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TECHNICAL NOTE

No. 1519

THE BUCKLING OF A COLUMN ON EQUALLY SPACED DEFLECTIONAL AND ROTATIONAL SPRINGS

By Bernard Budiansky, Paul Seide, and Robert A. Weinberger

Langley Memorial Aeronautical Laboratory
Langley Field, Va.

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Page 20, line 3: Insert $\frac{1}{2}$ before $\frac{\pi}{d^2}$ on left-hand side of equation and before $\frac{\pi^3}{\left(\frac{L}{j}\right)^2}$ on right-hand side of equation.

Page 21, equation (B25): In the first set of brackets, denominator of second expression, change $2\left(\frac{L}{j}\right)^2$ to $2\left(\frac{L}{j}\right)^3$. In the last set of brackets, insert $\frac{1}{2}$ in the numerator so that the numerator will be $\frac{1}{2} \sin \pi\frac{q}{N}\left(1 - \cos \frac{L}{j}\right)$.

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Page 24, equation (C5): In the first line of this equation $\sin \frac{L}{j}$ should be inserted after the term $2\frac{L}{j}S\left(1 - \cos \pi\frac{q}{N}\right)$ so that the term will be $2\frac{L}{j}S\left(1 - \cos \pi\frac{q}{N}\right) \sin \frac{L}{j}$.

Page 24, equation (C6): In the second term of this equation change the plus sign in the numerator to a minus sign so that the numerator will be $\frac{L}{j} - \sin \frac{L}{j}$.

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SUMMARY

A solution is presented for the problem of the buckling of a column on equally spaced deflectional and rotational springs. Useful charts, which relate deflectional spring stiffness, rotational spring stiffness, and buckling load, are given for columns having two, three, four, and an infinite number of spans.

INTRODUCTION

A problem that arises in the analysis of aircraft structures is the determination of the buckling load of a column which is supported at points along its span by other structural members. In general, the supporting members restrain the column elastically against both deflection and rotation. It is therefore convenient to consider that the elastic restraints come from deflectional and rotational springs at the points of support.

By solving the column differential equation, Klemperer and Gibbons (reference 1) found the buckling load of simply supported columns subdivided into two, three, and four spans by equally spaced intermediate deflectional springs of equal stiffness. Zahorski (reference 2), using the same approach, extended these results for columns with two and three spans by also considering intermediate rotational springs of equal stiffness. The method of solving the column differential equation is unduly laborious, however, for columns having many spans since each possible buckling configuration must be considered separately; consequently, a solution to the case of an infinite number of spans was not obtained.

By using difference equations, Ratzersdorfer (reference 3) and Tu (reference 4) obtained an expression for the buckling load of columns with any number of spans on deflectional springs alone (fig. 1(a)) and, in addition, were able to solve for the case of an infinite number of spans. In the present paper, the Rayleigh-Ritz energy method is used to extend the results by considering, in addition to deflectional springs, intermediate rotational springs of equal stiffness and end rotational springs of half the stiffness of the intermediate springs (fig. 1(b)). The special end-support conditions specified for the present problem facilitate an exact solution for the case of any number of spans and permit the derivation of a limiting expression for the case of an infinite number of spans.

RESULTS AND DISCUSSION

The results of this paper are presented in terms of the following three nondimensional parameters:

$\frac{PL^2}{EI}$ buckling-load parameter

$\frac{CL^3}{EI}$ deflectional-stiffness parameter

$\frac{KL}{EI}$ rotational-stiffness parameter

where

P buckling load

L length between supports

EI column bending stiffness

C deflectional spring constant, force per unit deflection

K rotational spring constant, torque per unit rotation

The curves of figures 2 to 5 show the relationships among these parameters for columns of two, three, four, and an infinite number of spans. The curves were obtained from the exact stability equations derived by the Rayleigh-Ritz energy method in appendixes B and C.

The discontinuities of the slopes of the curves in figures 2 to 4 correspond to sudden changes in the type of buckling pattern; the number of buckles q corresponding to each region between these discontinuities is given in these figures. The curves for the infinite-span column (fig. 5) are smooth because the buckling configuration varies continuously with changes in deflectional support stiffness. The horizontal parts of each curve of figures 2 to 5 correspond to buckling with no deflection of the supports and with the number of buckles equal to the number of spans. (See fig. 6(a).) The buckling load is then independent of the deflectional spring stiffness.

For the infinite-span column (fig. 5), parts of the curves for $\frac{KL}{EI} = 20$ and $\frac{KL}{EI} = 50$ are seen to be coincident with the curve for $\frac{KL}{EI} = \infty$. These parts correspond to buckling with the column deflection curve horizontal at the supports (see fig. 6(b)) so that the buckling load is no longer dependent on the rotational spring stiffness. In the finite-span columns this independence of rotational spring stiffness never occurs but is approximated more and more as the number of spans increases; this approximation is shown by the increasing proximity of the curves for $\frac{KL}{EI} = 20$, 50, and ∞ in figures 2 to 4. A discussion of this phenomenon is given in appendix C.

The curves for the infinite case may be used to obtain a close approximation, on the conservative side, to the buckling load of a column with more than four spans. The error involved, shown by figure 7 to be less than 10 percent for the four-span case, decreases as the number of spans increases.

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APPENDIX A

SYMBOLS

x	distance along column (fig. 1(b))
y	deflection of column (fig. 1(b))
y_c	deflection of support
N	number of spans
L	length between supports
EI	column bending stiffness
P	buckling load
$j = \sqrt{\frac{EI}{P}}$	
L/j	dimensionless buckling-load parameter $\left(\sqrt{\frac{PL^2}{EI}}\right)$
C	deflectional spring constant, force per unit deflection
S	dimensionless deflectional-stiffness parameter $\left(\frac{CL^3}{EI}\right)$
K	rotational spring constant, torque per unit rotation
T	dimensionless rotational-stiffness parameter $\left(\frac{KL}{EI}\right)$
k, m, n, p, r, s	integers
c	integer defining location of a support $\left(x_c = cL\right)$
q	number of buckles
δ_{mn}	Kronecker delta (1 if $m = n$; 0 if $m \neq n$)

APPENDIX B

DERIVATION OF STABILITY CRITERIONS

The following development of the stability criterions for a column on equally spaced deflectional and rotational supports is based on the Rayleigh-Ritz energy method. A Fourier series is chosen to represent the deflection curve of the buckled column, and the potential energy expression is minimized with respect to each of the unknown Fourier coefficients. The resulting equations are separated into independent sets, each set containing the coefficients corresponding to a particular buckling mode. A general expression for the stability criterion for each buckling mode is derived.

Energy Expressions

The deflection curve of the buckled column may be represented by the Fourier series

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{NL} \tag{B1}$$

When the initially straight column buckles, the bending energy stored in the column is

$$V_b = \frac{EI}{2}\int_0^{NL}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$= \frac{\pi^4}{4}\,\frac{EI}{(NL)^3}\sum_{n=1}^{\infty} n^4 a_n^{\,2} \tag{B2}$$

The energy stored in the deflectional springs is

$$V_d = \sum_{c=1}^{N-1}\frac{Cy_c^{\,2}}{2}$$

$$= \frac{C}{2}\sum_{c=1}^{N-1}\left(\sum_{n=1}^{\infty} a_n \sin \frac{n\pi c}{N}\right)^2 \tag{B3}$$

The energy stored in the rotational springs is

$$V_r = \frac{1}{2}\,\frac{K}{2}\left[\left(\frac{dy}{dx}\right)_{x=0}\right]^2 + \sum_{c=1}^{N-1}\frac{1}{2}K\left[\left(\frac{dy}{dx}\right)_{x=cL}\right]^2 + \frac{1}{2}\,\frac{K}{2}\left[\left(\frac{dy}{dx}\right)_{x=NL}\right]^2$$

$$= \frac{\pi^2}{2}\,\frac{K}{(NL)^2}\sum_{c=0}^{N}\left(\sum_{n=1}^{\infty} na_n \cos\frac{n\pi c}{N}\right)^2 \frac{1}{1 + \delta_{0c} + \delta_{Nc}} \tag{B4}$$

The ends of the column move toward each other and the work done by the buckling load is

$$W = \frac{P}{2}\int_0^{NL}\left(\frac{dy}{dx}\right)^2 dx = \frac{\pi^2}{4}\,\frac{P}{NL}\sum_{n=1}^{\infty} n^2 a_n^{\,2} \tag{B5}$$

The buckling load may be found by minimizing the energy expression

$$F = V_b + V_d + V_r - W \tag{B6}$$

with respect to the a's. Substitution of equations (B2) to (B5) into equation (B6) gives

$$F = \frac{\pi^4}{4}\,\frac{EI}{(NL)^3}\left\{\sum_{n=1}^{\infty}\left[n^4 - \left(\frac{NL}{\pi j}\right)^2 n^2\right]a_n^{\,2} + \frac{2N^3S}{\pi^4}\sum_{c=1}^{N-1}\left(\sum_{n=1}^{\infty} a_n \sin\frac{n\pi c}{N}\right)^2\right.$$

$$\left. + \frac{2NT}{\pi^2}\sum_{c=0}^{N}\left(\sum_{n=1}^{\infty} na_n \cos\frac{n\pi c}{N}\right)^2 \frac{1}{1 + \delta_{0c} + \delta_{Nc}}\right\} \tag{B7}$$

Minimization

Minimizing F with respect to the a's yields

$$\frac{\partial F}{\partial a_n} = 0$$

$$= \left[n^4 - \left(\frac{NL}{\pi j}\right)^2 n^2\right] a_n + \frac{2N^3S}{\pi^4} \sum_{m=1}^{\infty} a_m \sum_{c=1}^{N-1} \sin\frac{m\pi c}{N} \sin\frac{n\pi c}{N}$$

$$+ \frac{2NT}{\pi^2} \sum_{m=1}^{\infty} mna_m \sum_{c=0}^{N} \cos\frac{m\pi c}{N} \cos\frac{n\pi c}{N} \frac{1}{1 + \delta_{0c} + \delta_{Nc}} \qquad \text{(B8)}$$

$$(n = 1,2,3,\ldots)$$

Consider the summations

$$\sum_{c=1}^{N-1} \sin\frac{m\pi c}{N} \sin\frac{n\pi c}{N}$$

and

$$\sum_{c=0}^{N} \cos\frac{m\pi c}{N} \cos\frac{n\pi c}{N} \frac{1}{1 + \delta_{0c} + \delta_{Nc}}$$

