}_r - M_{cr} \\ &\dot{B}_{y1} - M_{cy1} \\ &\dot{B}_z - M_{cz} \end{aligned}\right\} \tag{68}$$

Equations of Oscillation

The equations of oscillation are formed by equating the inertia moments (equations (68)) to the aerodynamic, centrifugal, and damping moments. Thus, by using equations (65) and (67c), the oscillation equations are seen to be:

$$\dot{B}_r \equiv \ddot{\theta}\sigma\overline{I}_r - \ddot{\beta}\zeta_o\sigma\overline{I}_{zs} + \ddot{\zeta}\beta_o\sigma\overline{I}_H = M_{x1} + \Delta_d M_{x1} \tag{69}$$

$$\dot{B}_{y1} \equiv -\ddot{\beta}\left(\sigma\overline{I}_H + \sigma\overline{I}_{ys}\right) + \ddot{\zeta}\theta_o\sigma\overline{I}_{12} + \ddot{\theta}\zeta_o\sigma\overline{I}_{zs} + 2\dot{\zeta}\beta_o\sigma\Omega\overline{I}_H = M_{y1} + \Delta_d M_{y1} \tag{70}$$

$$\dot{B}_{z1} \equiv -\ddot{\zeta}\left(\sigma\overline{I}_H + \sigma\overline{I}_{zs}\right) - \ddot{\theta}\beta_o\sigma\overline{I}_H + \ddot{\beta}\theta_o\ \overline{I}_{12} - 2\dot{\beta}\beta_o\sigma\Omega\overline{I}_H = M_{z1} + \Delta_d M_{z1} \tag{71}$$

In these equations θ, β, and ζ are the total values, as given by relations $\theta = \theta_o + \overline{\theta}$, and so forth (equations (34)), and the unknowns in the foregoing equations are the deviations $\overline{\theta}$, $\overline{\beta}$, and $\overline{\zeta}$ from the steady state of equilibrium.

Equations (69) to (71) can be written in slightly simpler form as follows: From the definition of the quasi-elastic moments (given in the section FORCES AND MOMENTS DUE TO SMALL OSCILLATORY DISPLACEMENTS) it follows that, for the aerodynamic and centrifugal moments

$$M = M_{qe} + M_o \tag{72}$$

where M is the total moment (that is, $\theta = \theta_o + \overline{\theta}$, etc.), M_{qe} is the quasi-elastic moment, and M_o is the moment in the steady state. Moreover, for the damping moments,

$$\Delta M = \Delta_{\dot{\overline{\theta}}\ldots} M + \Delta_{\dot{\theta}_o\ldots} M \tag{73}$$

where, similarly to equation (72), ΔM is the total damping moment ($\theta = \theta_o + \overline{\theta}$, etc.), $\Delta_{\dot{\overline{\theta}}\ldots} M$ is the total damping moment minus the damping moment $\Delta_{\dot{\theta}_o\ldots} M$ for the steady state ($\theta = \theta_o$, etc.).

Hence, using the relations (34), (72), and (73) and observing that the steady-state moments must by themselves add up to zero for each axis yields the following simplified set of equations of oscillation:

$$\ddot{\overline{\theta}}\sigma\overline{I}_r - \ddot{\overline{\beta}}\zeta_o\sigma\overline{I}_{zs} + \ddot{\overline{\zeta}}\beta_o\sigma\overline{I}_H = \left(M_{x1}\right)_{qe} + \Delta_{\dot{\overline{\theta}}\ldots} M_{x1} \tag{74}$$

$$-\ddot{\overline{\beta}}\left(\sigma\overline{I}_H + \sigma\overline{I}_{ys}\right) + \ddot{\overline{\zeta}}\theta_o\sigma\overline{I}_{12} + \ddot{\overline{\theta}}\zeta_o\sigma\overline{I}_{zs} + 2\dot{\overline{\zeta}}\beta_o\sigma\Omega\overline{I}_H = \left(M_{y1}\right)_{qe} + \Delta_{\dot{\overline{\theta}}\ldots} M_{y1} \tag{75}$$

$$-\ddot{\overline{\zeta}}\left(\sigma\overline{I}_H + \sigma\overline{I}_{zs}\right) - \ddot{\overline{\theta}}\beta_o\sigma\overline{I}_H + \ddot{\overline{\beta}}\theta_o\sigma\overline{I}_{12} - 2\dot{\overline{\beta}}\beta_o\sigma\Omega\overline{I}_H = \left(M_{z1}\right)_{qe} + \Delta_{\dot{\overline{\theta}}\ldots} M_{z1} \tag{76}$$

The right-hand sides of these equations can be obtained from equations (35) to (37) and from equations (49) to (51).

Relations among Geometric Constants of a Blade

For purposes of simplicity, it will be convenient to use certain relations among the geometric properties (for example, moments of inertia) of a blade appearing in the foregoing equations and elsewhere. With the data of the helicopter heretofore assumed, the following relations can be derived (see appendix G):

$$\left.\begin{aligned}
&\overline{I}_{ys} \ll \overline{I}_{zs}; \quad \overline{I}_r \approx \overline{I}_{12} \approx \overline{I}_{zs} \equiv \overline{I}, \text{ say} \\
&\overline{I}/\overline{I}_H \approx 0.00386, \text{ so that } \overline{I}_H + \overline{I}_{ys} \approx \overline{I}_H + \overline{I}_{zs} \approx \overline{I}_H \\
&\overline{S} = 2\,\overline{I}_H/R, \quad \overline{I}_H + \overline{S}e = \overline{I}_H\left(1 + 2\eta_e\right), \quad \frac{R^4 c_1}{\overline{I}_H} = 796(1 - s_1)
\end{aligned}\right\} \tag{77}$$

STEADY STATE IN HOVERING AND IN LOW-SPEED TRAVELING

In order to solve the equations of oscillation, it is necessary to determine first the values θ_o, β_o, and ζ_o of the pitch, flapping, and lagging angles, respectively, of the blades in the steady state. They will be calculated herein for the cases of hovering and low-speed ratio of traveling.

If in the steady state the load-carrying condition (equation (19)) for θ, that is, its value θ_o, is by some means enforced, then β and ζ, as long as no kinematic constraints are imposed on them, will adjust themselves freely, so that the moments M_{y1} and M_{z1} are zero. If kinematic constraints are imposed it is still possible, by certain preadjustments, to make these moments zero for the same steady-state values β_o and ζ_o as without constraints.

In traveling it has been shown that it is necessary to change the pitch angle θ along the circumference of the path of a blade $(0 \leq \psi \leq 2\pi)$ in such a way as to keep constant the force component perpendicular to the plane of rotation. This aim, however, will lead to difficulties if the blade is continued too far inward, that is, to a value $s_i = \frac{r_i}{R}$ so small that this inner part of the blade, when on the retreating side, does not find a positive relative inflow velocity. If this case cannot be avoided, then an additional sideways tilting $\Delta\gamma$ of the axis of rotation would be necessary to counteract the moment $\Delta L_z e$ by a moment of the weight $W\Delta\gamma h$ (h = distance from center of gravity of structure to center of hub). A force component transverse to the direction of flight would arise. A loss of total lift L_z would also result, and therefore it shall be assumed in the following discussion that L_z can be kept constant along the circumference by means of a periodic change of the pitch angle θ.

It shall not be discussed in this paper in which manner, that is, by which mechanism, cam device, or tilted swash plate, the variation of the pitch angle θ may be realized. It may be assumed that this has been accomplished in some way.

Hovering

For hovering, the steady-state values of β and ζ can be readily obtained from the expressions (31b) and (32b) for the moments $\left(M_{y1}\right)_c$ and $\left(M_{z1}\right)_c$. Thus

$$\beta_c = \frac{32}{315}\left(1 + \frac{1}{2} s_1\right) \pi \frac{\rho}{\sigma} \frac{R^4 c_1}{\bar{I}_H + \bar{S}e} \theta_c - \frac{\bar{S}g}{\Omega^2\left(\bar{I}_H + \bar{S}e\right)} \tag{78}$$

$$\zeta_c = \frac{32}{315} \pi \frac{\rho}{\sigma} \frac{R^4 c_1}{\bar{S}e} \theta_c \left(\alpha_p + \frac{1}{5} \theta_c\right) \tag{79}$$

Formulas (78) and (79) can be interpreted also as an indication that it is possible to regulate automatically the equilibrium value of the pitch angle θ_c by the equilibrium values of the flapping and the lagging angles β_c and ζ_c.

Low-Speed Traveling

Although the angles θ, β, and ζ will be constant in hovering, they will, in traveling, have to vary periodically during each revolution under the action of the weight of the system and the inertia, damping, and Coriolis forces of the masses of the blades. It is necessary to calculate these angles because they appear as variable coefficients in the differential equations of oscillation.

This calculation, however, may be simplified by assuming the speed ratio μ_e to be a quantity sufficiently small so that second and higher powers of it may be neglected. In the calculations it will be treated as a first-order small quantity. Such a simplification is advisable in order to gain a first access to the behavior of the blade system in the transition from hovering to traveling. From the knowledge gained by such a first analysis, it will then later be possible to proceed to the extension to higher speed ratios.

The steady-state pitch angle θ_o must be determined from equation (21), which, reduced to first powers of μ_e, assumes the form:

$$\theta_o = \theta_c - k_2\mu_e\left(2\theta_c \sin\psi + \beta_c \cos\psi - \gamma\right) + \frac{\beta_o''}{6C_2} \tag{80}$$

where

$$C_2 \equiv \frac{4}{315}\left(1 + \frac{1}{2} s_1\right) \pi \frac{\rho}{\sigma} \frac{R^4 c_1}{\bar{I}_H}$$

and where β_o'' is supposed to be also developed up to first powers of μ_e. (See equations (81) and (85).) It may be remarked that the denominator of equation (21) as far as it concerns the term with β_o'' can be taken as unity, since β_o'' is already itself a second-order small quantity. (The damping term in β_o' given by equation (41c) has been neglected in this section but was later found to be of the nonnegligible order 2. In the section INFLUENCE OF DAMPING ON STEADY-STATE ANGLES IN TRAVELING this term has been incorporated into the θ_o, β_o expressions. It can thus be finally stated that β_o'' and β_o' are the only dynamic terms which must appear to obtain equations (95) and (96).)

Equation (80), however, is not sufficient to determine θ_o, because the value of the term β_o'' appearing in it depends on the equilibrium of moments about the y_1-axis. The moment M_{y1} is given by equation (31a), which, strictly speaking, must contain in addition to the inertia term $\sigma\Omega^2 I_H \beta''$ a Coriolis term $M_{cy1} = -2\Omega\sigma I_H \beta_o \dot{\zeta}_o$ (see equation (67c)), a damping term (equation (50)) proportional to $\dot{\zeta}_o \theta_o$, and two inertia terms (equation (65)) proportional to $\ddot{\zeta}_o \theta_o$ and to $\ddot{\theta}_o \zeta_o$. Each of these additional terms, however, is smaller than the second order and may therefore be neglected in equation (31a). (See, however, additional term in $\dot{\beta}_o$ from equation (50) in (91).) Equation (31a) for $M_{y1} = 0$ (without (91)) gives now (by using equation (77)):

$$\beta_o'' + \beta_o\left(1 + 2\eta_e\right) = C_2\left[8\theta_o + 12\mu_e\left(\beta_c \cos \psi + 2\theta_c \sin \psi - \gamma\right)\right] - \frac{2g}{\Omega^2 R} \tag{81}$$

By eliminating β_o'' from equations (80) and (81), the following relation between θ_o and β_o is obtained:

$$\left(1 + 2\eta_e\right)\beta_o = C_2\left[2\theta_o + 6\theta_c + \left(12 - 6k_2\right)\mu_e\left(\beta_c \cos \psi + 2\theta_c \sin \psi - \gamma\right)\right] - \frac{2g}{\Omega^2 R} \tag{82}$$

By double differentiation of equation (82) and then substitution for β_o'' in equation (80), a differential equation in θ_o alone appears, namely,

$$-\frac{1}{3}\theta_o'' + \theta_o = \theta_c + k_2\mu_e\gamma - 4\mu_e\theta_c \sin \psi - 2\mu_e\beta_c \cos \psi \tag{83}$$

The only integral of equation (83) not containing free oscillations and periodic in terms of sin ψ and cos ψ can be readily shown to be:

$$\theta_o = \theta_c + k_2\mu_e\gamma - 3\mu_e\theta_c \sin \psi - \frac{3}{2}\mu_e\beta_c \cos \psi \tag{84}$$

This relation must therefore be assumed as the steady-state variation of the pitch angle θ_o.

If equation (84) is put into equation (82), an analogous expression is obtained for the steady-state variation of the flapping angle β_o, namely,

$$\begin{aligned}(1 + 2\eta_e)\beta_o = C_2\Big[8\theta_c &+ \left(8k_2 - 12\right)\mu_e\gamma - 6\mu_e\left(2k_2 - 3\right)\theta_c \sin \psi \\ &- 3\mu_e\left(2k_2 - 3\right)\beta_c \cos \psi\Big] - \frac{2g}{\Omega^2 R}\end{aligned} \tag{85}$$

Finally, the expression for ζ_o must be obtained from the moment equilibrium about the z_1-axis. From equation (32a) for M_{z1} and from expressions (65), (67c), and (50) for the inertia, Coriolis, and damping moments, it is seen that up to second-order small quantities, the moment M_{z1} is

$$M_{z1} = -\frac{32}{315}\pi\rho\Omega^2 R^4 c_1 \theta_c\left(\alpha_p + \frac{1}{5}\theta_c\right) + \sigma\zeta_o e\Omega^2\overline{S} + \sigma\Omega^2\frac{\partial^2\zeta_o}{\partial\psi^2}\overline{I}_H \tag{86}$$

By setting $M_{z1} = 0$, the particular integral (without oscillation) of the resulting equation is seen to be

$$\zeta_o = \frac{32}{315}\pi\frac{\rho}{\sigma}\frac{\overline{R^4 c_1}}{\overline{S}e}\theta_c\left(\alpha_p + \frac{1}{5}\theta_c\right) = \zeta_c \tag{87}$$

Influence of Steady-State Inertia Terms

It is interesting to observe the effect the dynamic forces and moments in the transition from hovering to traveling have on the values of the angles θ_o, β_o, and ζ_o in the steady state. If the dynamic terms were neglected, then from equations (80) and (81) the values of θ_o and β_o would be found to be (primes are used to distinguish from the correct values)

$$\theta_o' = \theta_c + k_2\mu_e\gamma - 2k_2\mu_e\,\theta_c\sin\psi - k_2\mu_e\beta_c\cos\psi \tag{88}$$

$$\beta_o'\left(1 + 2\eta_e\right) = C_2\left[8\theta_c + \left(8k_2 - 12\right)\mu_e\gamma + \left(12 - 8k_2\right)\mu_e\left(\beta_c\cos\psi + 2\theta_c\sin\psi\right)\right] - \frac{2g}{\Omega^2 R} \tag{89}$$

When equation (88) is compared with equation (84) and it is remembered that with the numerical values used in this report $k_2 = \frac{7}{4}$, it is seen that θ_o' is the same as θ_o except for some slight differences in the coefficients of $\sin\psi$ and $\cos\psi$. Thus the dynamic terms appear to have little effect on the value of the steady-state pitch angle. Comparison of equation (89) with equation (85) shows that, as for the pitch angle, the value of β_o' differs from that of β_o only in the coefficients of $\sin\psi$ and $\cos\psi$. These coefficients in β_o' (without the effect of the inertia terms) are four-thirds times those of β_o (with the inertia terms). The steady-state value ζ_o of the lagging angle, on the other hand, is not affected by the dynamic terms (cf. equations (86) and (87)).

Influence of Damping on Steady-State Angles in Traveling

In relations (16) and (21) for the pitch angle θ, the inertia terms of the blade appearing in the steady state of traveling have been taken into account; whereas the damping terms have been neglected. It seemed advisable, on second thought, to consider also the influence of these damping terms on the steady-state angles θ_o, β_o, and ζ_o. (This influence, however, will not appear in the results of the oscillation analysis for low speed ratio μ_e, as it would only lead to terms of higher than second order. For higher speed ratios, nevertheless, the damping terms would have an influence on the final equations of oscillation.

Damping terms given by equation (41c) must first be added to the expression (equation (10b)) for the total lift gradient L_z'. In view of orders of magnitude, these terms may be reduced to the single term

$$L_z' = -\pi\rho\Omega^2 r^2 c\,\frac{\partial\beta}{\partial\psi} \tag{90}$$

Addition of this term has the consequence of adding to the numerator of the expression (21) for $\Delta\beta$ the additional term

$$\frac{\partial\beta_o}{\partial\psi} \tag{91}$$

and to the expression (31a) for the steady-state value of M_{y1} the term

$$\frac{32}{315}\left(1 + \frac{1}{2}s_1\right)\pi\rho\Omega^2R^4c_1\,\frac{\partial\beta_o}{\partial\psi} \tag{92}$$

Comparing expression (92) with the inertia expression appearing in equation (31a) shows that both are of the same order of magnitude. As has been seen in equation (87), the value of ζ_o will remain ζ_c to second-order small terms. In order to find the angle variables θ_o and β_o of the steady state for $\mu_e \neq 0$ (but μ_e small) the equilibrium conditions (equations (80) and (81)) of the force L_z and of the moment M_{y1} must be complemented by the foregoing terms. Thus, instead of equations (80) and (81), there is obtained

$$\theta_o = \theta_c - k_2\mu_e\left(2\theta_c \sin\psi + \beta_c \cos\psi - \gamma\right) + \frac{\beta_o''}{6C_2} + \beta_o' \tag{93}$$

$$\theta_o = -\frac{3}{2}\mu_e\left(2\theta_c \sin\psi + \beta_c \cos\psi - \gamma\right) + \frac{\beta_o''}{8C_2} + \beta_o'$$

$$+ \beta_o\frac{1 + 2\eta_e}{8C_2} + \frac{2g}{\Omega^2R8C_2} \tag{94}$$

Writing

$$\theta_o = \theta_c + \mu_e\,\theta_1,\quad \beta_o = \beta_c + \mu_e\beta_1 \tag{95}$$

equating the right-hand sides of equations (93) and (94), and calculating, as before, the appropriate particular integral, it is found that the value of β_o remains the same as in equation (85), but that θ_o as given by equation (84) must be changed by the addition of the term

$$\mu_e\beta_1' = 3\mu_e C_2\left(3 - 2k_2\right)\left(2\theta_c \cos\Psi - \beta_c \sin\Psi\right) \tag{96}$$

The damping term thus appears to have an appreciable influence on the steady-state value θ_o of the blade angle in forward flight, although it does not affect the value of the steady-state flapping angle β_o.

OSCILLATIONS OF BLADE SYSTEM IN HOVERING

General Explicit Equations

For the state of hovering $(\mu_e = 0)$ the right sides of equations (74), (75), and (76) are appreciably simplified.

