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THE USE OF SOURCE-SINK AND DOUBLET DISTRIBUTIONS EXTENDED TO THE SOLUTION OF ARBITRARY BOUNDARY VALUE PROBLEMS IN SUPERSONIC FLOW

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SUMMARY

A direct analogy is established between the use of source-sink and doublet distributions in the solution of arbitrary boundary value problems in subsonic wing theory and the corresponding problems in supersonic theory. The concept of the "finite part" of an integral is introduced and used in the calculation of the improper integrals associated with supersonic doublet distributions. The general equations developed are shown to include several previously published results and particular examples are given for the loading on rolling and pitching triangular wings with supersonic leading edges.

INTRODUCTION

The problem of finding pressure distributions over airfoils of arbitrary shape and plan form or of finding airfoils which have arbitrary pressure distributions is one of the most fundamental problems in aerodynamic theory. At the present time the most important and satisfactory approach to problems of this type is provided by the methods of so-called thin-airfoil theory. The essential assumptions in this theory are that the perturbation velocities induced by the airfoil are small relative to the free-stream velocity and that the boundary conditions can be specified in a fixed reference plane.

Under the assumptions of thin-airfoil theory the theoretical analysis of a problem in wing theory resolves itself into the task of determining the solution of a second-order linear partial differential equation with prescribed boundary conditions. In the case of purely subsonic flow, Laplace's equation in three dimensions must be considered, while in purely supersonic flow the differential equation which arises is algebraically equivalent to the two-dimensional wave equation of mathematical physics. The classical solutions of these two equations have been developed along two distinct lines: first by use of orthogonal functions which can be derived in terms of the boundary conditions, and alternatively by means of Green's theorem which in turn utilizes a known particular solution of the partial differential equation together with the given boundary conditions.

One particular solution associated with Laplace's equation and subsonic aerodynamics has been found to be outstanding in its mathematical usefulness and, when identified with the velocity potential, has a physical interpretation which can supply, in direct application, added insight into the nature of the problem. This function is referred to as the "fundamental solution" and can be developed from the concept of a so-called source. A concomitant development to the source potential is the doublet potential, and appropriate distributions of these functions are known to be sufficient for the solution of all problems in subsonic wing theory.

The extension of the use of the fundamental solution to problems in purely supersonic flow introduces mathematical difficulties which differ essentially from those encountered at low speeds. Both the source and the doublet potentials possess singularities on their conical characteristic surfaces or Mach cones and, in the case of the doublet, the singularity is of higher order than can be treated by elementary mathematical methods. In the historical development of the solutions of the wave equation this trouble was circumvented by replacing the source potential by other particular solutions of the differential equation. As an example, Volterra (reference 1) introduced the integral of the fundamental solution and in that way reduced the order of the singularities involved. The analytical development of Volterra's theory presents no inherent difficulties (e.g., reference 2) but the physical significance of the particular solution is lost, the direct analogy with subsonic theory no longer exists, and a certain amount of mathematical inefficiency arises since, after using the integral of the source potential, it is found necessary to resort at the end of the analysis to taking a final derivative.

In this report, following methods introduced by Hadamard (reference 3), a general solution to the thin-airfoil problem in supersonic theory will be given in terms of the distribution of sources and doublets over the given reference plane. Furthermore, a discussion of the nature of the boundary values required will be given. For properly set problems in wave theory it has been found necessary to specify, usually, both the required function and its derivative with respect to time along the boundary considered. In aerodynamic applications of the wave equation associated with lifting surface theory and thickness distributions it will be shown that only a knowledge of the unknown function or its normal derivative along the boundary is needed since a relationship between the two functions will be established on the boundary surface.

In the theoretical portion of the report a brief presentation will be made of the differential equations involved and the two forms of the fundamental solution. An outline is then given of the types of boundary value problems encountered and, since the purpose of the report is to extend the concepts of thin-airfoil theory which are used in subsonic theory to problems arising in supersonic theory, a discussion will be given of the subsonic development as a basis for the analogy which exists between the methods of solution corresponding to the two regimes of flow. In the discussion of the purely supersonic case it will be shown that the introduction of the concept of "finite part" will provide a technique whereby the improper integrals arising from the use of doublets may be evaluated in a straightforward manner. The applications of the theoretical developments will include the rederivation of some previously published results and will also contain the calculation of load distributions for rolling and pitching triangular wings with leading edges swept ahead of the Mach cone from the vertex of the triangles.

SYMBOLS

b	span of wing
c	chord of wing
M	free-stream Mach number
n	normal to arbitrary surface
n_1, n_2, n_3	direction cosines of normal n
p	static pressure
P	rate of roll about X-axis
q	free-stream dynamic pressure
Q	rate of pitch about Y-axis
$\frac{1}{r}$	fundamental solution of equation (3) $[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2]^{-\frac{1}{2}}$
$\frac{1}{r_o}$	fundamental solution of equation (4) $[(x-x_1)^2-(y-y_1)^2-(z-z_1)^2]^{-\frac{1}{2}}$

R	arbitrary region of integration
S	surface enclosing region R
u,v,w	perturbation velocities in direction of X-, Y-, and Z-axes, respectively
V	free-stream velocity
X,Y,Z	Cartesian coordinates in original space variables
x,y,z	transformed system of coordinates
β	$\sqrt{\lvert M^2-1 \rvert}$
ϵ	infinitesimal used in analysis
λ	surface along which stream enters induced field of wing
υ	conormal to arbitrary surface
$\upsilon_1,\upsilon_2,\upsilon_3$	direction cosines of conormal
σ	variable representing either acceleration potential, velocity potential, or any of the three perturbation velocity components
τ	surface on which boundary conditions are given
Φ	perturbation velocity potential
Ω	variable representing either acceleration potential, velocity potential, or any of the three perturbation velocity components
$\frac{\Delta p}{q}$	pressure coefficient
ΔC_p	load coefficient
C_l	rolling-moment coefficient $\left(\frac{\text{moment about X-axis}}{\text{qb} \times \text{wing area}}\right)$
C_{l_p}	$\partial C_l/\partial(Pb/2V)$
∇^2	differential operator $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$

$\square^2$ differential operator $\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)$

$\ulcorner$ sign denoting "finite part" of integral

Subscripts

u subscript denoting value of variable on upper surface of wing

l subscript denoting value of variable on lower surface of wing

1 subscript denoting variable of integration

c subscript on r denoting fundamental solution in supersonic flow

Superscript

superscript denoting value of variable on opposite side of τ from fixed point (x,y,z)

THEORETICAL DEVELOPMENT

Linearized Equations and Boundary Conditions

The linearization of the second-order differential equation for compressible fluid flow is developed under the assumptions of thin-airfoil or small-perturbation theory. If the velocity vector of the free stream is parallel to and in the direction of the positive X-axis, the resulting differential equation is expressible in the form

$$(1-M^2)\frac{\partial^2\Omega}{\partial X^2} + \frac{\partial^2\Omega}{\partial Y^2} + \frac{\partial^2\Omega}{\partial Z^2} = 0 \tag{1}$$

where Ω represents a velocity potential, acceleration potential, or any one of the perturbation velocities while M is the constant value of the free-stream Mach number. Assuming the plane of symmetry of the airfoil to lie in the XY plane, the boundary conditions associated with equation (1) are given for $Z=0$. Moreover, if u, v, and w represent, respectively, the perturbation velocity components along the X, Y, and Z-axes, and if the velocity of the free stream is V, the direction cosines of any stream line are proportional to the point functions $V + u(X, Y, Z)$, $v(X, Y, Z)$, and $w(X, Y, Z)$ while pressure coefficient $\frac{\Delta p}{q}$ is given by the relation

$$\frac{\Delta p}{q} = -\frac{2u}{V} \tag{2}$$

Detailed discussions of these results may be found in reference 4.

Introducing the affine transformations

$$x = X$$

$$y = \sqrt{\pm(1-M^2)}\,Y$$

$$z = \sqrt{\pm(1-M^2)}\,Z$$

where the signs under the radicals are chosen so that real values result, it follows that in the subsonic case $(M<1)$ equation (1) reduces to

$$\frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2} + \frac{\partial^2\Omega}{\partial z^2} = 0 \tag{3}$$

while the supersonic case $(M>1)$ yields

$$\frac{\partial^2\Omega}{\partial x^2} - \frac{\partial^2\Omega}{\partial y^2} - \frac{\partial^2\Omega}{\partial z^2} = 0 \tag{4}$$

The fundamental solution associated with equation (3) is

$$\frac{1}{r} = \left[(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2\right]^{-\frac{1}{2}} \tag{5}$$

or, in terms of the original space variables,

$$\frac{1}{r} = \left[(X-X_1)^2 + \beta^2(Y-Y_1)^2 + \beta^2(Z-Z_1)^2\right]^{-\frac{1}{2}} \tag{5a}$$

where

$$\beta^2 = (1 - M^2)$$

When the wave equation is to be considered the fundamental solution takes the form

$$\frac{1}{r_c} = \left[(x-x_1)^2 - (y-y_1)^2 - (z-z_1)^2\right]^{-\frac{1}{2}} \tag{6}$$

or

$$\frac{1}{r_c} = \left[(X-X_1)^2 - \beta^2(Y-Y_1)^2 - \beta^2(Z-Z_1)^2\right]^{-\frac{1}{2}} \tag{6a}$$

where

$$\beta^2 = (M^2 - 1)$$

These fundamental solutions represent, respectively, in subsonic and supersonic flow the velocity potentials at the point (x,y,z) or (X,Y,Z) of unit sources situated at the point (x_1,y_1,z_1) or (X_1,Y_1,Z_1). The velocity potential of a doublet may be obtained by taking a directional derivative of the source potential, the direction of the axis of the doublet coinciding with the direction along which the derivative is taken. These two functions will be seen to be of paramount importance when Green's theorem is applied to the given boundary conditions.

It remains now to mention the types of boundary conditions which appear in problems associated with wing theory. As a convenience to the development of the theory the normalized forms (equations (3) and (4)) of equation (1) will be used and boundary conditions will be assumed known with respect to the x,y,z coordinate system. Retransformation to the X,Y,Z system of axes can be made quite simply wherever needed in application. In order to define the boundary conditions, two subscripts will be introduced: the first, u, denotes the value of the required function on the upper surface, that is the limit of the function as z approaches zero from the positive direction; the second, l, denotes the value on the lower surface, that is, the limit of the required function as z approaches zero from the negative direction.

Using these definitions the three boundary value problems of principal interest can be defined as follows:

1. Symmetrical nonlifting airfoil.- In this case $w_u = w_l = 0$ over all of the xy-plane except for the region occupied by the airfoil where $2w_u = -2w_l = \Delta w = f(x,y)$ the function being determined by the geometry of the wing. Over all of the xy-plane, $\Delta u = 0$.

2. Lifting plate with specified loading.- It is given that $\Delta u = u_u - u_l = 0$ over the xy-plane except for the region occupied

by the airfoil where $\Delta u = f(x,y)$, the function being determined by the specified loading. Moreover, $\Delta w = 0$ everywhere.

3. Lifting plate with specified camber, twist, and angle of incidence.- Over the xy-plane $\Delta w = 0$ everywhere. And, except for the region occupied by the airfoil, $\Delta u = 0$. Over the region occupied by the airfoil $w = f(x,y)$ where $f(x,y)$ is determined by the given camber, twist, and angle of incidence.

It should be pointed out that the problems considered here differ from the usual type of boundary value problem encountered. In the so-called Dirichlet or Neumann problems, which arise in connection with Laplace's equation, the value of the normal derivative of the function or of the function itself is specified along the boundary while the Cauchy problem for second-order partial differential equations involves the knowledge of both the function and a derivative. Except for one case in the aerodynamic problems listed above, no absolute values are given but rather the jump in the value of the function along the boundary is prescribed.

Boundary Value Problems in Purely Subsonic Flow

Since the purpose of this report is to extend the concepts of thin-airfoil theory which are used in subsonic theory to problems arising in supersonic theory, some discussion of the former will be given to provide lucidity as well as to furnish a basis for the analogy which will be shown to exist between the methods of solution arising in the two regimes of flow.

