NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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EFFECT OF SHEAR LAG ON BENDING VIBRATION OF BOX BEAMS

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SUMMARY

An analytical investigation is made of the effect of shear lag on the bending vibration of wings that are designed essentially as shallow box beams, and a procedure is outlined for incorporating this effect in the determination of bending modes and frequencies. Numerical examples show that shear-lag action in a box beam can have a large influence on its vibration characteristics. The calculations indicate that even though only a small shear-lag action may be observed in a simple static deflection test of the beam, reductions in the second and higher-mode frequencies may be relatively large.

INTRODUCTION

Vibration tests of airplane wings have shown that discrepancies often exist between observed and calculated natural frequencies of wings. Among the possible sources of these discrepancies are aerodynamic and structural damping, rotary inertia, shearing deflections, and shear-lag effects, all of which are neglected in the usual engineering theory for beam vibration. The present paper investigates briefly the shear-lag effects. The shear-lag theory upon which this paper is based is presented in references 1 and 2.

The strength element of many wings is essentially a shallow box beam in which the secondary strains arising from shear lag sometimes have a significant influence on the bending stiffness, which in turn affects the vibration characteristics of the wing. That the natural frequencies of wings can be appreciably reduced when shear lag is present is shown by the included numerical examples. A simple procedure is outlined for incorporating the effects of shear lag on bending stiffness in the determination of bending modes and frequencies. Although not presented here, a similar procedure can be used to incorporate the effect of the so-called bending stresses due to torsion on the torsional modes and frequencies of box beams.

SYMBOLS

L	length of beam
E	Young's modulus of elasticity
G	effective shear modulus of cover sheet of box beam
I	bending moment of inertia of beam cross section
m	mass of beam per unit length
P_{eq}	equivalent loading used in numerical integration
S	total shear force
M	bending moment
$Y_n^{(i)}$	deflection of nth mode of vibration after i iterations for both derived values of deflection and for values written in terms of unit tip deflection
$\frac{x}{L}$	stations along beam
ω_n	circular frequency of nth natural mode of vibration, radians per second
f_n	frequency of nth natural mode of vibration, cycles per second
j	number of equal-length bays into which beam is divided for the shear-lag and vibration analysis
λ	distance between stations dividing box beam into bays
X	statically indeterminate forces in corner flanges at each station due to shear lag
R	radius of curvature of elastic axis at each station due to shear-lag strain in one cover of box beam
K,p,q,γ	parameters used in shear-lag analysis
A_F	area of corner angle or flange plus one-sixth the area of vertical shear web
A_L	area of longitudinals and effective sheet material over one-hal the width of box (See reference 1.)

b	width from center line of section to centroid of corner flange
h	depth between centroids of top and bottom flanges
t	average thickness of cover sheet

MEANING OF SHEAR LAG

Before an attempt is made to describe the effect of shear lag on the vibration of beams, a brief explanation of the term "shear lag" will be given. In figure 1 the tensile-stress distribution at two locations is shown for a simple sheet-stringer panel under a given loading. In the region near the load, the tensile stress is large at the edges where the load is applied, but in regions remote from the load the tensile stress tends to become more uniformly distributed over the width of the panel because of the shearing stresses which also exist in the structure. If these shearing stresses caused no shearing deformation of the sheet in the panel, the tensile-stress distribution would necessarily be uniform at all points instead of varying as shown in figure 1. Shear lag is the term commonly used to describe the influence that shearing deformations have on the stress distribution.

The extent to which shear lag occurs in a structure is a function of the geometry of the structure and of the loading. In the elementary bending theory of beams, the influence of shearing deformations on the stress distribution is neglected because shearing deformations are generally small. In box beams of certain proportions, however, the shearing deformations cannot be ignored if the stresses and deflection are to be predicted accurately. A typical bending-stress variation over the cover sheet in the vicinity of the root of a cantilever box beam is shown in figure 2. Instead of a uniform longitudinal stress across the section, as would be predicted by elementary bending theory which considers the thin cover sheet infinitely stiff in shear, the longitudinal stresses are increased near the edges and are decreased at the center of the beam. This change in stress distribution causes a change in the beam deflection. If cut-outs or certain load concentrations were present, similar shear-lag effects would occur. The deformation of wing structures, including the effect of shear lag, is discussed in reference 2.