They may be written explicitly as follows:

$$\ddot{\bar{\theta}}\,\bar{I}_r - \ddot{\bar{\beta}}\,\zeta_c\bar{I}_{zs} + \ddot{\bar{\zeta}}\beta_c\bar{I}_H + \frac{E\left(1 + s_i\right)}{12\Omega}\,\frac{c_i}{R}\left[\dot{\bar{\theta}}\,\frac{c_i}{R}\,\frac{2}{35}\left(2 + s_i + 7\eta_e\right) + \dot{\bar{\beta}}f\left(1 + 2\eta_e\right) + \dot{\bar{\zeta}}\left(2f\,\theta_c - \frac{\left|C_{M_{ac}}\right|}{\pi}\right)\right] - \bar{\theta}\left[\frac{E\left(1 + s_i\right)}{12}\,f\,\frac{c_i}{R} - \Omega^2\bar{I}_{12}\right] = \frac{M_{x1}}{\sigma} \tag{97a}$$

$$-\ddot{\bar{\beta}}\left(\bar{I}_H + \bar{I}_{ys}\right) + \ddot{\bar{\zeta}}\theta_c\bar{I}_{12} + \ddot{\bar{\theta}}\zeta_c\bar{I}_{zs} + 2\dot{\bar{\zeta}}\beta_c\Omega\bar{I}_H - \frac{32}{315}\,\frac{E}{\Omega}\left[\dot{\bar{\beta}}\left(1 + \frac{1}{2}s_i + \frac{3}{2}\eta_e\right) + 2\,\dot{\bar{\zeta}}\,\theta_c\left(1 + \frac{1}{2}s_i\right)\right] + \Omega^2\left[\bar{\theta}\,\frac{32}{315}\,\frac{E}{\Omega^2}\left(1 + \frac{1}{2}s_i\right) - \bar{\beta}\left(\bar{I}_H + \bar{S}_e\right)\right] = \frac{M_{y1}}{\sigma} \tag{97b}$$

$$
- \ddot{\bar{\zeta}}\left(\bar{I}_H + \bar{I}_{zs}\right) - \ddot{\bar{\theta}}\beta_c\bar{I}_H + \ddot{\bar{\beta}}\,\theta_c\bar{I}_{12} + \dot{\bar{\beta}}\Omega\left[\frac{32}{315}\frac{E}{\Omega^2}\left(1 + \frac{1}{2}s_1\right)\left(\frac{4}{5}\theta_c - \alpha_p\right)\right.
$$

$$
\left. - 2\beta_c\,\bar{I}_H\right] + \Omega^2\left[\bar{\theta}\,\frac{32}{315}\,\frac{E}{\Omega^2}\left(\frac{2}{5}\theta_c + \alpha_p\right) - \bar{\zeta}\,\bar{S}e\right] = \frac{M_{z1}}{\sigma} \qquad (97c)
$$

where $E = \pi \frac{\rho}{\sigma} \Omega^2 R^4 c_1$.

The external moments acting on the hinge system and transmitted through the hub are M_{x1}, M_{y1}, and M_{z1}. They have to satisfy certain conditions in order not to violate the equations of equilibrium. One of the possible sets of conditions is, for instance, $\bar{\theta} = 0(\theta = \theta_c)$ with $\bar{\beta}$ and $\bar{\zeta}$ free. In this case $M_{x1}(\neq 0)$ gives the moment to be enforced by the pitch-changing lever or gear, and M_{y1} and M_{z1} are zero.

Another condition might be suggested by such kinematic constraints between θ and β, and θ and ζ that the equilibrium positions θ_c, β_c, and ζ_c are obtained automatically without an external pitch-changing gear. A further condition may consist in fixing θ by an external kinematic constraint (pitch-changing mechanism) but using a kinematic constraint between β and ζ.

It is also possible to introduce into equations (97a), (97b), and (97c) friction constraints, as will be shown in one of the examples. The choice between these and other possibilities will be made from the following points of view.

On the one hand, it is desirable to keep the natural frequencies away from resonance with the circular frequency Ω of the drive and at the same time to achieve a sufficient damping decrement. On the other hand, it will be necessary to make sure that the kinematic conditions do not interfere with the transition to traveling, that is to $\mu_e \neq 0$. The effect of internal kinematic conditions (constraints) will most conveniently be determined by Lagrange multipliers.

The kinematic conditions can always be expressed by equations between the coordinate variables, preferably in such a way that the aforementioned desirable features are achieved.

The materialization of such a kinematic equation between any two or three of the coordinates is the problem of the design engineer, who would have to decide whether to use linkages, gear wheels, cams, or hydraulic connections, and so on. This detail-designing problem is beyond the scope of this report.

Friction forces at the hinges, particularly at the lagging hinge ζ, have been tried for the purpose of damping excessive lagging oscillations, but with the detrimental effect of producing high bending moments at the blade root. Such devices can also be readily calculated by equations (97a), (97b), and (97c).

The four cases of kinematic and friction constraints previously enumerated will now be discussed in detail both for hovering $(\mu_e = 0)$ and for small speed ratio $(\mu_e \neq 0)$.

Case A: θ (Pitch) Fixed, β (Flapping) and ζ (Lagging) Free

In this case, $\overline{\theta} = 0$. The moments M_{y1} and M_{z1} will be zero, but not the twisting moment M_{x1}, which will be taken by the pitch-holding and pitch-changing mechanism or by counter flyweights. Two equations, namely (97b) and (97c) with the right-hand sides equal to zero, must therefore be solved for two unknown variables $\overline{\beta}$ and $\overline{\zeta}$.

Neglecting $\overline{I}_{ys}$ and $\overline{I}_{zs}$ in comparison with $\overline{I}_H$ (cf. equations (77)) and observing in accordance with expression (78) for β_c that in equation (97b) the Coriolis term and the aerodynamic damping term in $\dot{\overline{\zeta}}$ partly cancel each other, whereas in equation (97c) the Coriolis term and the aerodynamic damping term in $\dot{\overline{\beta}}$ can be combined as one term, the equations of oscillation (97b) and (97c) become simplified to

$$-\ddot{\overline{\beta}}\overline{I}_H + \ddot{\overline{\zeta}}\,\theta_c\overline{I}_{12} - \frac{E}{\Omega}\dot{\overline{\beta}}\,\frac{32}{315}\left(1 + \frac{1}{2}s_1 + \frac{3}{2}\eta_e\right) - 2\,\frac{\overline{S}g}{\Omega}\,\dot{\overline{\zeta}} - \Omega^2\overline{I}_H\left(1 + 2\eta_e\right)\overline{\beta} = 0 \qquad (98a)$$

$$\ddot{\overline{\beta}}\,\theta_c\overline{I} - \ddot{\overline{\zeta}}\,\overline{I}_H - \dot{\overline{\beta}}\,\frac{E}{\Omega}\,\frac{32}{315}\left(1 + \frac{1}{2}s_1\right)\left(\frac{6}{5}\,\theta_c + \alpha_p\right) - \Omega^2\overline{S}e\,\overline{\zeta} = 0 \qquad (98b)$$

Equations (98a) and (98b) are a system of linear, homogeneous, differential equations with constant coefficients. The complete integral of these equations can, as is well known, be built up by particular integrals of the form

$$\overline{\beta} = Fe^{p\psi}, \quad \overline{\zeta} = De^{p\psi} \qquad (99)$$

where F and D are real or complex constants (amplitudes) and $\psi = \Omega t$. The values of the real or complex constant (frequency) p must be determined, as usual, from the condition that equations (98a) and (98b) have solutions different from $\overline{\beta} = 0$ and $\overline{\zeta} = 0$. (This means that their solutions are also different from $F = 0$ and $D = 0$.) These two equations will also determine the ratios D/F of the amplitudes of oscillation. Putting equations (99) into equations (98a) and (98b), noting that $\frac{d\psi}{dt} = \Omega$, and dividing through by Ω^2 yield the following homogeneous equations in F and D:

$$-F\left[\bar{I}_H\left(p^2 + 1 + 2\eta_e\right) + p\frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{1}{2}s_1 + \frac{3}{2}\eta_e\right)\right] + D\left(p^2\,\theta_c\bar{I} - p\,\frac{2\,\bar{S}\,g}{\Omega^2}\right) = 0 \tag{100a}$$

$$F\left[\theta_c p^2\,\bar{I} - p\frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{1}{2}s_1\right)\left(\frac{6}{5}\theta_c + \alpha_p\right)\right] - D\bar{I}_H\left(p^2 + 2\eta_e\right) = 0 \tag{100b}$$

As anticipated before, the determinant of the coefficients F and D must vanish. Hence p must be determined from:

$$\bar{I}_H^{\,2}\left(p^2 + 1 + 2\eta_e\right)\left(p^2 + 2\eta_e\right) + \bar{I}_H\left(p^2 + 2\eta_e\right)\frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{1}{2}s_1 + \frac{3}{2}\eta_e\right)p$$
$$+\left(p\,\frac{2\,\bar{S}\,g}{\Omega^2} - p^2\theta_c\bar{I}\right)\left[p^2\theta_c\bar{I} - p\frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{1}{2}s_1\right)\left(\frac{6}{5}\theta_c + \alpha_p\right)\right] = 0 \tag{101a}$$

or

$$\left[1 - \left(\theta_c\frac{\bar{I}}{\bar{I}_H}\right)^2\right]p^4 + \frac{E}{\Omega^2\bar{I}_H}\frac{32}{315}\left(1 + \frac{1}{2}s_1 + \frac{3}{2}\eta_e\right)\left[1 + \theta_c\left(\frac{6}{5}\theta_c + \alpha_p\right)\frac{\bar{I}}{\bar{I}_H}\left(1 - \frac{3}{2}\eta_e\right)\right]p^3$$
$$+\frac{2}{\bar{I}_H}\frac{\bar{S}g}{\Omega^2}\theta_c\frac{\bar{I}}{\bar{I}_H}p^3 - \frac{2\,\bar{S}\,g}{\bar{I}_H\,\Omega^2}\frac{E}{\Omega^2\bar{I}_H}\frac{32}{315}\left(1 + \frac{1}{2}s_1\right)\left(\frac{6}{5}\theta_c + \alpha_p\right)p^2 + \left(1 + 4\eta_e\right)p^2$$
$$+ 2\eta_e\frac{E}{\Omega^2\bar{I}_H}\frac{32}{315}\left(1 + \frac{1}{2}s_1 + \frac{3}{2}\eta_e\right)p + 2\eta_e\left(1 + 2\eta_e\right) = 0 \tag{101b}$$

The value of θ_c is given by equation (20). For example, if $W = 4000$ pounds, $\rho = 0.00238$ slug per cubic foot, $R = 25$ feet, $\Omega = 20$ radians per second, $\alpha_p = 0.020$ (angle of attack about 0° for Clark Y airfoil),

$$\eta_e = 0.05, \quad c_1 = \frac{1}{6} \times 25 = 4.17 \text{ feet}, \quad n = 4 \text{ blades}, \quad \text{and} \quad s_1 = 0.2, \tag{102}$$

then the value of θ_c will be

$$\theta_c = 0.0306 \tag{103}$$

Taking the value of $\frac{\bar{I}}{\bar{I}_H}$, moreover, from equation (77), namely

$$\frac{\bar{I}}{\bar{I}_H} = 0.00386$$

gives

$$\left.\begin{aligned}
\left(\theta_c \frac{\bar{I}}{\bar{I}_H}\right)^2 &= 1.39 \times 10^{-8} \\
\frac{2\bar{S}g}{\bar{I}_H \Omega^2} \theta_c \frac{\bar{I}}{\bar{I}_H} &= 1.52 \times 10^{-6} \\
\theta_c\left(\frac{6}{5}\theta_c + \alpha_p\right)\frac{\bar{I}}{\bar{I}_H}\left(1 - \frac{3}{2}\eta_e\right) &= 0.618 \times 10^{-5} \\
\frac{2\bar{S}g}{\bar{I}_H \Omega^2}\frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{1}{2}s_1\right)\left(\frac{6}{5}\theta_c + \alpha_p\right) &= 4.06 \times 10^{-4}
\end{aligned}\right\} \tag{104}$$

Consideration of the four quantities of equation (104) which appear in equation (101b) as negligible in comparison with unity is therefore justified. The second and fourth terms of equation (104) represent the influence of the weight of a blade, and this influence will, in accordance with the foregoing considerations, be henceforth everywhere neglected. Equation (101b) may hence be reduced to:

$$p^4 + Tp^3 + \left(1 + 4\eta_e\right)p^2 + 2\eta_e Tp + 2\eta_e\left(1 + 2\eta_e\right) = 0 \tag{105}$$

where

$$T \equiv \frac{E}{\Omega^2 \overline{I}_H} \frac{32}{315}\left(1 + \frac{1}{2} s_1 + \frac{3}{2}\eta_e\right) \equiv \pi \frac{\rho}{\sigma} \frac{R^4 c_i}{\overline{I}_H} \frac{32}{315}\left(1 + \frac{1}{2} s_1 + \frac{3}{2}\eta_e\right)$$

Because of the assumption (see appendix G) of a fairly high relative thickness t_b of a blade $\left(\frac{t_b}{c} = \frac{1}{8}\right)$, it must be assumed in the calculations that the blade material is of wood. An average value of the ratio ρ/σ will then be (for $\rho = 0.0765$ lb/cu ft, $\sigma = 30$ lb/cu ft)

$$\frac{\rho}{\sigma} = 0.0025$$

(For a hollow, built-up blade design the average density ratio may be even somewhat smaller.) Hence, by using equation (77), the numerical value of T is found to be

$$T = 0.595 \tag{106}$$

As can be seen from equation (101a), equation (105) can be readily solved by writing it in the form

$$\left(p^2 + 2\eta_e\right)\left(p^2 + Tp + 1 + 2\eta_e\right) = 0 \tag{107}$$

The solutions of equation (107) are:

$$\left.\begin{aligned} p_{1,2} &= \pm i \sqrt{2\eta_e} \\ p_{3,4} &\approx -\frac{T}{2} \pm i \left(1 + \eta_e\right) \end{aligned}\right\} \tag{108}$$

These solutions, which are of the form

$$p = -R_e \pm i\, S_i \tag{109}$$

have, in accordance with equation (99), the following physical significance. The logarithmic decrement $\log \frac{A_n}{A_{n+1}}$, defined as the logarithm of the ratio of the amplitude of one cycle to the amplitude of the succeeding cycle, will be

$$\log \frac{A_n}{A_{n+1}} = 2\pi \frac{R_e}{S_i} \tag{110}$$

The natural (real) frequency of oscillation q in cycles per second will be

$$q = \frac{\Omega}{2\pi} S_i \tag{111}$$

where Ω is in radians per second.

Thus, the solution (108) with the numerical data of equations (102) shows that in case A there will be two natural frequencies: namely,

$$q_{1,2} = \frac{\Omega}{2\pi} \sqrt{2\eta_e} = 1.005 \text{ cycles per second}$$

$$q_{3,4} = \frac{\Omega}{2\pi} \left(1 + \eta_e\right) = 3.34 \text{ cycles per second}$$

The oscillations corresponding to the higher frequency $q_{3,4}$ will be very highly damped, the logarithmic decrement being

$$\left(\log \frac{A_n}{A_{n+1}}\right)_{3,4} = \pi \frac{T}{1 + \eta_e} = 1.78$$

Therefore, despite the proximity of $q_{3,4}$ to the rotational frequency $\frac{\Omega}{2\pi}$, the oscillations corresponding to $q_{3,4}$ (flapping) will, because of the high damping, present no danger of resonance and will be quite stable. (In order to get a more exact insight into the influence of resonance, it would be advisable to determine the ratio of thrust fluctuation to flapping amplitude. This determination requires a study of fluctuation of engine torque and speed. This question of resonance will not appear in the case (B) of appropriate kinematic constraint between flapping and lagging.) On the other hand, the oscillations corresponding to the low natural frequency $q_{1,2}$ (lagging) will be practically undamped.

The ratio of amplitudes can be obtained by putting equation (108) into equation (100a). Thus, corresponding to the complex frequency $p_{1,2}$,

$$\left(\frac{F}{D}\right)_{1,2} = -\frac{2\eta_e\theta_c \frac{\bar{I}}{I_H}}{1 \pm i\,T\sqrt{2\eta_e}} = -1.18 \times 10^{-5}(1 \mp 0.181i)$$

(Equation (100b), because of the (nevertheless close) approximation for p, would give zero.) Quite generally a complex value of F/D can be interpreted as follows: If

$$p = -R_e \pm iS_1$$

and if, correspondingly,

$$\frac{F}{D} = a \pm bi$$

then, with the arbitrary amplitudes H_1 and H_2, either $\bar{\zeta}$ or $\bar{\beta}$ (say, $\bar{\zeta}$) will have the form

$$\bar{\zeta} = e^{-R_e\psi}\left[H_1 \cos\left(S_1\psi\right) + H_2 \sin\left(S_1\psi\right)\right] \tag{112a}$$

and, from the amplitude ratio F/D, $\bar{\beta}$ will be given by

$$\bar{\beta} = e^{-R_e\psi}\left[a\bar{\bar{\zeta}}(\psi) + b\bar{\bar{\zeta}}\left(\psi + \frac{\pi}{2S_1}\right)\right] \tag{112b}$$

where $\bar{\bar{\zeta}}(\psi)$ is the expression in brackets appearing in equation (112a). A complex value of a ratio F/D of amplitudes therefore indicates a difference in phase between flapping (β) and lagging (ζ) oscillations, the real part giving the magnitude of the component of $\bar{\beta}$ in phase with $\bar{\zeta}$ and the imaginary part the magnitude of the component of $\bar{\beta}$ one-quarter of a period out of phase with $\bar{\zeta}$.

Because of the actual small absolute value of $(F/D)_{1,2}$, it is seen that the undamped oscillations, corresponding to the rather low natural frequency $\left(\frac{\Omega}{2\pi}\right)\sqrt{2\eta_e}$, will occur practically only about the z_1-axis; that is, only the lagging angle will oscillate.

Following from the complex frequency $p_{3,4}$ the ratio D/F will, from equation (100b), be

$$\left(\frac{D}{F}\right)_{3,4} = \frac{\theta_c \frac{\bar{I}}{\bar{I}_H} p^2{}_{3,4} - \frac{E}{\bar{I}_H \Omega^2} \frac{32}{315}\left(1 + \frac{1}{2}s_1\right)\left(\frac{6}{5}\theta_c + \alpha_p\right)p_{3,4}}{p^2{}_{3,4} + 2\eta_e} = 0.01005 \pm 0.0295i$$

From the fairly small absolute value of the ratio $(D/F)_{3,4}$ it is seen that the highly damped oscillations of the natural frequency $\frac{\Omega}{2\pi}\left(1 + \eta_e\right)$ will occur practically only about the y_1-axis; that is, practically only the flapping angle will oscillate.

The results for both frequencies, implying that the lagging and flapping oscillations are practically independent of each other, show that the slow undamped lagging oscillation is very sensitive to disturbances and that the high restoring forces in flapping have no component which might oppose the lagging deviations.

Case A_1: θ Fixed, β Free, ζ under Friction Constraint

The unfavorable result concerning the lagging oscillation leads to the attempt to improve the stability by introducing a friction constraint acting on the lagging angle by means of a moment $\bar{I}_H K\dot{\zeta}\Omega$ at the root of the blade, where K denotes a constant of relative energy dissipation the value of which can be chosen according to the required degree of damping. This constraint could be accomplished, for instance, by a rod leading from the root of the blade to the hub and provided with a telescopic fluid brake.

The term $-K\dot{\zeta}$ must then be added to the left side of the dynamic equation of the lagging acceleration, that is, to equation (97c). Then by the same procedure as before the amplitude ratios (see equations (100a) and (100b)), the frequencies, and the damping terms (see equation (103)) can be determined. It is considered sufficient here to calculate only the new complex frequency consisting in the damping and the real-frequency contributions.

