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DETERMINATION OF BENDING MOMENTS IN PRESSURE-LOADED RINGS OF ARBITRARY SHAPE WHEN DEFLECTIONS ARE CONSIDERED

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DETERMINATION OF BENDING MOMENTS IN PRESSURE-LOADED RINGS OF ARBITRARY SHAPE WHEN DEFLECTIONS ARE CONSIDERED

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SUMMARY

An analytical method has been derived for determining bending-moment distribution in rings of arbitrary shape under internal pressure loads, with the change of geometric shapes caused by the load being considered. For the purpose of clarity, the method developed was applied only to double-symmetrical shapes. A differential-integral equation has been derived for this purpose and its solution obtained in the form of a trigonometric series.

Charts have been provided for two specific families of rings of various proportions and flexibilities. Tests conducted on rings of both families agree very well with the analytical calculations. For rings belonging to or close to these families, the bending-moment distributions and the deflections of the ring can be read directly from the curves. For rings of entirely different shapes, an average of 20 hours is necessary for the complete solution of the problem. Examples have been given to show the method of obtaining the curves.

On comparing the present results with the results of solutions in which deflections have been neglected, it is seen that the bending moments previously computed have always been much too conservative. The error introduced is considerable when the rings become more and more flexible.

It is believed that by the use of this method curves can be drawn for a typical frame in a fuselage and from these curves the bending moments and deflections can be estimated for all similar frames.

INTRODUCTION

Numerous papers have been written regarding the design of the rings for monocoque fuselages. All of them were based on the assumption that the deflections of the ring caused by the loads are so small that the internal bending moments and shear and axial loads will be unchanged by any small change in the geometric shape of the ring. However, the error introduced by neglecting the change in the geometric shape becomes more and more important as the size of the airplane increases and more flexible rings are used to save weight. It is the object of this paper, therefore, to take into account deflections when calculating the bending moments in rings.

The fact that the change of geometric shape has an important effect on the final bending-moment distribution in the rings can be illustrated by the beam-column analogy of an initially curved bar under axial tension loads. For the bar as illustrated in figure 1, the initial bending moment is $M = Py$. If the bar is stiff and deflections are small enough to be neglected, the final bending moment is the same. But if the bar is flexible and the deflections are comparably large, the final bending moment becomes

$$M = P(y - \delta)$$

The difference between these two equations depends on the flexibility of the bar and can be considerable for flexible bars.

In this paper two expressions for the bending-moment distribution are derived. The first expression is obtained by considering the action of the applied pressure and the final shape of the deflected ring. The second expression is obtained from the change of curvature of the ring caused by the loads. If these two expressions are set equal, a differential-integral equation is obtained. The solution of this equation gives the radial deflections of the ring and the bending-moment distribution. Another formula is derived from which the angular displacements can be determined when the radial deflections are known.

The rings discussed in this paper are assumed to be under only internal pressure. The method recommended herein can be extended to any system of external loads and is also applicable to rings of any shape although only shapes of double symmetry are discussed in the present paper. The following assumptions are made:

(1) The rings have regular and smooth shapes of the type encountered in fuselage frames.

(2) The thickness of the ring is small compared with its radius. Consequently the following formula can be used with an extremely small error. (See reference 1.)

$$\frac{1}{\rho_1} - \frac{1}{\rho_0} = -\frac{M_i}{EI}$$

where

$1/\rho_0$ original curvature at certain point of ring

$1/\rho_1$ final curvature at corresponding point of ring after ring is deflected

M_i internal bending moment built up because of deflections of ring

EI bending flexibility

(3) The ring is regarded as inextensible.

(4) The deflections of the ring are defined by its radial deflections w and angular displacement ϕ. Both w and ϕ are assumed to be large enough to be significant but small when compared with the radius of the ring. Therefore, all terms can be neglected containing second or higher powers or products of w/r or ϕ. Also all terms containing second or higher powers or products of the following items are neglected: $\frac{dw}{d\phi}\Big/r$, $\frac{d^2w}{d\phi^2}\Big/r$, $\frac{d\phi}{d\theta}$, and $\frac{d^2\phi}{d\theta^2}$, where r and θ are the polar coordinates of the ring.