EFFECTS OF SHEAR LAG ON VIBRATION

Beam deflections are usually considered to be a function of the loading, the manner of support, and the beam stiffness which is calculated from the geometry of the cross sections and the modulus of elasticity of the material. The stiffness of box beams, however, is also influenced by shearing deformation of the thin cover sheets. For beams in which this shearing deformation is appreciable, the increases in stress and, consequently, in strain in the corner flanges result in larger deflections for the beam as a whole than would be predicted by elementary beam theory. Such beams are essentially less stiff than similar beams (same EI variation) without shear lag. In problems dealing with shear-lag beams, therefore, the usual concept of beam stiffness must be replaced by an effective stiffness concept, which takes into account the shear-lag strains present. With such a concept, the effective stiffness is a function of the beam loading, since the amount and distribution of the shear-lag strains along the beam vary for each different loading condition.

Under the inertia loads occurring in vibration, the effective stiffness often differs appreciably from the geometrical stiffness. Since the stiffness characteristics of the beam change the vibration characteristics also change. Thus, shear-lag effects cannot be neglected in the determination of the bending modes and frequencies of some box beams.

ANALYTICAL PROCEDURE FOR INCLUDING SHEAR-LAG EFFECTS IN THE DETERMINATION OF NATURAL MODES AND FREQUENCIES OF WINGS

When bending modes and frequencies of beams are determined by the iteration procedure given in reference 3, the vertical shear forces and bending moments in the beam are found by direct integration of an inertia loading, $m(x)Y^{(0)}(x)$, where $Y^{(0)}(x)$ is the assumed deflection. The bending moments are converted into curvature by dividing by the stiffness EI; the slope and deflection are then found by integration of the curvature. This process is repeated, the newly-found deflection being used to compute the inertia loading for the next iteration, until two successively computed deflections bear a constant ratio to each other. This ratio is the square of the natural frequency ω^2.

For a box beam with shear lag, the process is essentially the same except that the curvature cannot be found simply by dividing the bending-moment variation by the "geometrical" stiffness variation along the beam. To the basic $\frac{M}{EI}$-curvature, a correction must be applied which takes into account the additional curvature caused by the secondary strains in the corner flanges due to shear lag. The addition of this curvature correction to the basic $\frac{M}{EI}$-curvature is simply the process of properly taking into account the effective reduction in stiffness of the beam. The resulting total curvature would be the same as if the bending moment had been divided by the effective stiffness. Since the stiffness term EI that appears in the differential equation of vibration for a beam is the stiffness that governs the curvature, the analysis of reference 3 still applies to the beams considered herein. In reference 3 the geometrical stiffness EI computed from the geometry of the cross sections of the beam was used, whereas in the present paper an effective stiffness which incorporates the effect of shear lag is used.

An outline of the procedure used in this paper to determine the shear-lag curvature corrections will be given to clarify the numerical examples which are presented. The procedure is an adaptation of the more general shear-lag analysis presented in references 1 and 2. For a doubly symmetrical box beam, the steps are as follows:

(1) Divide the box into j equal bays. (See fig. 3.) For each bay, determine the constants A_F, A_L, b, h, and t for the simplified cross section (fig. 4), as used in the shear-lag analysis presented in reference 1.

(2) From these simplified cross sections, compute the following shear-lag parameters:

$$K^2 = \frac{2Gt}{Eb}\left(\frac{1}{A_F} + \frac{1}{A_L}\right)$$

$$p = \frac{K}{t \tanh K\lambda}$$

$$q = \frac{K}{t \sinh K\lambda}$$

$$\gamma = \frac{1}{2} \frac{SA_L}{ht\left(A_F + A_L\right)}$$

(In reference 1, p, q, and γ are defined with a factor G in the denominator; but, since G is a constant that can be factored from the equations given in step (4), the factor is omitted in these formulas.)

(3) From an assumed inertia loading $mY^{(0)}$ on the beam in the first iteration of the vibration analysis compute the average vertical shear force existing in the webs in each bay (one-half the total vertical shear for a given bay).

(4) From the average vertical shear in the webs of each bay compute the self-equilibrated groups of X-forces at each station. They are defined by the equations from reference 1

$$\left.\begin{aligned}
X_j q_j - X_{j-1}\left(p_j + p_{j-1}\right) + X_{j-2} q_{j-1} &= -\gamma_j + \gamma_{j-1} \\
X_{j-1} q_{j-1} - X_{j-2}\left(p_{j-1} + p_{j-2}\right) + X_{j-3} q_{j-2} &= -\gamma_{j-1} + \gamma_{j-2} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\
X_1 q_1 - X_0 p_1 &= -\gamma_1
\end{aligned}\right\} \qquad (1)$$

The subscripts on the X-forces refer to the stations and the subscripts on the parameters p, q, and γ refer to the bays as they are numbered in figure 3.