Instead of equation (105), the following frequency equation appears:

$$p^4 + (T + K)p^3 + \left(1 + 4\eta_e\right)p^2 + 2\eta_e(T + K)p + Kp + 2\eta_e\left(1 + 2\eta_e\right) = 0 \qquad (113)$$

In order to determine the effect of the constant K in general terms on the damping and on the frequency, a simple approximate general solution was obtained by Newton's method based on the assumption that K is small. The first approximation here was taken as the exact solution to equation (113) when the term Kp is neglected. This first approximation is then the same as the solution given by equation (108) with T now replaced by (T + K). The second approximation is then found to be the following:

$$\left.\begin{aligned} p_{1,2} &= -\frac{4K\eta_e}{\left[K - 4\eta_e(T + K)\right]^2 + 8\eta_e} \pm i\,\frac{\left\{1 - \left[K - 4\eta_e(T + K)\right]\right\}\sqrt{2\eta_e}}{\left[K - 4\eta_e(T + K)\right]^2 + 8\eta_e} \\ p_{3,4} &= -\left[\frac{(T + K)}{2} - K\,\frac{\frac{1}{2}(T + K)A + 2(1 + 2\eta_e)}{A^2 + 4\left(1 + 2\eta_e\right)}\right] \pm i\left(1 + \eta_e\right)\left[1 - \frac{A - T - K}{A^2 + 4\left(1 + 2\eta_e\right)}\right] \end{aligned}\right\} \qquad (114a)$$

where

$$A \equiv \frac{(T + K)^3}{4} + (T + K)\left(2 + 4\eta_e\right) + K$$

For an example, the same numerical data as assumed in the preceding section (see also appendix G) together with the value of $K = 0.1$ were introduced. Equation (114a) then gave the following results:

$$\left.\begin{aligned} p_{1,2} &= -0.05 \pm 0.319i \\ p_{3,4} &= -0.310 \pm 1.035i \end{aligned}\right\} \qquad (114b)$$

The validity of the approximations given by equation (114a) was checked by putting the same numerical data into equation (113) and solving it exactly (by Ferrari's method) to three significant figures. The results were:

$$\left.\begin{aligned} p_{1,2} &= -0.0525 \pm 0.321i \\ p_{3,4} &= -0.295 \pm 0.972i \end{aligned}\right\} \qquad (114c)$$

Comparison of equations (114b) and (114c) shows that with values of K not appreciably greater than 0.1 the comparatively simple formulas in equation (114a) are sufficiently exact for most practical purposes, so that it is unnecessary to formulate an exact, but more involved, general solution of the quartic equation (113) here.

The natural frequencies and the logarithmic decrements corresponding to the solution (114c) are (see equations (110) and (111)):

$$q_{1,2} = 1.01 \text{ cycles per second}$$

$$q_{3,4} = 3.30 \text{ cycles per second}$$

$$\left(\log \frac{A_n}{A_{n+1}}\right)_{1,2} = 1.029$$

$$\left(\log \frac{A_n}{A_{n+1}}\right)_{3,4} = 1.91$$

where, temporarily, A_n denotes a real amplitude.

The following conclusions for case A_1 can be drawn from these results of the numerical example. It is possible to achieve fairly high damping of the lagging oscillations by introducing moderate fluid friction. This friction, moreover, will have little influence on the two values of the natural frequency and on the high damping which is associated with what originally were practically the flapping oscillations. It must be observed, however, that because of the low natural frequency $(q_{1,2})$ corresponding to the lagging oscillations, the restoring force in these oscillations will also be low, and may, in fact, not be sufficient to return the blade to its normal position. The damping introduced by the fluid friction would in that case be of no avail.

Case B: θ Fixed, β and ζ under Kinematic Constraint

It has been seen (case A) that with the pitch angle θ fixed and the flapping and lagging angles β and ζ free, there will be the disadvantage of an absence of damping for lagging oscillations. The possibility of overcoming this disadvantage by introducing kinematic constraints between the variables will now be investigated.

First an appropriate constraint between flapping and lagging will be introduced. As previously explained, this constraint should be such that the conditions of steady flight are not violated; that is, the constraint, as represented by an equation, should satisfy the condition that when $\zeta = \zeta_c$, then $\beta = \beta_c$, or when $\overline{\zeta} \equiv \zeta - \zeta_c = 0$, then $\overline{\beta} \equiv \beta - \beta_c = 0$. This condition will not be violated by a constraint of the form

$$\overline{\zeta} = \kappa\overline{\beta} \tag{115}$$

where κ is a constant to be chosen in accordance with requirements of stability and of avoidance of resonance. In order to achieve materially such a constraint, a preadjustment of the angle β or the angle ζ is necessary. For example, the lagging angle ζ may be preadjusted to a value ζ_1, where ζ_1 is the value of ζ when $\beta = 0$. From equation (115), this value is $\zeta_1 = \zeta_c - \kappa\ \beta_c$.

As in case A, the solution for $\overline{\beta}$ and $\overline{\zeta}$ will be of the exponential form given by equations (99). When equations (99) are then put into equation (115), the ratio of the amplitudes is obviously

$$\frac{D}{F} = \kappa \tag{116}$$

Condition (115) can mathematically be taken into account by means of a Lagrange multiplier L_m and a Lagrange function ϕ, where, according to equation (115),

$$\phi = \overline{\zeta} - \kappa\overline{\beta} = 0 \tag{117}$$

The equations for D and F corresponding to equations (100a) and (100b) then become:

$$-F\left[\bar{I}_H\left(p^2 + 1 + 2\eta_e\right) + p\frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{1}{2}s_i + \frac{3}{2}\eta_e\right)\right] + Dp^2\theta_c\bar{I} + L_m\frac{\partial\phi}{\partial\bar{\beta}} = 0 \qquad (118a)$$

$$F\left[p^2\bar{I}\theta_c - p\frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{1}{2}s_i\right)\left(\frac{6}{5}\theta_c + \alpha_p\right)\right] - D\bar{I}_H\left(p^2 + 2\eta_e\right) + L_m\frac{\partial\phi}{\partial\bar{\zeta}} = 0 \qquad (118b)$$

From equation (117),

$$\frac{\partial\phi}{\partial\bar{\beta}} = -\kappa, \quad \text{and} \quad \frac{\partial\phi}{\partial\bar{\zeta}} = 1 \qquad (119)$$

Putting equations (116) and (119) into equations (118a) and (118b), and eliminating L_m from these equations gives

$$F\left\{-\bar{I}_H\left(p^2 + 1 + 2\eta_e\right) - p\frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{1}{2}s_i + \frac{3}{2}\eta_e\right) + p^2\kappa\,\theta_c\,\bar{I}\right.$$
$$\left. + \kappa\left[p^2\bar{I}\theta_c - p\frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{1}{2}s_i\right)\left(\frac{6}{5}\theta_c + \alpha_p\right)\right] - \kappa^2\bar{I}_H\left(p^2 + 2\eta_e\right)\right\} = 0 \qquad (120)$$

For equation (120) to have a solution other than $F = 0$, p must have the values satisfying the equation

$$-p^2\left(1 + \kappa^2 - 2\kappa\theta_c\frac{\bar{I}}{\bar{I}_H}\right) - pa' - \left[1 + 2\eta_e\left(1 + \kappa^2\right)\right] = 0 \qquad (121)$$

where

$$a' = \frac{E}{\Omega^2\bar{I}_H}\frac{32}{315}\left(1 + \frac{1}{2}s_i + \frac{3}{2}\eta_e\right)\left[1 + \kappa\left(1 - \frac{3}{2}\eta_e\right)\left(\frac{6}{5}\theta_c + \alpha_p\right)\right]$$

Hence, neglecting the quantities $2\,\theta_c\frac{\bar{I}}{\bar{I}_H}$ in comparison with $\left(1 + \kappa^2\right)$ and also neglecting a'^2 in comparison with $4\left(1 + \kappa^2\right)\left[1 + 2\eta_e\left(1 + \kappa^2\right)\right]$,

$$p \approx -\frac{a'}{2\left(1 + \kappa^2\right)} \pm i\sqrt{\frac{1}{1 + \kappa^2} + 2\eta_\theta} \tag{122}$$

It may be observed that in this case of geometric constraints, the drag has a slight, but noticeable, effect on the damping; whereas in the case of free oscillations, the influence of the drag on the damping was negligible. For ordinary values of κ, the damping will be quite large, and the stability therefore very great. For example, if

$$\kappa = 1$$

then

$$p = -\frac{a'}{4} \pm \sqrt{\frac{1}{2} + 2\eta_\theta}\ i \tag{123}$$

By use of the same numerical values as before (equations (102) and (103)) there is obtained

$$p = -0.158 \pm 0.775i \tag{124}$$

which gives a natural frequency of $q = \frac{20}{2\pi} 0.775 = 2.47$ cycles per second and a logarithmic decrement (see equation (110)) of $2\pi \frac{0.158}{0.775} = 1.282$. From the point of view of stability this case therefore appears quite satisfactory because there is considerable damping and there is little danger of resonance from the drive, that is, from such probable disturbing frequencies as $\Omega \left(= \frac{20}{2\pi} = 3.19 \text{ cps}\right)$ or 2Ω.

Case C: θ, β, and ζ under Kinematic Constraint

In the cases thus far discussed, the pitch angle θ has been assumed fixed. The effect of allowing a certain freedom of pitch change $\overline{\theta}$ in accordance with geometric conditions (constraints) among the angles θ, β, and ζ will now be considered. For this purpose, constraints similar to that used in case A (equation (115)) will be assumed, namely,

$$\left.\begin{aligned}\bar{\beta} &= \lambda\bar{\theta}\\ \bar{\zeta} &= \mu\bar{\theta}\end{aligned}\right\} \tag{125}$$

where λ and μ are constants which will be selected in accord with stability requirements as will be discussed.

Solutions to equations (97a) to (97c) will have the same form as given by equations (99), namely,

$$\bar{\theta} = Be^{p\psi}, \quad \bar{\beta} = Fe^{p\psi}, \quad \bar{\zeta} = De^{p\psi} \tag{126}$$

From equations (125) and (126) it follows that

$$\left.\begin{aligned}F &= \lambda B\\ D &= \mu B\end{aligned}\right\} \tag{127}$$

By putting equation (126) into equations (97a), (97b), and (97c), equations of the following form are obtained:

$$\left.\begin{aligned}BP_{1b} + FP_{1f} + DP_{1d} &= \frac{M_{x1}e^{-p\psi}}{\sigma\Omega^2}\\ BP_{2b} + FP_{2f} + DP_{2d} &= \frac{M_{y1}e^{-p\psi}}{\sigma\Omega^2}\\ BP_{3b} + FP_{3f} + DP_{3d} &= \frac{M_{z1}e^{-p\psi}}{\sigma\Omega^2}\end{aligned}\right\} \tag{128}$$

where P indicates polynomials of second and lower degree in p. As in case B, the constraint conditions (equations (125)) can be taken into account by means of Lagrange multipliers L_{m1} and L_{m2} and kinematic conditions ϕ_1 and ϕ_2, where

$$\phi_1 = \bar{\beta} - \lambda\bar{\theta} = 0$$

$$\phi_2 = \bar{\zeta} - \mu\bar{\theta} = 0$$

thus, by using equations (127), equations (128) become:

$$\left.\begin{aligned} B\left(P_{1b} + \lambda P_{1f} + \mu P_{1d}\right) + L_{m1}\frac{\partial \phi_1}{\partial \overline{\theta}} + L_{m2}\frac{\partial \phi_2}{\partial \overline{\theta}} = 0 \\ B\left(P_{2b} + \lambda P_{2f} + \mu P_{2d}\right) + L_{m1}\frac{\partial \phi_1}{\partial \overline{\beta}} + L_{m2}\frac{\partial \phi_2}{\partial \overline{\beta}} = 0 \\ B\left(P_{3b} + \lambda P_{3f} + \mu P_{3d}\right) + L_{m1}\frac{\partial \phi_1}{\partial \overline{\zeta}} + L_{m2}\frac{\partial \phi_2}{\partial \overline{\zeta}} = 0 \end{aligned}\right\} \quad (129)$$

where

$$\frac{\partial \phi_1}{\partial \overline{\theta}} = -\lambda, \quad \frac{\partial \phi_2}{\partial \overline{\theta}} = -\mu, \quad \frac{\partial \phi_1}{\partial \overline{\beta}} = 1 = \frac{\partial \phi_2}{\partial \overline{\zeta}}, \quad \frac{\partial \phi_2}{\partial \overline{\beta}} = 0 = \frac{\partial \phi_1}{\partial \overline{\zeta}}$$

Eliminating L_{m1} and L_{m2} from the three expressions of equation (129) results in the following single equation:

$$B\left[P_{1b} + \lambda P_{1f} + \mu P_{1d} + \lambda\left(P_{2b} + \lambda P_{2f} + \mu P_{2d}\right) + \mu\left(P_{3b} + \lambda P_{3f} + \mu P_{3d}\right)\right] = 0 \quad (130a)$$

The value of p is then determined by setting the factor of B in equation (130a) equal to zero. Equation (130a) can hence be written as:

$$P_{1b} + \lambda\left(P_{1f} + P_{2b}\right) + \mu\left(P_{1d} + P_{3b}\right) + \lambda^2 P_{2f} + \mu^2 P_{3d} + \lambda\mu\left(P_{3f} + P_{2d}\right) = 0 \quad (130b)$$

When the expressions obtained for P (see appendix H) are substituted in accordance with the definitions (equations (128)) and the expression (78) for β_c, as well as the geometric properties (equations (77)) of the blade body, are used, equation (130b) becomes the following quadratic in p:

$$p^2\left(\lambda^2 + \mu^2\right) + p\left\{\frac{E}{\Omega^2\bar{I}_H}\frac{32}{315}\mu\left[\left(1 + \frac{1}{2}s_1\right)\left(\frac{4}{5}\theta_c - \alpha_p\right)\lambda - \alpha_p - \frac{2}{5}\theta_c\right] - \mu\frac{M}{\bar{I}_H} - \frac{H}{\bar{I}_H}\right.$$

$$\left. + \lambda^2\frac{E}{\Omega^2\bar{I}_H}\frac{32}{315}\left(1 + \frac{1}{2}s_1 + \frac{3}{2}\eta_e\right) - \lambda\frac{E}{\Omega^2\bar{I}_H}\frac{\left(1 + s_1 + 2\eta_e\right)}{12} f\frac{c_1}{R}\right\}$$

$$+ \left[\frac{J}{\bar{I}_H} + \lambda^2\left(1 + 2\eta_e\right) + 2\eta_e\mu^2 - \lambda\frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{1}{2}s_1\right)\right] = 0 \qquad (131)$$

where

$$M \equiv \frac{E}{\Omega^2}\frac{\left(1 + s_1\right)}{12}\left(2\frac{c_1}{R}f\quad\theta_c - \left|C_{M_{ac}}\right|\frac{c_1}{\pi R}\right)$$

$$H \equiv \frac{E}{\Omega^2}\frac{\left(1 + s_1\right)}{12}\left(\frac{c_1}{R}\right)^2\frac{2}{35}\left(2 + s_1 + 7\eta_e\right)$$

$$J \equiv \frac{E}{\Omega^2}\frac{\left(1 + s_1\right)}{12}\frac{c_1}{R}f - \bar{I}$$

With the numerical values consistently used in this report, equation (131) reduces to (see appendix H):

$$p^2 + 2a_1p + a_o = 0 \qquad (132)$$

where

$$a_1 = \left(\lambda^2 + \mu^2\right)^{-1}\left(0.299\lambda^2 - 0.0068\lambda + 0.00126\lambda\mu - 0.0082\mu - 0.00084\right)$$

$$a_o = \left(\lambda^2 + \mu^2\right)^{-1}\left(1.1\lambda^2 - 0.560\lambda + 0.1\mu^2 + 0.00656\right)$$

The solution to equation (132) is

$$p = -a_1 \pm i\sqrt{a_o - a_1^2} \qquad (133)$$

For stability it is necessary and sufficient that the roots of equation (132) be either real and negative or complex with the real part negative. These conditions will be satisfied if and only if a_1 and a_o are both positive (or zero). The values of λ and μ can now be so chosen that these conditions will be satisfied. For example, if

$$\lambda = 0.7, \quad \mu = 1$$

then the complex relative frequency p will be

$$p = -0.0882 \pm i \times 0.403 \tag{134}$$

giving a logarithmic decrement of $2\pi \frac{0.0882}{0.403} = 1.370$, and, with $\Omega = 20$ radians per second, a natural absolute frequency of $\frac{20}{2\pi} 0.403 = 1.287$ cycles per second. From the point of view of stability, case C, as well as case B, is therefore satisfactory for hovering.

OSCILLATIONS OF BLADE SYSTEM IN LOW-SPEED TRAVELING

The same assumptions in regard to the (first order small) value of the speed ratio μ_e and the neglect of third-order terms, as made for the steady state in the section STEADY STATE IN HOVERING AND IN LOW-SPEED TRAVELING, will also be made in this section for the oscillations in traveling.

General Explicit Equations

As in hovering, the oscillation equations (74) to (76) must be solved, except that now the expressions for θ_o, β_o, and ζ_o will be not only different from the hovering values θ_c, β_c, and ζ_c, but also variable, and in addition the equations will contain more terms than in hovering.

The expressions for θ_o, β_o, and ζ_o are given by equations (84) with (96), (85), and (87). By using equations (35) to (37) and (49) to (51) for the quasi-elastic and the damping moments and rejecting all terms smaller than the second order, the dynamic equations (74) to (76) $\left(E \equiv \frac{\pi\rho}{\sigma} R^4 c_i \Omega^2 \text{ as in the foregoing section}\right)$ become:

$$\ddot{\bar{\theta}}\,\bar{I}_r - \ddot{\bar{\beta}}\zeta_c\bar{I}_{zs} + \ddot{\bar{\zeta}}\beta_c\bar{I}_H + \frac{E\left(1 + s_i\right)}{12}\left[\frac{\dot{\bar{\theta}}}{\Omega}\left(\frac{c_i}{R}\right)^2 \frac{2}{35}\left(2 + s_i + 7\eta_e + 7\mu_e \sin\psi\right)\right.$$

$$\left. + \frac{\dot{\bar{\zeta}}}{\Omega}\left(f\frac{c_i}{R}2\theta_c - \left|C_{M_{ac}}\right|\frac{c_i}{\pi R}\right) + \frac{\dot{\bar{\beta}}}{\Omega} f \frac{c_i}{R}\left(1 + 2\eta_e + 2\mu_e \sin\psi\right)\right]$$

$$- \bar{\theta}\left[\frac{E\left(1 + s_i\right)}{12}\frac{c_i}{R} f - \Omega^2\bar{I}_{12} + \frac{E\left(1 + s_i\right)}{3}\frac{c_i}{R} f\mu_e \sin\psi\right]$$

$$- \bar{\beta}\,\frac{E\left(1 + s_i\right)}{6}\frac{c_i}{R} f\mu_e \cos\psi = \frac{M_{x1}}{\sigma} \tag{135a}$$

$$\ddot{\bar{\theta}}\zeta_c\bar{I}_{zs} - \ddot{\bar{\beta}}\left(\bar{I}_H + \bar{I}_{ys}\right) + \ddot{\bar{\zeta}}\theta_c\bar{I}_{12} - \dot{\bar{\beta}}\,\frac{E}{\Omega}\frac{32}{315}\left(1 + \frac{1}{2}s_i\right)\left(1 + \frac{3}{2}\eta_e + \frac{3}{2}\mu_e \sin\psi\right)$$

$$+ \bar{\theta}\,\frac{32}{315}\left(1 + \frac{1}{2}s_i\right)E\left(1 + 3\mu_e \sin\psi\right) - \bar{\beta}\left[\Omega^2\left(\bar{I}_H + \bar{S}e\right) - \frac{48}{315}\left(1 + \frac{1}{2}s_i\right)E\mu_e \cos\psi\right] = \frac{M_{y1}}{\sigma} \tag{135b}$$

$$- \ddot{\bar{\theta}}\,\beta_c\bar{I}_H + \ddot{\bar{\beta}}\,\theta_c\bar{I}_{12} - \ddot{\bar{\zeta}}\left(\bar{I}_H + \bar{I}_{zs}\right) - \dot{\bar{\beta}}\left[2\beta_c\Omega\bar{I}_H - \frac{E}{\Omega}\frac{32}{315}\left(1 + \frac{1}{2}s_i\right)\left(\frac{4}{5}\theta_c - \alpha_p\right)\right]$$

$$+ \frac{32}{315}E\left(\alpha_p + \frac{2}{5}\theta_c\right)\bar{\theta} - \bar{\zeta}\Omega^2\,\bar{S}e = \frac{M_{z1}}{\sigma} \tag{135c}$$

When equations (135a) to (135c) are compared with the corresponding set (equations (97a) to (97c)) for hovering, it is seen that both sets are the same except for four additional terms in equation (135a) for M_{x1} and three additional terms in equation (135b) for M_{y1}. Each of these additional terms has the periodic coefficient $\sin\psi$ or $\cos\psi\ (\psi = \Omega t)$ but of relatively small magnitude. It will therefore not be necessary in this case to apply the theory of Mathieu functions, but it will be sufficient to use a method of successive approximation starting from the hovering state as a first approximation. This method may be expressed by the propositions

$$\bar{\theta} = \bar{\theta}_1 + \mu_e \bar{\theta}_2, \quad \bar{\beta} = \bar{\beta}_1 + \mu_e \bar{\beta}_2, \quad \bar{\zeta} = \bar{\zeta}_1 + \mu_e \bar{\zeta}_2$$

The validity of this method of solution will be checked by the results obtained by its application, for it is necessary that these results remain within the order of magnitude assumed in advance; that is, in this case the terms proportional to μ_e $\left(\text{e.g.,}\ \mu_e \bar{\theta}_2\right)$ must not be larger than second order small, so as to be of a higher order small than the corresponding hovering solutions $\left(\text{e.g.,}\ \bar{\theta}_1\right)$.