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SYMBOLS

a,b major and minor axis, respectively, of elliptical ring

$a_n,b_n,c_n,d_n,e_n,f_n,g_n,h_n$ Fourier coefficients

$k^2 = 1 - \left(\frac{b}{a}\right)^2$

$k' = \frac{H}{R}$

q applied pressure load per unit length

r,θ polar coordinates of original ring

r_1,θ_1 polar coordinates of deflected ring

r_A radius from origin to point A on original ring

s arc length along circumference of ring

t	nondimensional parameter (EI/qa^3)
t'	nondimensional parameter (EI/qR^3)
u,v	displacements of a point on ring parallel to x- and y-axis, respectively
w	radial deflection of ring
A_m	Fourier coefficient for radial-deflection function
EI	bending flexibility of ring
F_1,F_2,F_3	certain functions
H	height above x-axis of straight-line portion of rings of family II
H_A	axial stress at point A on ring
J	polar moment of inertia of equivalent ring of elastic weight referred to original ring
J_1	polar moment of inertia of equivalent ring of elastic weight referred to deflected ring
K_{mn}	certain constants
M	bending moment
M_0	bending moment in ring if change of geometric shape of ring is neglected
M_A	bending moment at point A on ring
M_a	bending-moment expression derived from consideration of applied pressure load and final shape of deflected ring
M_i	bending-moment expression derived from consideration of internal stress due to change of curvature of ring
M_θ	bending moment in ring at point defined by θ
Q_n	certain constants
R	radius of circular arc of rings of family II
S	area of equivalent ring of elastic weight referred to original ring

S_1 area of equivalent ring of elastic weight referred to deflected ring

α angle of rotation of cross section of ring

ρ_0 radius of curvature at a point on original ring

ρ_1 radius of curvature at point on deflected ring corresponding to ρ_0

ϕ angular displacement

THEORETICAL ANALYSIS

In the following analysis two expressions for the bending-moment distributions are derived. The first expression is obtained from consideration of the action of the applied pressure load on the final shape of the deflected ring. The second expression is obtained from consideration of the internal stress distribution due to change of curvature caused by loading of the ring. Hereinafter the bending moment corresponding to the first expression is designated by M_a and the bending moment corresponding to the second expression is designated by M_i. By equating the two expressions, a differential-integral equation which represents the equilibrium condition of the final deflected ring is obtained. The solution of this differential-integral equation determines the radial deflections of the ring and the bending-moment distribution. From the radial deflections, the angular displacements can be found from the nonextension theory, which is treated in detail in appendix A.

Expression for M_a

The moment M_a is the bending moment determined from consideration of the applied pressure load and the final deflected position of the ring. Before the expression for M_a is derived, a typical method which has been used to determine the bending moment in the ring without consideration of the change of geometrical shape of the ring is discussed.

Let the shape of the given ring be defined by

$$r = r(\theta)$$

The ring is assumed to be symmetrical with respect to both the x- and y-axes. This simplification does not affect the validity of the method, which can be used for any shape. The intensity of the internal pressure on the ring is designated by q.

An imaginary cut is assumed at point A (see fig. 2) and two unknowns H_A and M_A are introduced, where H_A is the axial force at A (positive if in tension) and M_A is the bending moment (positive if the outside fiber is under compression). There is no transverse shear at point A because of symmetry. (See reference 2.) The ring is now statically determined.

The bending moment at any point C, defined by θ, as shown in figure 2, can be expressed as

$$M_\theta = M_A - H_A\left(r_A - r_\theta \sin \theta\right) + \frac{q}{2}l^2$$

Substituting the expression

$$l^2 = {r_A}^2 + {r_\theta}^2 - 2r_A r_\theta \sin \theta$$

into the foregoing equation, there results

$$M_\theta = A + Br_\theta \sin \theta + C{r_\theta}^2 \tag{1}$$

where

$$\left.\begin{aligned} A &= M_A - H_A r_A + \frac{q}{2}{r_A}^2 \\ B &= H_A - qr_A \\ C &= \frac{q}{2} \end{aligned}\right\} \tag{2}$$

The terms A, B, and C of equation (2) are independent of θ.

The angle of rotation α of the cross section at C and the horizontal displacement u of point C, both relative to point A, are given by the equations:

$$\alpha_C = \int_A^C \frac{M_\theta}{EI} ds \tag{3}$$

$$u_C = \int_A^C \frac{M_\theta}{EI}\left(r_A - r_\theta \sin \theta\right) ds \tag{4}$$