The method whereby the solutions of the given problems can be effected is provided by Green's theorem which relates a volume integral over a region R to a surface integral over the surface S enclosing R. If σ, Ω are any two functions which, together with their first and second derivatives, are finite and single-valued throughout R, then for the subsonic case

$$\int_S\int\left[\sigma\frac{\partial\Omega}{\partial n} - \Omega\frac{\partial\sigma}{\partial n}\right]dS = \iint_R\int\left[\sigma\nabla^2\Omega - \Omega\nabla^2\sigma\right]dR \tag{7}$$

where the Laplacian operator, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, is

introduced and the directional derivatives on the left side are taken along the normal n, drawn inward, to the surface S. Identifying now the function σ with the fundamental solution $\frac{1}{r}$ and specifying that Ω satisfies Laplace's equation, equation (7) simplifies to give

$$\int_S\int\left[\frac{1}{r}\left(\frac{\partial\Omega}{\partial n}\right)-\Omega\,\frac{\partial(1/r)}{\partial n}\right]dS = 0 \qquad (8)$$

where

$$\frac{1}{r} = \left[(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2\right]^{-\frac{1}{2}}$$

The variables of integration in the equation are x_1, y_1, z_1, while x, y, z are the coordinates of a point P either inside or outside of the region of integration.

If the point P is assumed to be inside the region of integration, it is evident that the function $\frac{1}{r}$ becomes infinite at P, and it is necessary to exclude this point from the region if formula (8) is to apply. Describing a spherical surface Σ with radius ϵ about the point P, and considering the integral over the two surfaces Σ and S which enclose the region, it can be shown that in the limit as $\epsilon \longrightarrow 0$ equation (8) becomes

$$\Omega\,(x,y,z) = -\frac{1}{4\pi}\int_S\int\left[\frac{1}{r}\left(\frac{\partial\Omega}{\partial n}\right)-\Omega\,\frac{\partial(1/r)}{\partial n}\right]dS \qquad (9)$$

The physical significance of this last relation follows immediately: the term $\frac{1}{r}$ represents a fluid source and the term $\frac{\partial(1/r)}{\partial n}$ represents a doublet with its axis lying along the normal to S, both source and doublet being situated at the surface point x_1, y_1, z_1. The value of the function Ω at the point x, y, z is therefore given as an integral of a source and doublet distribution, the strengths of the two being determined directly from the respective boundary values of Ω and $\frac{\partial\Omega}{\partial n}$.

Equation (9) expresses the value of Ω in terms of the surface values of Ω and $\frac{\partial\Omega}{\partial n}$ but this relation does not imply that a knowledge of both these variables is necessary for the determination of Ω. As can be shown easily, another condition may be established which relates the two surface values. Applying equation (8) to the case where P lies outside the region of integration, it follows that the integral is equal to zero and that Ω and $\frac{\partial\Omega}{\partial n}$ on the surface are functionally dependent.

Sufficient information is now at hand to provide a solution for the thin-airfoil-boundary value problems. Consider the region R bounded by the xy-plane and a hemispherical dome of infinite radius lying above this plane. For all problems to which the results will be applied, the value of Ω may be assumed equal to zero at infinity. The contribution of the surface integral over the hemisphere is thus zero and, from equation (9)

$$\Omega = -\frac{1}{4\pi}\iint\left[\frac{1}{r}\frac{\partial\Omega_u}{\partial z} - \Omega_u\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]dx_1\,dy_1$$

where the integration extends over the entire plane. The directional derivatives are necessarily in the direction of the positive z-axis and subscripts are again introduced to denote conditions on the upper side of the plane. Keeping P fixed and integrating over the lower side of the xy-plane, it follows that

$$0 = -\frac{1}{4\pi}\iint\left[\frac{1}{r}\frac{\partial\Omega_l}{\partial z} - \Omega_l\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]dx_1\,dy_1$$

where the negative direction of the normal may be ignored since the integral is equal to zero. Subtracting these two equations gives the expression

$$\Omega(x,y,z) = -\frac{1}{4\pi}\iint_\tau\left[\frac{1}{r}\left(\frac{\partial\Omega_u}{\partial z} - \frac{\partial\Omega_l}{\partial z}\right) - (\Omega_u - \Omega_l)\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]dx_1\,dy_1 \qquad (10)$$

the integral extending now only over the area τ for which the integrand does not vanish. Equation (10) is the basic equation from which all solutions in subsonic wing theory will be developed. It should be pointed out that the derivation proceeded from the assumption that the point (x,y,z) lay above the xy-plane. When (x,y,z) lies below the xy-plane, however, the derivation can be carried through in exactly the same manner. Such a development reveals that equation (10) is general and that no restriction need be imposed on the position of (x, y, z) relative to the reference plane.

As a particular application of equation (10) consider a thin symmetrical airfoil at zero angle of attack and set $\Omega = \Phi$ where Φ is the perturbation velocity potential. Conditions of symmetry demand that $\Omega_u = \Phi_u = \Phi_l = \Omega_l$ while $\frac{\partial\Omega_u}{\partial z} = w_u$ and $\frac{\partial\Omega_l}{\partial z} = w_l$.

Thus, if $w_u - w_l = \Delta w$

$$\Phi = -\frac{1}{4\pi}\iint_{\tau} \Delta w \frac{1}{r} dx_1 dy_1 \tag{11}$$

and the velocity potential is given by a distribution of source potentials. This distribution can be immediately related to the slope of the basic section by means of the equation

$$\frac{w_u}{V} = -\frac{w_l}{V} = \left(\frac{dy}{dx}\right)_u$$

The symmetric airfoil can also be treated by replacing Ω by the perturbation velocity w and in the case of the thin lifting surface with given loading the function Ω can be set equal to u. Employing, respectively, conditions of symmetry and irrotationality, it follows that $\frac{\partial \Omega_u}{\partial z} - \frac{\partial \Omega_l}{\partial z}$ vanishes and, after setting $\Delta\Omega = \Omega_u - \Omega_l$ equation (10) becomes

$$\Omega(x,y,z) = \frac{1}{4\pi}\iint_{\tau} \Delta\Omega \frac{\partial}{\partial z}\left(\frac{1}{r}\right) dx_1 dy_1 \tag{12}$$

Boundary Value Problems in Purely Supersonic Flow

Applications of Green's Theorem.- The problem to be discussed at this point is the extent to which an analogue to equation (10) can be developed for supersonic flow fields. The first step in the presentation is, once more, the introduction of Green's theorem for equation (1) after it has been modified to the form given by equation (4). Employing the operator

$$\square^2 = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

Green's theorem now becomes

$$\iint_S \left(\sigma \frac{\partial \Omega}{\partial \nu} - \Omega \frac{\partial \sigma}{\partial \nu}\right) dS = \iiint_R \left(\Omega \square^2 \sigma - \sigma \square^2 \Omega\right) dR \tag{13}$$

where υ is the so-called conormal to the surface S and has direction cosines equal to $\upsilon_1, \upsilon_2, \upsilon_3$ such that

$$\upsilon_1 = -n_1, \ \upsilon_2 = n_2, \ \upsilon_3 = n_3 \tag{14}$$

where n_1, n_2, n_3 are the direction cosines of the normal to the surface S (fig. 1). (The conormal at any point x_1, y_1, z_1 of a surface is the mirror image in the plane $x = x_1$ of the normal through the same point.) If σ and Ω are perfectly arbitrary functions, aside from satisfying the usual conditions of single-valuedness, etc., equation (13) represents an identity and this fact will be useful at a later time. For immediate purposes, however, σ and Ω will be chosen as solutions of the differential equation under consideration so that

$$\Box^2 \sigma = \Box^2 \Omega = 0$$

and, consequently,

$$\iint_S \left(\Omega \frac{\partial \sigma}{\partial \upsilon} - \sigma \frac{\partial \Omega}{\partial \upsilon} \right) dS = 0 \tag{15}$$

The use of equations (13) and (15) depends upon an understanding of the physical nature of supersonic flow fields. The essential feature of such flow is the presence of Mach cones which correspond to the characteristic cones arising in the mathematical study of the wave equation. In accordance with these concepts a disturbance in the flow field can affect the flow only within its aftercone, that is, the cone with vertex at the point of disturbance and with axis extending in the direction of the undisturbed stream velocity vector; conversely, a point in the flow field can be affected only by disturbances which emanate from points within its forecone.

When the disturbances are generated by a wing it is, moreover, necessary to speak more specifically about the nature of the leading edge of the wing. For all cases considered here the assumption will be made that the plan form is a polygon, that is, is composed of straight line segments. If the wing is swept ahead of the foremost Mach cone, the cones arising at the leading edge will have as envelope a wedge-shaped surface passing through and extending back from the leading edge, while if the wing is swept back of the foremost Mach cone this cone will be the surface along which the air first experiences perturbations or disturbances. Thus, a point P with coordinates x, y, z is affected by all disturbances lying within its forecone Γ and at the same time behind the forward surface λ, the nature of the latter surface being dictated by the leading edge. In figures 2(a) and 2(b) these surfaces, along with the disturbance plane τ, are indicated for two different wing plan forms. In the applications of equation (13) the volume integral is limited to the portion of space common to the surfaces Γ, λ, and τ and the surface integral involves a discussion of conditions on these surfaces.

Up to this point the analogy between the subsonic and supersonic cases, insofar as the use of Green's theorem is concerned, is quite apparent. The principal difference which occurred was brought about by the use of the true normal in the subsonic field together with the fact that the xy-plane was covered by a hemispherical dome of infinite radius; whereas, in the supersonic field, the concept of the conormal was introduced and the volume to be considered was that enclosed within a finite region. In continuing the analogy, however, far more formidable obstacles arise. To begin with, the discussion of σ and Ω over the surface in the subsonic case was relatively simple. Thus, with no limitations of generality Ω could be assumed zero at infinity and $\Delta\Omega$ was specified completely in the xy-plane. But in the supersonic case, although $\Delta\Omega$ can be assumed known in the xy-plane and, as will be seen later, Ω may be evaluated on the forward boundary of the region, nothing is known of Ω on the forecone Γ. Hence σ must be chosen properly so that the knowledge of Ω is unnecessary on Γ. The most obvious choice of σ would be a particular solution of equation (4) which would make $\sigma = 0$ on Γ and this is in fact the choice used by Volterra (reference 1) and applied to aerodynamic problems in reference 2. However, if the analogy is to be maintained the choice of σ is not arbitrary but must be the three-dimensional supersonic source corresponding to the fundamental solution $\frac{1}{r}$ in subsonic theory. But such a solution, $\frac{1}{r_c} = \left[(x-x_1)^2 - (y-y_1)^2 - (z-z_1)^2\right]^{-\frac{1}{2}}$ becomes infinite along the forecone Γ which has the equation $(x-x_1)^2 - (y-y_1)^2 - (z-z_1)^2 = 0$.

It is just this difficulty which apparently invalidates any furtherance of the analogy and the prediction in advance of an aerodynamic shape from a distribution of sources and doublets in supersonic flow. However, it is also precisely this difficulty which is overcome by Hadamard's general methods.

Extension of analogy by Hadamard's Method.- The full development of Hadamard's methods cannot be given here, but a rough sketch of his reasoning will perhaps be useful. The basis of his arguments stems from equation (13). First it is admitted that the right-hand side of equation (13) will tend to infinity as the surface S approaches Γ so that $\frac{1}{r_c}$ is not a regular solution to $\Box^2\Omega = 0$ on Γ. However, as has been mentioned, equation (13) still must hold whether or not σ or Ω satisfy the wave equation and thus it still provides an equality. Hence, if the surface integral tends to infinity so also must the volume integral. Further, equation (13) implies that these infinite portions just cancel since the difference of the two integrations must always give zero. To deal with such a problem quantitatively by the usual mathematical techniques would require the study of a limiting process for each new boundary value problem. Hadamard's contribution was the introduction and justification of a concept which removed the necessity for studying the infinite portions involved. This concept is best presented by means of a new notation, thus the sign $\overline{\Big|\qquad\qquad}$ is used and is to be read "the finite part of."