Case A: θ Guided, $\theta = \theta_0 (\Psi)$; β and ζ Free

This case is the same as case A for hovering, except that now instead of keeping θ fixed at the steady-state value θ_c for hovering, the blade angle is guided so that at all times $\theta = \theta_o$, where θ_o is a function of $\Psi(=\Omega t)$ and therefore of the time and is given by equations (84) and (96). The deviation of θ from the steady-state values will therefore be zero, so that, as for case A in hovering,

$$\bar{\theta} = 0$$

and only two equations, namely equations (135b) and (135c), need be considered to determine the natural frequency and the damping.

Equations (135b) and (135c) can be solved as indicated above by writing the solution in the form

$$\bar{\theta} = 0, \quad \bar{\beta} = \bar{\beta}_1 + \mu_e \bar{\beta}_2, \quad \bar{\zeta} = \bar{\zeta}_1 + \mu_e \bar{\zeta}_2 \qquad (136)$$

where $\bar{\beta}_1$ and $\bar{\zeta}_1$ are the solutions for hovering, which have already been obtained in the section OSCILLATIONS OF BLADE SYSTEM IN HOVERING.

Inasmuch as equations (135b) and (135c) are a set of differential equations of the second order in the two unknowns $\bar{\beta}$ and $\bar{\zeta}$, the complete solution to these equations must contain exactly four arbitrary constants, which must satisfy any given initial conditions of position and velocity. Moreover, any solution containing four arbitrary constants and satisfying equations (135b) and (135c) will be the complete solution of those two equations (if it does not violate the condition that the results remain within the order of magnitude assumed in advance). Now, from equations (99) and (108),

$$\bar{\beta}_1 = \sum_{n=1}^{4} F_n e^{p_n \psi}, \quad \bar{\zeta}_1 = \sum_{n=1}^{4} F_n' e^{p_n \psi} \tag{137}$$

where F_n is an arbitrary constant, whereas F_n' is a constant depending on F_n. Therefore, from equation (136), it follows that, to obtain a complete solution for $\bar{\beta}$ and $\bar{\zeta}$ in traveling, it is sufficient to obtain only a particular integral of the differential equations (138b) and (138c) in the unknowns $\bar{\beta}_2$ and $\bar{\zeta}_2$. In obtaining a particular integral it will be sufficiently exact to determine $\bar{\beta}_2$ and $\bar{\zeta}_2$ to only first-order small quantities, because in the final solution they must be multiplied by the first-order small quantity μ_e and will therefore yield only second-order small terms.

Thus, putting equations (136) into equations (135b) and (135c), dividing through by $\mu_e \bar{I}_H$, using the expression (equation (78)) for β_c, and rejecting all terms smaller than the first order result in the following equations:

$$-\ddot{\bar{\beta}}_2 - \dot{\bar{\beta}}_2 \frac{E}{\Omega} \frac{32}{315}\left(1 + \frac{1}{2}s_1 + \frac{3}{2}\eta_e\right) - \bar{\beta}_2 \Omega^2 = \frac{48}{315} \frac{E}{\bar{I}_H} \left(1 + \frac{1}{2}s_1\right)\left[-\bar{\beta}_1 \cos\psi + \frac{\dot{\bar{\beta}}_1}{\Omega} \sin\psi\right] \tag{138b}$$

$$\ddot{\bar{\zeta}}_2 = 0 \tag{138c}$$

A particular integral of equation (138c) is obviously

$$\bar{\zeta}_2 = 0 \tag{139}$$

A particular integral of equation (138b) can be obtained by first observing the following relations:

$$\left.\begin{aligned} &\equiv \frac{d}{dt} = \Omega \frac{d}{d\psi} \\ &\cos \psi = \frac{e^{i\psi} + e^{-i\psi}}{2} \\ &\sin \psi = -\frac{i}{2}\left(e^{i\psi} - e^{-i\psi}\right) \end{aligned}\right\} \tag{140}$$

By putting equations (137) and (140) into equation (138b), the equation for $\overline{\beta}_2$ becomes:

$$\overline{\beta}_2'' + T\overline{\beta}_2' + \overline{\beta}_2 = \frac{3}{4}T\left(1 - \frac{3}{2}\eta_e\right) \sum_{n=1}^{4} F_n\left[\left(1 + ip_n\right)e^{\left(p_n+i\right)\psi} + \left(1 - ip_n\right)e^{\left(p_n-i\right)\psi}\right] \tag{141}$$

where

$$' \equiv \frac{d}{d\psi}, \quad T \equiv \frac{E}{\Omega^2 I_H} \frac{32}{315}\left(1 + \frac{3}{2}\eta_e + \frac{1}{2}\epsilon_i\right) \doteq 0.595. \quad \text{(Cf. equation (105).)}$$

A particular integral of equation (141) will be the sum of four pairs of terms, each pair corresponding to a given p_n. Thus, for each p_n,

$$\left(\overline{\beta}_2\right)_n = A_n e^{\left(p_n+i\right)\psi} + A_n' e^{\left(p_n-i\right)\psi} \tag{142}$$

The ratios of the constants A_n and A_n' to F_n are readily obtained by putting equation (142) into equation (141). Thus, by writing (cf. equation (109))

$$p_n \equiv R_{en} + iS_{in}$$

the value of A_n is seen to be given by

$$\frac{4}{3T\left(1 - \frac{3}{2}\eta_e\right)} \frac{A_n}{F_n} = \frac{1 + ip_n}{\left(p_n + i\right)^2 + T\left(p_n + i\right) + 1}$$

$$= \frac{\left(1 - S_{in}\right) + iR_{en}}{\left[R_{en}^2 - \left(S_{in} + 1\right)^2 + TR_{en} + 1\right] + i\left(S_{in} + 1\right)\left(2R_{en} + T\right)}$$

Rationalizing the denominator yields

$$A_n = \frac{3}{4}T\left(1 - \frac{3}{2}\eta_e\right)F_n \frac{\left[\left(1 - S_{in}\right)C + R_{en}D\right] + i\left[R_{en}C - \left(1 - S_{in}\right)D\right]}{C^2 + D^2} \tag{143a}$$

where

$$C = R_{en}^2 - \left(S_{in} + 1\right)^2 + TR_{en} + 1$$

$$D = \left(S_{in} + 1\right)\left(2R_{en} + T\right)$$

Similarly,

$$A_n' = \frac{3}{4}TF_n \frac{\left[\left(1 + S_{in}\right)C' - R_{en}D'\right] - i\left[R_{en}C' + \left(1 + S_{in}\right)D'\right]}{C'^2 + D'^2} \tag{143b}$$

where

$$C' = R_{en}^2 + TR_{en} + 1 - \left(S_{in} - 1\right)^2$$

$$D' = \left(2R_{en} + T\right)\left(S_{in} - 1\right)$$

Taking the values of R_{en} and S_{in} from the hovering solution (equations (108)) and the definition (equation (109)), and observing that the numerical calculations are simplified by the fact that, for $n = 1,2$, $R_{en} = 0$ and, for $n = 3,4$, $D = D' = 0$, yields, for the numerical values of A_n/F_n and A_n'/F_n:

$$\left.\begin{aligned} &A_1/F_1 = -a - bi, \quad A_1'/F_1 = c + di, \quad A_2/F_2 = c - di, \quad A_2'/F_2 = -a + bi \\ &A_3/F_3 = e + fi, \quad A_3'/F_3 = g + hi, \quad A_4/F_4 = g - hi, \quad A_4'/F_4 = e - fi \end{aligned}\right\} \qquad (144)$$

where from equations (143a) and (143b)

$$a = 0.182, \quad b = 0.197, \quad c = 0.641, \quad d = 0.486$$

$$e = 0.0064, \quad f = 0.0348, \quad g = 0.947, \quad h = 0.126$$

The numerical values of equation (144) show that $\overline{\beta}_2$ actually is a first-order small quantity (that is, of the same order of magnitude as $\overline{\beta}_1$) and that therefore the method of solution used here is valid, inasmuch as $\overline{\beta}_2$ always appears in the form $\mu_e \overline{\beta}_2$.

The foregoing results can be mathematically interpreted for each pair of conjugate values of p_n as follows:

With

$$p_1 = -R_{e1} + S_{11}i, \quad p_2 = -R_{e1} - S_{11}i$$

for example, and with a, b, c, and d defined as in equations (144), it can be easily shown that $\left(\beta_1\right)_{1,2}$ is of the form

$$\left(\overline{\beta}_1\right)_{1,2} = e^{-R_{e1}\psi}\,\overline{\overline{\beta}}_{1,2}\left(S_{11}\psi\right) \qquad (145a)$$

where

$$\bar{\bar{\beta}}_{1,2}\left(S_{11}\psi\right) = H_1 \cos\left(S_{11}\psi\right) + H_2 \sin\left(S_{11}\psi\right)$$

and H_1 and H_2 are constants.

It can be readily proved, moreover, that the additional solution $(\beta_2)_{1,2}$ will then have the form

$$\left(\bar{\beta}_2\right)_{1,2} = e^{-R_{e1}\psi}\left[-a\bar{\bar{\beta}}_{1,2}\left(S_{11} + 1, \psi\right) + c\bar{\bar{\beta}}_{1,2}\left(S_{11} - 1, \psi\right) + b\bar{\bar{\beta}}_{1,2}\left(S_{11} + 1, \psi + \frac{\pi}{2\left(S_{11} + 1\right)}\right) + d\bar{\bar{\beta}}_{1,2}\left(S_{11} - 1, \psi + \frac{\pi}{2\left(S_{11} - 1\right)}\right)\right] \tag{145b}$$

(Cf. equations (112a) and (112b).)

Comparison of equation (145b) with equation (145a) shows that a complex value for the ratio of amplitudes indicates a difference in phase between hovering vibrations $\bar{\beta}_1$ and the additional vibrations $\bar{\beta}_2$. In fact, for each of the frequencies $\left(S_{11} + 1\right)\frac{\Omega}{2\pi}$ and $\left(S_{11} - 1\right)\frac{\Omega}{2\pi}$, $(\bar{\beta}_2)_{1,2}$ consists in two components one-quarter of a period out of phase with each other.

The physical significance of the results can be stated as follows: In regard to the frequencies, equations (142) and (108) show that one new frequency is added which is practically double the frequency of rotation Ω. Three other frequency roots appear which, however, are not significantly different from the frequencies of the roots $\sqrt{2\eta_e}\,\frac{\Omega}{2\pi}$ and $\left(1 + \eta_e\right)\frac{\Omega}{2\pi}$ of hovering.

The additional amplitudes given by equation (144) in terms of the amplitudes in hovering, and depending on initial disturbances, are small in comparison with the hovering amplitudes, as long as the value of μ_e is small. This result, in fact, is the proof that the method of integration is consistent.

The logarithmic decrements remain practically unchanged in the transition to traveling, and the solution corresponding to the frequency $\sqrt{2\eta_e}\,\frac{\Omega}{2\pi}$, which has no damping in hovering, therefore is still very sensitive against disturbances in the transition to traveling.

Case B: θ Guided, β and ζ under Kinematic Constraint

It was seen that in case A of traveling there is the same danger of lack of stability due to the absence of damping as in case A of hovering.

Case B of hovering with an appropriate kinematic condition between $\overline{\beta}$ and $\overline{\zeta}$ made it possible to obtain better damping and also frequency values of no resonance danger. Whether a device of the same kind will serve the same purpose in traveling will now be determined.

For case B of $\mu_e \neq 0$ with its kinematic constraint between $\overline{\beta}$ and $\overline{\zeta}$ it will also be advisable to avoid mutual bending moments in the constraint mechanism in the steady state of motion by a preadjustment $\beta_{pr} = (\beta)_{\zeta=0}$ or $\zeta_{pr} = (\zeta)_{\beta=0}$. (During the oscillation, however, such mutual moments, though small, can again not be avoided.) Such a preadjustment for the case of traveling must, however, be periodic, as can be seen by the following consideration. The kinematic condition may again be expressed by

$$\overline{\zeta} = \kappa\overline{\beta} \quad \text{or} \quad \zeta - \zeta_o = \kappa(\beta - \beta_o) \tag{146}$$

where κ is a constant and $\zeta_o(\psi)$ and $\beta_o(\psi)$ are periodic functions of the angle of position ψ, given by equations (85) and (87). From equation (146) it follows, then, that the preadjustment must be $\beta_{pr} = \beta_o - \frac{\zeta_o}{\kappa}$ or (equivalently) $\zeta_{pr} = \zeta_o - \kappa\beta_o$. Some means, for instance, a cam plate, will be required to enforce such a periodic condition.

The constraint (146) can be treated, as in the case of hovering, by means of a Lagrange multiplier L_m. Thus, equations (135b) and (135c) become:

$$-\ddot{\overline{\beta}}\,\overline{I}_H + \ddot{\overline{\zeta}}\,\theta_c\overline{I} - \dot{\overline{\beta}}\,\frac{E}{\Omega}\,\frac{32}{315}\left(1 + \frac{1}{2}s_1\right)\left(1 + \frac{3}{2}\eta_e + \frac{3}{2}\mu_e\,\sin\psi\right)$$

$$-\overline{\beta}\left[\Omega^2\left(\overline{I}_H + \overline{S}e\right) - \frac{48}{315}\left(1 + \frac{1}{2}s_1\right)E\mu_e\,\cos\psi\right] - L_m\kappa = 0 \tag{147b}$$

$$\ddot{\overline{\beta}}\,\theta_c\overline{I} - \ddot{\overline{\zeta}}\,\overline{I}_H - \dot{\overline{\beta}}\left[2\beta_c\Omega\overline{I}_H - \frac{E}{\Omega}\,\frac{32}{315}\left(1 + \frac{1}{2}s_1\right)\left(\frac{4}{5}\,\theta_c - \alpha_p\right)\right]$$

$$-\overline{\zeta}\Omega^2\overline{S}e + L_m = 0 \tag{147c}$$

Eliminating L_m from equations (147b) and (147c) and thus obtaining a single equation, expressing $\bar{\zeta}$ by $\bar{\beta}$ according to equation (146), setting

$$\bar{\beta} = \bar{\beta}_1 + \mu_e\bar{\beta}_2 \tag{148}$$

where $\bar{\beta}_1$ is the solution for $\bar{\beta}$ in hovering $(\mu_e = 0)$, rejecting terms smaller than the second order, and finally dividing through by $\mu_e\bar{I}_H$ result in the following differential equation for $\bar{\beta}_2$:

$$-\ddot{\bar{\beta}}_2(1 + \kappa^2) - T\dot{\bar{\beta}}_2 - \bar{\beta}_2\Omega^2 = \frac{3}{2}T\left(1 - \frac{3}{2}\eta_e\right)\left(\dot{\bar{\beta}}_1 \sin\psi - \bar{\beta}_1 \cos\psi\right) \tag{149}$$

where T is as defined in equation (105). It will be observed that equation (149) is similar to equation (138b) of case A except for the coefficient $(1 + \kappa^2)$ of $\ddot{\bar{\beta}}_2$ in equation (149) and for the fact that $\bar{\beta}_1$ must now be taken from case B of hovering instead of from case A. Thus, from equations (99) and (123),

$$\bar{\beta}_1 = \sum_{n=1}^{2} F_n e^{p_n\psi} \tag{150}$$

For a complete solution for $\bar{\beta}$ it is sufficient, as explained in the previous case, to obtain only a particular integral of equation (149). This particular integral can be obtained in the same manner as shown in case A for equation (138b). Thus,

$$\bar{\beta}_2 = A_1e^{(p_1+i)\psi} + A_1'e^{(p_1-i)\psi} + A_2e^{(p_2+i)\psi} + A_2'e^{(p_2-i)\psi} \tag{151}$$

where A must be determined in terms of F_n by substitution for $\bar{\beta}_2$ into equation (149). By use of the expressions for $\sin\psi$, $\cos\psi$, and $\bar{\beta}_1$ (equations (140) and (150)), equation (149) can be written in the form (cf. equation (141)):

$$\left(1 + \kappa^2\right)\bar{\beta}_2'' + \bar{\beta}_2'T + \bar{\beta}_2 = \frac{3}{4}T\left(1 - \frac{3}{2}\eta_e\right)\sum_{n=1}^{2}\left[\left(1 + ip_n\right)e^{(p_n+i)\psi} + \left(1 - ip_n\right)e^{(p_n-i)\psi}\right]$$

When equation (151) is put into equation (152), is is found, for

$$\kappa = 1$$

(which gave satisfactory results in regard to stability in hovering), that:

$$A_n = \frac{3}{4} T\left(1 - \frac{3}{2}\eta_e\right)F_n \frac{\left[\left(1 - S_{in}\right)C + R_{en}D\right] + i\left[R_{en}C - \left(1 - S_{in}\right)D\right]}{C^2 + D^2} \qquad (153a)$$

where

$$C = 2R_{en}^{\ 2} + TR_{en} + 1 - 2\left(S_{in} + 1\right)^2$$

$$D = \left(4R_{en} + T\right)\left(S_{in} + 1\right)$$

$$A_n' = \frac{3}{4} T\left(1 - \frac{3}{2}\eta_e\right)F_n \frac{\left[\left(1 + S_{in}\right)C' - R_{en}D'\right] - i\left[R_{en}C' + \left(1 + S_{in}\right)D'\right]}{C'^2 + D'^2} \qquad (153b)$$

where

$$C' = 2R_{en}^{\ 2} + TR_{en} + 1 - 2\left(S_{in} - 1\right)^2$$

$$D' = \left(4R_{en} + T\right)\left(S_{in} - 1\right)$$

With the values of R_{en} and S_{in} given by the solution (124) for hovering, equations (153a) and (153b) yield the following results:

$$\left.\begin{aligned} A_1 &= F_1(-a + bi), \quad A_1' = F_1(c + di) \\ A_2 &= F_2(c - di), \quad A_2' = F_2(-a - bi) \end{aligned}\right\} \qquad (154)$$

where (from equations (153a) and (153b))

$$a = 0.0175, \quad b = 0.01245$$

$$c = 0.860, \quad d = 0.0678$$

These results can be physically interpreted in a manner similar to that in case A. In the transition to traveling, two new natural frequencies are added to the frequency $0.775 \frac{\Omega}{2\pi}$ for hovering. These frequencies are $(1 + 0.775) \frac{\Omega}{2\pi}$ and $(1 - 0.775) \frac{\Omega}{2\pi}$ or $1.775 \frac{\Omega}{2\pi}$ and $0.225 \frac{\Omega}{2\pi}$. Neither of these frequencies appears to present any particular danger of resonance, which might otherwise be caused by exciting disturbances having a frequency connected with the frequency of rotation, that is, a multiple of $\frac{\Omega}{2\pi}$. The damping decrement, moreover, remains practically the same in low-speed traveling as in hovering (log decrement = 1.222), and this means that the rotor system will remain quite stable in the transition to traveling.