Here C can be any point on the ring. If A' denotes the other end of the ring at the cut, equations (3) and (4) should also hold at point A'; and since there is no rotation nor horizontal displacement at the cut, the following relations are true:

$$\alpha_{A'} = \oint \frac{M_\theta \, ds}{EI} = 0 \tag{5}$$

$$u_{A'} = \oint \frac{M_\theta}{EI}\left(r_A - r_\theta \sin \theta\right) ds$$

$$= - \oint \frac{M_\theta r_\theta}{EI} \sin \theta \, ds = 0 \tag{6}$$

The substitution of equation (1) into equations (5) and (6) yields the following equations:

$$A \oint \frac{ds}{EI} + B \oint \frac{r_\theta \sin \theta}{EI} ds + C \oint \frac{{r_\theta}^2}{EI} ds = 0 \tag{7}$$

$$A \oint \frac{r_\theta \sin\theta \, ds}{EI} + B \oint \frac{{r_\theta}^2 \sin^2\theta \, ds}{EI} + C \oint \frac{{r_\theta}^3 \sin\theta \, ds}{EI} = 0 \qquad (8)$$

Because of the property of double symmetry,

$$\left.\begin{aligned} &\oint r_\theta \sin\theta \frac{ds}{EI} = 0 \\ &\oint {r_\theta}^3 \sin\theta \frac{ds}{EI} = 0 \end{aligned}\right\} \qquad (9)$$

Therefore

$$\left.\begin{aligned} A &= -\frac{\oint \frac{{r_\theta}^2 \, ds}{EI}}{\oint \frac{ds}{EI}} C \\ B &= 0 \end{aligned}\right\} \qquad (10)$$

Or, with the following designation,

$$\left.\begin{aligned} J &= \oint \frac{{r_\theta}^2 \, ds}{EI} \\ S &= \oint \frac{ds}{EI} \end{aligned}\right\} \qquad (11)$$

it follows that

$$\left.\begin{aligned} A &= -\frac{J}{S} C \\ B &= 0 \end{aligned}\right\} \qquad (12)$$

After substituting equation (12) into equation (1), the expression for the bending moment becomes

$$M_\theta = C\left(r_\theta^2 - \frac{J}{S}\right)$$

$$= \frac{q}{2}\left(r_\theta^2 - \frac{J}{S}\right) \tag{13}$$

The expressions for J and S can be easily memorized because they are equivalent to the polar moment of inertia and area of the corresponding ring of elastic weight, respectively. (See appendix B for more details.)

It should be noticed that the bending moment given by equation (13) is a function of r. Any change of shape of the ring changes the values of r and consequently the bending moment also is different. All papers in the past have neglected this change and have called M_θ, given by equation (13), the final bending-moment distribution. It is shown later in the examples that the error is considerable for flexible rings. The derivation of equation (13), however, leads to the establishment of the expression for M_a, which is based on the final deflected shape of the ring rather than the original shape.

The final position of the deflected ring is defined by $r_1 = r_1(\theta)$. Because of the assumption that the ring is inextensible, or in other words, ds = Constant, there immediately follows from equation (13) the following expression for M_a which is referred to the final deflected ring:

$$M_a = \frac{q}{2}\left(r_1^2 - \frac{J_1}{S_1}\right) \tag{14}$$

where

$$\left.\begin{aligned} J_1 &= \oint r_1^2 \frac{ds}{EI} \\ S_1 &= \oint \frac{ds}{EI} = S \end{aligned}\right\} \tag{15}$$

Equation (14) gives the expression for the bending-moment distribution from consideration of the applied pressure load and referred to the final deflected position of the ring.

If the shape of the final deflected position of the ring is known, the bending-moment distribution can be obtained immediately from equation (14). The problem now is to find a method for determining the final deflected position of the ring, which, in turn, depends on the bending-moment distribution.

Expression for M_i

The moment M_i is the bending moment built up from the fiber stresses resulting from the changes of curvature of the ring when it is deflected. To set up the relationship existing between the bending moment M_i and the deflections is rather difficult, especially when deflections in both directions (two-dimensional) are to be considered. Timoshenko presents the derivation of a differential equation that gives the relationship between the radial deflections and the internal bending moment. (See reference 3.) The differential equation is

$$\frac{d^2w}{d\theta^2} + w = \frac{M_i}{EI}r^2$$

where w is the radial deflection of the ring. This equation is based on circular rings and can be used only for rings that are nearly circular. In the following paragraphs, a differential equation is derived which is more general than the one given by Timoshenko.