Using this concept it is possible to show that if σ were set equal to $\frac{1}{r_c}$, then equation (13) could be written

$$\overline{\Big|\iiint_R \left(\frac{1}{r_c}\,\Box^2\Omega - \Omega\,\Box^2\frac{1}{r_c}\right)dR}$$

$$= \overline{\Big|\iint_S \left[\Omega\,\frac{\partial}{\partial\nu}\left(\frac{1}{r_c}\right) - \frac{1}{r_c}\,\frac{\partial\Omega}{\partial\nu}\right]dS} \qquad (16)$$

so that the "finite parts" of each side of the equation would be equal. Such a notation would, of course, in general be meaningless since in discarding arbitrarily a part which tended to infinity it would be possible, by proper combinations, to obtain as a remainder <u>any</u> finite value. The fact is, however, that the integrals involved in equation (16) tend to infinity only at a limit of the integration and this limit always involves the forecone Γ. It was consequently possible to devise a manipulative technique to handle equation (16) without considering the singularities individually. It might be mentioned, without stressing the correspondence, that a treatment of improper integrals is also employed in the use of Cauchy's principal value. In subsonic thin-airfoil theory and lifting-line theory integrals of the latter type are well known in the form

$$I_1 = \int_0^c \frac{f(x_0)dx_0}{x-x_0} \qquad 0< x <c$$

I_1 certainly tends to infinity as x_0 approaches x but the use of Cauchy's principal value allows the very large values of the integrand obtained when x_0 is on either side of x to just cancel in such a way that I_1 is finite and unique. So again the integral

$$I_2 = \left|\overline{\int_a^{x_0} \frac{A(x)dx}{(x_0-x)^{3/2}}}\right.$$

is finite and unique and given by Hadamard in the form

$$I_2 = \int_a^{x_0} \frac{A(x)-A(x_0)}{(x_0-x)^{3/2}}\,dx - \frac{2A(x_0)}{(x_0-a)^{1/2}}$$

It is actually possible to generalize the idea of "finite part" to the case when the exponent in the denominator is of the form $j + \frac{1}{2}$ where j is a positive integer but such a generalization is not needed for aerodynamic applications and will therefore be omitted.

In actual calculation, the evaluation of the integral I_2 can be shortened considerably. Thus, if the indefinite integral of

$$\int \frac{A(x)dx}{(x_0-x)^{3/2}}$$

is written in the form $f(x)+C$ then

$$I_2 = \lim_{x \to x_0} \left[\int_a^x \frac{A(x_1)-A(x_0)}{(x_0-x_1)^{3/2}}\,dx_1 - \frac{2A(x_0)}{(x_0-a)^{1/2}} \right]$$

$$= \lim_{x \to x_0} \left[F(x)+C-F(a)-C- \frac{2A(x_0)}{(x_0-x)^{1/2}} \right]$$

It follows that if C is chosen so that

$$C = \lim_{x \to x_0} \left[\frac{2A(x_0)}{\sqrt{x_0-x}} - F(x) \right]$$

then the expression for I_2 may be written $I_2 = -[F(a)+C]$. When C is chosen in this manner, the notation for the calculation may thus be modified to the form

$$\left|\int_a^{x_0} \frac{A(x)dx}{(x_0-x)^{3/2}} = \int_a^{*} \frac{A(x)dx}{(x_0-x)^{3/2}}\right.$$

where the asterisk indicates that the upper limit is not substituted into the indefinite integral, $F(x) + C$.

The technique for the calculation of the finite part has therefore been reduced to three simple steps: first, the indefinite integral $F(x) + C$ is determined, second, the constant C in the indefinite integral is evaluated by means of a limiting process, third, the lower limit of the integral is substituted into the indefinite integral and a minus sign prefixed. As an example, consider the integral

$$\left|\int_a^{x_0} \frac{xdx}{(x_0{}^2-x^2)^{3/2}} = \int_a^{*} \frac{xdx}{(x_0{}^2-x^2)^{3/2}}\right.$$

In this case $F(x) + C = \frac{1}{(x_0{}^2-x^2)^{1/2}} + C$ and

$$C = \lim_{x \to x_0} \left[\frac{1}{(2x_0)^{1/2}(x_0-x)^{1/2}} - \frac{1}{(x_0{}^2-x^2)^{1/2}} \right] = 0$$

so that, finally,

$$\int_a^{*} \frac{xdx}{(x_0{}^2-x^2)^{3/2}} = -[F(a) + C] = \frac{-1}{(x_0{}^2-a^2)^{1/2}}$$

With the aid of this artifice the analogy between the subsonic and supersonic cases can be continued with relative ease. Thus, in equation (16) the right-hand member is zero provided we exclude the point P from the volume of integration. This can be done most easily by limiting the integration to the x_1 = constant plane, a distance ϵ upstream from P. The portion of this plane intersected by the cone, and thus the section over which the integration must be carried, will be denoted by Σ (fig. 3).

As drawn, figure 3 shows a cross section in a y_1 = constant plane for the special case when P is located directly behind and above the foremost disturbance. Applying equation (16) to the regions above and below the disturbance surface τ (plane of the airfoil) yields the two equations

$$\iint_{\lambda+\Gamma+\tau}\left[\Omega\frac{\partial}{\partial\upsilon}\left(\frac{1}{r_c}\right)-\frac{1}{r_c}\frac{\partial\Omega}{\partial\upsilon}\right]dS$$

$$=-\iint_{\Sigma}\left[\Omega\frac{\partial}{\partial\upsilon}\left(\frac{1}{r_c}\right)-\frac{1}{r_c}\frac{\partial\Omega}{\partial\upsilon}\right]dS \tag{17}$$

and

$$\iint_{\lambda+\Gamma+\tau}\left[\Omega'\frac{\partial}{\partial\upsilon'}\left(\frac{1}{r_c}\right)-\frac{1}{r_c}\frac{\partial\Omega'}{\partial\upsilon'}\right]dS=0 \tag{18}$$

where the prime indicates the surface value of Ω on the opposite side of τ from P.

The integration over Σ can be computed for ϵ very small. For convenience, consider P to be the origin; then it follows that since the conormal is in the x_1 direction and the area element can be written $\gamma\, d\gamma\, d\theta$ where $\theta = \text{arc tan}\,\frac{z_1}{y_1}$ and $\gamma = \sqrt{y_1{}^2 + z_1{}^2}$, the right side of equation (17) will give

$$-\left|\iint_{\Sigma}\left[\Omega\frac{\partial}{\partial\upsilon}\left(\frac{1}{r_c}\right)-\frac{1}{r_c}\frac{\partial\Omega}{\partial\upsilon}\right]dS = \lim_{\epsilon\to 0}\left\{\left|\Omega(x,y,z)\int_0^{2\pi}d\theta\int_0^{\epsilon}\frac{-\epsilon\,\gamma\,d\gamma}{[\epsilon^2-\gamma^2]^{3/2}}\right.\right.$$

$$\left.+\left|\frac{\partial\Omega(x,y,z)}{\partial x_1}\int_0^{2\pi}d\theta\int_0^{\epsilon}\frac{\gamma\,d\gamma}{\sqrt{\epsilon^2-r^2}}\right.\right\}$$

$$=\lim_{\epsilon\to 0}\left\{2\pi\Omega(x,y,z)\int_0^{*}\frac{-\epsilon\,\gamma\,d\gamma}{[\epsilon^2-\gamma^2]^{3/2}}+2\pi\frac{\partial\Omega(x,y,z)}{\partial x_1}\int_0^{\epsilon}\frac{\gamma\,d\gamma}{\sqrt{\epsilon^2-\gamma^2}}\right\}=2\pi\Omega(x,y,z)$$

Hence the value of Ω at the point P, $\Omega(x,y,z)$ can be determined from equation (17) with the restriction implied by equation (18). Further, since only the "finite part" is considered, the integration over Γ yields zero and the two equations combine to give

$$\Omega(x,y,z) = -\frac{1}{2\pi}\left|\iint_{\tau}\left[\frac{1}{r_c}\left(\frac{\partial\Omega}{\partial z}-\frac{\partial\Omega'}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{1}{r_c}\right)(\Omega-\Omega')\right]dx_1\,dy_1\right.$$

$$-\frac{1}{2\pi}\left|\iint_{\lambda}\left[\Omega\frac{\partial}{\partial\upsilon}\left(\frac{1}{r_c}\right)-\frac{1}{r_c}\frac{\partial\Omega}{\partial\upsilon}\right]dS\right. -\frac{1}{2\pi}\left|\iint_{\lambda}\left[\Omega'\frac{\partial}{\partial\upsilon'}\left(\frac{1}{r_c}\right)-\frac{1}{r_c}\frac{\partial\Omega'}{\partial\upsilon'}\right]dS\right. \tag{19}$$

The only remaining difference between the subsonic solution for the distribution of sources and doublets, equation (10), and the supersonic solution, equation (19), is the integration over surface λ. The detailed discussion of the contribution of the surface integrals over λ will be deferred to Appendix A. It must suffice for the present to remark that in all applications the integrals over λ in equation (19) are either zero or combine to give zero. It therefore follows that

$$\Omega(x,y,z) = -\frac{1}{2\pi} \left| \int\int_{\tau} \left[\frac{1}{r_c} \left(\frac{\partial \Omega_u}{\partial z} - \frac{\partial \Omega_l}{\partial z} \right) - (\Omega_u - \Omega_l) \frac{\partial}{\partial z} \left(\frac{1}{r_c} \right) \right] dx_1 \, dy_1 \right. \tag{20}$$

and the complete analogue to equation (10) has been established through the use of the concept of finite part.

APPLICATIONS

Interpretation of Previous Results

As a means of indicating the various problems to which equation (20) can be applied, three previously published results will be discussed. These applications include, first, the expression for the perturbation velocity potential of a symmetrical nonlifting airfoil (reference 6), second, the calculation of pressure distribution over a semi-infinite wedge with leading edge swept back of the foremost Mach cone (reference 5) and, third, the integral equation method for determining the load distribution over a lifting surface of arbitrary slope (reference 7).

As in the case of equation (11) for subsonic flow, let Ω represent velocity potential Φ and consider the case of a symmetrical wing at zero angle of attack.

Then $\frac{\partial \Omega_u}{\partial z} = w_u$ and $\frac{\partial \Omega_l}{\partial z} = w_l$, where w_u and w_l are induced vertical velocities on the upper and lower surfaces, respectively. Moreover, $\Phi_u - \Phi_l = 0$ for the symmetrical case so that, since $w_u - w_l = 2\, w_u$,

$$\Phi = -\frac{1}{\pi} \left| \int\int_{\tau} \frac{w_u}{r_c} \, dx_1 \, dy_1 \right.$$

The integral in this equation is finite at Γ so the finite part sign may be disregarded and

$$\Phi = -\frac{1}{\pi} \int\int_{\tau} \frac{w_u}{r_c} \, dx_1 \, dy_1 \tag{21}$$

This equation agrees with results given by Puckett in reference 6.