From the fact that the values a, b, c, and d in equations (153a) and (153b) must be applied with the small factor μ_e in accordance with equation (148), it is seen that the additional amplitudes in the transition to traveling are again small in comparison with the amplitudes in the state of hovering. Thus, it appears that case B (of constraint β, ζ) remains satisfactory in the transition from hovering to traveling.

Case C: θ, β, and ζ under Kinematic Constraint

This case has (as in hovering) the practical advantage over the others in that the blade angle need not be guided but will automatically adjust itself to the proper value to support the weight of the helicopter. The same constraints for β and ζ as in hovering will be assumed here, namely

$$\left.\begin{aligned} \overline{\beta} &= \lambda\overline{\theta} \\ \overline{\zeta} &= \mu\overline{\theta} \end{aligned}\right\} \qquad (155)$$

where λ and μ are constants.

These constraint conditions can be realized practically, for a correct pitch-angle function θ_o, by means of a variable preadjustment of the angles β and ζ. This preadjustment is given by

$$\left.\begin{aligned}\beta_{pr} &= (\beta)_{\theta=0} = \beta_o - \lambda\theta_o \\ \zeta_{pr} &= (\zeta)_{\theta=0} = \zeta_o - \mu\theta_o\end{aligned}\right\} \quad (156)$$

These preadjustment values can be found explicitly by substituting the expressions for θ_o, β_o, and ζ_o given by equations (84) with (96), (85), and (87), respectively, into equation (156). They will appear in the form

$$\beta_{pr} = f_1 + \mu_e\left(f_2 + f_3 \sin\psi + f_4 \cos\psi\right)$$

$$\zeta_{pr} = d_1 + \mu_e\left(d_2 + d_3 \sin\psi + d_4 \cos\psi\right)$$

where f and d are constants found from equations (84), (96), (85), and (87).

Thus β_{pr} and ζ_{pr} will be fairly simple functions of the angular position ψ of a blade, and can be materialized by, for example, a cam plate.

The stability of the rotor system with the constraints (155) can be considered in the same manner as case C in hovering. Thus, using Lagrange multipliers L_{m1} and L_{m2} and denoting by $D_1(\overline{\theta}, \overline{\beta}, \overline{\zeta}, \psi)$, D_2, and D_3 the left-hand sides of equations (135a), (135b), and (135c), respectively, yields, for these equations:

$$D_1(\overline{\theta}, \overline{\beta}, \overline{\zeta}, \psi) + L_{m1}\frac{\partial\phi_1}{\partial\overline{\theta}} + L_{m2}\frac{\partial\phi_2}{\partial\overline{\theta}} = 0 \quad (157a)$$

$$D_2(\overline{\theta}, \overline{\beta}, \overline{\zeta}, \psi) + L_{m1}\frac{\partial\phi_1}{\partial\overline{\beta}} + L_{m2}\frac{\partial\phi_2}{\partial\overline{\beta}} = 0 \quad (157b)$$

$$D_3(\overline{\theta}, \overline{\beta}, \overline{\zeta}, \psi) + L_{m1}\frac{\partial\phi_1}{\partial\overline{\zeta}} + L_{m2}\frac{\partial\phi_2}{\partial\overline{\zeta}} = 0 \quad (157c)$$

where ϕ_1 and ϕ_2 are the functions given by equations (155). Substituting the values of the derivatives of ϕ_1 and ϕ_2, solving for L_{m1} and L_{m2} by means of equations (157b) and (157c), and then substituting into equation (157a) yields the following single equation:

$$D_1 + \lambda D_2 + \mu D_3 = 0 \tag{158}$$

By taking the expressions for D_1, D_2, and D_3 from the left-hand sides of equations (135a) to (135c), eliminating $\overline{\beta}$ and $\overline{\zeta}$ from equation (158) by means of equations (155), setting

$$\overline{\theta} = \overline{\theta}_1 + \mu_e \overline{\theta}_2 \tag{159}$$

where $\overline{\theta} = \overline{\theta}_1$ is the solution of equation (158) in hovering, rejecting all terms in equation (158) smaller than the second order (remembering that μ_e is assumed first order small), and finally dividing through by $\mu_e \overline{\overline{I}}_H$ the following differential equation is obtained for $\overline{\theta}_2$:

$$\left.\begin{aligned} \ddot{\overline{\theta}}_2 a_2 + \dot{\overline{\theta}}_2 a_1 \Omega + \overline{\theta}_2 a_o \Omega^2 &= \frac{E_1}{\overline{\overline{I}}_H}\left[\left(-\frac{\dot{\overline{\theta}}_1}{\Omega} b_1 + \overline{\theta}_1 b_o\right) \sin\psi + \overline{\theta}_1 b_o' \cos\psi\right] \\ E_1 &= \frac{E\left(1 + s_i\right)}{12} \frac{c_i}{R} \end{aligned}\right\} \tag{160a}$$

where

$$\left.\begin{aligned} a_2 &= \frac{\overline{\overline{I}}}{\overline{\overline{I}}_H} - \left(\lambda^2 + \mu^2\right) \\ a_1 &= \frac{E_1}{\Omega^2 \overline{\overline{I}}_H}\left[\frac{4}{35}\frac{c_i}{R}\left(1 + \frac{1}{2}s_i + \frac{7}{2}\eta_e\right) + \lambda f\left(1 + 2\eta_e\right) - \lambda^2 12 \frac{R}{c_i}\frac{32}{315}\left(1 - \frac{1}{2}s_i + \frac{3}{2}\eta_e\right)\right] \\ a_o &= -\frac{E_1 f}{\overline{\overline{I}}_H \Omega^2} + \frac{\overline{\overline{I}}}{\overline{\overline{I}}_H} + \frac{32}{315}\lambda \frac{E}{\overline{\overline{I}}_H \Omega^2}\left(1 + \frac{1}{2}s_i\right) - \lambda^2 \\ b_1 &= \frac{14}{35}\frac{c_i}{R} + 2\lambda f - \lambda^2 12 \frac{R}{c_i}\left(1 - \frac{1}{2}s_i\right)\frac{48}{315} \\ b_o &= 4f - \frac{32}{315}\left(1 - \frac{1}{2}s_i\right) 36 \frac{R}{c_i}\lambda \\ b_o' &= 2f\lambda - \lambda^2 \frac{48}{315}\left(1 - \frac{1}{2}s_i\right) 12 \frac{R}{c_i} \end{aligned}\right\} \tag{160b}$$

The expression for $\overline{\theta}_1$ is

$$\overline{\theta}_1 = \sum_{n=1}^{2} B_n e^{p_n \psi} \tag{161}$$

where p_n is given by equation (133) and B_n is an arbitrary constant.

As already explained in cases A and B, it is sufficient in solving equation (160a) to obtain only a particular integral of this differential equation, and this integral can be obtained in the same manner as in cases A and B. Thus, by using equation (161), noting that $\psi = \Omega t$, and also noting the expressions (140) for $\sin \psi$ and $\cos \psi$ as exponential functions, equation (160a) may be written in the form $\left(\text{where } ' \equiv \frac{d}{d\psi}\right)$:

$$a_2\overline{\theta}_2'' + a_1\overline{\theta}_2' + a_o\overline{\theta}_2 = \sum_{n=1}^{2} \frac{E_1}{2\Omega^2\overline{I}_H}\left\{\left[i\left(p_n b_1 - b_o\right) + b_o'\right]e^{\left(p_n+i\right)\psi} + \left[i\left(-p_n b_1 + b_o\right) + b_o'\right]e^{\left(p_n-i\right)\psi}\right\} \tag{162}$$

As in cases A and B, set

$$\overline{\theta}_2 = \sum_{n=1}^{2}\left[A_n e^{\left(p_n+i\right)\psi} + A_n' e^{\left(p_n-i\right)\psi}\right] \tag{163}$$

where A_n and A_n' must be determined in terms of B_n by substitution of equation (163) into equation (162). Thus,

$$A_n = \frac{E_1}{2\Omega^2\overline{I}_H} B_n \frac{b_o' + i\left(p_n b_1 - b_o\right)}{a_2\left(p_n + i\right)^2 + \left(p_n + i\right)a_1 + a_o} \tag{164}$$

By putting $p_n \equiv R_{en} + iS_{in}$, the following expression for A_n can be readily derived from equation (164):

$$A_n = B_n \frac{E_1}{2\Omega^2 I_H} \frac{\left[\left(b_o' - b_1 S_{in}\right)C + D\left(b_1 R_{en} - b_o\right)\right] + i\left[\left(b_1 R_{en} - b_o\right)C + D\left(S_{in} b_1 - b_o'\right)\right]}{C^2 + D^2} \tag{165a}$$

where

$$C = a_2 R_{en}{}^2 + a_1 R_{en} + a_o - a_2\left(S_{in} + 1\right)^2$$

$$D = \left(S_{in} + 1\right)\left(2a_2 + a_1\right)$$

Similarly,

$$A_n' = B_n \frac{E_1}{2\Omega^2 I_H} \frac{\left[\left(b_o' + b_1 S_{in}\right)C' - D'\left(b_1 R_{en} - b_o\right)\right] - i\left[\left(b_1 R_{en} - b_o\right)C' + D'\left(b_o' + b_1 S_{in}\right)\right]}{C'^2 + D'^2} \tag{165b}$$

where

$$C' = a_2 R_{en}{}^2 + a_1 R_{en} + a_o - a_2\left(S_{in} - 1\right)^2$$

$$D' = \left(S_{in} - 1\right)\left(2a_2 + a_1\right)$$

For the special case that was treated in case C (hovering), which gave satisfactory results in regard to stability, the values of λ and μ were taken to be

$$\lambda = 0.7, \quad \mu = 1$$

With these values of λ and μ and with the numerical data (summarized in appendix G) which have been consistently used in this text, the values of the constants appearing in equations (165a) and (165b) are, in accordance with equations (160b), found to be:

$$a_2 = -1.49, \quad a_1 = -0.280, \quad a_o = -0.108$$

$$b_1 = -4.55, \quad b_o = -13.22, \quad b_o' = -4.63, \quad \frac{E_1}{2\bar{I}_H\Omega^2} = 0.0417$$

From equation (134), p_n is given by:

$$p_n = -0.0882 \pm 0.4031$$

Hence $R_{e1} = R_{e2} = -0.0882$, $S_{i1} = 0.403$, and $S_{i2} = -0.403$.

Substitution into equations (165a) and (165b) leads then to the following results:

$$\left.\begin{aligned} A_1 &= B_1(-a + bi), \quad A_1' = B_1(-c + di) \\ A_2 &= B_2(-c - di), \quad A_2' = B_2(-a - bi) \end{aligned}\right\} \qquad (166)$$

where

$$a = 0.102, \quad b = 0.0372$$

$$c = 0.308, \quad d = 0.0695$$

As explained in cases A and B, the fact that the amplitude factors of $\overline{\theta}_2\mu_e$ are of the same order of magnitude as the amplitudes in hovering shows that the method employed here of obtaining a solution in traveling is valid.

The results can be physically interpreted again in a manner quite analogous to that in case A and in case B. Two new natural frequencies appear in addition to the hovering frequency $0.403 \frac{\Omega}{2\pi}$, namely, $(1 + 0.403) \frac{\Omega}{2\pi}$ and $(1 - 0.403) \frac{\Omega}{2\pi}$, neither of which appears to present any particular danger of resonance. The damping decrement remains essentially the same as in hovering, and this shows that the system in case C will be equally stable in low-speed travel and in hovering. The natural modes in this case can be expressed analogously to the relations (equations (145a) and (145b)) of case A (traveling).

CONCLUSIONS

Following the geometry, the statics, and the dynamics of the motion of a hinged blade system, the parameters of pitch angle θ, flapping angle β, and lagging angle ζ in the hovering and the traveling (that is, forward, backward, and sideways) steady state of flight were determined. The geometric part of this problem, particularly for the case of traveling, consisted in the determination of the angle between the direction of flight and the zero-lift lines of the blade sections, which are rotating on a conical surface the axis of which is tilted toward the direction of flight. This involved the determination of the components of the relative inflow velocity both in the planes of the cross sections and in the direction of the blade axis.

The influence of the induced inflow on the total inflow velocity was calculated only in regard to the direction of this total velocity, and the always very small induced change of magnitude of this resultant velocity was neglected. The three-dimensional Kutta-Joukowski theorem was then applied to the calculation of the lift-force vectors and their periodic deviations from the planes of the cross sections. In this way the velocity components both along a blade axis and in a plane perpendicular to the axis were taken into account in the vector product $\overline{\Gamma} \times \overline{V}$, whereas the value of the circulation Γ was determined by the transverse velocity component only. In all previous publications the radial (blade-axis) component of the velocity had been neglected.

The pitch angle θ was expressed for hovering as a constant depending on the total weight of the helicopter and for traveling as a fractional function in terms of the first and second powers of the speed ratio μ_e, of the sines and cosines of the circumferential angles ψ and 2ψ of blade position, and of the acceleration of the flapping angle β.

Although the effect of the induced downwash on the lift forces has not been treated explicitly, this effect can, in accordance with the assumptions made in the present analysis, be considered to be contained implicitly. The induced angle α_i, in fact, as given by equation (15) can be written in the form $\alpha_i = \frac{1}{5}\alpha$. Therefore, if substitution of $(\alpha - \alpha_i)$ for α is made in equation (7), giving the magnitude Γ of the circulation, it follows that the effect of the induced angle on the lift loads will simply diminish these loads by a constant factor, 4/5. The numerical results given in this analysis will then actually remain unchanged if one assumes the gross weight W of the helicopter to be four-fifths of the value originally assumed.

In the derivation of the steady-state values of the flapping and lagging angles based on the equilibrium of moments about the hinges, the inertia moments of the angle accelerations $\ddot{\theta}$, $\ddot{\beta}$, and $\ddot{\zeta}$ also had to be taken into account for the case $\mu_e \neq 0$. The damping moments proportional to $\dot{\theta}$, $\dot{\beta}$, and $\dot{\zeta}$ were also considered. This required the integration of differential equations in order to determine in the section STEADY STATE IN HOVERING AND IN LOW-SPEED TRAVELING the steady-state values of the blade-position angles.

The forces and moments due to small oscillatory displacements, velocities, and accelerations, necessary for the analysis of small oscillations about a state of steady motion, were determined in the section INERTIA FORCES AND MOMENTS AND EQUATIONS OF OSCILLATION. The inertial moments especially were expressed by means of the moment of momentum vector, and the Coriolis forces were obtained by the use of a rotating reference system. In this way, the complete system of the equations of small oscillations about a state of steady motion was established (equations (74), (75), and (76)). In the hovering state this system of differential equations has constant coefficients but in the traveling state the coefficients have periodic additional terms.

The integration was performed first in general terms, with results for frequencies, logarithmic decrements, and amplitude ratios given by simple formulas. These results were then applied for the following set of plausible numerical design data. Four different cases in hovering and three corresponding cases in traveling have been discussed:

Case A. Pitch angle θ fixed, flapping angle β and lagging angle ζ free.

Case A_1. Pitch angle θ fixed, flapping angle β free, lagging angle ζ constrained by fluid friction (dashpot).

Case B. Pitch angle θ fixed, β and ζ connected by a frictionless kinematic constraint.

Case C. θ, β, and ζ externally free but internally connected by two (frictionless) kinematic conditions.

The numerical data for these cases were assumed as follows:

Total weight $W = 4000$ pounds

Tip radius $R = 25$ feet

Rotational speed $\Omega = 20$ radians per second $= \frac{20}{2\pi} = (3.19$ cycles per second$)$

Chord $c = c_i\left(\frac{1 - s}{1 - s_i}\right)^{1/2}$, $c_i = \frac{25}{6}$ feet

Number of blades $n = 4$

Inner cross section of blade at $\frac{r_i}{R} = s_i = 0.2$

Hinge eccentricity $\frac{e}{R} = \eta_e = 0.05$

Thickness ratio of cross section of blade $\frac{t_b}{c} = \text{Constant} = \frac{1}{8}$

Average density ratio of air to blade material $\frac{\rho}{\sigma} = 0.0025$

Parasite drag angle $\alpha_p = 0.02$

In case A it was found that the oscillatory motion of the rotor system can be considered as consisting approximately of oscillations of only the flapping angle β and of independent oscillations of only the lagging angle ζ. The natural frequency of the flapping oscillations is $q_{3,4} = \frac{\Omega}{2\pi}\left(1 + \eta_e\right) = 3.34$ cycles per second. Although this frequency is quite close to the rotational frequency $\frac{\Omega}{2\pi} = 3.19$ cycles per second, there will be little danger of resonance because of the high logarithmic decrement, namely $\pi T/(1 + \eta_e) = 1.78$, associated with the flapping oscillation.[1] The lagging oscillations will have the low natural frequency $q_{1,2} = \frac{\Omega}{2\pi}\sqrt{2\eta_e} = 1.005$ cycles per second but will be practically undamped, and therefore sensitive to disturbances. Inasmuch as the flapping and lagging oscillations are practically independent of

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[1]For definition of T, see SYMBOLS.

each other, it follows as plausible that any phase difference between lagging and flapping which might arise on account of a separation of the flapping hinge from the lagging hinge would be quite small. It can, moreover, be observed from the formulas that both the natural frequencies and the logarithmic decrement are only slightly affected by the location η_e of the hinge, especially in flapping.

In case A_1, the stability of the lagging oscillations of case A is very much improved by the introduction of fluid friction, producing a damping moment $k\bar{I}_H\Omega\zeta$ at the root of the blade; k is a constant of relative energy dissipation which may be adjusted to suit requirements of operation. For the case of $k = 0.1$ it was found that the new natural frequencies will be practically the same as those in case A, whereas the new logarithmic decrement corresponding to the flapping oscillations of case A remains practically unchanged. The logarithmic decrement corresponding to the lower natural frequency, however, is now no longer zero, but fairly high. The numerical results for the natural frequencies and logarithmic decrements were:

$$q_{1,2} = 1.01 \text{ cycles per second, } \log\left(\frac{A_n}{A_{n+1}}\right)_{1,2} = 1.029$$

$$q_{3,4} = 3.30 \text{ cycles per second, } \log\left(\frac{A_n}{A_{n+1}}\right)_{3,4} = 1.91$$

For any other values of k and any other numerical data, the results can be obtained by determining the complex frequencies p from either the biquadratic equation (113) or the approximate general solution (114). The logarithmic decrements and natural frequencies can then be determined directly from equations (109), (110), and (111).

Because of the low natural frequency, with the consequently small restoring forces, of the independent lagging oscillations, the friction damping may prove insufficient for stability. As will be seen in the following cases, however, the damping can be successfully enforced with an appropriate kinematic constraint between, for example, lagging and flapping.

Case B was worked out in detail for a kinematic constraint (geometric condition) of the form

$$\bar{\zeta} = \kappa\bar{\beta}$$

where κ is a constant for small oscillations. In cases B and C the method of Lagrange multipliers has been used to satisfy such geometric conditions. These multipliers also have a physical significance, for they give the forces acting in the constraint connections.

With such a constraint there will be only one natural frequency. (See equation (123).) For $\kappa = 1$ and for the foregoing numerical data, the natural frequency of oscillation was found to be 2.47 cycles per second, with a logarithmic decrement of 1.282. From the point of view of stability and avoidance of resonance this case appears quite satisfactory. For any other data, but for the same form of constraint condition, the frequency and the damping can be determined either by solving the quadratic equation (121) or by substituting in the approximate general solution to this equation (equation (122)).

It may be noted that in cases A, A_1, and B the aerodynamic loads had practically no influence on the natural frequencies of oscillation. As may have been expected, moreover, the natural frequencies are only little affected by the damping terms.