Referring to figure 3, let the original shape of the ring be defined by $r = r(\theta)$. An arbitrary point A, after loading, experiences an angular rotation ϕ and a radial deflection $w = BD$. Its new position is completely defined by ϕ and w, both of which are functions of θ. The deflected shape of the ring is then represented by the equation

$$r_1 = g(\theta_1) \tag{16}$$

where

$$\left.\begin{aligned} r_1 &= r + w \\ \theta_1 &= \theta + \phi \end{aligned}\right\} \tag{17}$$

From the calculus, the original curvature of the ring at point A is given by

$$\frac{1}{\rho_0} = \frac{r^2 - r\frac{d^2r}{d\theta^2} + 2\left(\frac{dr}{d\theta}\right)^2}{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}} \tag{18}$$

and the curvature of the deflected ring C_1 at corresponding point D is given by

$$\frac{1}{\rho_1} = \frac{{r_1}^2 - r_1\frac{d^2r_1}{d{\theta_1}^2} + 2\left(\frac{dr_1}{d\theta_1}\right)^2}{\left[{r_1}^2 + \left(\frac{dr_1}{d\theta_1}\right)^2\right]^{3/2}} \tag{19}$$

Since

$$\frac{dr_1}{d\theta_1} = \frac{\frac{dr_1}{d\theta}}{\frac{d\theta_1}{d\theta}} = \frac{\frac{d}{d\theta}(r + w)}{\frac{d}{d\theta}(\theta + \phi)} = \frac{r' + w'}{1 + \phi'} \tag{20}$$

$$\frac{d^2r_1}{d{\theta_1}^2} = \frac{\frac{d}{d\theta}\left(\frac{dr_1}{d\theta_1}\right)}{\frac{d\theta_1}{d\theta}} = \frac{r'' + w''}{(1 + \phi')^2} - \frac{r' + w'}{(1 + \phi')^3}\phi'' \tag{21}$$

where r', w', and so forth indicate derivatives with respect to θ, equation (19) becomes

$$\frac{1}{\rho_1} = \frac{(r + w)^2 - (r + w)\left[\frac{r'' + w''}{(1 + \phi')^2} - \frac{r' + w'}{(1 + \phi')^3}\phi''\right] + 2\left(\frac{r' + w'}{1 + \phi'}\right)^2}{\left[(r + w)^2 + \left(\frac{r' + w'}{1 + \phi'}\right)^2\right]^{3/2}} \tag{22}$$

From the assumption that the ring is inextensible, there exist certain relationships between the angular displacements ϕ and the radial deflections w. These relationships, as given in equations (23) and (24), are derived from the nonextension theory. (See appendix A.)

$$\phi' = -\frac{w}{r} - \frac{w'}{r}\frac{r'}{r} \tag{23}$$

$$\phi'' = \frac{w}{r}\left(\frac{r'}{r}\right) + \frac{w'}{r}\left[-1 - \frac{r''}{r} + 2\left(\frac{r'}{r}\right)^2\right] - \frac{w''}{r}\frac{r'}{r} \tag{24}$$

The foregoing relations, when applied to circular rings for which $r = \text{Constant}$ and $r' = 0$, become

$$\left.\begin{aligned} \phi' &= -\frac{w}{r} \\ \phi'' &= -\frac{w'}{r} \end{aligned}\right\} \tag{25}$$

which are the equations given by Timoshenko (see reference 3, p. 208) for inextensible circular rings.

Substituting equations (23) and (24) into equation (22) and simplifying (see appendix C for details), the following equation is obtained:

$$\frac{1}{\rho_1} = \frac{\left[1 - \frac{r''}{r} + 2\left(\frac{r'}{r}\right)^2\right] - \left[1 + \left(\frac{r'}{r}\right)^2\right]\left(\frac{w}{r} + \frac{w''}{r}\right)}{r\left[1 + \left(\frac{r'}{r}\right)^2\right]^{3/2}} \qquad (26)$$

Now, from equation (18),

$$\frac{1}{\rho_0} = \frac{1 - \frac{r''}{r} + 2\left(\frac{r'}{r}\right)^2}{r\left[1 + \left(\frac{r'}{r}\right)^2\right]^{3/2}}$$

Therefore

$$\frac{1}{\rho_1} - \frac{1}{\rho_0} = -\frac{\left[1 + \left(\frac{r'}{r}\right)^2\right]\left(\frac{w}{r} + \frac{w''}{r}\right)}{r\left[1 + \left(\frac{r'}{r}\right)^2\right]^{3/2}}$$

$$= -\frac{w + w''}{r^2\sqrt{1 + \left(\frac{r'}{r}\right)^2}} \qquad (27)$$

But as already mentioned (assumption (2) of the INTRODUCTION) the following equation is true:

$$\frac{1}{\rho_1} - \frac{1}{\rho_0} = -\frac{M_i}{EI}$$

Therefore

$$w'' + w = \frac{M_i}{EI} r^2 \sqrt{1 + \left(\frac{r'}{r}\right)^2} \tag{28}$$

This equation expresses the relationship between the bending moment at a certain point on the ring and the radial deflection of the ring at the same point.