As another example consider the solution used by R.T. Jones in reference 5 for a nonlifting symmetrical wing. Setting Ω equal to w in equation (20) and using the fact that w and u are related by the expression

$$u = \frac{\partial}{\partial x} \int_{\infty}^{z} w \, dz$$

it follows that

$$u = \frac{1}{\pi} \frac{\partial}{\partial x} \int_{\infty}^{z} dz \left[\iint_{\tau} w_u \frac{\partial}{\partial z} \left(\frac{1}{r_c} \right) dx_1 \, dy_1 \right] \tag{22}$$

For a wedge swept behind the forward Mach cone and having as the equation of its leading edge the relation $x_1 = my_1$, the expression for u may be written in the form

$$u = \frac{w_u}{\pi} \frac{\partial}{\partial x} \int_{\sqrt{x^2 - y^2}}^{z} dz \int_{0}^{Y_1} dy_1 \int_{\frac{y_1}{m}}^{*} \frac{z \, dx_1}{[(x-x_1)^2 - (y-y_1)^2 - z^2]^{3/2}}$$

where

$$Y_1 = \frac{\frac{x}{m} - y - \sqrt{\left(x - \frac{y}{m}\right)^2 + z^2 \left(\frac{1}{m^2} - 1\right)}}{\frac{1}{m^2} - 1}$$

Performing the integration with respect to x_1, it follows that

$$u = \frac{w_u}{\pi} \frac{\partial}{\partial x} \int_{\sqrt{x^2 - y^2}}^{z} z \, dz \int_{0}^{Y_1} \frac{-\left(x - \frac{y_1}{m}\right) dy_1}{[(y-y_1)^2 + z^2] \sqrt{\left(x - \frac{y_1}{m}\right)^2 - (y-y_1)^2 - z^2}}$$

and, after reversing the order of integration

$$u = \frac{w_u}{\pi}\frac{\partial}{\partial x}\int_0^{Y_1} dy_1 \int^z \frac{-\left(x - \frac{y_1}{m}\right) z\,dz}{\sqrt{\left(x - \frac{y_1}{m}\right)^2 - (y-y_1)^2}\;\left[(y-y_1)^2 + z^2\right]\sqrt{\left(x - \frac{y_1}{m}\right)^2 - (y-y_1)^2 - z^2}}$$

$$= k\,\frac{w_u}{2\pi}\frac{\partial}{\partial x}\int_0^{Y_1} - \ln\left[\frac{\sqrt{\left(x - \frac{y_1}{m}\right)^2 - (y-y_1)^2 - z^2} - k\left(x - \frac{y_1}{m}\right)}{\sqrt{\left(x - \frac{y_1}{m}\right)^2 - (y-y_1)^2 - z^2} + k\left(x - \frac{y_1}{m}\right)}\right] dy_1$$

where $k = 1$ for $x > \frac{y_1}{m}$ and -1 for $x < \frac{y_1}{m}$. Taking the partial derivative with respect to x and noting that the value of the logarithm at the upper limit is zero, the value of the induced velocity is

$$u = -\frac{w_u}{\pi}\int_0^{Y_1} \frac{dy_1}{\sqrt{\left(x - \frac{y_1}{m}\right)^2 - (y - y_1)^2 - z^2}}$$

and integration yields the final result

$$u = \frac{w_u}{\pi}\frac{m}{\sqrt{1 - m^2}}\ln\frac{(x - my) - \sqrt{(1 - m^2)(x^2 - y^2 - z^2)}}{\sqrt{(mx - y)^2 + z^2(1 - m^2)}}$$

Denoting pressure coefficient $\frac{\Delta p}{q}$ by $-\frac{2u}{V}$ and setting $\frac{w_u}{V}$ equal to $\left(\frac{dz}{dx}\right)_o$ the slope of the surface, this may be written

$$\frac{\Delta p}{q} = \frac{2}{\pi}\left(\frac{dz}{dx}\right)_o \frac{m}{\sqrt{1 - m^2}}\ln\frac{(x - my) - \sqrt{(1 - m^2)(x^2 - y^2 - z^2)}}{\sqrt{(mx - y)^2 + z^2(1 - m^2)}} \qquad (23)$$

Equation (23) gives the pressure coefficient at any point in the field produced by a wedge swept behind the Mach cone. When z is set equal to zero the pressure distribution over the wedge itself is determined and the equation corresponds exactly with equation (12) of reference 5.

When loading is to be prescribed over a thin lifting surface, Ω may be assumed equal to the perturbation velocity u. A direct consequence of this assumption is that in equation (20)

$$\frac{\partial \Omega_u}{\partial z} = \frac{\partial \Omega_l}{\partial z}$$

since, from conditions of irrotationality,

$$\frac{\partial u_u}{\partial z} = \frac{\partial w_u}{\partial x} = \frac{\partial w_l}{\partial x} = \frac{\partial u_l}{\partial z}$$

By definition

$$\frac{\Delta p_u}{q} = -\frac{2u_u}{V}, \quad \frac{\Delta p_l}{q} = -\frac{2u_l}{V}$$

and load distribution in coefficient form, ΔC_p, is given by the relation

$$\Delta Cp = \frac{\Delta p_l}{q} - \frac{\Delta p_u}{q}$$

so that

$$\Delta Cp = -\frac{2u_l}{V} + \frac{2u_u}{V} = -2\frac{\Delta u}{V}$$

Equation (20) can therefore be written in the form

$$\frac{u}{V} = -\frac{1}{4\pi}\left[\iint_T \Delta Cp \frac{\partial}{\partial z}\left(\frac{1}{r_c}\right) dx_1\, dy_1\right]$$

$$= \frac{1}{4\pi}\left[\iint_T \Delta Cp \frac{z\, dx_1\, dy_1}{\sqrt{[(x-x_1)^2 - (y-y_1)^2 - z^2]^3}}\right] \tag{24}$$

If equation (24) is transformed to the original space variables, the relation for u is

$$\frac{u(X,Y,Z)}{V} = \left| \frac{\beta^2}{4\pi} \iint_{\tau} \frac{\Delta Cp\ Z\ dX_1\ dY_1}{\left\{(X-X_1)^2 - \beta^2\left[(Y-Y_1)^2 + Z^2\right]\right\}^{3/2}} \right. \tag{25}$$

Equation (25) is valid for arbitrary plan forms with known load distributions. Particular examples which may be worked out with relative ease are the lifting surfaces carrying constant load. Once u is known the value of w can be determined from the integral

$$w = \frac{\partial}{\partial z} \int_{\infty}^{x} u dx$$

and from w the ordinate z of the surface as a function of x and y is given by

$$z = \int_{l.e.}^{x} \frac{w}{V} dx_o$$

where $l.e.$ denotes the leading edge. A discussion of trapezoidal, rectangular, and triangular plan forms with constant loading is given in reference 2 although the method of derivation is different.

Interest in constantly loaded wings has been based primarily on the fact that in certain cases they can be combined to produce surfaces of given camber. Thus, a superposition of trapezoidal plan forms of variable rake, the constant loading over each trapezoid being a function of its rake angle, can be used to produce a flat plate of trapezoidal or rectangular plan form at an arbitrary angle of attack. In this case the loading as a function of rake angle is determined so that induced vertical velocity is kept constant. For problems in conical flow a lifting element can be constructed by subtracting from a constantly loaded right triangle with angle of sweep equal to δ the constantly loaded right triangle with sweep angle equal to $\delta - d\delta$. The resultant element carries a constant load and has a sweep angle equal to δ. By summing these elements it is possible to find the load distribution as a function of δ such that certain flat lifting surfaces at angles of attack are formed. In reference 2, triangular wings swept back of the Mach cone were studied by this method for arbitrary angles of yaw. Brown (reference 7) has used this same lifting element to study the more restrictive case of the symmetrical triangular wing.

A brief discussion of differences existing between the methods for producing the swept-back lifting element will shed some light on the various lines of attack. The approach used in reference 2 is essentially mathematical in that a particular solution of the partial differential equation is used in conjunction with Green's theorem to satisfy the boundary conditions of the problem. The principal criticism of such a method is that the physical interpretation is missing. The use of equation (25), however, removes all such criticisms for precisely as in the case of incompressible flow the lifting element is created by distributing doublets over the wing. In Brown's solution it was necessary for him to determine first a line of sources by means of an integration along the line and then to form the doublet line by differentiating along the normal to the line. The order of differentiation for incompressible flow is immaterial, since the limits in the integral are independent of the position of the point P at X, Y, Z. Supersonic flow destroys this property and it is only after the introduction of the concept of "finite part" that the derivative of an integral may be written as the integral of the differential coefficient of the integrand. Equation (25) thus simplifies the analysis and at the same time maintains the analogy with previous work.

Load Distribution for Rolling Wing

The usefulness of equation (20) is not at all restricted to a synthesis of previously known solutions. As an example of its generality consider its application to the problem of the rolling wing with leading edge swept ahead of the Mach cone. Figure 4 shows the boundary conditions involved. The value of w is specified over the wing and, since the Mach cone is behind the leading edge, the value of the perturbation velocities u, v, and w are of course zero ahead of the leading edge. Assume for the moment that a symmetrical body at zero angle of attack is considered. It follows that if $\Omega_u = w_u$ and $\Omega_l = w_l$ then equation (20) can be written in the form

$$w = \frac{1}{\pi} \,\Big|\!\!\!\! \iint_{\tau} w_u \frac{\partial}{\partial z}\left(\frac{1}{r_c}\right) dx_1 \, dy_1 \qquad (26)$$

since, for reasons of symmetry, the normal gradients of w on the two surfaces are equal. Using now the fact that the Mach cone is behind the leading edge then the pressure over the upper surface is independent of the shape of the lower surface and equation (26) may be applied directly to the rolling flat plate if w_u is determined from the given induced velocity on either the upper or lower surface. This method of approach, of course, limits the solution to cases where the leading edge is ahead of the Mach cone.

If the rate of roll is given as P radians per second then $2w_u = 2PY_1$ and equation (26) becomes

$$w = \frac{P}{\pi} \left| \iint_\tau \frac{\beta^2 Y_1 Z \, dY_1 \, dX_1}{[(X-X_1)^2 - \beta^2 (Y-Y_1)^2 - \beta^2 Z^2]^{3/2}} \right. \quad (27)$$

The area τ in equation (27) is that contained between the leading edge and the trace of the forecone on the XY-plane. Figure 5(a) shows the configuration for three traces corresponding to forecones from the points P_1, P_2, and P_3. The region containing the point P_2 is distinguished from that containing P_1 and P_3 by the fact that τ for P_2 lies ahead of the Mach cone from the apex and, furthermore, entirely on the right of the x_1 axis. The regions corresponding to P_1 and P_3 differ in the fact that when integrating from $+\infty$ to z to find u, the upper limit of the integral in the first case is the Mach cone $X^2 - \beta^2 Y^2 - \beta^2 Z^2 = 0$ whereas in the latter case the upper limit is the leading-edge wedge $Z = \frac{mX - Y}{\sqrt{m^2\beta^2 - 1}}$ (fig. 5(b)).

The solution must be carried out separately for each of these regions but only the details for the region corresponding to P_1 need be given here since the others are similar.

It follows that the induced velocity u at the point P is given as the sum of the triple integrals.

$$u = \frac{\partial}{\partial X} \frac{P}{\pi} \int_{\frac{1}{\beta}\sqrt{X^2-\beta^2 Y^2}}^{Z} dZ_o \int_{A(1,Z_o)}^{o} Y_1 dY_1 \int_{-\frac{Y_1}{m}}^{*} \frac{\beta^2 Z_o dX_1}{[(X-X_1)^2 - \beta^2 (Y-Y_1)^2 - \beta^2 Z_o^2]^{3/2}}$$

$$+ \frac{\partial}{\partial X} \frac{P}{\pi} \int_{\frac{1}{\beta}\sqrt{X^2-\beta^2 Y^2}}^{Z} dZ_o \int_{o}^{A(-1,Z_o)} Y_1 dY_1 \int_{\frac{Y_1}{m}}^{*} \frac{\beta^2 Z_o dX_1}{[(X-X_1)^2 - \beta^2 (Y-Y_1)^2 - \beta^2 Z_o^2]^{3/2}}$$

$$= \sum_{k=-1,1} \frac{\partial}{\partial X} \frac{P}{\pi} \int_{\frac{1}{\beta}\sqrt{X^2-\beta^2 Y^2}}^{Z} dZ_o k \int_{A(k,Z_o)}^{o} \frac{Z_o Y_1 \left(X + k \frac{Y_1}{m}\right) dY_1}{[(Y-Y_1)^2 + Z_o^2] \sqrt{\left(X + k\frac{Y_1}{m}\right)^2 - \beta^2 (Y-Y_1)^2 - \beta^2 Z_o^2}}$$

(28)

where $A(k, Z_o)$ is the value of Y_1 determined by the intersection of the forecone with the leading edge on the right and left sides with k equal respectively to -1 and $+1$. See fig. 5(a).) Thus

$$A(k,Z_o) = \frac{-(k\frac{X}{m} + \beta^2 Y) + k\beta\sqrt{(X + k\frac{Y}{m})^2 + Z_o^2(\frac{1}{m^2} - \beta^2)}}{\frac{1}{m^2} - \beta^2}$$