(5) Compute the curvature correction at each station from the X-forces by the relation

$$\frac{1}{R} = \frac{X}{EA_F h} \qquad (2)$$

where A_F and h are computed at the station points. Equation (2) gives the correction due to shear lag in one cover.

(6) To the $\frac{M}{EI}$-curvature at each station computed in the vibration analysis, add the shear-lag curvature correction for both covers. Complete the remaining steps of an iteration.

New curvature corrections should be computed in each succeeding iteration; but if the inertia loading computed from the assumed deflection in the first iteration is reasonably representative of the true loading, the curvature corrections computed from this loading are accurate enough in most cases. Furthermore, precise computations are not justified because the accuracy of the shear-lag theory for loadings of the type that occur in vibration is unknown. In those cases for which curvature corrections are computed from assumed deflections differing widely from the derived shape, however, a second set of curvature corrections may improve the accuracy of the derived mode and frequency.

NUMERICAL EXAMPLES

In order to show the effect of shear lag on bending vibration, frequency and mode calculations have been performed for the two uniform box beams shown in figures 5 and 6. Beam A (fig. 5) was designed to show only a small amount of shear-lag action in static loading as shown by the fact that under a concentrated tip load the calculated tip deflection of this beam is only two percent greater by shear-lag theory than by ordinary beam theory. Beam B (fig. 6) was designed to show an exaggerated shear-lag action in static loading tests. Under a concentrated tip load, the calculated increase in tip deflection due to shear lag in this beam is about 19 percent. The following table gives a comparison of some of the natural bending frequencies of the two beams computed both with and without taking shear lag into account:

Beam	Mode	Frequency (cps)		Percent change
		Shear lag	No shear lag	
A	1	45.7	46.65	2.1
	2	260	292	12.3
	3	672	817	21.6
B	1	34.6	43.8	21.0

The calculations indicate that, even though a box beam shows little shear-lag action under a simple static loading (for example, beam A), appreciable reductions in the second and higher-mode frequencies of vibration might be expected. For a box beam that shows large shear-lag action under a simple static loading (for example, beam B), a correspondingly large reduction in even the fundamental bending frequency might be expected. A brief discussion of the calculation for each beam is presented in the following sections.

Beam A

Fundamental mode.- The procedure followed in the determination of the fundamental mode and frequency is indicated in the upper half of table 1. The first iteration is shown in detail and each step follows closely the equivalent-load method shown in table 1 of reference 3, except for the insertion of the shear-lag curvature corrections (column 7), which are added to the $\frac{M}{I}$-curvatures in column 8. (The mass per unit length m is carried as a common factor in each column because it is constant over the length of the beam.) The shear-lag curvature corrections are calculated from the X-forces which are defined by the system of equations in the lower half of the table. The shear data in column 4 are used to compute the values of the parameter γ, which form the constant terms in the equations (equations (1)). The coefficients of the X-forces are determined from the shear-lag parameters p and q. The X-forces are solved for from this system of equations. The curvature corrections E/R are then calculated from these X-forces by the use of the equation shown in the table. $(12/m\lambda^2$ is a factor used to convert the curvature corrections to the same units as the $\frac{M}{I}$-curvature and the factor 2 takes care of the correction for both covers of the symmetrical box beam.) The curvature corrections are then inserted in the iteration procedure (column 7).

The fundamental mode, as determined from one more iteration, is listed in column 15. Shear lag has little effect on the shape of the fundamental mode, as shown in figure 7(a). The frequency of vibration with shear lag taken into account is computed in the table. If no shear lag were present, the fundamental frequency of this beam would be the same as that for a uniform cantilever, or

$$\omega_1^2 = \overline{3.52}^2 \frac{EI}{mL^4}$$

$$= 85,700 \text{ (radians/sec)}^2$$

Second and third modes.- The procedure for finding the second and third modes and frequencies is the same as that given for higher-mode determination in reference 3 except for the shear-lag curvature corrections, which are introduced in the same manner as illustrated for the first mode in table 1. The computations show that shear lag accounted for about 39 and 47 percent of the total curvature at the root station of the beam in the second and third modes of vibration, respectively. The curvature at the other stations was affected to a lesser but still significant extent. The effect of these curvature corrections on mode shape is illustrated in figure 7. It is evident that conclusions regarding the frequency change associated with these mode shapes cannot readily be drawn from comparison of the deflections alone because the curvature differences in the mode shapes, which also influence the frequency, are hidden.