Case C has the advantage that here the pitch angle β is automatically controlled. The two constraint conditions were assumed to be of the form

$$\overline{\beta} = \lambda\overline{\theta}, \quad \overline{\zeta} = \mu\overline{\theta}$$

where λ and μ are constants. This condition could be realized by a preadjustment of flapping and lagging angles to the following values: $\beta_{pr} = \beta_c - \lambda\theta_c$, $\zeta_{pr} = \zeta_c - \mu\theta_c$. The method of Lagrange multipliers led to the quadratic equation (131) for the complex frequency p. This equation can be used to determine p for any given data. For the preceding data and for

$$\lambda = 0.7, \quad \mu = 1.0$$

the natural frequency of oscillation was found to be 1.287 cycles per second, with a logarithmic decrement of 1.370. These results appeared quite satisfactory in regard to stability.

It may be remarked that in the formulas of all the cases treated, the drag terms (induced plus parasite) had only a small influence on the stability characteristics of the rotor system. This shows the lack of necessity of determining the aerodynamic drag any more exactly than by the simplifying assumptions made in this analysis.

The differential equations of oscillation for any finite constant value of the speed ratio μ_e remain linear, as in hovering, but they now have variable coefficients (periodic in ψ, i.e., in Ωt) instead of constant. In order to investigate the stability conditions in the transition from hovering to traveling, the speed ratio μ_e was assumed to be a first-order small quantity. Solutions to the differential equations could then be obtained by using the solutions in hovering as a first approximation and then making the corrections in accordance with the consistent procedure used in this report, that is, neglecting terms smaller than the second order. The correction in each case consisted in the addition of a particular integral of a non-homogeneous linear differential equation with constant coefficients. These additions were, in all cases treated, found to be small in comparison with the corresponding solution in hovering. (A general development in powers of larger μ_e but < 1 might also prove convergent.) Cases A, B, and C of hovering were by this method treated for low-speed traveling, with the following results.

In case A, the solution was found to be of the form

$$\bar{\theta} = 0, \quad \bar{\beta} = \bar{\beta}_1 + \mu_e \bar{\beta}_2, \quad \bar{\zeta} = \bar{\zeta}_1$$

where $\bar{\beta}_1$ and $\bar{\zeta}_1$ were the solutions for case A in hovering. In accordance with equations (99) and (108), $\bar{\beta}_1$ had the form

$$\bar{\beta}_1 = \sum_{n=1}^{4} F_n e^{p_n \psi}$$

where F_n was an arbitrary constant. The expression for $\bar{\beta}_2$ was then found to have the form

$$\bar{\beta}_2 = \sum_{n=1}^{4} \left[A_n e^{(p_n+i)\psi} + A_n' e^{(p_n-i)\psi} \right]$$

where A_n and A_n' are constants which depend on F_n, given in general terms by equations (143a) and (143b) and for the numerical example by equations (144). The physical significance of these results is that four new natural frequencies appear, obtained by adding and subtracting $\Omega/2\pi$ to and from each of the two natural frequencies $\sqrt{2\eta_e}\,\frac{\Omega}{2\pi}$ and $(1 + \eta_e)\,\frac{\Omega}{2\pi}$ in hovering. It can be seen that three of the new frequencies will not be significantly different from the hovering frequencies, but that one new natural frequency appears which is approximately double the rotational frequency $\Omega/2\pi$. The expression for $\bar{\beta}_2$ also shows that the logarithmic decrements remain unchanged in the transition from hovering to traveling. Therefore the solution which indicated no damping in hovering corresponding to the frequency $\sqrt{2\eta_e}\,\frac{\Omega}{2\pi}$ still indicates a danger of instability in low-speed traveling.

In case B, the constraint condition was again assumed to be of the form

$$\bar{\zeta} = \kappa\bar{\beta}$$

As in the previous case, the solution for $\bar{\beta}$ was of the form

$$\bar{\beta} = \bar{\beta}_1 + \mu_e\bar{\beta}_2$$

where $\bar{\beta}_1$ is the solution for case B in hovering. From equations (99) and (122) $\bar{\beta}_1$ is of the form

$$\bar{\beta}_1 = \sum_{n=1}^{2} F_n e^{p_n\psi}$$

The angle $\bar{\beta}_2$ was then found to be of the form

$$\bar{\beta}_2 = \sum_{n=1}^{2}\left[A_n e^{(p_n+i)\psi} + A_n' e^{(p_n-i)\psi}\right]$$

where A_n and A_n' are functions of F_n given by equations (153a) and (153b) in general terms. For the numerical example, A_n/F_n and A_n'/F_n are given by equations (154). The expression for $\bar{\beta}_2$ shows that two new natural frequencies are added to the frequency $0.775\,\frac{\Omega}{2\pi}$ for hovering. These frequencies are $1.775\,\frac{\Omega}{2\pi}$ and $0.225\,\frac{\Omega}{2\pi}$, neither of which appears to present any particular danger of resonance. As in case A, the damping remains the same as in hovering; that is, the logarithmic decrement remains 1.282. This indicates satisfactory stability in the transition to traveling.

In case C, the two kinematic constraints were also assumed as in hovering, namely,

$$\bar{\beta} = \lambda\bar{\theta}, \quad \bar{\zeta} = \mu\bar{\theta}$$

Proceeding as in the two previous cases, the solution for $\bar{\theta}$ was assumed in the form

$$\bar{\theta} = \bar{\theta}_1 + \mu_e\bar{\theta}_2$$

where $\bar{\theta}_1$ was the solution for case C in hovering, and was given by

$$\bar{\theta}_1 = \sum_{n=1}^{2} B_n e^{p_n\psi}$$

The constant B_n is arbitrary and p_n is given by equation (131) in general and by equation (134) for the numerical example calculated. Then $\bar{\theta}_2$ was seen to have the form

$$\bar{\theta}_2 = \sum_{n=1}^{2}\left[A_n e^{(p_n+i)\psi} + A_n' e^{(p_n-i)\psi}\right]$$

where A_n and A_n' are given as functions of B_n by equation (165a) and (165b) generally, and by equation (166) for the numerical example. The expression for $\bar{\theta}_2$ again shows that two new natural frequencies appear in the transition from hovering to traveling and that these can be obtained by adding (algebraically) $\pm \frac{\Omega}{2\pi}$ to the hovering natural frequency. For the numerical example treated $(\lambda = 0.7, \quad \mu = 1)$, the hovering frequency was $0.403 \frac{\Omega}{2\pi}$, and the two new frequencies in low-speed traveling are therefore $1.403 \frac{\Omega}{2\pi}$ and $0.597 \frac{\Omega}{2\pi}$ cycles per second. The logarithmic decrement, as in cases A and B, remains the same as in hovering, namely 1.370 for case C. Thus the system appears in this case to remain quite stable in the transition from hovering to traveling.

It will be observed that the effect on the hovering oscillations of the transition to traveling is essentially the same for all the cases treated.

In regard to the reaction of the blade system on the fuselage and to the strength of the hinge structure, the question of external and internal forces arising from the effect of external or internal constraints is of interest. The external constraint moment caused by the fixation of the pitch angle was given by the static and dynamic equations of the θ component, whereas the internal constraint moments acting on the linkages between the hinge axes were given by the Lagrange multipliers of the derivatives of the kinematic conditions. Simple preadjustments were indicated between the angles θ_o, β_o, and ζ_o, by which the steady-state internal moments can be eliminated.

In this paper the problems of vertical climbing, inclined travel direction, large speed ratios, disturbing external forces, and elastic vibrations have not yet been discussed. These problems can be solved, however, by the basic geometric, static, and dynamic procedure presented and by the introduction of the appropriate inflow velocities and inertia forces.

Polytechnic Institute of Brooklyn
Brooklyn, N. Y., May 5, 1946

APPENDIX A

VELOCITY COMPONENTS IN PLANE OF CROSS SECTION OF BLADE

By definition of the blade angle or pitch angle θ, the line c_1 of zero lift makes an angle of θ with the line h, where h is perpendicular to the z-axis and to the blade center line r. Let l, m, and n denote the direction cosines of any line with respect to the x-, y-, and z-axes, respectively. Then, from figure 1, the direction cosines of r are:

$$\left.\begin{aligned} l_r &= \cos\beta\cos\zeta \\ m_r &= -\cos\beta\sin\zeta \\ n_r &= \sin\beta \end{aligned}\right\} \tag{A1}$$

Moreover, the direction cosines of line h, which lies in the xy-plane and is perpendicular to the projection of r on that plane, are:

$$\left.\begin{aligned} l_h &= -\sin\zeta \\ m_h &= -\cos\zeta \\ n_h &= 0 \end{aligned}\right\} \tag{A2}$$

The direction cosines of c_1 can now be determined as follows: Since c_1 is perpendicular to r, it follows from equation (A1) that

$$l_{c_1}\cos\beta\cos\zeta - m_{c_1}\cos\beta\sin\zeta + n_{c_1}\sin\beta = 0 \tag{A3}$$

Also, since the angle between c_1 and h is θ, it follows from equation (A2) that

$$\cos\theta = -l_{c_1}\sin\zeta - m_{c_1}\cos\zeta \tag{A4}$$

Moreover,

$$l_{c_1}^{\ 2} + m_{c_1}^{\ 2} + n_{c_1}^{\ 2} = 1 \tag{A5}$$

Equations (A3), (A4), and (A5) can be solved for the three unknowns l_{c_1}, m_{c_1}, and n_{c_1}. For example, m_{c_1} and n_{c_1} can both be put in terms of l_{c_1} by means of equations (A4) and (A3). Substitution into equation (A5) then gives a quadratic in l_{c_1}. The results are:

$$\left.\begin{aligned} l_{c_1} &= -\cos\theta\sin\zeta \pm \sin\beta\sin\theta\cos\zeta \\ m_{c_1} &= -\cos\zeta\cos\theta \mp \sin\beta\sin\theta\sin\zeta \\ n_{c_1} &= \mp\cos\beta\sin\theta \end{aligned}\right\} \tag{A6}$$

The lower alternative signs in parentheses, which really mean replacing θ by $-\theta$, must be rejected, because, as may be verified later (cf. equation (10a) for L_z'), they would give negative lift in hovering. To second-order small quantities, assuming in addition to relations (4) that θ is first order small, that is

$$\theta \ll 1$$

equation (A6) can be written as:

$$\left.\begin{aligned} l_{c_1} &= -\zeta + \beta\theta \\ m_{c_1} &= -\left(1 - \frac{\zeta^2 + \theta^2}{2}\right) \\ n_{c_1} &= -\theta \end{aligned}\right\} \tag{A7}$$

The components V_{c_1} and V_n in the direction of c_1 and in the direction of n normal to c_1 and r will be:

$$\left.\begin{aligned} V_{c_1} &= V_x l_{c_1} + V_y m_{c_1} + V_z n_{c_1} \\ V_n &= V_x l_n + V_y m_n + V_z n_n \end{aligned}\right\} \qquad \text{(A8)}$$

All quantities in equation (A8) have already been obtained except the direction cosines of n. These can be obtained by considering n as the vector product of unit vectors $\bar{c}_1/c_1$ and $\bar{r}/r$, inasmuch as n is perpendicular to both c_1 and r. Hence

$$\left.\begin{aligned} l_n &= m_{c_1} n_r - m_r n_{c_1} \\ m_n &= n_{c_1} l_r - n_r l_{c_1} \\ n_n &= l_{c_1} m_r - l_r m_{c_1} \end{aligned}\right\} \qquad \text{(A9)}$$

From equations (A1), (A7), and (A9), the direction cosines of n are found to be:

$$\left.\begin{aligned} l_n &= -\beta - \zeta\theta \\ m_n &= -\theta + \beta\zeta \\ n_n &= 1 - \frac{\beta^2 + \theta^2}{2} \end{aligned}\right\} \qquad \text{(A10)}$$

By putting equations (5), (A7), and (A10) into equation (A8), the expressions for V_{c_1} and V_n given by equations (8a) and (8b) in the text are obtained.

APPENDIX B

DETERMINATION OF LIFT COMPONENTS

If it is assumed that the circulation vector $\overline{\Gamma}$ lies along the direction r of the centerline of the blade, the vector product $\overline{\Gamma} \times \overline{V}$ (equation (6)) can be written as follows:

$$\overline{L}' = \rho\overline{\Gamma}\left[i\left(V_y n_r - V_z m_r\right) + j\left(V_z l_r - V_x n_r\right) + k\left(V_x m_r - V_y l_r\right)\right] \qquad \text{(B1)}$$

Therefore, from equations (5) and (A1), the lift components per unit length will be:

$$\left.\begin{aligned} L_x' &= -\rho\Gamma\Omega r\left[(1 + \eta + \mu \sin \psi)\beta + \mu\gamma\zeta\right] \\ L_y' &= -\rho\Gamma\Omega r\left[\mu(\gamma - \beta \cos \psi) - \beta\zeta\right] \\ L_z' &= \rho\Gamma\Omega r\left[1 + \eta - \beta^2 + \mu\zeta \cos \psi + \mu\left(1 - \frac{\gamma^2 + \beta^2 + \zeta^2}{2}\right)\sin \psi\right] \end{aligned}\right\} \qquad \text{(B2)}$$

By substituting the expression (9b) for the magnitude of Γ, equations (10a) of the text are obtained.

APPENDIX C

EXPRESSIONS FOR HINGE MOMENTS

The Moment M_{x1}

The first term in equation (22b) for the moment M_{x1} can be expressed by the use of equation (8a) for V_{c_1} and equation (18) for c. Thus,

$$\frac{\rho}{2}\int_{r_1}^{R} dr\, C_{M_{ac}}\, c^2 V_{c_1}{}^2 = \frac{\rho}{2}\,\Omega^2 R^3 c_1{}^2\, C_{M_{ac}} \int_{s_1}^{1} ds\, \frac{1 - s}{1 - s_1}\left(\frac{V_{c_1}}{\Omega R}\right)^2 \qquad \text{(C1)}$$

where

$$\left(\frac{V_{c_1}}{\Omega R}\right)^2 = s^2 + 2s\left(\mu_e \sin\psi + \eta_e + \mu_e \zeta \cos\psi\right)$$
$$+ \frac{1}{2}\mu_e{}^2(1 - \cos 2\psi + 2\zeta \sin 2\psi) + 2\mu_e \eta_e \sin\psi \qquad \text{(C2)}$$

to first-order small terms.

Putting equation (C2) into equation (C1) and integrating gives

$$\int_{s_1}^{1} ds\, \frac{1 - s}{1 - s_1}\left(\frac{V_{c_1}}{\Omega R}\right)^2 = \frac{1 + s_1 + s_1{}^2}{3} + (1 + s_1)\left(\mu_e \sin\psi + \eta_e + \mu_e\zeta \cos\psi\right)$$
$$+ \frac{1}{2}\mu_e{}^2(1 - \cos 2\psi + 2\zeta \sin 2\psi) + 2\mu_e\eta_e \sin\psi - \frac{(1 + s_1)(1 + s_1{}^2)}{4}$$
$$- \frac{2}{3}\left(1 + s_1 + s_1{}^2\right)\left(\mu_e \sin\psi + \eta_e + \mu_e\zeta \cos\psi\right)$$
$$- \frac{1 + s_1}{4}\left[\mu_e{}^2(1 - \cos 2\psi + 2\zeta \sin 2\psi) + 4\mu_e\eta_e \sin\psi\right]$$
$$= \frac{1}{12}\left[1 + s_1 + 4\eta_e + 3\mu_e{}^2(1 - s_1) + 4\mu_e(1 + s_1 + 3\eta_e)\sin\psi\right.$$
$$\left. + 4\mu_e\zeta \cos\psi + 6\mu_e{}^2\zeta \sin 2\psi - 3\mu_e{}^2(1 - s_1)\cos 2\psi\right] \qquad \text{(C3)}$$

when higher powers of s_i than the first and products of s_i and smaller quantities are neglected.

From equation (10b), the second term in equation (22b), to first-order small quantities, becomes:

$$f\int_{r_i}^{R} L_z' \, c \, dr = f\pi\rho\Omega^2\int_{r_i}^{R} dr \, r^2c^2(1 + \mu \sin \psi)\left[\theta + \mu(\beta \cos \psi + \theta \sin \psi - \gamma)\right] \tag{C4}$$

where

$$f \equiv \frac{l_{cg} - l_{ac}}{c}$$

Substituting equation (18) for c, equation (C4), in the dimensionless variable $s\left(\equiv \frac{r}{R}\right)$, becomes:

$$f\int_{r_i}^{R} L_z' \, c \, dr = f\pi\rho\Omega^2R^3c_i^2\int_{s_i}^{1} ds \, \frac{1 - s}{1 - s_i}\left(s + \mu_e \sin \psi\right)\left[\theta s + \mu_e(\beta \cos \psi\right.$$

$$\left. + \theta \sin \psi - \gamma)\right] = f \, \pi\rho\Omega^2R^3c_i^2\left\{\frac{\theta}{12}\left(1 + s_i\right) - \mu_e \frac{\gamma}{6}\left(1 + s_i\right) + \mu_e\left[\frac{1}{3}\left(1 + s_i\right)\theta\right.\right.$$

$$\left. - \frac{\mu_e}{2} \gamma\left(1 - s_i\right)\right] \sin \psi + \frac{\beta}{6}\left(1 + s_i\right)\mu_e \cos \psi + \frac{\mu_e^2}{4}\left(1 - s_i\right)\beta \sin 2\psi$$

$$\left. + \frac{\mu_e^2}{4}\left(1 - s_i\right) \theta(1 - \cos 2\psi)\right\} \tag{C5}$$

Addition of equations (C3) and (C5), together with the third term of equation (22b) yields the expression for M_{x1} given by equation (30a) in the text.

The Moment M_{y1}

By use of equations (10b) and (18), the first term in the expression (equation (23)) for M_{y1} can be written as follows, to first-order small quantities:

$$-\int_{r_1}^{R} L_z r\,dr = -\pi\rho\Omega^2R^4c_1\int_{s_1}^{1} ds\,\frac{\sqrt{1-s}}{\sqrt{1-s_1}}\Big[s^3\theta + s^2\mu_e(\beta\cos\psi + 2\theta\sin\psi - \gamma)$$

$$+ s\mu_e^2(\beta\cos\psi + \theta\sin\psi - \gamma)\sin\psi\Big] \tag{C6}$$

The integral in equation (C6) can be evaluated by means of the change of variables

$$\sqrt{1-s} = u, \quad ds = -2u\,du, \quad s = 1 - u^2$$

and the result is given by equation (31a) of the text.

The Moment M_{z1}

The moment M_{z1} as given by equation (28a) will be second order small. Hence it is necessary here to use terms of the second order. From equations (10a) and (14),

$$L_{y1}' + D_{y1}' = -\pi\rho c\Omega^2r^2\Bigg\{\frac{\alpha_p\theta}{k} + \frac{\theta^2}{5k} + \mu\Big[\theta(\gamma - \beta\cos\psi) + \alpha_p(A + \theta\sin\psi)$$

$$+ \frac{2}{5k}A\theta\Big] + \mu^2\Big[A(\gamma - \beta\cos\psi) + \frac{A\alpha_p}{k}\sin\psi + \frac{A^2}{5k}\Big]\Bigg\} \tag{C7}$$

where $A \equiv \beta\cos\psi + \theta\sin\psi - \gamma$. Thus, by using equation (18), the first term of equation (28a) for M_{z1} becomes:

$$\int_{r_1}^{R}\left(L_{y1}' + D_{y1}'\right)r\,dr = -\pi\rho\Omega^2R^4c_1\int_{s_1}^{1} ds\left(\frac{1-s}{1-s_1}\right)^{1/2}\Bigg[s^3\left(\frac{\alpha_p\theta}{k} + \frac{\theta^2}{5k}\right)$$

$$+ \mu_e s^2 M + \mu_e^2 sN\Bigg] \tag{C8}$$

where M and N are the coefficients of μ and μ^2, respectively, in equation (C7). The integral of equation (C8) can be evaluated in the same way as that of equation (C6), and the result is given by equation (32a) in the text.