After reversing the order of integration and integrating with respect to Z_o, it follows that

$$u = \sum_{k=-1,1} k\frac{P}{2\pi}\frac{\partial}{\partial X}\int_{A(k,Z)}^{o} Y_1 dY_1 \ln\left[\frac{(X+k\frac{Y_1}{m}) - \sqrt{(X+k\frac{Y_1}{m})^2 - \beta^2(Y-Y_1)^2 - \beta^2 Z^2}}{(X+k\frac{Y_1}{m}) + \sqrt{(X+k\frac{Y_1}{m})^2 - \beta^2(Y-Y_1)^2 - \beta^2 Z^2}}\right]$$

Moreover, since the integrand is zero at $Y_1 = A(k,Z)$ the derivative with respect to X can be taken inside the integral and

$$u = -\sum_{k=-1,1} \frac{P}{\pi} k \int_{A(k,Z)}^{o} \frac{Y_1\, dY_1}{\sqrt{(X + \frac{Y_1}{m}k)^2 - \beta^2\,(Y-Y_1)^2 - \beta^2 Z^2}}$$

Integrating in this equation and combining terms it follows that induced velocity u is given by the expression

$$u = \frac{P}{\pi} m^2 \frac{m\beta^2 Y + X}{(m^2\beta^2-1)^{3/2}}\left[\arcsin\left(\frac{m\beta^2 Y + X}{\beta\sqrt{(mX+Y)^2 + (1-m^2\beta^2)Z^2}}\right) - \frac{\pi}{2}\right]$$

$$- \frac{P}{\pi} m^2 \frac{m\beta^2 Y - X}{(m^2\beta^2-1)^{3/2}}\left[\arcsin\left(\frac{m\beta^2 Y - X}{\beta\sqrt{(mX-Y)^2 + (1-m^2\beta^2)Z^2}}\right) + \frac{\pi}{2}\right]$$

(29)

Setting $Z = 0$ in equation (29), pressure coefficient $\frac{\Delta p}{q} = -\frac{2u}{V}$ is given by the expression

$$\frac{\Delta p}{q} = \frac{2P}{\pi V} m^2 \left\{ \frac{m\beta^2 Y - X}{(m^2\beta^2 - 1)^{3/2}} \text{arc sin} \left[\frac{m\beta^2 Y - X}{\beta(mX - Y)} \right] \right.$$

$$\left. - \frac{m\beta^2 Y + X}{(m^2\beta^2 - 1)^{3/2}} \text{arc sin} \left[\frac{m\beta^2 Y + X}{\beta(mX + Y)} \right] + \frac{m\beta^2 Y \pi}{(m^2\beta^2 - 1)^{3/2}} \right\} \quad (30)$$

when $X > \beta Y$.

This solution holds in both regions containing the points P_1 and P_3 . However, in the region ahead of the Mach cone but still on the wing (region corresponding to P_2) it is easy to show that

$$\frac{\Delta p}{q} = \frac{2Pm^2}{V(m^2\beta^2 - 1)^{3/2}} (m\beta^2 Y - X) \quad (31)$$

where $\frac{Y}{m} < X < \beta Y$. Figure 6 shows a spanwise plot of $\frac{\Delta p}{q}$ for $m = 2$ and $\beta = 1$.

Equations (30) and (31) provide sufficient information for the calculation of the stability derivative for damping in roll, C_{l_p}.

Integration of the load distribution yields the result that

$$C_{l_p} = \frac{\partial C_l}{\partial (Pb/2V)} = -\frac{1}{3\beta}$$

Load Distribution for Pitching Wing

Another simple application of equation (20) is found in the solution to the problem of the pitching wing. Figure 7 shows the boundary condition involved which is that the vertical induced velocity be a linear function of X_1. If Q is the rate of pitch in radians per second, then $w = QX_1$ on the wing. Again the solution is obtained only for wings which have leading edges swept ahead of the Mach cone. (Although solutions can be obtained for leading edges swept behind the Mach cone, they involve integral equations and do nothing to illustrate further the direct methods of this report.)

In the rolling wing case, Ω was set equal to perturbation velocity w and as a result a distribution of doublets was used in equations (27) and (28). As an example of the manner in which source-sink distributions may be used for the same type of problem, equation (22) will be applied in the present case. Since the wing is swept ahead of the foremost Mach cone, induced effects on the upper and lower surface are independent and

$$\Phi = -\frac{1}{\pi} \iint_T \frac{QX_1 \, dX_1 \, dY_1}{\sqrt{(X-X_1)^2 - \beta^2 (Y-Y_1)^2 - \beta^2 Z^2}} \tag{32}$$

Again three regions containing the points P_1, P_2, and P_3 are distinguished (fig. 5(a)) and the solutions will be derived only for the region containing P_1. Integrating first with respect to Y_1 and then differentiating with respect to X yields

$$u = \frac{\partial\Phi}{\partial X} = -\frac{Q}{\beta}(X-\beta Z) + \frac{Q}{\pi\beta}\sum_{k=-1,1} k \int_0^{B(k,Z)} X_1 \frac{\partial}{\partial X} \text{arc sin} \frac{Y-kmX_1}{\sqrt{(X-X_1)^2 - \beta^2 Z^2}} \, dX_1 \tag{33}$$

where

$$B(k,Z) = m \frac{X - \beta^2 Ykm - \beta k \sqrt{(Xm-kY)^2 + Z^2 (1-m^2\beta^2)}}{1 - m^2\beta^2}$$

Considering the limit as $Z \longrightarrow 0$ and integrating gives:

(since $\frac{\Delta p}{q} = -\frac{2u}{V}$)

$$\left(\frac{\Delta p}{q}\right)\frac{V}{2Q}(m^2\beta^2-1)^{3/2} = \frac{2m}{\pi}\sqrt{(X^2-\beta^2Y^2)(m^2\beta^2-1)} - mX(2-m^2\beta^2)$$

$$-\frac{mX(2-m^2\beta^2)-Y}{\pi}\text{ arc sin}\left(\frac{X-\beta^2Ym}{\beta Y-\beta mX}\right)$$

$$+\frac{mX(2-m^2\beta^2)+Y}{\pi}\text{ arc sin}\left(\frac{X+\beta^2Ym}{\beta Y+\beta mX}\right) \qquad (34)$$

Formula (34) is valid for the regions P_1 and P_3 of figure 5(a). For the region P_2 the solution is:

$$\left(\frac{\Delta p}{q}\right)\frac{V}{2Q}(m^2\beta^2-1)^{3/2} = Y - 2Xm + Xm^3\beta^2 \qquad (35)$$

Equations (34) and (35) provide sufficient information for the calculation of the stability derivative for the damping in pitch m_q. Values of $\left(\frac{\Delta p}{q}\right)$ $\left(\frac{1/Qb}{2V}\right)$ as determined from these equations are plotted in figure 8 for $m = 2$ and $\beta = 1$.

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APPENDIX A

Discussion of Conditions on Surface λ

By definition λ is the surface on which the streamlines of the flow first experience pressure disturbances, that is, the surface along which the stream first becomes aware of the existence of the wing. Figures 2(a) and 2(b) were introduced to show the nature of the configurations involved for two different plan forms. It is apparent that when the wing is swept ahead of the foremost Mach cone the wedge-like form of λ is comparable to the wedge appearing in purely two-dimensional problems while the wing swept back of the Mach cone has for its surface a conical surface and thus may be thought of as involving a purely three-dimensional problem.

In order to determine the value of Φ on λ it is sufficient to impose the condition that the tangential component of velocity is continuous across λ . Such a condition represents no essential restriction since it is an immediate consequence of continuity of mass flow and continuity of the tangential component of momentum across the surface. As a result of this condition, however, it follows that the tangential component of the perturbation velocity is zero on the downstream surface of λ since it is obviously zero on the upstream surface. Moreover, velocity being equal to the gradient of the velocity potential the perturbation-velocity potential must be equal to a constant on λ . But an arbitrary constant can be added or subtracted from the velocity potential so that with no loss of generality the value of Φ on λ can be assumed equal to zero and, since the conormal lies on the surface λ , $\frac{\partial \Phi}{\partial \upsilon}$ is also zero.

The complete analogue to equation (10) has now been developed for $\Omega = \Phi$ so that

$$\Omega(x,y,z) = -\frac{1}{2\pi}\sqrt{\iint_{\tau}\left[\frac{1}{r_c}\left(\frac{\partial \Omega_u}{\partial z} - \frac{\partial \Omega_l}{\partial z}\right)\right] - (\Omega_u - \Omega_l)\frac{\partial}{\partial z}\left(\frac{1}{r_c}\right)dx_1\, dy_1} \tag{A1}$$

When Ω is equal to one of the perturbation-velocity components, it is obvious that boundary conditions over λ and τ cannot be considered to be absolutely arbitrary since it is necessary to include the added restriction that the resultant potential Φ also satisfies the equation

$$\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Considering first the case where the wing is swept behind the Mach cone, it follows that

$$\Phi = \int_{\sqrt{z^2+y^2}}^{x} u(x_1,y,z)\,dx_1 = \int_{\sqrt{x^2-z^2}}^{y} v(x,y_1,z)\,dy_1$$

$$= \int_{\sqrt{x^2-y^2}}^{z} w(x,y,z_1)\,dz_1$$

and, after evaluating the partial derivatives of Φ and substituting in the given differential equation, direct calculation leads to the conclusion that on λ the following differential equations hold

$$2y \frac{\partial u_1}{\partial y} + 2z \frac{\partial u_1}{\partial z} + u_1 = 0$$

$$2x \frac{\partial v_1}{\partial x} + 2z \frac{\partial v_1}{\partial z} + v_1 = 0$$

$$2x \frac{\partial w_1}{\partial x} + 2y \frac{\partial w_1}{\partial y} + w_1 = 0$$

where u_1, v_1, and w_1 are the values of u, v, and w on λ. The general solutions of these linear partial differential equations can be written as follows

$$u_1 = \frac{1}{\sqrt{y^2+z^2}} f_1\left(\frac{y}{z}\right), \quad v_1 = \frac{1}{\sqrt{x^2+z^2}} f_2\left(\frac{z}{x}\right), \quad w_1 = \frac{1}{\sqrt{x^2+y^2}} f_3\left(\frac{x}{y}\right)$$

It has been stated, however, that the tangential component of the total perturbation-velocity vector vanishes on λ , or, in analytical terms

$$l u_1 + m v_1 + n w_1 = 0$$

where l, m, n are direction numbers of any tangent to λ and therefore satisfy the relation

$$l x - m y - n z = 0$$

Substituting the known expressions for u_1, v_1, w_1, it follows that

$$\frac{l}{y} \frac{y}{\sqrt{y^2+z^2}} f_1\left(\frac{y}{z}\right) + \frac{m}{z} \frac{z}{\sqrt{x^2+z^2}} f_2\left(\frac{z}{x}\right) + \frac{n}{x} \frac{x}{\sqrt{x^2+y^2}} f_3\left(\frac{x}{y}\right) = 0$$

or, using a different notation,

$$\frac{l}{y} F_1 \left(\frac{y}{z}\right) + \frac{m}{z} F_2 \left(\frac{z}{x}\right) + \frac{n}{x} F_3 \left(\frac{x}{y}\right) = 0 \tag{A2}$$

Consider now the special case when $l = 0$ and $m = -\frac{nz}{y}$. Under these conditions

$$-\frac{n}{y} F_2 \left(\frac{z}{x}\right) + \frac{n}{x} F_3 \left(\frac{x}{y}\right) = 0$$

so that

$$F_2 \left(\frac{z}{x}\right) = \frac{y}{x} F_3 \left(\frac{x}{y}\right)$$

Since the variables $\frac{z}{x}$ and $\frac{x}{y}$ are separated, the solution of this equation may be written

$$F_2 \left(\frac{z}{x}\right) = K \quad \text{and} \quad F_3 \left(\frac{x}{y}\right) = K \frac{x}{y}$$

where K is a constant. Returning now to the case where l, m, and n are in the ratio x: y: z, direct substitution into equation (A2) gives

$$\frac{x}{y} F_1 \left(\frac{y}{z}\right) + \frac{y}{z} F_2 \left(\frac{z}{x}\right) + \frac{z}{x} F_3 \left(\frac{x}{y}\right) = 0$$

so that

$$\frac{x}{y} F_1 \left(\frac{y}{z}\right) + K \frac{y}{z} + \frac{z}{x} K \frac{x}{y} = 0$$

and

$$F_1 \left(\frac{y}{z}\right) = -K \frac{y}{x} \left(\frac{z}{y} + \frac{y}{z}\right)$$

This equation can, however, be written in the form

$$\frac{F_1\left(\frac{y}{z}\right)}{\frac{z}{y} + \frac{y}{z}} = -K\frac{y}{x}$$

from which it follows that $K = 0$ and $F_1 = F_2 = F_3 = 0$. All perturbation velocity components are seen to be zero, consequently equation (A1) is valid for all cases in which the wing is swept back of the Mach cone.