Beam B

The length-width ratio, as well as the A_F/A_L ratio, for beam B is considerably smaller than that for beam A and, as might be expected from the static-load deflection comparison presented previously for these two beams, a more pronounced shear-lag action was noticed for beam B in the fundamental-mode calculations. The calculations showed that shear-lag action accounted for about 48 percent of the total root curvature. The resulting sizeable reduction in frequency (21 percent) would not be expected in a box beam of ordinary design, but this reduction does indicate that the effect of shear lag can be appreciable even on the fundamental mode of vibration.

CONCLUDING DISCUSSION

The investigation has shown that, even though the shear lag in a given box beam had a relatively insignificant effect on the static deflection of the beam under a tip load and on the fundamental frequency of vibration, it caused an appreciable reduction in the higher frequencies. This increasing effect of shear lag on the higher modes can be explained by the fact that the relatively greater rates of change of the bending moment over the span in these modes of vibration are accompanied by increased shear deformation of the cover sheets and result in a reduced flexural stiffness of the beam. It has been pointed out that shearing deformations in the thin cover sheets effectively change the stiffness, and hence the vibration characteristics, of the beam.

Shear deformation of the cover also takes place around discontinuities and abrupt changes in cross sections, such as cut-outs, and around points of load concentration. In an actual wing, shear-lag effects due to these disturbances may be of more importance than the simple effect treated in this paper and should therefore be investigated when a determination of the modes and frequencies is being made. The analytical procedure for introducing shear-lag corrections for more complicated structures differs little from the procedure presented herein.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., February 2, 1948

REFERENCES

1. Kuhn, Paul: A Procedure for the Shear-Lag Analysis of Box Beams. NACA ARR, Jan. 1943.

2. Kuhn, Paul: Deformation Analysis of Wing Structures. NACA TN No. 1361, 1947.

3. Houbolt, John C., and Anderson, Roger A.: Calculation of Uncoupled Modes and Frequencies in Bending or Torsion of Nonuniform Beams. NACA TN No. 1522, 1948.

TABLE I

THE DETERMINATION OF THE FUNDAMENTAL MODE AND FREQUENCY OF BOX BEAM A

INCLUDING THE EFFECTS OF SHEAR LAG

$\left[E = 10{,}500{,}000 \text{ psi};\ G = 4{,}000{,}000 \text{ psi};\ L = 75 \text{ in.};\ \lambda = 7.5 \text{ in.};\ I = 13.11 \text{ in.}^4;\ m = 0.00063 \frac{\text{lb-sec}^2}{\text{sq in.}}\right]$

Fundamental mode and frequency computations

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Station	$Y_1^{(0)}$	P_{eq}	S	M	M/I	E/R	$(M/I)_{eff}$	$(M/I)_{eq}$	Slope	$Y_1^{(1)}$	$Y_1^{(1)}$	$Y_1^{(2)}$	ω_1^2 (a)	$Y_1^{(2)}$
Common factor →	m	$\frac{m\lambda}{12}$	$\frac{m\lambda}{12}$	$\frac{m\lambda^2}{12}$	$\frac{m\lambda^2}{12}$	$\frac{m\lambda^2}{12}$	$\frac{m\lambda^2}{12}$	$\frac{m\lambda^3}{144}$	$\frac{m\lambda^3}{144E}$	$\frac{m\lambda^4}{144E}$	m	$\frac{m\lambda^4}{144E}$		
10	1.00	5.67	5.67	0	0		0		1204.7	8984	1.000	9232	82,200	1.000
9	.84	10.10	15.77	5.67	.43	-0.57	-.14	-0.4	1205.1	7779	.866	7993		.866
8	.70	8.42	24.19	21.44	1.64	-.65	.99	12.7	1192.4	6574	.732	6754	82,200	.732
7	.58	6.95	31.04	45.63	3.48	-.58	2.90	35.4	1157.0	5382	.599	5528		.599
6	.45	5.43	36.47	76.67	5.85	-.47	5.38	65.0	1092.0	4225	.470	4339	82,200	.470
5	.35	4.19	40.66	113.14	8.63	-.36	8.27	99.6	992.4	3133	.348	3216		.348
4	.24	2.89	43.55	153.80	11.71	-.22	11.49	138.2	854.2	2141	.238	2196	82,200	.238
3	.14	1.71	45.26	197.35	15.05	-.04	15.01	180.4	673.8	1287	.143	1319		.1430
2	.07	.86	46.12	242.61	18.50	.31	18.81	226.4	447.4	613	.0681	628	82,200	.0680
1	.02	.27	46.39	288.73	22.03	1.29	23.32	281.9	165.5	166	.0184	169		.0183
0	0			335.12	25.55	4.30	29.85	165.5		0	0	0		0