APPENDIX D

INCREMENTS IN LIFT COMPONENTS DUE TO DAMPING

From equation (9a), the total circulation (i.e., including the velocities $\dot{\beta}$ and $\dot{\zeta}$) will be

$$\Gamma = \pi c\left(V_n + \Delta V_n\right) \tag{D1}$$

where V_n is given by equation (8b), and ΔV_n is the increment in V_n due to $\dot{\beta}$ and $\dot{\zeta}$. This increment can be obtained from the relation (cf. equations (A8))

$$V_n = l_n\,\Delta V_x + m_n\,\Delta V_y + n_n\,\Delta V_z \tag{D2}$$

where, for example, $\Delta V_x = V_{x\dot{\beta}} + V_{x\dot{\zeta}}$. Thus, by using equations (38), (39), and (A10), the expression for ΔV_n becomes:

$$\frac{\Delta V_n}{\Omega r} = -(\beta + \zeta\theta)\left(\beta\frac{\dot{\beta}}{\Omega} + \zeta\frac{\dot{\zeta}}{\Omega}\right) - (\theta - \beta\zeta)\frac{\dot{\zeta}}{\Omega} - \left(1 - \frac{\beta^2 + \theta^2}{2}\right)\frac{\dot{\beta}}{\Omega}$$

which to second-order small terms reduces to

$$\frac{\Delta V_n}{\Omega r} = -\theta\frac{\dot{\zeta}}{\Omega} - \frac{\dot{\beta}}{\Omega} \tag{D3}$$

The total circulation will therefore be

$$\Gamma = \pi c\Omega r\left\{\theta(1 + \eta) + \mu\left[(\beta + \zeta\theta)\cos\psi + (\theta - \beta\zeta)\sin\psi - \gamma\right] - \theta\frac{\dot{\zeta}}{\Omega} - \frac{\dot{\beta}}{\Omega}\right. \tag{D4}$$

From the Kutta-Joukowski relation (equation (B1)), the direction cosines (equations (A1)) of the line $\overline{r}$, and the velocity increments (equations (38) and (39)), it is seen that the expressions (equations (B2)) for the lift components per unit length must be changed by the following increments:

$$\left.\begin{aligned}
\frac{\Delta L_x'}{\rho\Gamma\Omega r} &= \frac{\dot{\zeta}}{\Omega}\beta - \frac{\dot{\beta}}{\Omega}\zeta \equiv \Delta\kappa_x \\
\frac{\Delta L_y'}{\rho\Gamma\Omega r} &= -\frac{\dot{\beta}}{\Omega} \equiv \Delta\kappa_y \\
\frac{L_z'}{\rho\Gamma\Omega r} &= -\frac{\dot{\zeta}}{\Omega} \equiv \Delta\kappa_z
\end{aligned}\right\} \qquad \text{(D5)}$$

The total lift components, for example the x-components, may then be written as follows:

$$L_x' = \rho\Omega r\left(\Gamma + \Delta\Gamma\right)\left(\kappa_x + \Delta\kappa_x\right) \qquad \text{(D6)}$$

where Γ is given by equation (9b), $\Delta\Gamma$ (see equation (D3)) by

$$\Delta\Gamma = -\pi c\Omega r\left(\theta\frac{\dot{\zeta}}{\Omega} + \frac{\dot{\beta}}{\Omega}\right) \qquad \text{(D7)}$$

and κ_x by

$$\kappa_x = \frac{L_x'}{\rho\Gamma\Omega r}$$

to be taken from equation (B2). By neglecting cross products of damping terms, the additional lift components can, according to equation (D6), be written in the form

$$\Delta L_x' = \pi\rho\Omega rc\left(\kappa_x\frac{\Delta\Gamma}{\pi c} + \frac{\Gamma}{\pi c}\Delta\kappa_x\right) \qquad \text{(D8)}$$

Thus, by the use of equations (B2), (D7), (9b), and (D5), equation (D8) leads to the results given by equations (41a), (41b), and (41c) in the text.

APPENDIX E

DAMPING-MOMENT INCREMENT ABOUT BLADE AXIS

The following relations follow readily from figure 6, where the velocities (including $\dot{\theta}s$ in the vector diagram) shown are those of the relative wind:

$$V = V_{c_1} - \dot{\theta}s\ \alpha_o, \quad V^2 = V_{c_1}{}^2 - 2V_{nc_1}\dot{\theta}s\ \alpha_o$$

$$\Delta\alpha = \frac{\dot{\theta}s \cos \alpha_o}{V_{c_1} - \dot{\theta}s\ \alpha_o} \approx \frac{\dot{\theta}s}{V_{c_1}}$$

$$dM = dFs = \frac{\rho}{2} 2\pi\left(\alpha_o - \Delta\alpha\right) ds \left(V_{c_1}{}^2 - 2V_{c_1}\ \dot{\theta}s\ \alpha_o\right)s$$

$$= 2\pi\ \alpha_o \frac{\rho}{2} V_{c_1}{}^2 \left(1 - \frac{\Delta\alpha}{\alpha_o}\right)\left(1 - \frac{2\dot{\theta}s\alpha_o}{V_{c_1}}\right) s\ ds$$

$$= 2\pi\ \alpha_o \frac{\rho}{2} V_{c_1}{}^2 \left(1 - \frac{\dot{\theta}s}{V_{c_1}\alpha_o} - \frac{2\dot{\theta}s\alpha_o}{V_{c_1}}\right) s\ ds$$

$$M = 2\pi\ \alpha_o \frac{\rho}{2} V_{c_1}{}^2 \int_{-\frac{c}{2}}^{\frac{c}{2}} ds\ s\left(1 - \frac{\dot{\theta}s}{V_{c_1}\alpha_o}\right), \quad \alpha_o \ll 1$$

Therefore

$$\left(M_{x1}\right)_{\dot{\beta}} = -\frac{\pi}{12} \rho\dot{\theta}\int_{r_i}^{R} dr\ V_{c_1} c^3$$

where V_{c_1} can be taken from equation (8a), and

$$c = c_i\sqrt{\frac{1 - s}{1 - s_i}}, \quad s \equiv \frac{r}{R}$$

The integration gives

$$M = -\frac{\pi}{12}\rho\dot{\theta}\Omega R^2 c_i^3 \frac{2}{35}\left(1 + s_1\right)\left[2 + s_1 + 7\eta_e + 7\mu_e(\sin\psi + \zeta\cos\psi)\right]$$

APPENDIX F

DAMPING-MOMENT INCREMENTS ABOUT HINGE AXES

Increment in M_{x1}

In order to evaluate the first term of equation (46), the increment ΔV_{c_1} must first be determined. This can be easily obtained from (cf. equation (A8))

$$\Delta V_{c_1} = l_{c_1}\ \Delta V_x + m_{c_1}\ \Delta V_y + n_{c_1}\ \Delta V_z \tag{F1}$$

The direction cosines l_{c_1}, m_{c_1}, and n_{c_1} are given by equation (A7), and the velocities ΔV_x, and so forth are given by the addition of equations (38) and (39). Thus, it is found that up to first-order small quantities

$$\Delta V_{c_1} = -\Omega r \frac{\dot{\zeta}}{\Omega} \tag{F2}$$

(Since $C_{M_{ac}}$ is already first order small, it suffices to obtain ΔV_{c_1} to only first orders in order to determine the first term of equation (46) up to second-order small quantities.)

By using equation (8a) for V_{c_1} and equation (18) for c it is seen that to first-order small terms

$$\int_{r_i}^{R} dr\ c^2 V_{c_1}\ \Delta V_{c_1} = -\Omega^2 R^3 c_i{}^2 \frac{\dot{\zeta}}{\Omega} \int_{s_i}^{1} ds\ \frac{1 - s}{1 - s_i} \left(s^2 + s\mu_e \sin \psi\right)$$

$$= -\Omega^2 R^3 c_i{}^2 \frac{\dot{\zeta}}{\Omega} \left(1 + s_i\right) \frac{1}{12} \left(1 + 2\mu_e \sin \psi\right) \tag{F3}$$

to first powers of s_i.

From equations (18) and (41c) the second term of equation (46) is seen to be:

$$\int_{r_i}^{R} L_z{}' c\, dr = -\pi\rho\Omega^2 \int_{r_i}^{R} dr\, c^2 r^2 \left\{ \frac{\dot{\beta}}{\Omega}(1 + \eta + \mu\zeta \cos\psi + \mu \sin\psi) + \frac{\dot{\zeta}}{\Omega}\left[2\theta + \mu\left(\beta\cos\psi + \theta\sin\psi - \gamma\right)\right] \right\}$$

$$= -\pi\rho\Omega^2 R^3 c_i{}^2 \int_{s_i}^{1} ds \left(\frac{1-s}{1-s_i}\right)\left(\frac{\dot{\beta}}{\Omega} s^2 + \frac{\dot{\beta}}{\Omega} As + \frac{\dot{\zeta}}{\Omega} Ds + 2\frac{\dot{\zeta}}{\Omega}\theta s^2\right) \tag{F4}$$

where, temporarily, the following **abbreviations** for terms, not containing the integration variable s (or r), have been introduced:

$$\left.\begin{aligned} A &\equiv \eta_e + \mu_e(\sin\psi + \zeta\cos\psi) \\ D &\equiv \mu_e(\beta\cos\psi + \theta\sin\psi - \gamma) \end{aligned}\right\} \tag{F5}$$

The integral in equation (F4) can be easily evaluated, with the result:

$$\int_{r_i}^{R} \Delta L_z{}' c\, dr = -\pi\rho\Omega^2 R^3 c_i{}^2 \left(1 + s_i\right) \left\{ \frac{\dot{\beta}}{\Omega}\frac{1}{12}\left[1 + 2\eta_e + 2\mu_e(\zeta\cos\psi + \sin\psi)\right] + \frac{\dot{\zeta}}{\Omega}\frac{1}{6}\left[\theta + \mu_e(\beta\cos\psi + \theta\sin\psi - \gamma)\right] \right\} \tag{F6}$$

The third term of equation (46), with the substitution of equation (8a) for V_{c_1} and equation (18) for c is:

$$\dot{\theta}\int_{r_i}^{R} V_{c_1} c^3\, dr = \frac{\dot{\theta}}{\Omega}\Omega^2 R^2 c_i{}^3 \int_{s_i}^{1} (s + A)\left(\frac{1-s}{1-s_i}\right)^{3/2} ds \tag{F7}$$

where A has the same meaning as in equation (F5).

The integral in equation (F7) can be evaluated by means of the change of variables

$$1 - s = u, \quad ds = -du, \quad s = 1 - u, \quad u_i = 1 - s_i \tag{F8}$$

Thus,

$$\dot{\theta}\int_{r_i}^{R} V_{c_1} c^3 \, dr = \frac{\dot{\theta}}{\Omega} \Omega^2 R^2 c_i^{\,3} \int_0^{u_i} (1 + A - u) \frac{u^{3/2}}{u_i^{\,3/2}} \, du$$

$$= \frac{\dot{\theta}}{\Omega} \Omega^2 R^2 c_i^{\,3} \left[\frac{2}{5}(1 + A)\left(1 - s_i\right) - \frac{2}{7}\left(1 - 2s_i\right)\right]$$

$$= \frac{\dot{\theta}}{\Omega} \Omega^2 R^2 c_i^{\,3} \frac{2}{35}\left(1 + s_i\right)\Big[2 + s_i + 7\left(1 - 2s_i\right)\eta_e$$

$$+ 7\left(1 - 2s_i\right)\mu_e \left(\sin\psi + \zeta \cos\psi\right)\Big] \tag{F9}$$

By putting equations (F3), (F6), and (F9) into equation (46), the result for the damping moment ΔM_{x1} as given by equation (49) of the text is obtained.

Damping Moment ΔM_{y1}

Putting equation (41c) into equation (47) shows the expression for the moment ΔM_{y1} to be:

$$-\int_{r_i}^{R} \Delta L_z{}' r \, dr = \pi\rho\Omega^2 R^4 c_i \int_{s_i}^{1} ds \left(\frac{1 - s}{1 - s_i}\right)^{1/2} \Bigg[s^2\left(\frac{\dot{\beta}}{\Omega} A + \frac{\dot{\zeta}}{\Omega} D\right)$$

$$+ s^3\left(\frac{\dot{\beta}}{\Omega} + 2\frac{\dot{\zeta}}{\Omega}\theta\right)\Bigg] \tag{F10}$$

where A and D are given by equations (F5). The integrals in equation (F10) can be evaluated by means of the change of variables (equation (F8)), with the results:

$$\int_{s_1}^{1} ds \left(\frac{1-s}{1-s_1}\right)^{1/2} s^2 = \frac{16}{105}\left(1 + \frac{1}{2}s_1\right)$$

$$\int_{s_1}^{1} ds \left(\frac{1-s}{1-s_1}\right)^{1/2} s^3 = \frac{32}{315}\left(1 + \frac{1}{2}s_1\right)$$

The use in equation (F10) of these values leads to the expression (equation (50)) for the damping moment ΔM_{y1} given in the text.

Damping Moment ΔM_{z1}

According to equation (48a),

$$\Delta M_{z1} = \int_{r_1}^{R} \Delta L_y{}' r\, dr + \int_{r_1}^{R} \Delta D_y{}' r\, dr \qquad \text{(F11)}$$

From the expressions for $\Delta L_y{}'$ (equation (41b) and for c (equation (18)) the first term of equation (F11) is seen to be

$$\int_{r_1}^{R} \Delta L_y{}' r\, dr = -\pi\rho\Omega^2 R^4 c_1 \int_{s_1}^{1} \left(\frac{1-s}{1-s_1}\right)^{1/2} ds\, \frac{\dot{\beta}}{\Omega}\left[\theta s^3 + s^2\mu_e(2\beta \cos \psi + \theta \sin \psi - \dot{\gamma})\right] \qquad \text{(F12)}$$

With the preceding values of the integrals, equation (F12) becomes:

$$\int_{r_1}^{R} \Delta L_y{}' r\, dr = -\pi\rho\Omega^2 R^4 c_1\left(1 + \frac{1}{2}s_1\right) \frac{\dot{\beta}}{\Omega}\left[\frac{32}{315}\theta + \frac{16}{105}\mu_e(2\beta \cos \psi + \theta \sin \psi - 2\gamma)\right] \qquad \text{(F13)}$$

From the expression (equation (44)) for $\Delta D_y{}'$, the second term of equation (F11) is seen to be

$$\int_{r_1}^{R} \Delta D_y{}' r\, dr = \int_{r_1}^{R} dr \left(\alpha_p + \frac{1}{5}\,\frac{\frac{D}{\mu_e}\mu + \theta}{1 + \mu \sin \psi}\right)(1 + \mu \sin \psi)\pi c\rho\Omega^2 r^3 \frac{\dot{\beta}}{\Omega} \qquad \text{(F14}$$

where D is given by equation (F5). Substituting equation (18) for c, integrals similar to those in equation (F12) appear, and the result is:

$$\int_{r_i}^{R} \Delta D_y{}' r \, dr = \pi\rho\Omega^2 R^4 c_i\left(1 + \frac{1}{2}s_i\right) \frac{1}{k} \frac{\dot{\beta}}{\Omega} \frac{16}{315}\left\{2\alpha_p + \frac{2}{5}\theta + 3\mu_e\left[\alpha_p \sin \psi + \frac{1}{5}(\beta \cos \psi + \theta \sin \psi - \gamma)\right]\right\} \quad \text{(F15)}$$

By adding equations (F13) and (F15), the expression (equation (51)) for the damping moment ΔM_{z1} is obtained.

APPENDIX G

PHYSICAL CONSTANTS OF BLADE SYSTEM

General Assumptions

The following is a summary of the data assumed in this report for the purpose of numerical calculation:

$W = 4000$ lb

$R = 25$ ft

$\Omega = 20$ radians/sec

$\eta_e \equiv \frac{e}{R} = 0.05$

$c = c_i\left(\frac{R - r}{R - r_i}\right)^{1/2}$

$\frac{c_i}{R} = \frac{1}{6}$

$s_i \equiv \frac{r_i}{R} = 0.2$

$\frac{t_b}{c}$ = Constant along span of blade $= \frac{1}{8}$

$n = 4$ blades

$\frac{\rho}{\sigma} = 0.0025$

$\alpha_p = 0.020$

Moments of Inertia

Because the pitch angle, with the foregoing data, will be quite small ($\theta_c = 0.0306$, equation (103)) it will be sufficiently accurate, for the purposes of calculating the moments of inertia, to neglect the rotation of axes due to the angle θ.

In general, for a blade,

$$I_1\left(\equiv I_{max}\right) = f_1 c^3 t_b, \qquad I_2\left(\equiv I_{min}\right) = f_2 t_b{}^3 c \tag{G1}$$

where f_1 and f_2 are constants, which for a solid Clark Y section have the values

$$f_1 = 0.0418, \quad f_2 = 0.0454$$

From equation (G1),

$$\frac{I_2}{I_1} = \frac{f_2}{f_1}\left(\frac{t}{c}\right)^2 = \frac{0.0454}{0.0418} \times \left(\frac{1}{8}\right)^2 = 0.017 \qquad \text{(G2)}$$

Neglect, for simplicity, of I_2 in comparison with I_1 will therefore be justified. Similarly, moreover, in accordance with the definitions in equations (61), the moment of inertia I_{ys} may be neglected in comparison with I_{zs}. It follows then, because for an airfoil section such as Clark Y, I_1 and I_{zs} will be practically the same, that

$$I_p \equiv \int \left(y_s{}^2 + z_s{}^2\right) dA \approx I_1 + I_2 \approx I_1 - I_2 \approx I_{zs}$$

Therefore, with the notations in equations (61),

$$\bar{\bar{I}}_r \approx \bar{\bar{I}}_{12} \approx \bar{\bar{I}}_{zs} \equiv \bar{\bar{I}} \qquad \text{(G3)}$$

The numerical value of $\bar{\bar{I}}$ can be readily determined by using equation (G1). Thus

$$\bar{\bar{I}} = \int_{r_i}^{R} f_1 c^3 t_b \, dr = \frac{f_1}{8}\int_{r_i}^{R} c^4 \, dr = \frac{f_1}{8\left(R - r_i\right)^2} c_i{}^4 \int_{r_i}^{R} (R - r)^2 \, dr$$

$$= \frac{f_1}{24} Rc_i{}^4\left(1 - s_i\right) \qquad \text{(G4)}$$

The value of $\bar{\bar{I}}_H$, by definition, will be

$$\bar{\bar{I}}_H = \int_{r_i}^{R} Ar^2 \, dr \qquad \text{(G5)}$$

where the cross-sectional area A may be given by

$$A = f_3 tc \qquad \text{(G6)}$$

and f_3 is a constant, which for a solid Clark Y section has the value

$$f_3 = 0.725$$

Hence

$$\overline{I}_H = f_3 \int_{r_1}^{R} t_b c r^2 \, dr = \frac{f_3}{8} \int_{r_1}^{R} c^2 r^2 \, dr$$

$$= \frac{f_3}{8} c_1^{\,2} \int_{r_1}^{R} r^2 \frac{(r - R)}{\left(r_1 - R\right)} dr = \frac{f_3}{8} R^3 c_1^{\,2} \frac{\left(1 + s_1\right)}{12}$$

$$= \frac{1}{6} \frac{f_3}{8} R^4 c_1 \frac{\left(1 + s_1\right)}{12} \qquad \text{(G7)}$$

From equations (G4) and (G7),

$$\frac{\overline{I}}{\overline{I}_H} \approx 24 f_1 \left(1 - 2s_1\right)\left(\frac{c_1}{R}\right)^3 = 24 \times 0.0418 \times 0.6 \times \left(\frac{1}{6}\right)^3 = 0.00386 \qquad \text{(G8)}$$

Therefore $\overline{I}$ may be considered negligible in comparison with $\overline{I}_H$.