Appendix D shows that the summations have the following values:

Condition		$\sum_{c=1}^{N-1} \sin\frac{m\pi c}{N} \sin\frac{n\pi c}{N}$	$\sum_{c=0}^{N} \cos\frac{m\pi c}{N} \cos\frac{n\pi c}{N} \frac{1}{1 + \delta_{0c} + \delta_{Nc}}$
$\frac{m+n}{2N}$	$\frac{m-n}{2N}$		
Not integer	Not integer	0	0
Integer	Integer	0	N
Integer	Not integer	$-\frac{N}{2}$	$\frac{N}{2}$
Not integer	Integer	$\frac{N}{2}$	$\frac{N}{2}$

For a given value of n, the condition that will apply for each value of m is indicated in the following table where p is a positive integer, r is a positive integer such that $r + p$ is even, and k_1 and k_2 are integers (plus or minus) yielding positive m:

Condition		$n = pN$	$n \neq pN$
$\frac{m+n}{2N}$	$\frac{m-n}{2N}$		
Not integer	Not integer	$m \neq rN$	$\begin{cases} m \neq 2k_1N - n \\ m \neq 2k_2N + n \end{cases}$
Integer	Integer	$m = rN$	Never
Integer	Not integer	Never	$\begin{cases} m = 2k_1N - n \\ m \neq 2k_2N + n \end{cases}$
Not integer	Integer	Never	$\begin{cases} m \neq 2k_1N - n \\ m = 2k_2N + n \end{cases}$

The infinite set of equations (B8), with the use of the values of the summations, may be divided into the following three independent infinite sets of equations:

$$\left[(pN)^4 - \left(\frac{NL}{\pi j}\right)^2 (pN)^2\right] a_{pN} + \frac{2N^2T}{\pi^2} pN \sum_{r=1,3,5}^{\infty} rNa_{rN} = 0 \qquad \text{(B9)}$$

$$(p = 1,3,5,\ldots)$$

$$\left[(pN)^4 - \left(\frac{NL}{\pi j}\right)^2 (pN)^2\right] a_{pN} + \frac{2N^2T}{\pi^2} pN \sum_{r=2,4,6}^{\infty} rNa_{rN} = 0 \qquad \text{(B10)}$$

$$(p = 2,4,6,\ldots)$$

$$\left[n^4 - \left(\frac{NL}{\pi j}\right)^2 n^2\right] a_n + \frac{N^4S}{\pi^4}\left(\sum_{k_2} a_{m_2} - \sum_{k_1} a_{m_1}\right)$$

$$+ \frac{N^2Tn}{\pi^2}\left(\sum_{k_2} m_2 a_{m_2} + \sum_{k_1} m_1 a_{m_1}\right) = 0 \qquad \text{(B11)}$$

$$(n = 1,2,3,\ldots)$$

$$(n \neq pN)$$

where $m_1 = 2k_1N - n$, $m_2 = 2k_2N + n$, and the summations are over all plus or minus integral values of k_1 and k_2 that yield positive m_1 and m_2.

Equations (B11) may be further subdivided into $N - 1$ independent sets. Consider one of equations (B11) for a particular n equal to q; the a's appearing in the summations will have the subscripts

$$m_1 = 2N - q, \quad 4N - q, \quad 6N - q, \ldots$$

and

$$m_2 = q, \quad 2N + q, \quad 4N + q, \quad 6N + q, \ldots$$

If equations (B11) are now written for n equal to these preceding values, a's having the same subscripts, and only these a's, will appear in the summations. Thus if

$$n = q, \quad 2N + q, \quad 4N + q, \quad 6N + q, \ldots$$

then

$$m_1 = 2N - q, \quad 4N - q, \quad 6N - q, \ldots$$

and

$$m_2 = q, \quad 2N + q, \quad 4N + q, \quad 6N + q, \ldots$$

If

$$n = 2N - q, \quad 4N - q, \quad 6N - q, \ldots$$

then

$$m_1 = q, \quad 2N + q, \quad 4N + q, \quad 6N + q, \ldots$$

and

$$m_2 = 2N - q, \quad 4N - q, \quad 6N - q, \ldots$$

Then, an infinite independent subset of equations (B11) is given by the following two groups of equations (equations (B12) and (B13)):

$$\left[(2sN + q)^4 - \left(\frac{NL}{\pi j}\right)^2 (2sN + q)^2\right] a_{2sN+q} + \frac{N^4 S}{\pi^4} \sum_{k=0}^{\infty} \left[a_{2kN+q} - a_{2(k+1)N-q}\right]$$

$$+ \frac{N^2 T}{\pi^2} (2sN + q) \sum_{k=0}^{\infty} \left\{(2kN + q) a_{2kN+q} + \left[2(k + 1)N - q\right] a_{2(k+1)N-q}\right\} = 0 \qquad \text{(B12)}$$

$$(s = 0,1,2,\ldots)$$

$$\left\{\left[2(s + 1)N - q\right]^4 - \left(\frac{NL}{\pi j}\right)^2 \left[2(s + 1)N - q\right]^2\right\} a_{2(s+1)N-q}$$

$$+ \frac{N^4S}{\pi^4} \sum_{k=0}^{\infty} \left[a_{2(k+1)N-q} - a_{2kN+q}\right]$$

$$+ \frac{N^2T}{\pi^2} \left[2(s + 1)N - q\right] \left\{\sum_{k=0}^{\infty} \left[2(k + 1)N - q\right] a_{2(k+1)N-q}\right.$$

$$\left. + (2kN + q) a_{2kN+q}\right\} = 0 \qquad \text{(B13)}$$

$$(s = 0,1,2,\ldots)$$

All the equations of (B11) are given by $N - 1$ sets obtained by letting q in equations (B12) and (B13) assume the values 1, 2, . . . N - 1.

Stability Criterions

It has been shown that equations (B8) can be broken up into $N + 1$ subsets, two of which are given by equations (B9) and (B10) and the remaining $N - 1$ by equations (B12) and (B13). Each set contains a's appearing in no other set; hence, each set of equations leads to an independent stability criterion corresponding to buckling in a particular mode. These criterions are derived as follows.

First consider equations (B9) which involve only the Fourier components a_N, a_{3N}, . . ., which correspond to buckling of the column with nodes at the supports and with a symmetrical buckling configuration in each bay. Solving for a_{pN} and multiplying through by pN gives

$$pNa_{pN} = -\frac{2N^2T}{\pi^2} \frac{(pN)^2}{(pN)^4 - \left(\frac{NL}{\pi j}\right)^2 (pN)^2} \sum_{r=1,3,5}^{\infty} rNa_{rN} \qquad \text{(B14)}$$

$$(p = 1,3,5,\ldots)$$

Summing over p yields

$$\sum_{p=1,3,5}^{\infty} pNa_{pN} = -\frac{2N^2T}{\pi^2} \sum_{p=1,3,5}^{\infty} \frac{(pN)^2}{(pN)^4 - \left(\frac{NL}{\pi j}\right)^2 (pN)^2} \sum_{r=1,3,5}^{\infty} rNa_{rN} \qquad \text{(B15)}$$

Since

$$\sum_{p=1,3,5}^{\infty} pNa_{pN} = \sum_{r=1,3,5}^{\infty} rNa_{rN}$$

$$\neq 0$$

$$\frac{1}{T} = -\sum_{p=1,3,5}^{\infty} \frac{2}{p^2\pi^2 - \left(\frac{L}{j}\right)^2} \qquad \text{(B16)}$$

which is the desired stability criterion.

Equations (B10), which contain only the Fourier coefficients a_{2N}, a_{4N}, a_{6N}, . . . , yield a criterion for buckling of the column with an antisymmetrical buckling configuration in each bay and with nodes at the supports. This buckling criterion need never be considered because it always gives a higher buckling load than does equation (B16).

In order to obtain the buckling criterions for the other modes, equations (B12) and (B13) are transposed as follows:

$$a_{2sN+q} = \frac{\frac{S}{\pi^4}}{\left(2s + \frac{q}{N}\right)^2 Q_{2s+\frac{q}{N}}} \sum_{k=0}^{\infty} \left[a'_{2kN+q} - a_{2(k+1)N-q}\right]$$

$$+ \frac{\frac{T}{\pi^2}}{\left(2s + \frac{q}{N}\right) Q_{2s+\frac{q}{N}}} \sum_{k=0}^{\infty} \left\{\left(2k + \frac{q}{N}\right) a_{2kN+q} + \left[2(k+1) - \frac{q}{N}\right] a_{2(k+1)N-q}\right\} \qquad \text{(B17)}$$

$$a_{2(s+1)N-q} = \frac{-\frac{S}{\pi^4}}{\left[2(s+1) - \frac{q}{N}\right]^2 Q_{2(s+1)-\frac{q}{N}}} \sum_{k=0}^{\infty} \left[a_{2kN+q} - a_{2(k+1)N-q}\right]$$

$$+ \frac{\frac{T}{\pi^2}}{\left[2(s+1) - \frac{q}{N}\right] Q_{2(s+1)-\frac{q}{N}}} \sum_{k=0}^{\infty} \left\{\left(2k + \frac{q}{N}\right) a_{2kN+q} + \left[2(k+1) - \frac{q}{N}\right] a_{2(k+1)N-q}\right\} \qquad \text{(B18)}$$

where

$$Q_{2s+\frac{q}{N}} = \left(\frac{L}{\pi j}\right)^2 - \left(2s + \frac{q}{N}\right)^2$$

$$Q_{2(s+1)-\frac{q}{N}} = \left(\frac{L}{\pi j}\right)^2 - \left[2(s+1) - \frac{q}{N}\right]^2$$

$$s = 0, 1, 2, \ldots$$

$$q = 1, 2, \ldots N - 1$$

For any value of q, summing equations (B17) and equations (B18) over s and subtracting equations (B18) from equations (B17) gives

$$\sum_{s=0}^{\infty}\left[a_{2sN+q} - a_{2(s+1)N-q}\right] = \frac{S}{\pi^4}\sum_{k=0}^{\infty}\left[a_{2kN+q} - a_{2(k+1)N-q}\right]\sum_{s=0}^{\infty}\left\{\frac{1}{\left(2s+\frac{q}{N}\right)^2 Q_{2s+\frac{q}{N}}} + \frac{1}{\left[2(s+1)-\frac{q}{N}\right]^2 Q_{2(s+1)-\frac{q}{N}}}\right\}$$

$$+ \frac{T}{\pi^2}\sum_{k=0}^{\infty}\left\{\left(2k+\frac{q}{N}\right)a_{2kN+q} + \left[2(k+1)-\frac{q}{N}\right]a_{2(k+1)N-q}\right\}\sum_{s=0}^{\infty}\left\{\frac{1}{\left(2s+\frac{q}{N}\right)Q_{2s+\frac{q}{N}}}\right.$$

$$\left. - \frac{1}{\left[2(s+1)-\frac{q}{N}\right]Q_{2(s+1)-\frac{q}{N}}}\right\} \qquad \text{(B19)}$$