A discussion of conditions on the surface λ will next be given for the case where the leading edge of the wing lies along the y-axis (fig. 2(a)) and where Ω represents u or w. The perturbation-velocity potential Φ may be given by the relation

$$\Phi(x,y,z) = \int_{\pm z}^{x} u(x_1,y,z)\, dx_1$$

where the plus and minus signs in the limit apply, respectively, to conditions above and below the xy-plane. Since Φ must satisfy the basic differential equation, an added restriction is imposed on u and as a result of this condition it can be shown that

$$\frac{\partial u_1}{\partial z} = 0$$

where u_1 is the value of u on either the lower or upper surface of the wedge. It follows that the values of u on the two surfaces are

$$u_1 = f_1(x,y) \quad \text{and} \quad u_1 = f_2(x,y)$$

and, since the solution is independent of z, and x is proportional to z in both cases, the final expressions are

$$u_1 = f_1(y) \quad \text{and} \quad u_1 = f_2(y)$$

If $\Phi(x,y,z)$ is defined as an integral involving w, the same type of analysis leads to the conclusion that w on the two surfaces can also be expressed as functions of y alone. Perturbation velocity v will not be considered for this type of leading edge since the inclusion of u and w covers all commonly used boundary conditions.

It remains to substitute the results just obtained into equation (19) in order to study the contribution of the integrals over λ . Apparently only one term in each integrand need be considered since the conormal is perpendicular to the y-axis and the gradient of Ω in that direction vanishes. As a preliminary step to setting up the integrals it is convenient to introduce a new coordinate system x'', y'', z'' which is obtained by rotating the axial system about the y-axis so that the x''- and z''-axes lie respectively in the lower and upper wedge planes while the y''-axis coincides with the y-axis. The transformation of variables is

$$x'' = \frac{1}{\sqrt{2}}(x_1 - z_1)$$

$$z'' = \frac{1}{\sqrt{2}}(x_1 + z_1)$$

When $\Omega = u$, the last two integrals in equation (19) may now be written

$$-\frac{1}{2\pi}\left|\int_{y-\sqrt{x^2-z^2}}^{y+\sqrt{x^2-z^2}} f_1(y_1)\,dy_1 \int_0^{\text{cone}} \frac{\partial}{\partial z'}\left(\frac{1}{r_c}\right) dz''\right.$$

$$-\frac{1}{2\pi}\left|\int_{y-\sqrt{x^2-z^2}}^{y+\sqrt{x^2-z^2}} f_2(y_1)\,dy_1 \int_0^{\text{cone}} \frac{\partial}{\partial x'}\left(\frac{1}{r_c}\right) dx''\right.$$

Substituting for r_c, this expression becomes

$$-\frac{1}{2\pi}\int_{y-\sqrt{x^2-z^2}}^{y+\sqrt{x^2-z^2}} f_1(y_1)dy_1\int_0^{*}\frac{\partial}{\partial z''}\frac{dz''}{\sqrt{\left(x-\frac{1}{\sqrt{2}}z''\right)^2-(y-y_1)^2-\left(z-\frac{1}{\sqrt{2}}z''\right)^2}}$$

$$-\frac{1}{2\pi}\int_{y-\sqrt{x^2-z^2}}^{y+\sqrt{x^2-z^2}} f_2(y_1)dy_1\int_0^{*}\frac{\partial}{\partial x''}\frac{dx''}{\sqrt{\left(x-\frac{1}{\sqrt{2}}x''\right)^2-(y-y_1)^2-\left(z+\frac{1}{\sqrt{2}}x''\right)^2}}$$

or

$$-\frac{1}{2\pi}\int_{y-\sqrt{x^2-z^2}}^{y+\sqrt{x^2-z^2}}\frac{f_1(y_1)dy_1}{\sqrt{x^2-(y-y_1)^2-z^2}}$$

$$-\frac{1}{2\pi}\int_{y-\sqrt{x^2-z^2}}^{y+\sqrt{x^2-z^2}}\frac{f_2(y_1)dy_1}{\sqrt{x^2-(y-y_1)^2-z^2}}$$

It is apparent that if $f_1(y) = -f_2(y)$ the integrals combine to give zero so that equation (A1) may again be used in all calculations; moreover, the same condition applies when $\Omega = w$. The assumption that $f_1(y) = -f_2(y)$ is equivalent to postulating that in all cases $f_1(y)$ and $f_2(y)$ are odd functions of y. In application, however, this property is always maintained.

It remains finally to consider the case when the leading edge of a wing is swept ahead of the Mach cone and when Ω is a perturbation-velocity component. As a means of avoiding unnecessary complication in treating the problem it is possible to substitute first the transformation (reference 5)

$$\xi = -y + mx$$
$$\eta = -x + my$$
$$\zeta = \sqrt{m^2-1}\,z$$

where the leading edge has the equation $y = mx$, $z = 0$, and $m > 1$. Direct calculation shows that the basic differential equation and the Mach cone are invariant under the transformation and that in the new oblique coordinate system the leading edge lies along the η-axis while the planes of the wedge become

$$\xi \mp \zeta = 0$$

Because of the invariant properties of the transformation, and the fact that the $z = 0$ plane is fixed, equation (19) is applicable directly to the boundary value problem for the swept wing in the new coordinate system. The treatment of the integrals over λ can therefore be developed algebraically in exactly the same manner that applied to the previous case, hence u and w are constant along the lines

$$\xi \mp \zeta = 0, \quad \eta = \text{const}$$

and, again, if conditions of skew symmetry are maintained above and below the $z = \zeta = 0$ plane the integration over the surfaces λ cancel. Thus equation (20) is seen to be valid for Ω equal to perturbation-velocity potential or perturbation velocity for all types of striaght leading-edge configurations. And this is the complete analogue of the subsonic theory.

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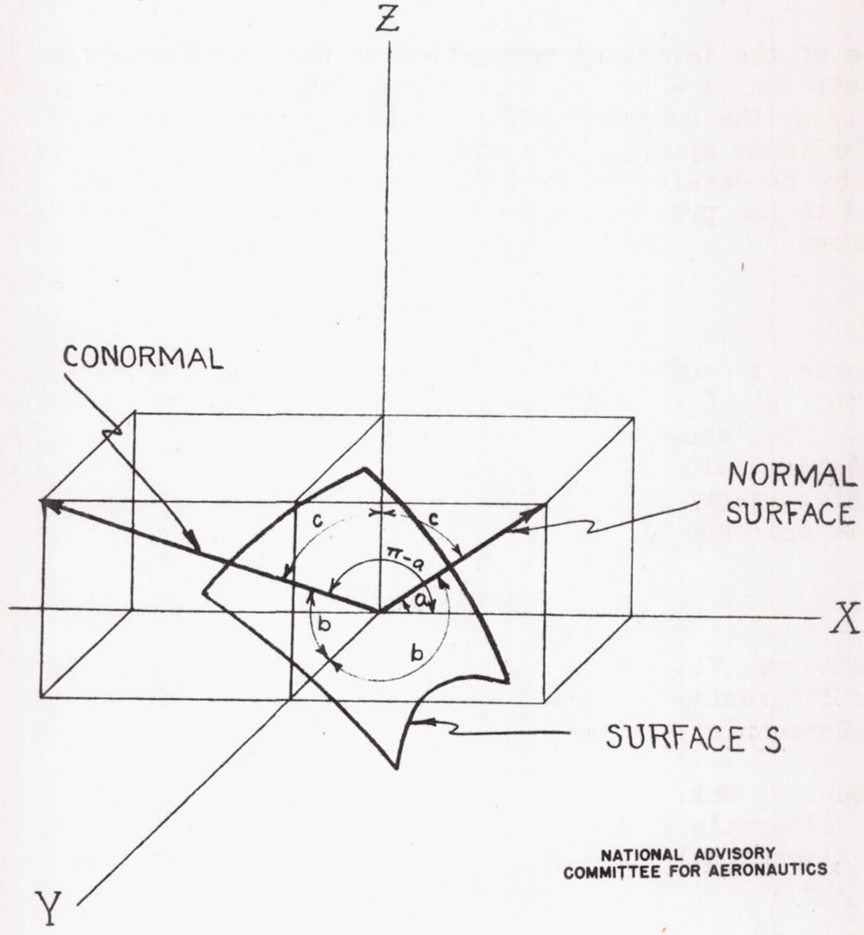
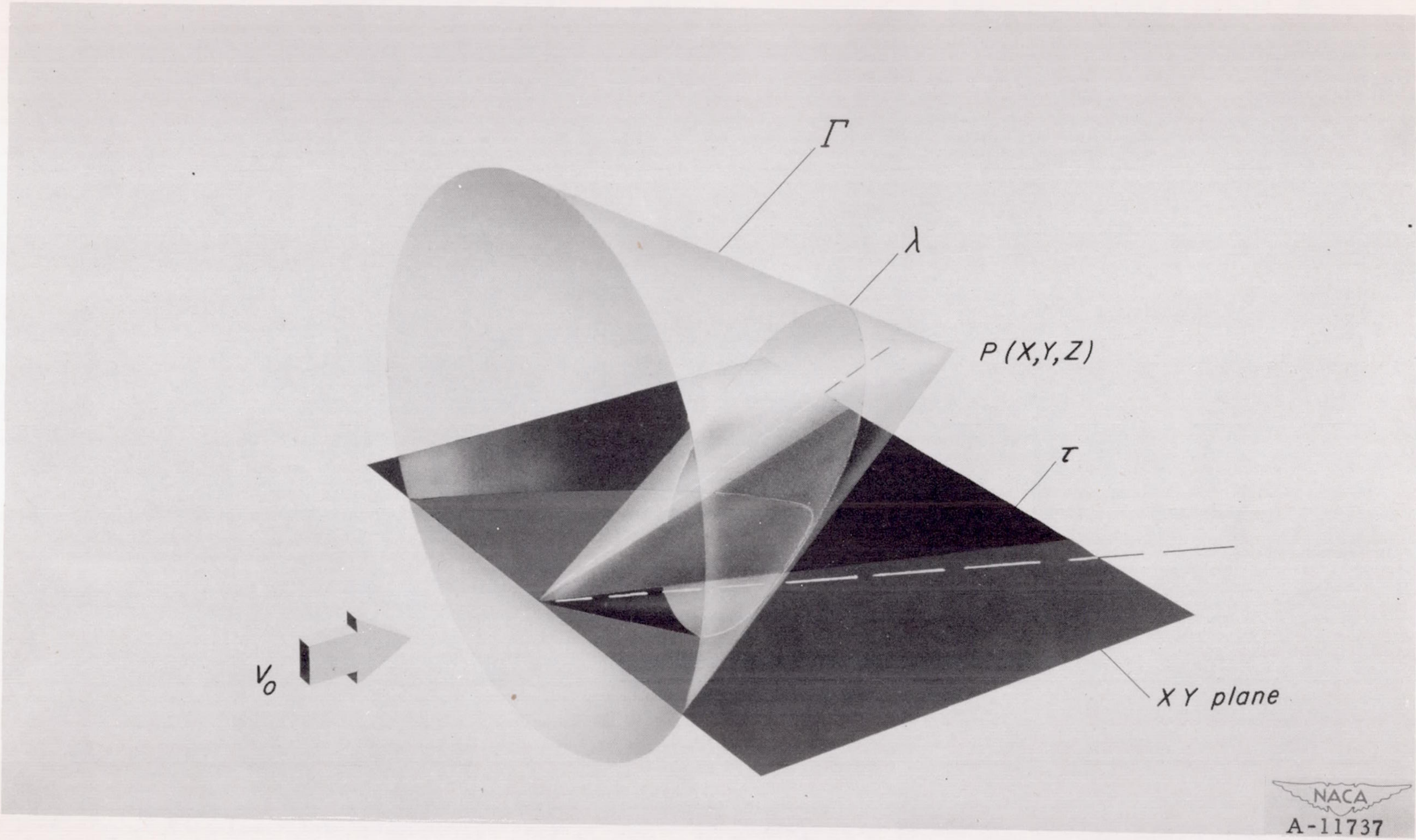