[a] $\omega_1^2 = \frac{144 \times 10500000}{0.00063 \times \overline{7.5}^4} \frac{Y_1^{(1)}}{Y_1^{(2)}}$

Calculations for shear-lag curvature corrections

(A_F = 0.237 sq in.; A_L = 0.320 sq in.; K = 0.1575; p = 4.76; q = 2.67; h = 4.506 in.; b = 9 in.; t = 0.040 in.; values are same for each bay because box beam is uniform)

$$\gamma = \frac{S}{2} \frac{m\lambda}{12} \frac{A_L}{ht(A_F + A_L)}$$

$$= 0.0005798$$

Station	γ	$-\gamma_j + \gamma_{j-1}$
10	0.00328	
9	.00914	0.00586
8	.01400	.00486
7	.01800	.00400
6	.02110	.00310
5	.02355	.00245
4	.02520	.00165
3	.02620	.00100
2	.02670	.00050
1	.02680	.00010
0		-.02680

$$\frac{E}{R} = 2\frac{12}{m\lambda^2} \frac{X}{hA_F}$$

$$= 635X$$

Station	E/R
10	
9	-0.57
8	-.65
7	-.58
6	-.47
5	-.36
4	-.22
3	-.04
2	.31
1	1.29
0	4.30

X_9	X_8	X_7	X_6	X_5	X_4	X_3	X_2	X_1	X_0	$-\gamma_j + \gamma_{j-1}$
-9.52	2.67									0.00586
2.67	-9.52	2.67								.00486
	2.67	-9.52	2.67							.00400
		2.67	-9.52	2.67						.00310
			2.67	-9.52	2.67					.00245
				2.67	-9.52	2.67				.00165
					2.67	-9.52	2.67			.00100
						2.67	-9.52	2.67		.00050
							2.67	-9.52	2.67	.00010
								2.67	-4.76	-.02680

Solution	
Station	X
9	-0.00090
8	-.00102
7	-.00091
6	-.00074
5	-.00056
4	-.00035
3	-.00006
2	.000499
1	.00203
0	.00677

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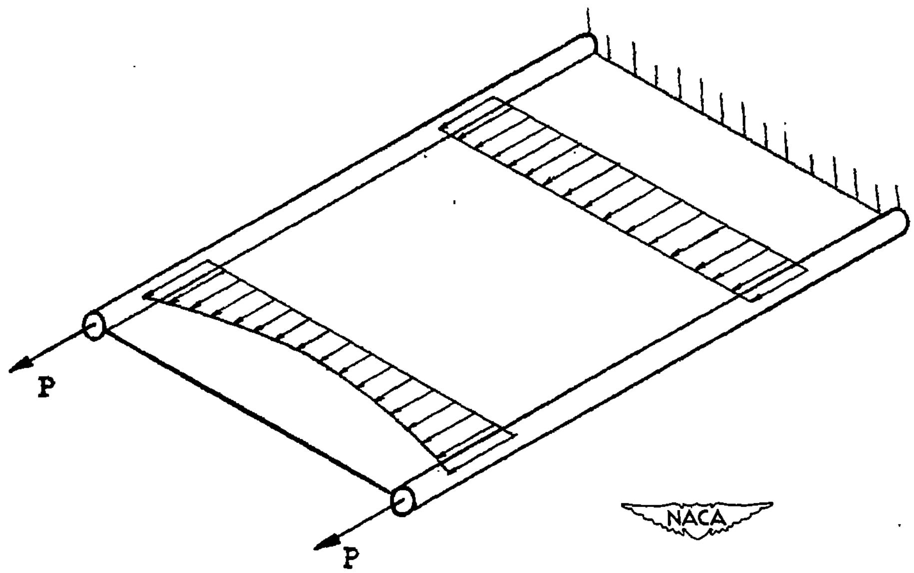

Figure 1.- Tensile-stress distribution in a panel with shear deformation of the sheet.