Multiplying equations (B17) by $2s + \frac{q}{N}$ and equations (B18) by $2(s + 1) - \frac{q}{N}$, summing over s, and adding the two equations yields

$$\sum_{s=0}^{\infty}\left\{\left(2s+\frac{q}{N}\right)a_{2sN+q}+\left[2(s+1)-\frac{q}{N}\right]a_{2(s+1)-\frac{q}{N}}\right\}=\frac{S}{\pi^4}\sum_{k=0}^{\infty}\left[a_{2kN+q}\right.$$

$$\left.-a_{2(k+1)N-q}\right]\sum_{s=0}^{\infty}\left\{\frac{1}{\left(2s+\frac{q}{N}\right)Q_{2s+\frac{q}{N}}}-\frac{1}{\left[2(s+1)-\frac{q}{N}\right]Q_{2(s+1)-\frac{q}{N}}}\right\}$$

$$+\frac{T}{\pi^2}\sum_{k=0}^{\infty}\left\{\left(2k+\frac{q}{N}\right)a_{2kN+q}+\left[2(k+1)-\frac{q}{N}\right]a_{2(k+1)N-q}\right\}\sum_{s=0}^{\infty}\left[\frac{1}{Q_{2s+\frac{q}{N}}}+\frac{1}{Q_{2(s+1)-\frac{q}{N}}}\right] \qquad \text{(B20)}$$

Denoting the left side of equation (B19) by X and the left side of equation (B20) by Y and rearranging the equations gives

$$X\left(\frac{\pi^4}{S}-\sum_{s=0}^{\infty}\left\{\frac{1}{\left(2s+\frac{q}{N}\right)^2Q_{2s+\frac{q}{N}}}+\frac{1}{\left[2(s+1)-\frac{q}{N}\right]^2Q_{2(s+1)-\frac{q}{N}}}\right\}\right)$$

$$-Y\left(\frac{\pi^2T}{S}\sum_{s=0}^{\infty}\left\{\frac{1}{\left(2s+\frac{q}{N}\right)Q_{2s+\frac{q}{N}}}-\frac{1}{\left[2(s+1)-\frac{q}{N}\right]Q_{2(s+1)-\frac{q}{N}}}\right\}\right)=0 \qquad \text{(B21)}$$

$$- X\left(\frac{S}{\pi^2 T}\sum_{s=0}^{\infty}\left\{\frac{1}{\left(2s+\frac{q}{N}\right)Q_{2s+\frac{q}{N}}} - \frac{1}{\left[2(s+1)-\frac{q}{N}\right]Q_{2(s+1)-\frac{q}{N}}}\right\}\right)$$

$$+ Y\left\{\frac{\pi^2}{T} - \sum_{s=0}^{\infty}\left[\frac{1}{Q_{2s+\frac{q}{N}}} + \frac{1}{Q_{2(s+1)-\frac{q}{N}}}\right]\right\} = 0 \qquad \text{(B22)}$$

Equating the determinant of the coefficients of X and Y to zero yields the $N - 1$ stability criterions corresponding to $q = 1,2,3,\ldots N - 1$

$$\left(\frac{\pi^4}{S} - \sum_{s=0}^{\infty}\left\{\frac{1}{\left(2s+\frac{q}{N}\right)^2 Q_{2s+\frac{q}{N}}} + \frac{1}{\left[2(s+1)-\frac{q}{N}\right]^2 Q_{2(s+1)-\frac{q}{N}}}\right\}\right)\left\{\frac{\pi^2}{T} - \sum_{s=0}^{\infty}\left[\frac{1}{Q_{2s+\frac{q}{N}}} + \frac{1}{Q_{2(s+1)-\frac{q}{N}}}\right]\right\}$$

$$- \left(\sum_{s=0}^{\infty}\left\{\frac{1}{\left(2s+\frac{q}{N}\right)Q_{2s+\frac{q}{N}}} - \frac{1}{\left[2(s+1)-\frac{q}{N}\right]Q_{2(s+1)-\frac{q}{N}}}\right\}\right)^2 = 0 \qquad \text{(B23)}$$

or

$$\left(\frac{\pi^4}{S} - A\right)\left(\frac{\pi^2}{T} - B\right) - C^2 = 0 \qquad \text{(B24)}$$

where A, B, and C denote the series of equations (B23).

These N - 1 equations, together with equation (B16)

$$\frac{1}{T} = - \sum_{p=1,3,5}^{\infty} \frac{2}{p^2\pi^2 - \left(\frac{L}{j}\right)^2} \qquad \text{(B16)}$$

constitute the complete set of stability criterions.

The Fourier expression for the column deflection curve corresponding to each of the criterions of equations (B23) contains only the coefficients

$$a_q, \quad a_{2N+q}, \quad a_{4N+2}, \quad a_{6N+q}, \ldots$$

and

$$a_{2N-q}, \quad a_{4N-q}, \quad a_{6N-q}, \ldots$$

Each of the criterions are satisfied by many different buckling loads for given values of S and T, the lowest of which will be obtained when the coefficient a_q is dominant.

Each criterion of equations (B23) for a given q therefore corresponds to a buckling configuration of q buckles. Equation (B16), as previously indicated, corresponds to buckling with no deflection of the supports in N buckles.

Closed Forms of Stability Criterions

Each of the series in equation (B24) may be evaluated and the stability criterions expressed in closed form. Series B and C are evaluated first since the results are necessary in the evaluation of series A.

<u>Series B</u>.- Let $\frac{q}{N} = b$ and $\frac{L}{\pi j} = d$. Then

$$\sum_{s=0}^{\infty}\left[\frac{1}{Q_{2s+\frac{q}{N}}} + \frac{1}{Q_{2(s+1)-\frac{q}{N}}}\right]$$

$$= \sum_{s=0}^{\infty}\left\{\frac{1}{d^2 - (2s + b)^2} + \frac{1}{d^2 - \left[2(s + 1) - b\right]^2}\right\}$$

$$= \frac{1}{d^2 - b^2} + \sum_{s=1}^{\infty}\left[\frac{1}{d^2 - (2s + b)^2} + \frac{1}{d^2 - (2s - b)^2}\right]$$

$$= \frac{1}{d^2 - b^2} + \frac{1}{2d}\sum_{s=1}^{\infty}\left[\frac{1}{2s + (d + b)} - \frac{1}{2s - (d - b)} + \frac{1}{2s + (d - b)} - \frac{1}{2s - (d + b)}\right]$$

$$= \frac{1}{d^2 - b^2} - \frac{1}{2d}\sum_{s=1}^{\infty}\left[\frac{2(d + b)}{4s^2 - (d + b)^2} + \frac{2(d - b)}{4s^2 - (d - b)^2}\right]$$

$$= \frac{1}{d^2 - b^2} - \frac{\pi}{4d}\sum_{s=1}^{\infty}\left\{\frac{2\pi\left(\frac{d + b}{2}\right)}{s^2\pi^2 - \left[\pi\left(\frac{d + b}{2}\right)\right]^2} + \frac{2\pi\left(\frac{d - b}{2}\right)}{s^2\pi^2 - \left[\pi\left(\frac{d - b}{2}\right)\right]^2}\right\}$$

With the use of equation (6.495) for cotangent in reference 5, the summation is equal to

$$\frac{1}{d^2 - b^2} + \frac{\pi}{4d}\left[\cot\frac{\pi}{2}(d + b) - \frac{2}{\pi(d + b)} + \cot\frac{\pi}{2}(d - b) - \frac{2}{\pi(d - b)}\right]$$

$$= \frac{\pi}{2d}\,\frac{\sin \pi d}{\cos \pi b - \cos \pi d}$$

$$= \frac{\pi^2}{2\frac{L}{j}}\,\frac{\sin\frac{L}{j}}{\cos \pi\frac{q}{N} - \cos\frac{L}{j}}$$

Series C.- For series C

$$\sum_{s=0}^{\infty}\left\{\frac{1}{\left(2s + \frac{q}{N}\right)Q_{2s+\frac{q}{N}}} - \frac{1}{\left[2(s + 1) - \frac{q}{N}\right]Q_{2(s+1)-\frac{q}{N}}}\right\}$$

$$= \frac{1}{b(d^2 - b^2)} + \sum_{s=1}^{\infty}\left\{\frac{1}{(2s + b)\left[d^2 - (2s + b)^2\right]} - \frac{1}{(2s - b)\left[d^2 - (2s - b)^2\right]}\right\}$$

$$= \frac{1}{b(d^2 - b^2)} + \frac{1}{d^2}\sum_{s=1}^{\infty}\left[\frac{1}{2s + b} - \frac{1}{2}\,\frac{1}{2s + (d + b)} - \frac{1}{2}\,\frac{1}{2s - (d - b)}\right.$$

$$\left. - \frac{1}{2s - b} + \frac{1}{2}\,\frac{1}{2s + (d - b)} + \frac{1}{2}\,\frac{1}{2s - (d + b)}\right]$$

$$= \frac{1}{b(d^2 - b^2)} - \frac{1}{d^2}\sum_{s=1}^{\infty}\left[\frac{2b}{4s^2 - b^2} + \frac{1}{2}\,\frac{2(d + b)}{4s^2 - (d + b)^2} - \frac{1}{2}\,\frac{2(d - b)}{4s^2 - (d - b)^2}\right]$$