$n_1 = \cos a$ $\qquad \nu_1 = \cos(\pi - a)$

$n_2 = \cos b$ $\qquad \nu_2 = \cos b$

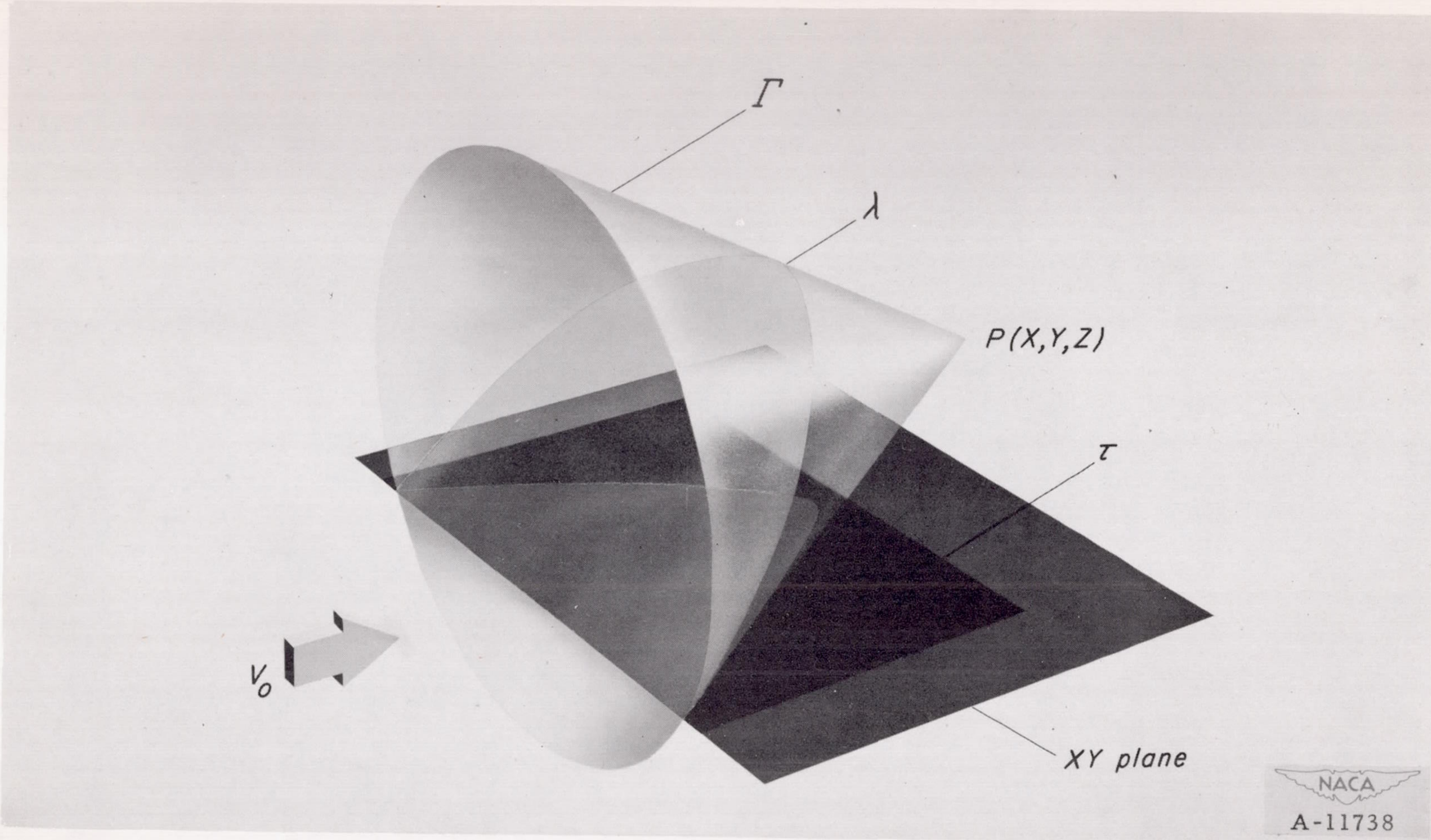
$n_3 = \cos c$ $\qquad \nu_3 = \cos c$

FIGURE 1 - GEOMETRIC RELATIONS BETWEEN DIRECTION COSINES (n_1, n_2, n_3) OF NORMAL AND DIRECTION COSINES (ν_1, ν_2, ν_3) OF CONORMAL TO SURFACE S.



(a) rectangular plan form

Figure 2.– Mach forecone from point P(X,Y,Z) intersecting surface τ



(b) triangular plan form

Figure 2.- Concluded.

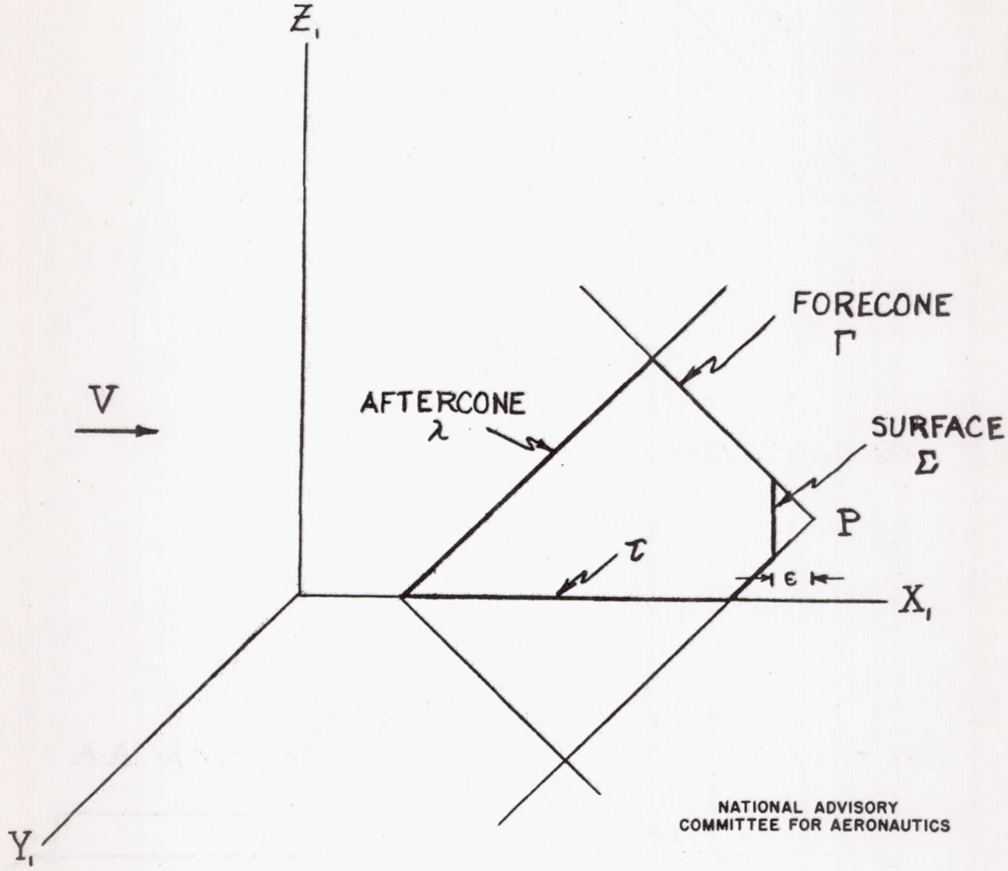
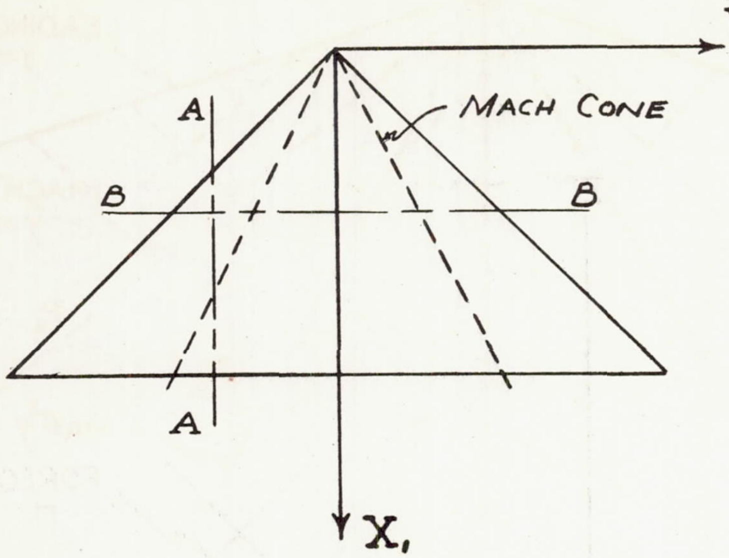
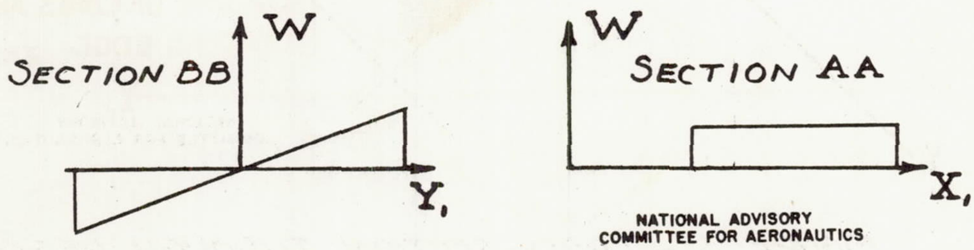


FIGURE 3 – CROSS SECTION THROUGH REGION OF INTEGRATION USED TO OBTAIN EQUATION 19.

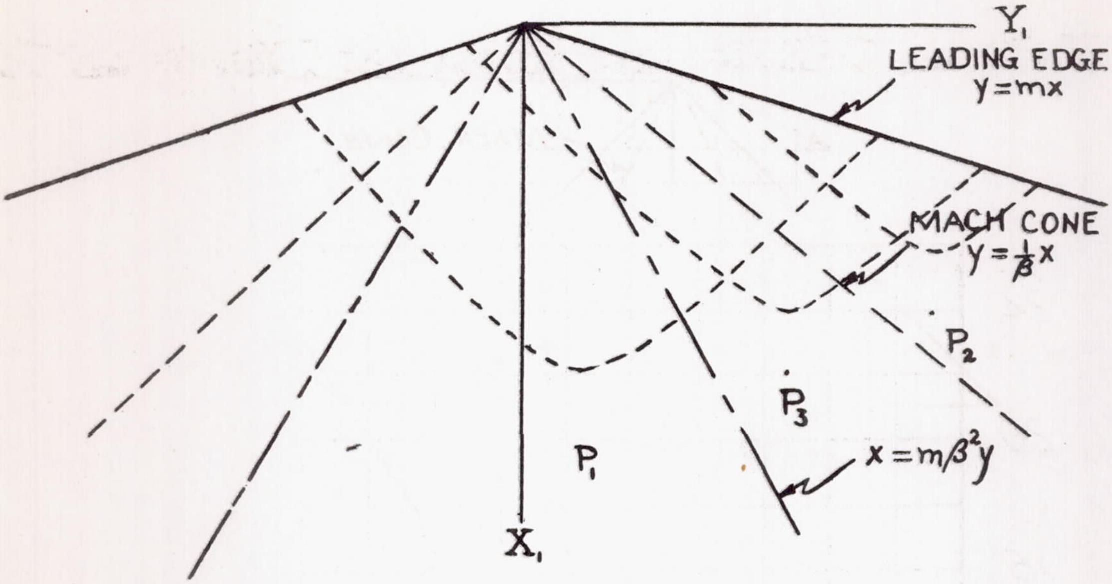


(a) PLANFORM.