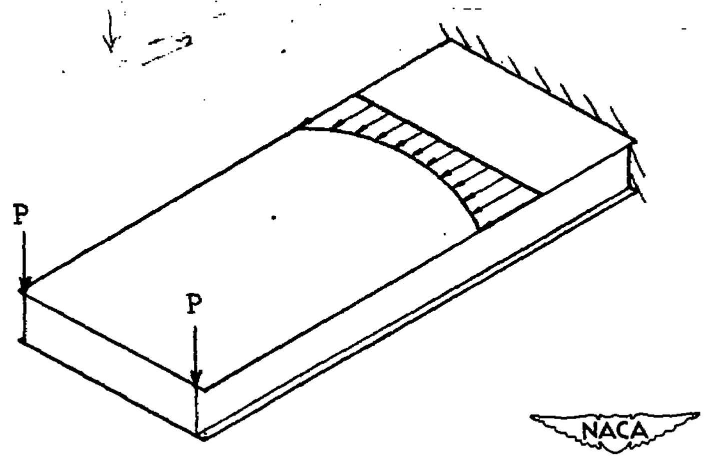

Figure 2.- Bending-stress distribution in a cantilevered box beam.

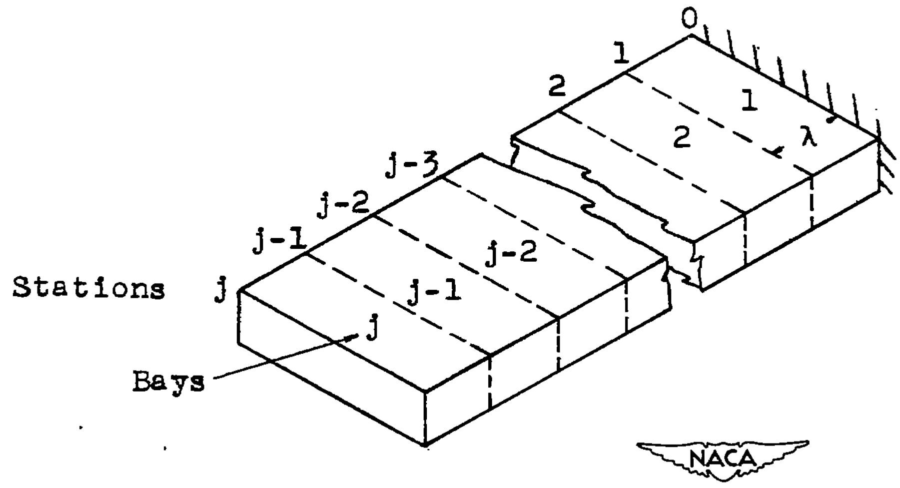

Figure 3.- Notation for bays and stations in shear-lag and vibration analysis.

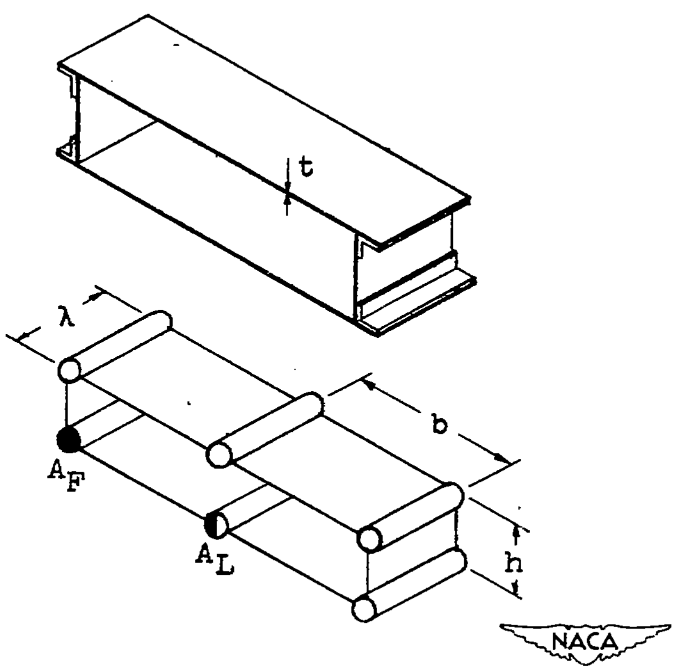

Figure 4.- Simplified cross section used in shear-lag analysis.