Using equation (6.495) for cotangent in reference 5 yields after simplifying the closed form

$$\frac{1}{2}\frac{\pi}{d^2}\frac{\sin \pi b(1 - \cos \pi d)}{(\cos \pi b - \cos \pi d)(1 - \cos \pi b)} = \frac{\pi^3}{\left(\frac{L}{j}\right)^2}\frac{\frac{1}{2}\sin \pi\frac{q}{N}\left(1 - \cos\frac{L}{j}\right)}{\left(\cos \pi\frac{q}{N} - \cos\frac{L}{j}\right)\left(1 - \cos \pi\frac{q}{N}\right)}$$

Series A.- For series A

$$\sum_{s=0}^{\infty}\left\{\frac{1}{\left(2s + \frac{q}{N}\right)^2 Q_{2s+\frac{q}{N}}} + \frac{1}{\left[2(s + 1) - \frac{q}{N}\right]^2 Q_{2(s+1)-\frac{q}{N}}}\right\}$$

$$= \frac{1}{b^2(d^2 - b^2)} + \sum_{s=1}^{\infty}\left\{\frac{1}{(2s + b)^2\left[d^2 - (2s + b)^2\right]} + \frac{1}{(2s - b)^2\left[d^2 - (2s - b)^2\right]}\right\}$$

$$= \frac{1}{b^2(d^2 - b^2)} + \frac{1}{d^2}\sum_{s=1}^{\infty}\left[\frac{1}{(2s + b)^2} + \frac{1}{d^2 - (2s + b)^2} + \frac{1}{(2s - b)^2} + \frac{1}{d^2 - (2s - b)^2}\right]$$

$$= \frac{1}{b^2(d^2 - b^2)} + \frac{1}{d^2}\sum_{s=1}^{\infty}\left\{\frac{2}{4s^2 - b^2} + \frac{4b^2}{(4s^2 - b^2)^2} + \left[\frac{1}{d^2 - (2s + b)^2} + \frac{1}{d^2 - (2s - b)^2}\right]\right\}$$

Using the results of the preceding evaluations and differentiating equation (6.495) for cotangent in reference 5 to evaluate the term $\sum_{s=1}^{\infty} \frac{4b^2}{(4s^2 - b^2)^2}$ yields after simplifying

$$\frac{\pi^2}{2d^2} \frac{1}{1 - \cos \pi b} + \frac{\pi}{2d^3} \frac{\sin \pi d}{\cos \pi b - \cos \pi d} = \frac{\pi^4}{2\left(\frac{L}{j}\right)^2} \frac{1}{1 - \cos \pi\frac{q}{N}} + \frac{\pi^4}{2\left(\frac{L}{j}\right)^3} \frac{\sin \frac{L}{j}}{\cos \pi\frac{q}{N} - \cos \frac{L}{j}}$$

Closed forms.- Substituting these results of the three series in equations (B23) gives

$$\left\{\frac{1}{S} - \left[\frac{1}{2\left(\frac{L}{j}\right)^2\left(1 - \cos \pi\frac{q}{N}\right)} + \frac{\sin \frac{L}{j}}{2\left(\frac{L}{j}\right)^3\left(\cos \pi\frac{q}{N} - \cos \frac{L}{j}\right)}\right]\right\}\left[\frac{1}{T} - \frac{\sin \frac{L}{j}}{2\frac{L}{j}\left(\cos \pi\frac{q}{N} - \cos \frac{L}{j}\right)}\right]$$

$$- \left[\frac{\frac{1}{2} \sin \pi\frac{q}{N}\left(1 - \cos \frac{L}{j}\right)}{\left(\frac{L}{j}\right)^2\left(\cos \pi\frac{q}{N} - \cos \frac{L}{j}\right)\left(1 - \cos \pi\frac{q}{N}\right)}\right]^2 = 0 \qquad \text{(B25)}$$

as the closed-form stability criterions for buckling in the modes where $q = 1, 2, \ldots N - 1$.

The series of equation (B16) may be evaluated as follows:

$$\frac{1}{T} = -\sum_{p=1,3,5}^{\infty} \frac{2}{p^2\pi^2 - \left(\frac{L}{j}\right)^2}$$

$$= -\frac{1}{2\frac{L}{j}} \sum_{p=1,3,5}^{\infty} \frac{8\frac{L}{2j}}{p^2\pi^2 - 4\left(\frac{L}{2j}\right)^2} \tag{B26}$$

From equation (6.495) for tangent in reference 5, the summation is equal to $\tan \frac{L}{2j}$; hence,

$$T = -\frac{2\frac{L}{j}}{\tan \frac{L}{2j}} \tag{B27}$$

which is the stability criterion for buckling in N buckles with nodes at the supports.

Equations (B25) and (B27) constitute the complete set of closed-form stability criterions. The correct criterion for any given values of S and T is that which yields the lowest buckling load.

APPENDIX C

STABILITY CRITERION FOR $N = \infty$

When N becomes infinite, q/N can assume any value between 0 and 1. Therefore, it becomes necessary to find the value of q/N that makes the buckling-load parameter L/j a minimum for given values of deflectional-stiffness parameter S and rotational-stiffness parameter T. The required condition is

$$\frac{d\left(\frac{L}{j}\right)}{d\left(\frac{q}{N}\right)} = 0 \qquad \text{(C1)}$$

However, L/j is defined implicitly by a function (see equations (B25))

$$f(S,\ T,\ L/j,\ q/N) = 0 \qquad \text{(C2)}$$

where

$$0 < \frac{q}{N} < 1$$

Taking the total derivative of equation (C2) and keeping S and T constant gives

$$\frac{d\left(\frac{L}{j}\right)}{d\left(\frac{q}{N}\right)} = -\frac{\partial(f)\Big/\partial\left(\frac{q}{N}\right)}{\partial(f)\Big/\partial\left(\frac{L}{j}\right)} \qquad \text{(C3)}$$

But $\dfrac{d\left(\frac{L}{j}\right)}{d\left(\frac{q}{N}\right)}$ must vanish. Therefore a required condition for minimization of L/J is

$$\frac{\partial(f)}{\partial\left(\frac{q}{N}\right)} = 0 \qquad \text{(C4)}$$

Expanding equations (B25), clearing of fractions, and dividing by $\left(1 - \cos \pi\frac{q}{N}\right)\left(\cos \pi\frac{q}{N} - \cos \frac{L}{j}\right)$ yields

$$f = 2\left(\frac{L}{j}\right)^2 S\left(\cos \pi\frac{q}{N} - \cos \frac{L}{j}\right) + 2\frac{L}{j}S\left(1 - \cos \pi\frac{q}{N}\right)\sin \frac{L}{j}$$

$$- ST\left[\frac{L}{j}\sin \frac{L}{j} - 2\left(1 - \cos \frac{L}{j}\right)\right] + 2\left(\frac{L}{j}\right)^3 T \sin \frac{L}{j}\left(1 - \cos \pi\frac{q}{N}\right)$$

$$- 4\left(\frac{L}{j}\right)^4\left(1 - \cos \pi\frac{q}{N}\right)\left(\cos \pi\frac{q}{N} - \cos \frac{L}{j}\right)$$

$$= 0 \qquad \text{(C5)}$$

when

$$0 < \frac{q}{N} < 1$$

Then, setting $\partial(f)/\partial\left(\frac{q}{N}\right)$ equal to 0 gives

$$\cos \pi\frac{q}{N} = \frac{1 + \cos \frac{L}{j}}{2} - \frac{S}{4}\,\frac{\frac{L}{j} - \sin \frac{L}{j}}{\left(\frac{L}{j}\right)^3} + \frac{T}{4}\,\frac{\sin \frac{L}{j}}{\frac{L}{j}} \qquad \text{(C6)}$$

when

$$0 < \frac{q}{N} < 1$$

Substituting equation (C6) in equation (C5) yields after simplifying

$$S^2\left(\frac{L}{j} - \sin\frac{L}{j}\right)^2 - 4S\left(\frac{L}{j}\right)^3\left(\frac{L}{j} + \sin\frac{L}{j}\right)\left(1 - \cos\frac{L}{j}\right)$$

$$+ 2ST\left(\frac{L}{j}\right)^2\left[\sin\frac{L}{j}\left(\frac{L}{j} + \sin\frac{L}{j}\right) - 4\left(1 - \cos\frac{L}{j}\right)\right]$$

$$- 4T\left(\frac{L}{j}\right)^5 \sin\frac{L}{j}\left(1 - \cos\frac{L}{j}\right)$$

$$+ T^2\left(\frac{L}{j}\right)^4 \sin^2\frac{L}{j} + 4\left(\frac{L}{j}\right)^6\left(1 - \cos\frac{L}{j}\right)^2 = 0 \qquad \text{(C7)}$$

which is the stability criterion for a column with an infinite number of spans when $0 < \frac{q}{N} < 1$.

When q/N is equal to its limiting value, 1, equation (C5) yields two independent criterions:

If $\frac{KL}{EI} = 50$

$$T = -\frac{2\frac{L}{j}}{\tan\frac{L}{2j}} = \frac{KL}{EI} \qquad \text{(C8)}$$

which corresponds to column buckling with no support deflection, and

$$\frac{L}{j} = \sqrt{\frac{PL^2}{EI}}$$

$$S = \frac{4\left(\frac{L}{j}\right)^3 \sin\frac{L}{j}}{\frac{L}{j}\sin\frac{L}{j} - 2\left(1 - \cos\frac{L}{j}\right)} = \frac{CL^3}{EI} \qquad \text{(C9)}$$

which corresponds to column buckling with no support rotation.