From equation (G7), it follows that

$$R^4 c_1 = \frac{576}{f_3} \left(1 - s_1\right) \overline{I}_H = \frac{576}{0.725} \left(1 - s_1\right) \overline{I}_H$$

$$= 796 \left(1 - s_1\right) \overline{I}_H \qquad \text{(G9)}$$

The value of $\overline{S}$ is, by definition,

$$\overline{S} = \int_{r_1}^{R} A r \, dr \qquad \text{(G10)}$$

Putting equation (G6) into equation (G10) shows the value of $\bar{S}$ to be:

$$\bar{S} = \frac{1}{6}\,\frac{f_3}{8}\,R^3 c_1\,\frac{\left(1 + s_1\right)}{6} \tag{G11}$$

Comparison of equations (G7) and (G11) shows that

$$\bar{S} = 2\frac{\bar{I}_H}{R} \tag{G12}$$

Thus all the relations in equations (77) of the text have been derived.

APPENDIX H

EXPLICIT DERIVATION OF OSCILLATIONS IN CASE C

By putting equation (126) into equations (97a), (97b), and (97c), the expressions for the P's, in accordance with the definitions (equation (128)), are found to be the following:

$$\left.\begin{aligned}
P_{1b} &= p^2\bar{I} + Hp - J, \quad P_{1f} = -p^2 \zeta_c \bar{I}_{zs} \\
P_{1d} &= p^2 \beta_c \bar{I}_H + p\left[M + \frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{3}{2}\eta_e\right)\right] \\
P_{2b} &= p^2 \zeta_c \bar{I}_{zs} + \frac{32}{315}\left(1 + \frac{1}{2}s_i\right)\frac{E}{\Omega^2} \\
P_{2f} &= -p^2\bar{I}_H - \bar{I}_H\left(1 + 2\eta_e\right) - p\frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{3}{2}\eta_e\right) \\
P_{2d} &= p^2 \theta_c \bar{I}, \quad P_{3b} = -p^2 \beta_c \bar{I}_H + \frac{32}{315}\frac{E}{\Omega^2}\left(\alpha_p + \frac{2}{5}\theta_c\right) \\
P_{3f} &= p^2 \theta_c \bar{I} - \frac{E}{\Omega^2}\left(1 + \frac{1}{2}s_i\right)\frac{32}{315}\left(\frac{4}{5}\theta_c - \alpha_p\right)p \\
P_{3d} &= -p^2\bar{I}_H - 2\eta_e \bar{I}_H
\end{aligned}\right\} \quad \text{(H1)}$$

where

$$H \equiv \frac{E}{\Omega^2}\frac{\left(1 + s_i\right)}{12}\left(\frac{c_i}{R}\right)^2 \frac{2}{35}\left(2 + s_i + 7\eta_e\right)$$

$$J \equiv \frac{E}{\Omega^2}\frac{\left(1 + s_i\right)}{12}\frac{c_i}{R}f - \bar{I}_{12}$$

$$M \equiv \frac{E}{\Omega^2}\frac{\left(1 + s_i\right)}{12}\left(2\frac{c_i}{R}f\,\theta_c - \left|C_{M_{ac}}\right|\frac{c_i}{\pi R}\right)$$

Equation (130b) therefore becomes:

$$p^2\bar{I} + Hp - J + \lambda \frac{E}{\Omega^2}\left[\frac{\left(1 + s_i\right)}{12} f \frac{c_i}{R}\left(1 + 2\eta_e\right) + \frac{32}{315}\left(1 + \frac{1}{2}s_i\right)\right]$$

$$+ \mu p\left[M + \frac{32}{315}\frac{E}{\Omega^2}\left(\alpha_p + \frac{2}{5}\theta_c\right)\right] - \lambda^2\left[p^2\bar{I}_H + p\frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{3}{2}\eta_e\right) + \bar{I}_H\left(1 + 2\eta_e\right)\right]$$

$$- \mu^2\left(p^2\bar{I}_H + 2\eta_e\bar{I}_H\right) + \lambda\mu\left[2p^2\,\theta_c\bar{I} - \frac{E}{\Omega^2}\left(1 + \frac{1}{2}s_i\right)\frac{32}{315}\left(\frac{4}{5}\theta_c - \alpha_p\right)p\right] = 0$$

or, rearranging in powers of p and dividing through by $\bar{I}_H$, gives

$$p^2\left(-\lambda^2 - \mu^2 + \frac{\bar{I}}{\bar{I}_H} + 2\lambda\mu\,\theta_c\frac{\bar{I}}{\bar{I}_H}\right) + p\left\{\frac{H}{\bar{I}_H} + \mu\left[\frac{M}{\bar{I}_H} + \frac{32}{315}\frac{E}{\Omega^2\bar{I}_H}\left(\alpha_p + \frac{2}{5}\theta_c\right)\right]\right.$$

$$+ \lambda\frac{E}{\Omega^2\bar{I}_H}\frac{\left(1 + s_i + 2\eta_e\right)}{12} f \frac{c_i}{R} - \lambda^2\frac{E}{\Omega^2\bar{I}_H}\frac{32}{315}\left(1 + \frac{3}{2}\eta_e\right)$$

$$\left. - \lambda\mu\frac{E}{\Omega^2\bar{I}_H}\frac{32}{315}\left(1 + \frac{1}{2}s_i\right)\left(\frac{4}{5}\theta_c - \alpha_p\right)\right\}$$

$$+ \left[\frac{J}{\bar{I}_H} - \lambda^2\left(1 + 2\eta_e\right) + \lambda\frac{E}{\Omega^2}\frac{32}{315}\left(1 + \frac{1}{2}s_i\right) - 2\eta_e\mu^2\right] = 0 \qquad \text{(H2)}$$

Equation (H2) is equivalent to equation (131) of the text, when $\left(\frac{\bar{I}}{\bar{I}_H} + 2\lambda\mu\,\theta_c\frac{\bar{I}}{\bar{I}_H}\right)$ has been taken as negligible in comparison with $\left(\lambda^2 + \mu^2\right)$.

Equation (H2) is most easily handled by using numerical values immediately. With the numerical data consistently assumed in this report (see appendix G) and with the use of equations (77), the following values are obtained, in accordance with the abbreviations in equation (131):

$$\frac{E}{\Omega^2\bar{I}_H}\frac{32}{315}\left[\left(1+\frac{1}{2}s_1\right)\left(\frac{4}{5}\theta_c-\alpha_p\right)\lambda-\alpha_p-\frac{2}{5}\theta_c\right]$$

$$=\pi\frac{\rho}{\sigma}\frac{R^4c_1}{\bar{I}_H}\frac{32}{315}\left(1.1\times 0.0045\lambda-0.020-\frac{2}{5}\times 0.0301\right)$$

$$=3.14\times 0.0025\times 796\times 0.8\times\frac{32}{315}(0.00495\lambda-0.032)$$

$$=0.00252\lambda-0.0163$$

$$\frac{M}{\bar{I}_H}=-\pi\frac{\rho}{\sigma}\frac{\left(1+s_1\right)}{12}\frac{R^4c_1}{\bar{I}_H}\left(\frac{2}{6}\times 0.15\times 0.0306-\frac{0.06}{6\pi}\right)$$

$$=-3.14\times\frac{0.0025}{12}\times 796\times 0.8\times 0.00165=-0.00069$$

$$\frac{H}{\bar{I}_H}=3.14\times 0.0025\times\frac{796\times 0.8}{12}\times\frac{1}{36}\times\frac{2}{35}\times 2.55=0.00168$$

$$\frac{J}{\bar{I}_H}=3.14\times 0.0025\times\frac{796\times 0.8}{12}\times\frac{1}{6}\times 0.15-0.00386=0.00656$$

$$\frac{E}{\Omega^2\bar{I}_H}\frac{32}{315}\left(1+\frac{3}{2}\eta_e\right)=3.14\times 0.0025\times 796\times 0.8\times\frac{32}{315}\times 1.075=0.548$$

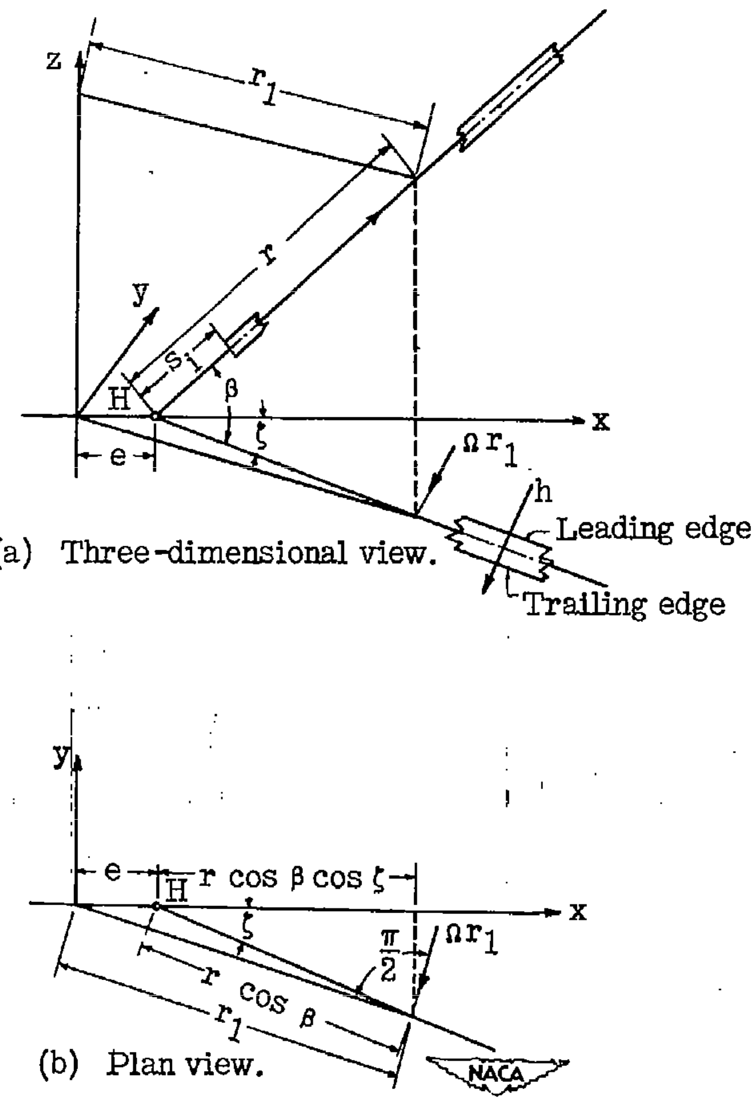
$$\frac{E}{\Omega^2\bar{I}_H}\frac{\left(1+s_1+2\eta_e\right)}{12}f\frac{c_1}{R}=3.14\times 0.0025\times 796\times 0.8\times\frac{1.3}{12}\times 0.15\times\frac{1}{6}=0.0136$$

Substitution of the foregoing numerical values into equation (H2), the changing of all signs there, and neglect of the quantity $\left(\frac{\bar{I}}{\bar{I}_H}+2\lambda\mu\theta_c\frac{\bar{I}}{\bar{I}_H}\right)$ in comparison with $-\left(\lambda^2+\mu^2\right)$ leads to the following equation:

$$p^2\left(\lambda^2 + \mu^2\right) + p(0.00252\lambda\mu - 0.0163\mu + 0.00069\mu - 0.00168 + 0.598\lambda^2 - 0.0136\lambda)$$

$$+ (0.00656 + 1.1\lambda^2 + 0.1\mu^2 - 0.560\lambda) = 0 \qquad \text{(H3)}$$

This equation is equivalent to equation (132) of the text.



(a) Three-dimensional view.

(b) Plan view.

Figure 1.- Blade position.

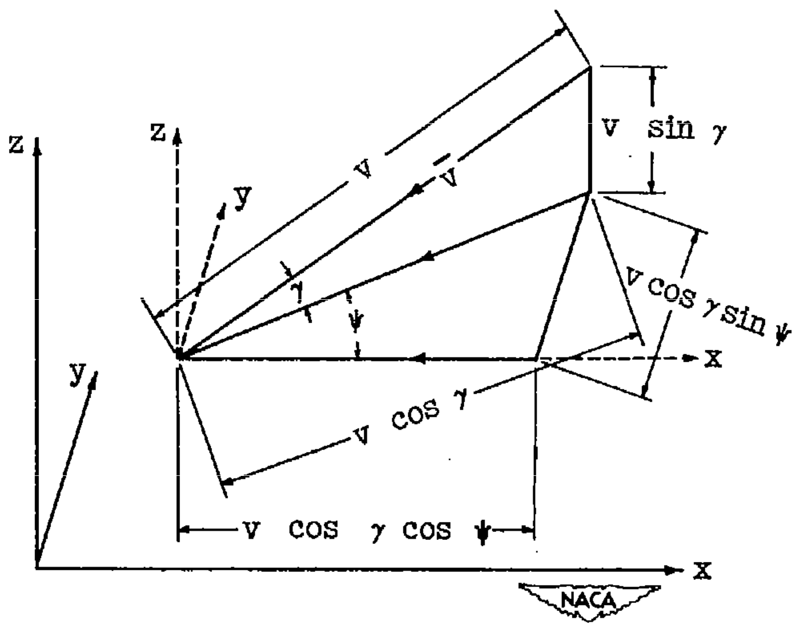


Figure 2.- Components of traveling velocity v.

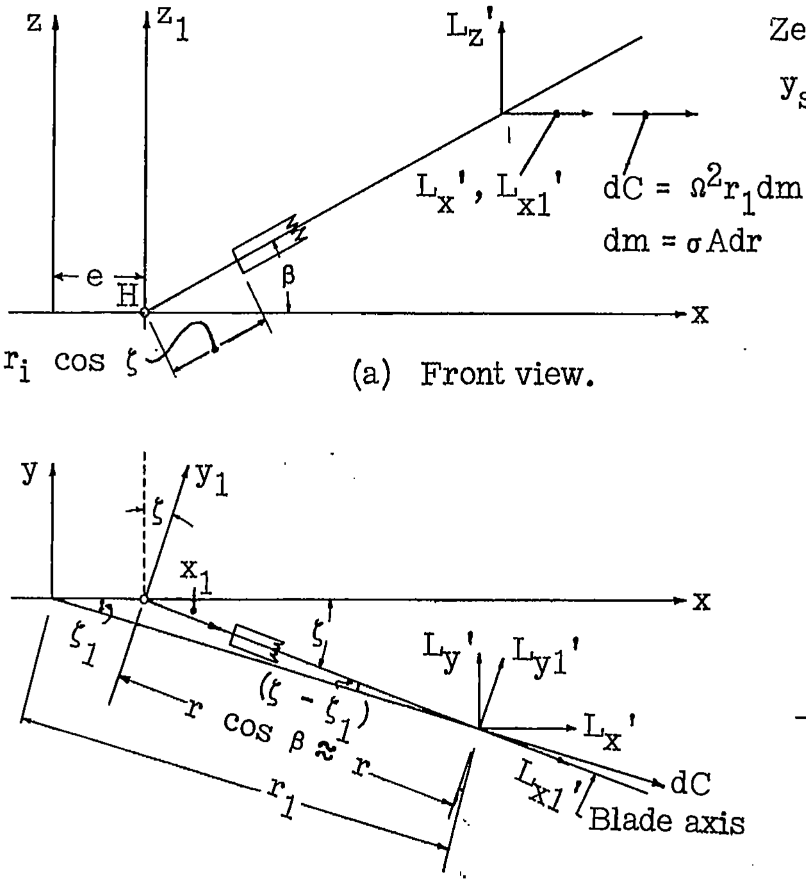


Figure 3.- Centrifugal force element on blade and axes x_1, y_1, z_1 of moments about hinges.

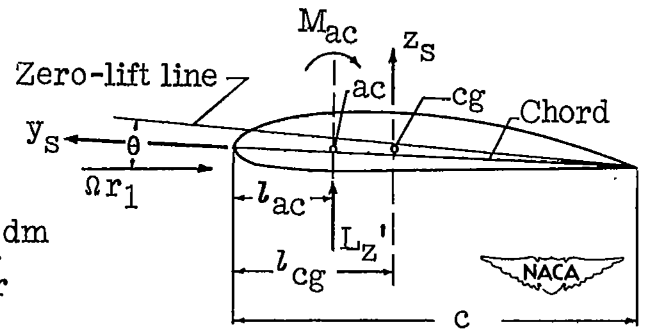


Figure 4.- Cross section of blade. $dM_{cg} = dM_{ac} + L_z' dr\,(l_{cg} - l_{ac})$.

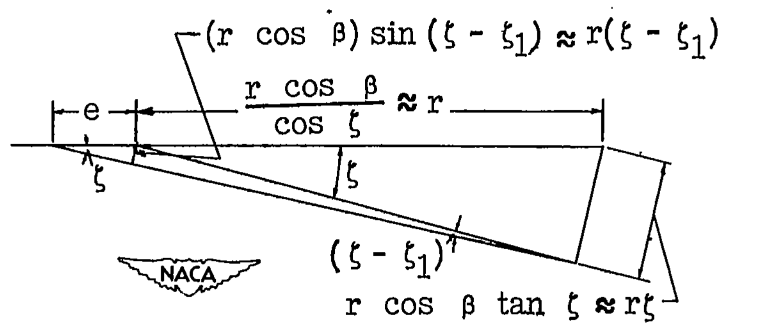


Figure 5.- Lever $[r(\zeta - \zeta_1)]$ of centrifugal force element $\frac{r(\zeta - \zeta_1)}{r} = \frac{e}{e + r}$ or $(\zeta - \zeta_1) \approx \zeta \frac{e}{r} = \zeta\eta$.

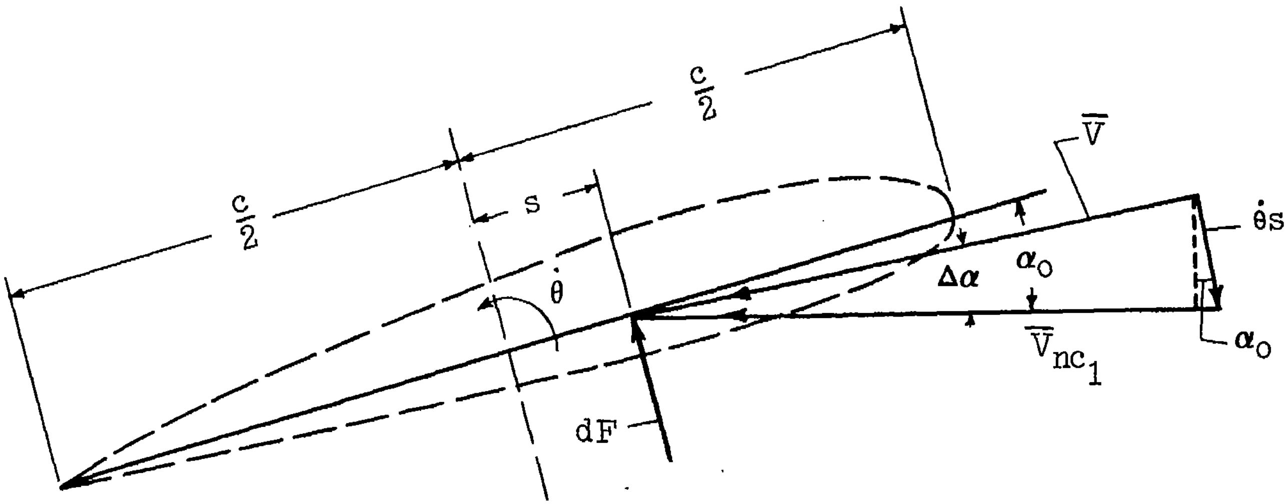


Figure 6.- Element dFs of damping moment due to $\dot{\theta}$.