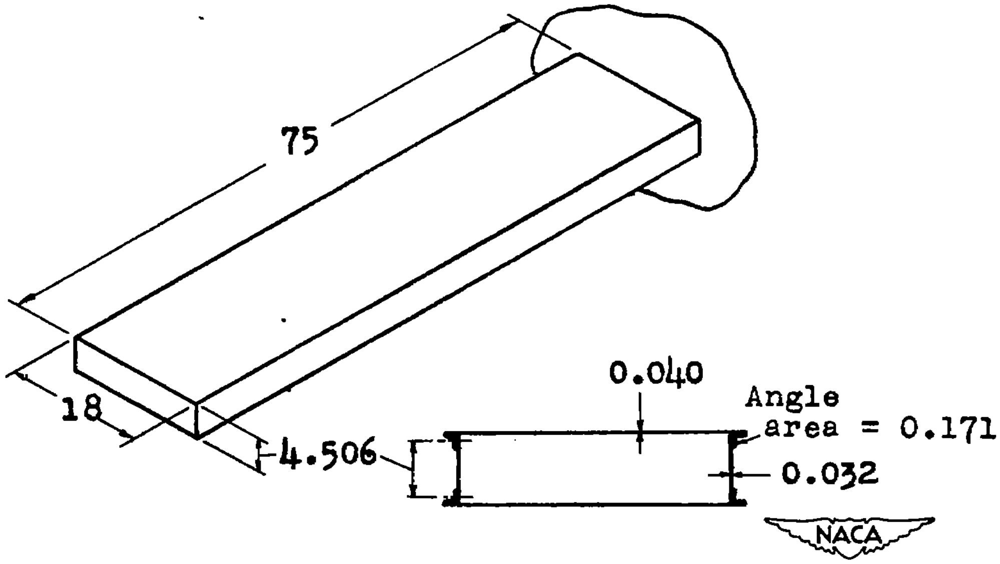

Figure 5.- Box beam A used in numerical examples.

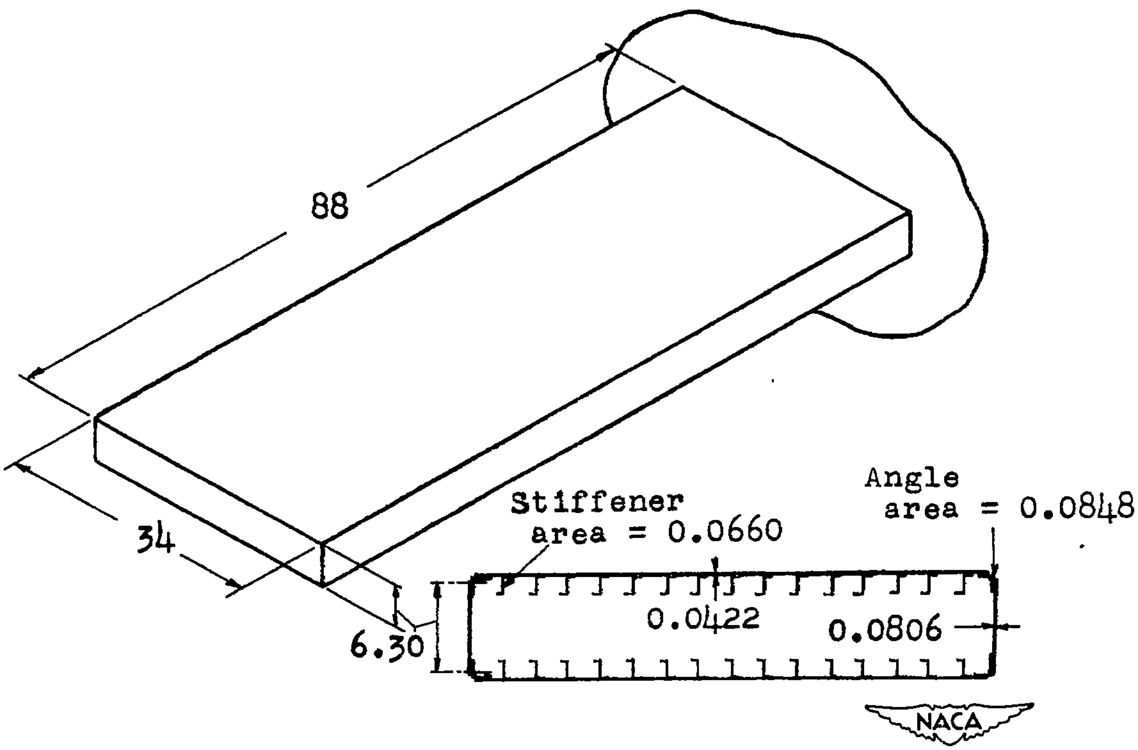

Figure 6.- Box beam B used in numerical examples.