In order to obtain the curves of figures 5 and 8, equations (C7) to (C9) must be carefully used in conjunction with each other. Thus, for example, along the curve for $\frac{KL}{EI} = 5$ in figure 5, equation (C7) is used up to $\frac{CL^3}{EI} = 58.7$, at which point equation (C8) is satisfied. For greater values of $\frac{CL^3}{EI}$, the combinations of $\frac{CL^3}{EI}$, $\frac{KL}{EI}$, and $\frac{PL^2}{EI}$ which satisfy equation (C7) will make q/N imaginary $\left(\cos \pi\frac{q}{N} < -1\right)$ in equation (C6) . Hence, beyond the limiting value of $\frac{CL^3}{EI}$, q/N remains equal to 1 and the buckling load remains constant. The dashed-line demarcation curve in figure 5, which gives the limiting value of $\frac{CL^3}{EI}$, is obtained by eliminating $\frac{KL}{EI}$ between equations (C7) and (C8).

Similarly, in figure 8, along the curve for $\frac{CL^3}{EI} = 25$, for example, equation (C7) is used up to $\frac{KL}{EI} = 14.0$, at which point equation (C9) is satisfied. For greater values of $\frac{KL}{EI}$, the combination of $\frac{CL^3}{EI}$, $\frac{KL}{EI}$, and $\frac{PL^2}{EI}$ which satisfy equation (C7) yields imaginary values of q/N. Beyond the limiting value of $\frac{KL}{EI}$, therefore, the buckling load remains at the value given by equation (C9). The dashed-line demarcation curve of figure 8 is obtained by eliminating $\frac{CL^3}{EI}$ between equations (C7) and (C9).

The peculiar shape of the demarcation curve of figure 8 accounts for the peculiarities of the behavior of the curves for $\frac{KL}{EI} = 20$ and $\frac{KL}{EI} = 50$ in figure 5. If $\frac{KL}{EI}$ is greater than 11.04 (the minimum value of $\frac{KL}{EI}$ on the demarcation curve) a constant $\frac{KL}{EI}$-line will intersect the demarcation curve in two points. Between these points the buckling loads are independent of the rotational spring stiffness and are equal to the buckling loads for $\frac{KL}{EI} = \infty$ which accounts for the fact that along parts of their length, the curves in figure 5 for $\frac{KL}{EI} = 20$ and $\frac{KL}{EI} = 50$ coincide with the curve for $\frac{KL}{EI} = \infty$.

It is of interest to note that buckling which is independent of the rotational spring stiffness cannot occur when the number of spans is finite, but does occur for the infinite case. For the buckling load to be independent of the rotational spring stiffness, the column deflection curve must be horizontal at each support. In the case of finite columns, this condition can obviously not be fulfilled at the end supports so long as the rotational spring stiffness is finite; in the infinite column, however, there is no end effect and the column can buckle as shown in figure 6(b).

APPENDIX D

EVALUATION OF SUMMATIONS ENCOUNTERED IN DERIVATION OF STABILITY CRITERIONS

Evaluation of $\sum_{c=1}^{N-1} \sin\frac{m\pi c}{N} \sin\frac{n\pi c}{N}$

In order to evaluate $\sum_{c=1}^{N-1} \sin\frac{m\pi c}{N} \sin\frac{n\pi c}{N}$ first make the substitutions

$$\sin\frac{m\pi c}{N} = \frac{e^{i\frac{m\pi c}{N}} - e^{-i\frac{m\pi c}{N}}}{2i} \tag{D1}$$

and

$$\sin\frac{n\pi c}{N} = \frac{e^{i\frac{n\pi c}{N}} - e^{-i\frac{n\pi c}{N}}}{2i} \tag{D2}$$

Then

$$\sum_{c=1}^{N-1} \sin\frac{m\pi c}{N} \sin\frac{n\pi c}{N}$$

$$= \sum_{c=0}^{N-1} \sin\frac{m\pi c}{N} \sin\frac{n\pi c}{N}$$

$$= -\frac{1}{4}\sum_{c=0}^{N-1}\left[e^{i\frac{\pi c}{N}(m+n)} + e^{-i\frac{\pi c}{N}(m+n)} - e^{i\frac{\pi c}{N}(m-n)} - e^{-i\frac{\pi c}{N}(m-n)}\right] \tag{D3}$$

Case 1: m + n even.- Consider the summation of the first term on the right-hand side of equation (D3)

$$\sum_{c=0}^{N-1} e^{i\frac{\pi c}{N}(m+n)} \doteq \sum_{c=0}^{N-1}\left(e^{\frac{2\pi c}{N}i}\right)^{\frac{m+n}{2}} \tag{D4}$$

According to reference 6, page 36, this summation is recognized as the sum of the $\left(\frac{m + n}{2}\right)$th powers of the N Nth roots of unity and the sum is N or 0 according as $\frac{m + n}{2}$ is or is not a multiple of N. The summation of the second general term in equation (D3) is also the sum of the $\left(\frac{m + n}{2}\right)$th powers of the N Nth roots of unity. The summations of the last two terms are the sum of the $\left(\frac{m - n}{2}\right)$th powers of the N Nth roots of unity. Hence, the following conclusions may be made: If neither m + n nor m - n is a multiple of 2N,

$$\sum_{c=1}^{N-1} \sin\frac{m\pi c}{N} \sin\frac{n\pi c}{N} = 0$$

If both m + n and m - n are multiples of 2N,

$$\sum_{c=1}^{N-1} \sin\frac{m\pi c}{N} \sin\frac{n\pi c}{N} = 0$$

If only m + n is a multiple of 2N,

$$\sum_{c=1}^{N-1} \sin\frac{m\pi c}{N} \sin\frac{n\pi c}{N} = -\frac{N}{2}.$$

If only m - n is a multiple of 2N,

$$\sum_{c=2}^{N-1} \sin\frac{m\pi c}{N} \sin\frac{n\pi c}{N} = \frac{N}{2}$$

<u>Case 2: $m + n$ odd</u>.- Consider the summation

$$\sum_{c=0}^{N-1} e^{i\frac{\pi c}{N}(m+n)} = 1 + e^{i\frac{\pi}{N}(m+n)} + \left[e^{i\frac{\pi}{N}(m+n)}\right]^2 + \ldots + \left[e^{i\frac{\pi}{N}(m+n)}\right]^{N-1}$$

$$= \frac{1 - e^{\pi i(m+n)}}{1 - e^{\frac{\pi i}{N}(m+n)}} \tag{D5}$$

Now

$$e^{\pi i(m+n)} = \cos \pi(m + n) + i \sin \pi(m + n)$$

$$= -1$$

since $m + n$ is odd. Hence

$$\sum_{c=0}^{N-1} e^{i\frac{\pi c}{N}(m+n)} = \frac{2}{1 - e^{\frac{\pi i}{N}(m+n)}} \tag{D6}$$

Performing similar operations on the other summations of equation (D3) yields

$$\sum_{c=1}^{N-1} \sin \frac{m\pi c}{N} \sin \frac{n\pi c}{N} = -\frac{1}{2}\left[\frac{1}{1 - e^{\frac{\pi i}{N}(m+n)}} + \frac{1}{1 - e^{-\frac{\pi i}{N}(m+n)}} - \frac{1}{1 - e^{\frac{\pi i}{N}(m-n)}} - \frac{1}{1 - e^{-\frac{\pi i}{N}(m-n)}}\right]$$

$$= 0 \tag{D7}$$

Since m + n is odd, neither m + n nor m - n is a multiple of 2N; and the results of case 2 may be included in the first conclusion for case 1.

Evaluation of $\sum_{c=0}^{N} \cos \frac{m\pi c}{N} \cos \frac{n\pi c}{N} \left(\frac{1}{1 + \delta_{0c} + \delta_{Nc}}\right)$

In the evaluation of the summation

$$\sum_{c=0}^{N} \cos \frac{m\pi c}{N} \cos \frac{n\pi c}{N} \left(\frac{1}{1 + \delta_{0c} + \delta_{Nc}}\right) = \frac{1}{2} + \sum_{c=0}^{N-1} \cos \frac{m\pi c}{N} \cos \frac{n\pi c}{N} + \frac{1}{2} \cos n\pi \cos m\pi$$

$$= \sum_{c=0}^{N-1} \cos \frac{m\pi c}{N} \cos \frac{n\pi c}{N} + \frac{1}{2}\left(-1^{m+n} - 1\right) \tag{D8}$$

make the substitutions

$$\cos \frac{m\pi c}{N} = \frac{e^{i\frac{m\pi c}{N}} + e^{-i\frac{m\pi c}{N}}}{2} \tag{D9}$$

and

$$\cos \frac{n\pi c}{N} = \frac{e^{i\frac{n\pi c}{N}} + e^{-i\frac{n\pi c}{N}}}{2} \tag{D10}$$

Then

$$\sum_{c=0}^{N} \cos\frac{m\pi c}{N} \cos\frac{n\pi c}{N}\left(\frac{1}{1 + \delta_{0c} + \delta_{Nc}}\right)$$

$$= \frac{1}{4}\sum_{c=0}^{N-1}\left[e^{i\frac{\pi c}{N}(m+n)} + e^{-i\frac{\pi c}{N}(m+n)} + e^{i\frac{\pi c}{N}(m-n)} + e^{-i\frac{\pi c}{N}(m-n)}\right] + \frac{1}{2}\left(-1^{m+n} - 1\right) \qquad \text{(D11)}$$

Case 1: $m + n$ even.- Applying the theorem of reference 6 regarding the sum of powers of the N Nth roots of unity and noting that $\frac{1}{2}\left(-1^{m+n} - 1\right) = 0$ results in the following conclusions: If neither $m + n$ nor $m - n$ is a multiple of $2N$

$$\sum_{c=0}^{N} \cos\frac{m\pi c}{N} \cos\frac{n\pi c}{N}\left(\frac{1}{1 + \delta_{0c} + \delta_{Nc}}\right) = 0$$

If both $m + n$ and $m - n$ are multiples of $2N$

$$\sum_{c=0}^{N} \cos\frac{m\pi c}{N} \cos\frac{n\pi c}{N}\left(\frac{1}{1 + \delta_{0c} + \delta_{Nc}}\right) = N$$

If only $m + n$ is a multiple of $2N$

$$\sum_{c=0}^{N} \cos\frac{m\pi c}{N} \cos\frac{n\pi c}{N}\left(\frac{1}{1 + \delta_{0c} + \delta_{Nc}}\right) = \frac{N}{2}$$

If only $m - n$ is a multiple of $2N$

$$\sum_{c=0}^{N} \cos\frac{m\pi c}{N} \cos\frac{n\pi c}{N}\left(\frac{1}{1 + \delta_{0c} + \delta_{Nc}}\right) = \frac{N}{2}$$

Case 2: m + n odd.- By use of the same evaluation procedure as for case 2 of the previous series, the summation on the right-hand side of equation (C11) is found to be equal to 1. However, $\frac{1}{2}\left(-1^{m+n} - 1\right)$ equals -1 when m + n is odd, and hence

$$\sum_{c=0}^{N} \cos \frac{m\pi c}{N} \cos \frac{n\pi c}{N} \left(\frac{1}{1 + \delta_{0c} + \delta_{Nc}}\right) = 0 \tag{D12}$$

This result may be included in the first conclusion for case 1 since, if m + n is odd, neither m + n nor m - n is a multiple of 2N.