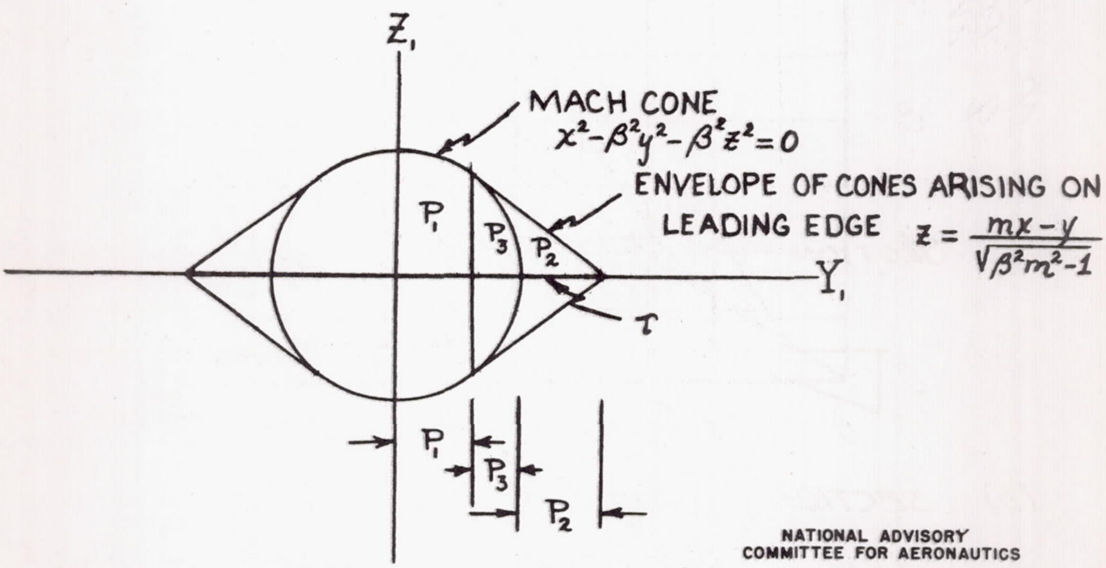


(b) SECTIONS SHOWING DISTRIBUTION.

FIGURE 4 – VERTICAL VELOCITY DISTRIBUTION FOR ROLLING WING.



(a) TRACES IN $Z_1=0$ PLANE.



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(b) TRACES IN X_1 = CONSTANT PLANE.

FIGURE 5 – INTEGRATION REGIONS USED IN DETERMINING LOADING OVER TRIANGULAR WING SWEPT AHEAD OF MACH CONE.

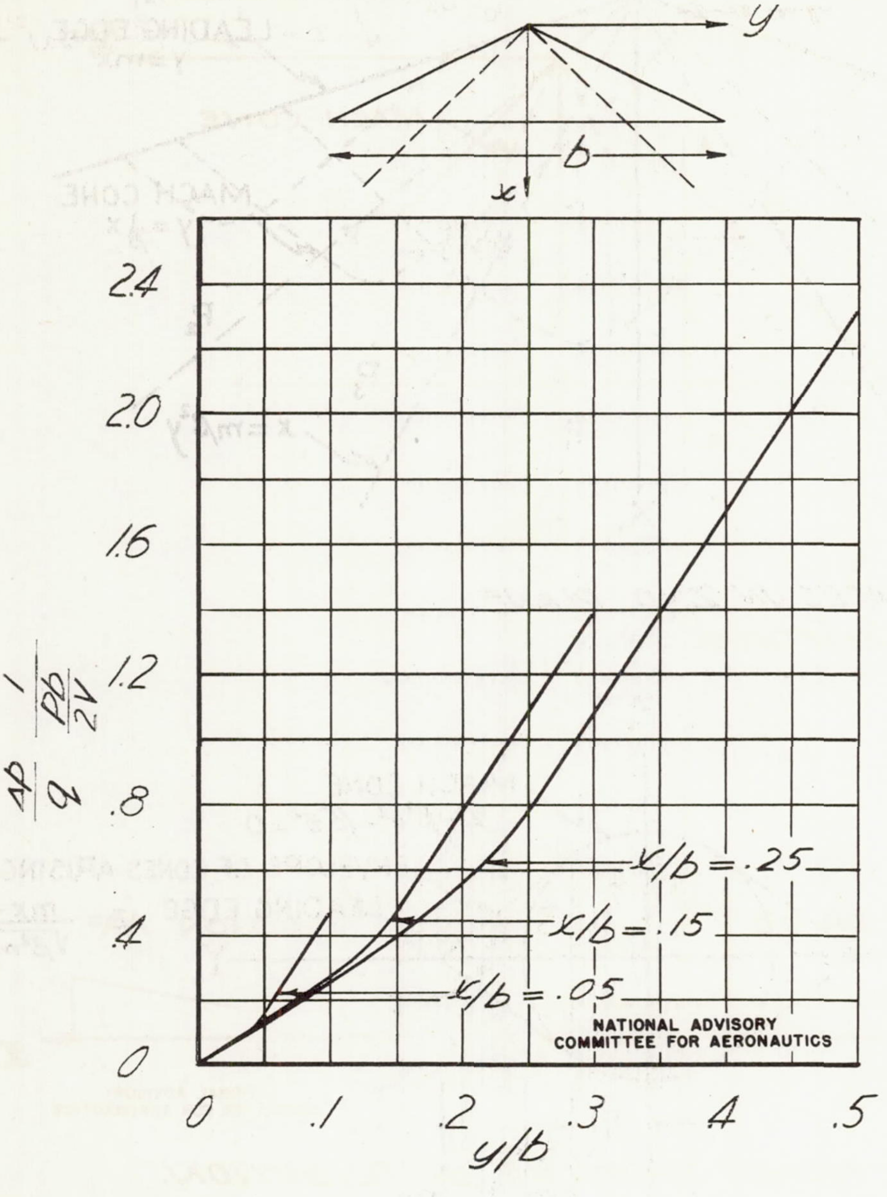
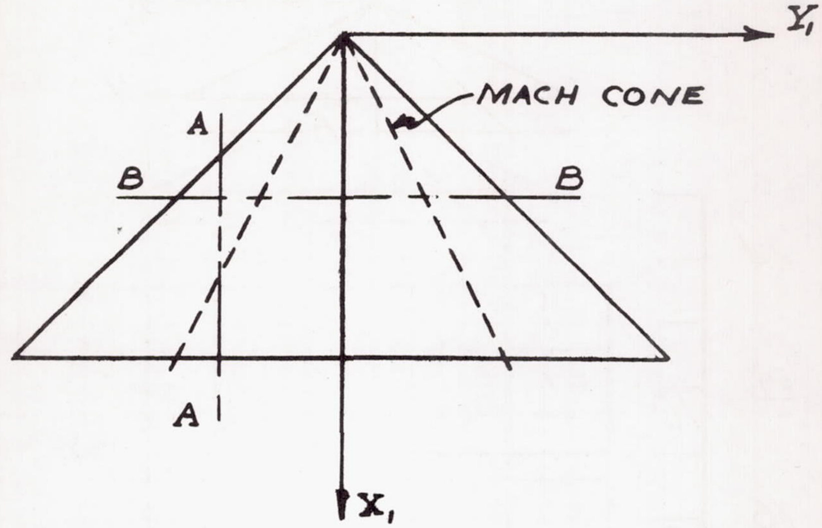
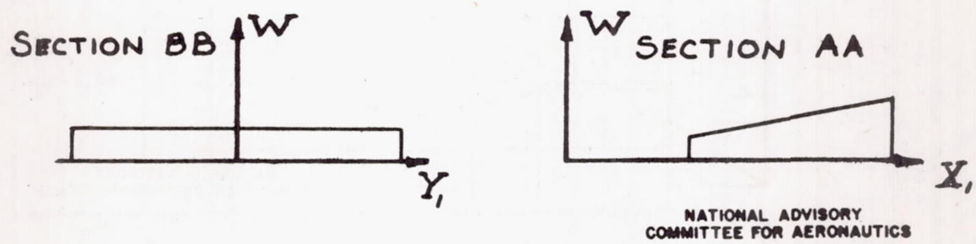


Figure 6 – Variation of pressure coefficient for rolling triangular wing swept ahead of mach cone.



(a) PLANFORM.



(b) SECTIONS SHOWING DISTRIBUTION.

FIGURE 7 - VERTICAL VELOCITY DISTRIBUTION FOR PITCHING WING.

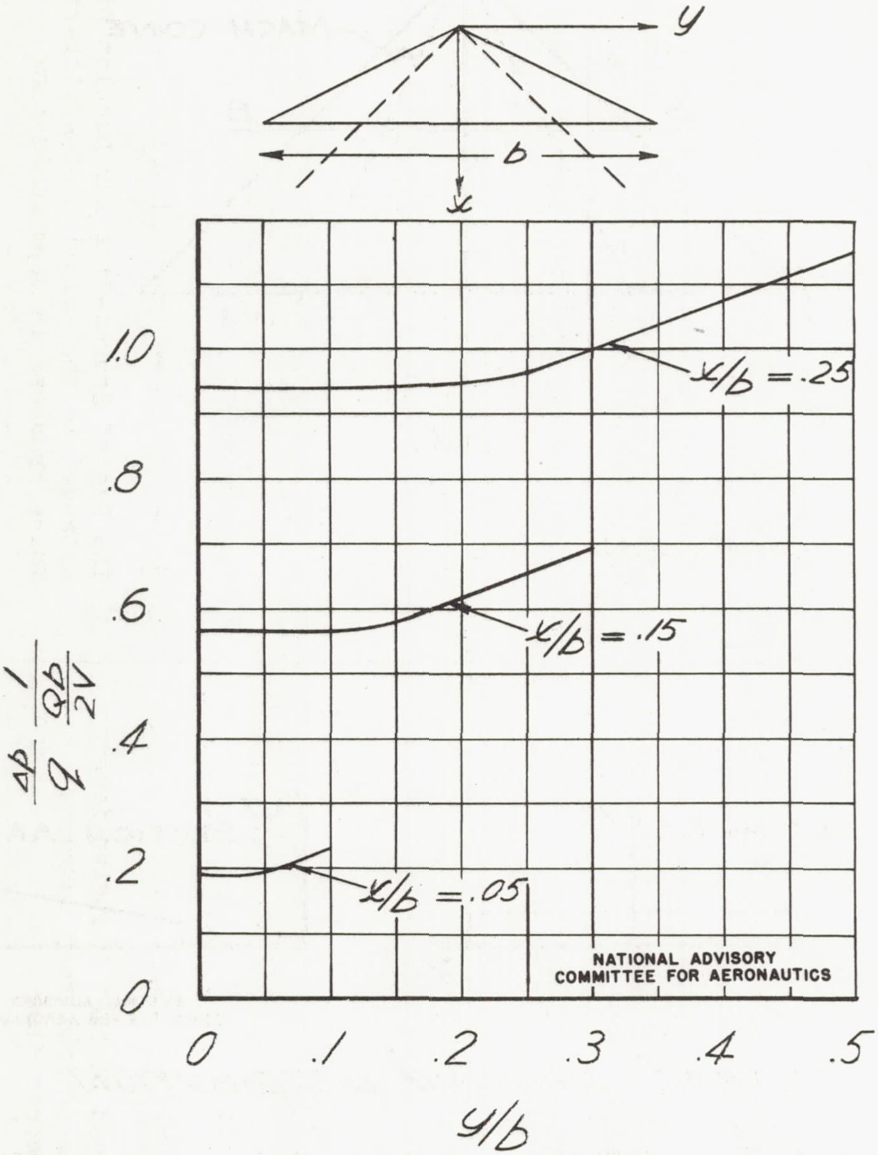


Figure 8 –Variation of pressure coefficient for pitching triangular wing swept ahead of mach cone.