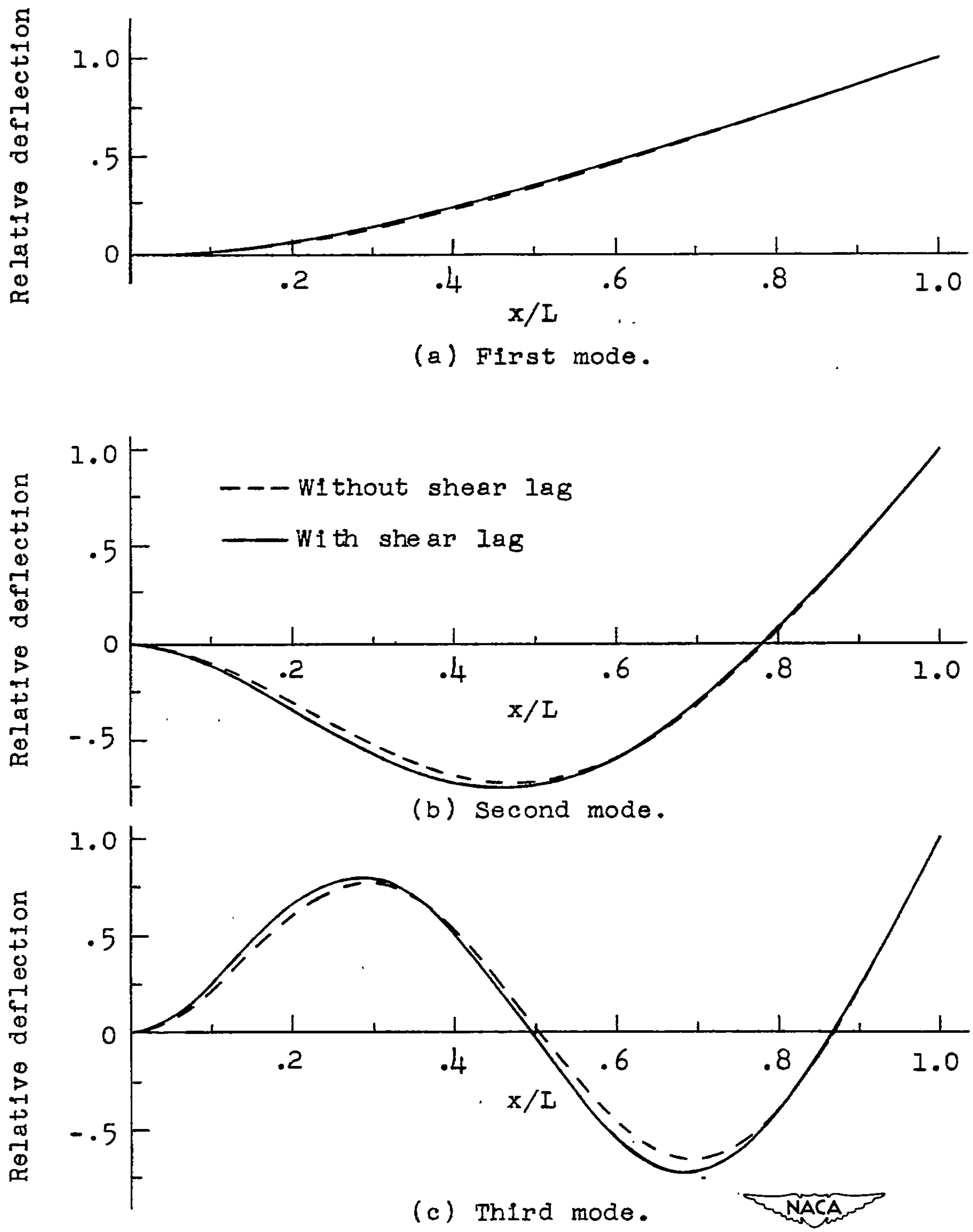

Figure 7.- Effect of shear lag on the bending modes of vibration of a uniform box beam.