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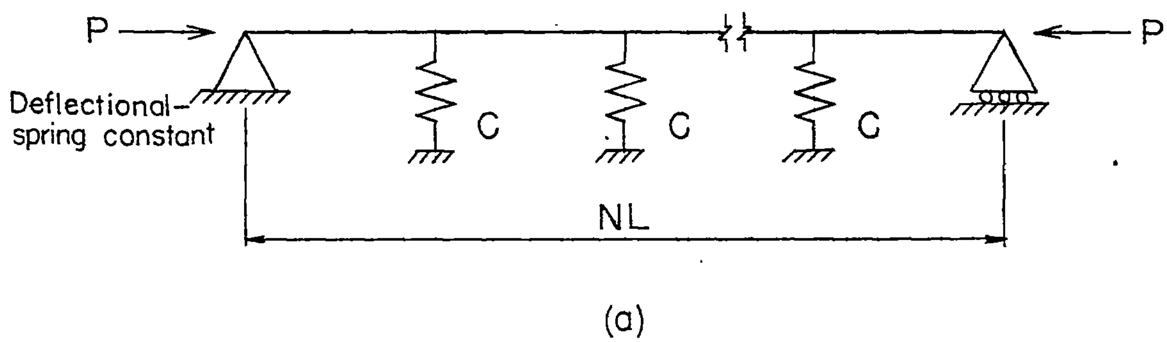


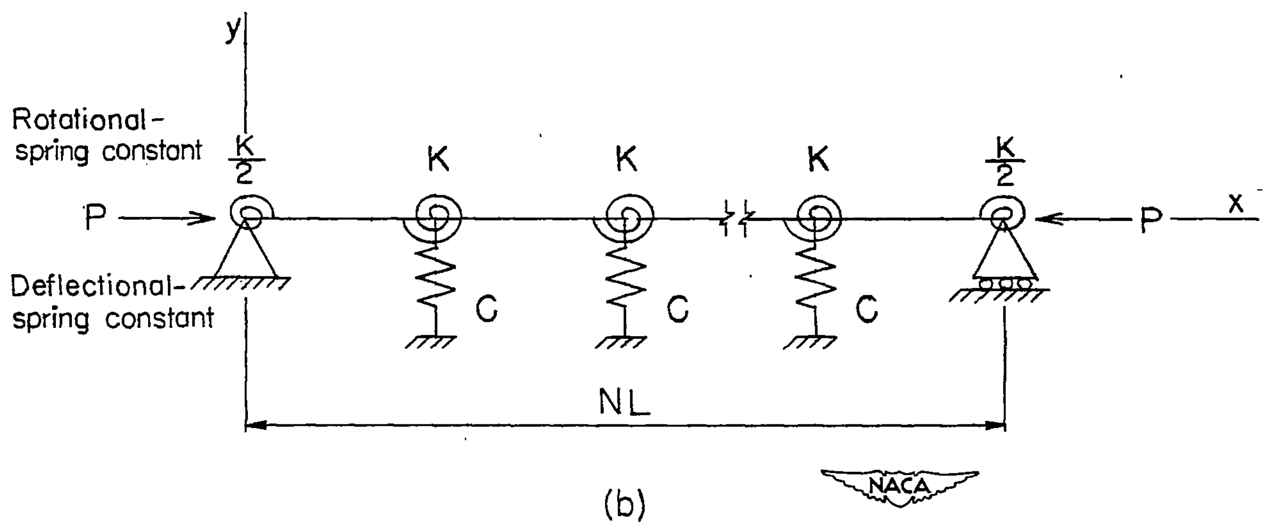


Figure 1.- Column on elastic supports.

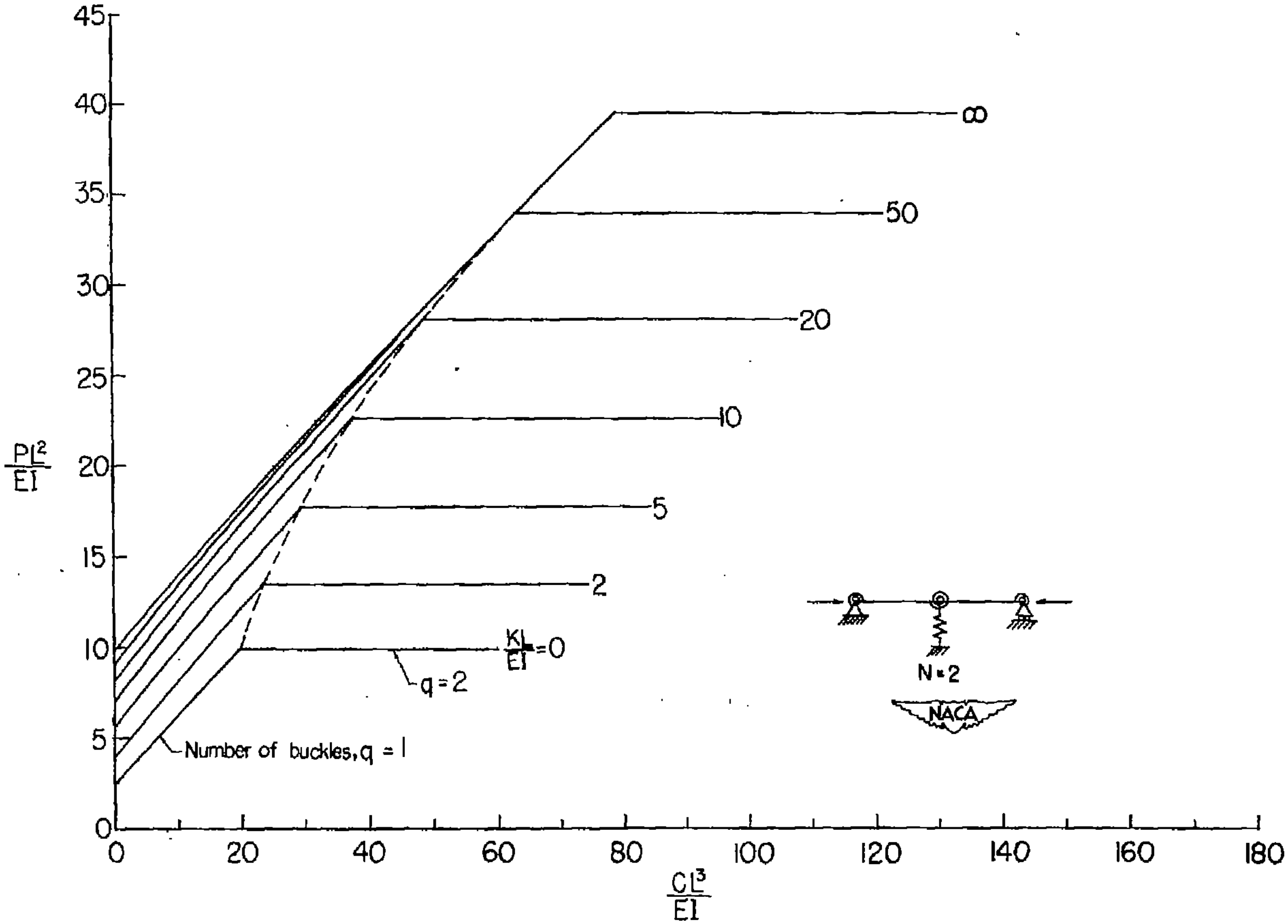


Figure 2.– Buckling curves for two-span column.

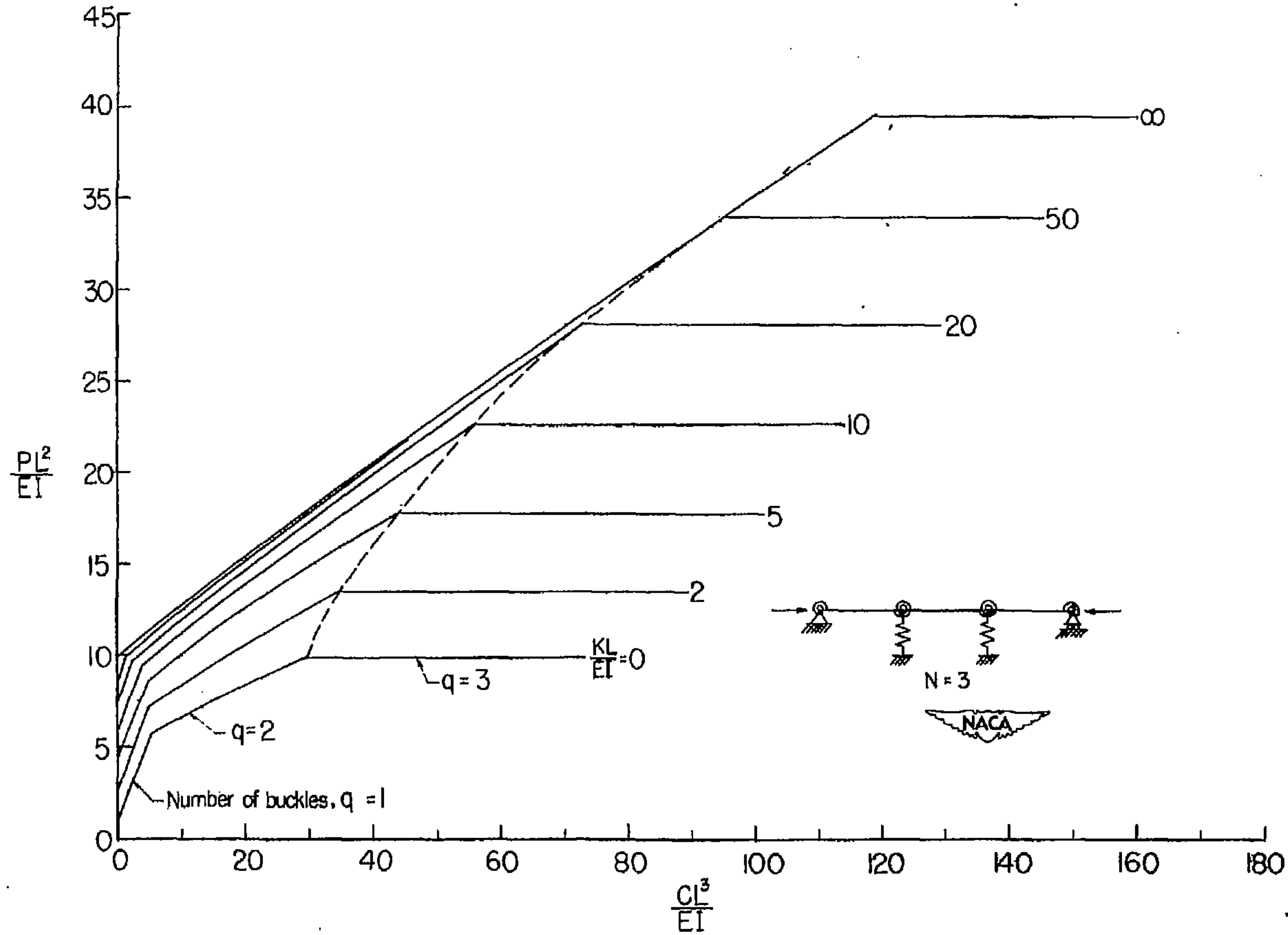


Figure 3.– Buckling curves for three-span column.

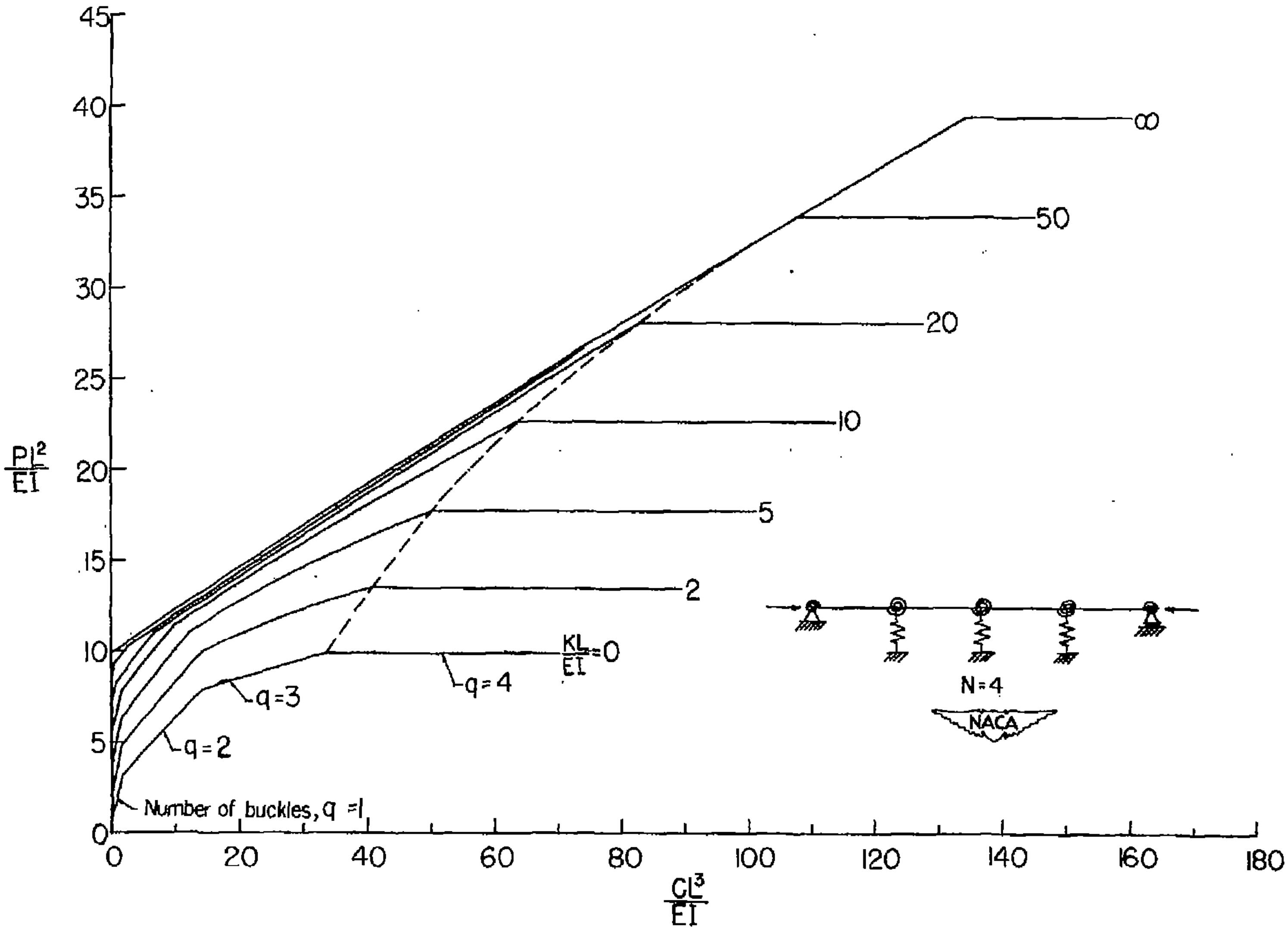


Figure 4.- Buckling curves for four-span column.

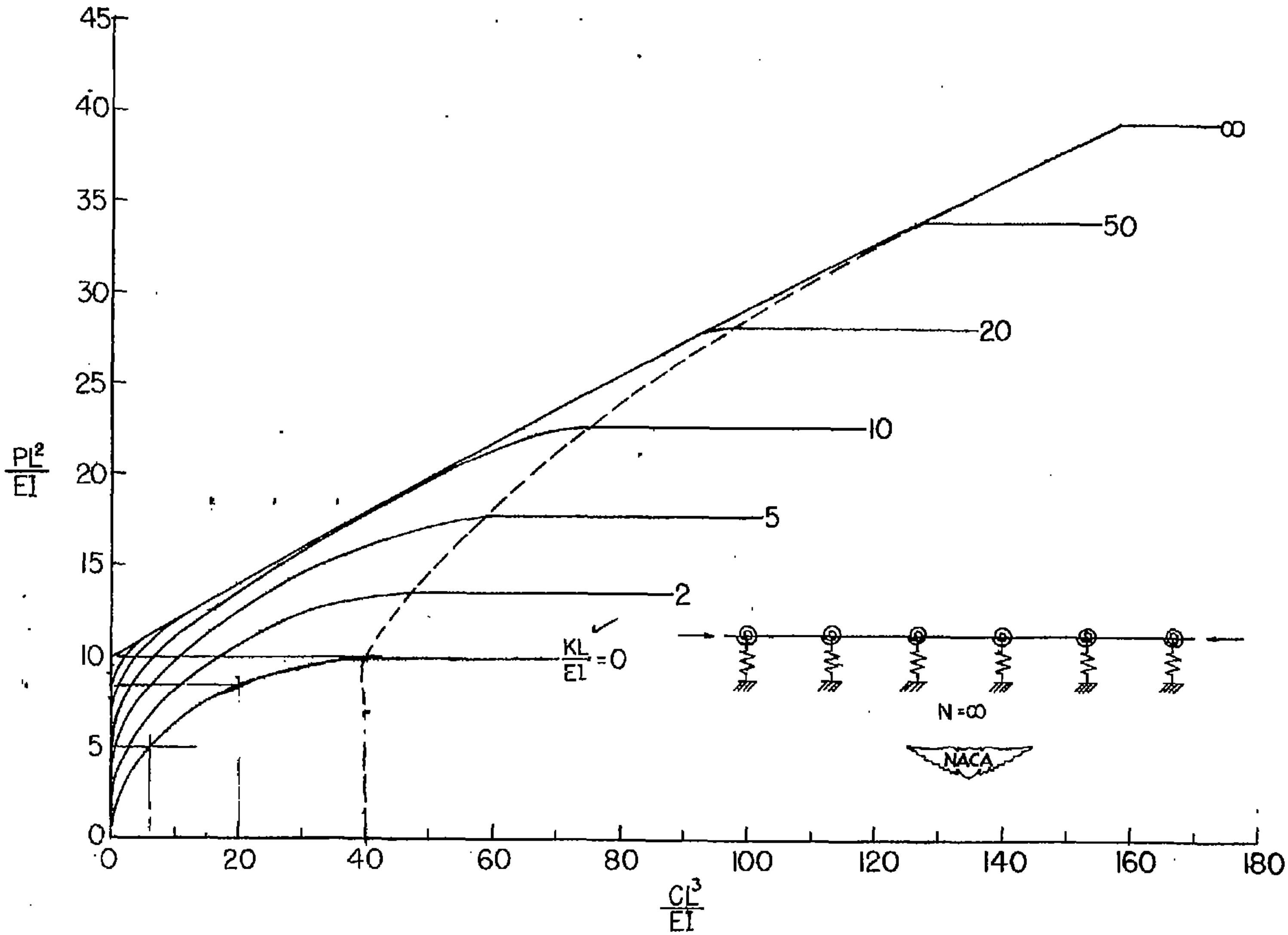
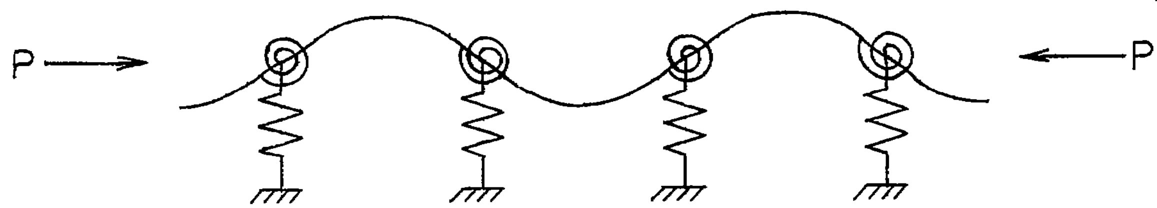
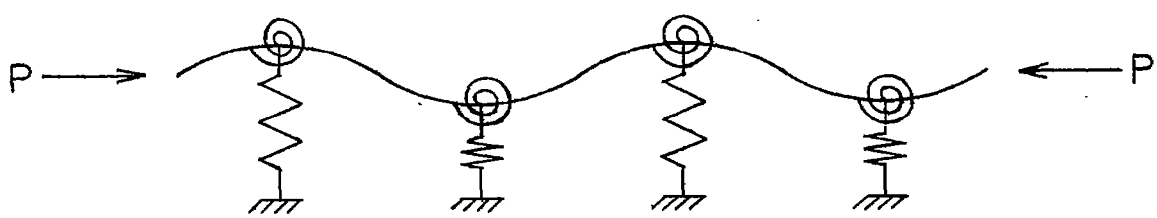


Figure 5.– Buckling curves for column with an infinite number of spans.



(a) $\frac{q}{N}=1$. No support deflection.



(b) $\frac{q}{N}=1$. No support rotation.

Figure 6.– Limiting buckling configurations.

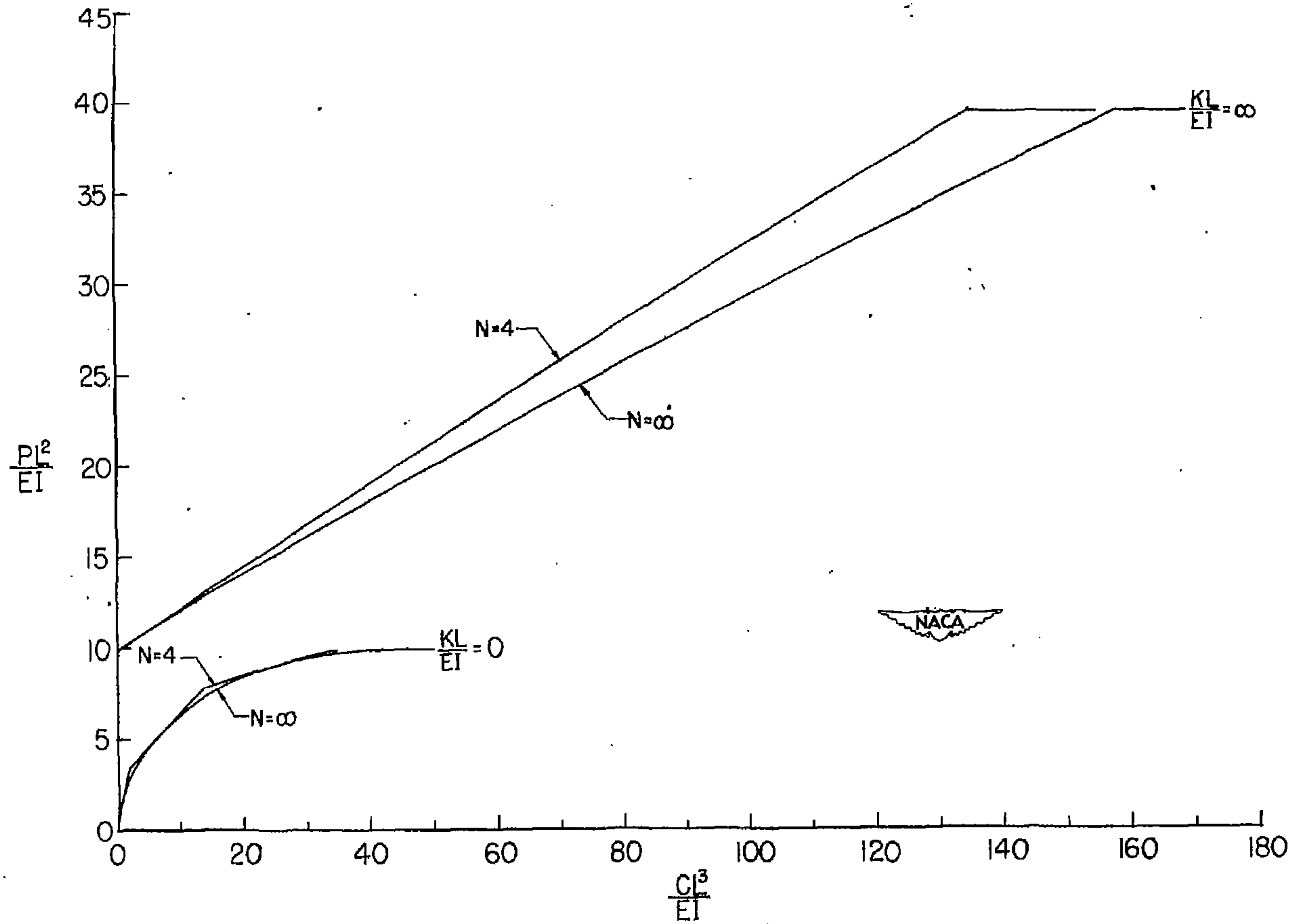


Figure 7.– Comparison of buckling curves for columns with four spans and an infinite number of spans.

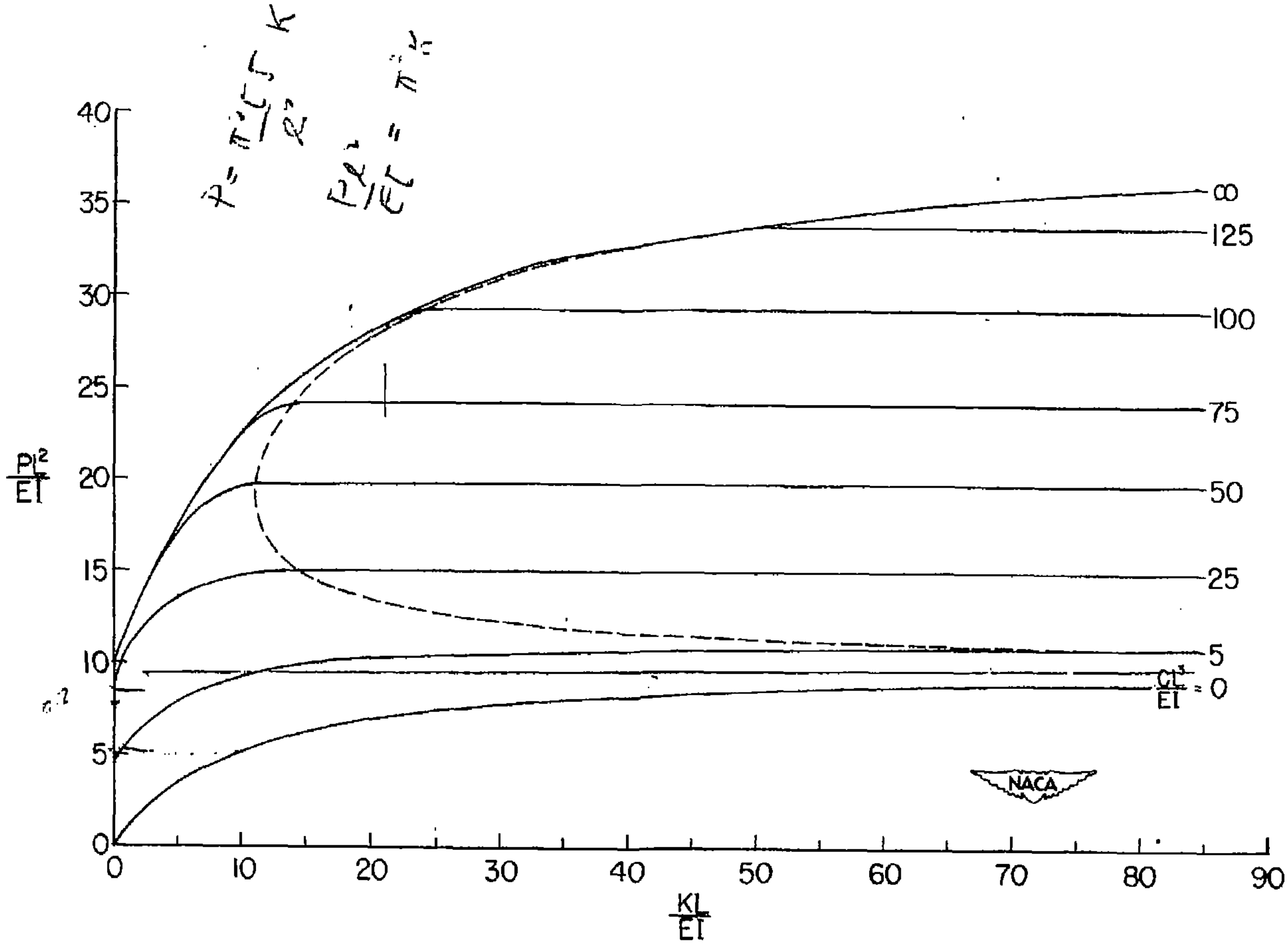


Figure 8.- Buckling curves for column with an infinite number of spans.