Equation (28) is valid not only for rings but can be applied to any curved bars provided that the contour of the bar is regular and smooth, the deflections are not too large, and the bar is inextensible. Two extreme cases are given in the following paragraphs for illustration.

First, for circular rings, r is constant and $r' = 0$. Equation (28) reduces to

$$w'' + w = \frac{M_i}{EI} r^2$$

which is exactly the formula given by Timoshenko.

Next, consider a straight beam. Let the origin be chosen at infinity so that $r = \infty$ and $r' = 0$ at all points on the beam. Equation (28) can be simplified as

$$\frac{d^2 w}{r^2\, d\theta^2} + 0 = \frac{M_i}{EI}\sqrt{1 + 0}$$

For the x,y-coordinate system used in beam deflection, $w = y$, $r\, d\theta = dx$. (See fig. 4.) Thus the foregoing equation becomes

$$\frac{d^2 y}{dx^2} = \frac{M_i}{EI}$$

which is the familiar straight-beam formula.

Differential-Integral Equation

Equation (14) gives the expression for M_a which is determined by considering the applied pressure load and the final deflected position of the ring. Equation (28) gives the expression for M_i. It is determined by considering the change of curvature of the ring resulting from the loading. At the final deflected position of the ring, these two bending moments M_a and M_i should be equal at every point on the ring; that is,

$$M_a = M_i$$

Therefore,

$$w'' + w = \frac{1}{2} \frac{q}{EI}\left(r_1^2 - \frac{J_1}{S}\right) r^2 \sqrt{1 + \left(\frac{r'}{r}\right)^2} \tag{29}$$

In this equation, r_1 and J_1 are referred to the final deflected position. (See equation (14).)

From equation (17),

$$\left.\begin{aligned} r_1 &= r + w \\ r_1^2 &= (r + w)^2 = r^2\left[1 + 2\frac{w}{r} + \left(\frac{w}{r}\right)^2\right] \end{aligned}\right\} \tag{30}$$

Therefore

$$J_1 = \oint \frac{r_1^2 \, ds}{EI} = \oint r^2\left[1 + 2\frac{w}{r} + \left(\frac{w}{r}\right)^2\right]\frac{ds}{EI} \tag{31}$$

Remembering the assumption that any term containing a power of w/r higher than second can be neglected, equations (30) and (31) become

$$r_1^2 = r^2 + 2rw \tag{32}$$

and

$$J_1 = \oint r^2 \frac{ds}{EI} + \oint 2wr \frac{ds}{EI}$$

$$= J + \oint 2wr \frac{ds}{EI} \qquad (33)$$

Putting these expressions into equation (29) yields

$$w'' + w = \frac{1}{2}\frac{q}{EI}\left(r^2 + 2rw - \frac{J}{S} - \frac{2}{S}\oint wr \frac{ds}{EI}\right)r^2\sqrt{1 + \left(\frac{r'}{r}\right)^2} \qquad (34)$$

Transferring all terms containing w to one side of the equation gives

$$w'' + w - \frac{qw}{EI}r^3\sqrt{1 + \left(\frac{r'}{r}\right)^2} + \frac{q}{EI}\frac{1}{S}r^2\sqrt{1 + \left(\frac{r'}{r}\right)^2}\oint wr \frac{ds}{EI}$$

$$= \frac{1}{2}\frac{q}{EI}\left[r^4\sqrt{1 + \left(\frac{r'}{r}\right)^2} - \frac{J}{S}r^2\sqrt{1 + \left(\frac{r'}{r}\right)^2}\right] \qquad (35)$$

This is the final differential-integral equation with the radial deflection w as the only variable in the equation. The solution of this differential equation and a method to make it more practical for applications is discussed in the following section.

Solution of Differential-Integral Equation

The solution of the foregoing differential-integral equation can be made by the following steps:

(a) All the known functions and unknown functions are expanded into Fourier series, with known and unknown Fourier coefficients.

(b) By comparing the Fourier coefficients of both sides of the equation, a system of simultaneous equations is obtained.

(c) The solution of the simultaneous equations gives the unknown Fourier coefficients.

In carrying out these steps, rewrite equation (35), using the following relationship from the elementary calculus:

$$ds = r\left[\sqrt{1 + \left(\frac{r'}{r}\right)^2}\right]d\theta \tag{36}$$

Equation (35) then becomes

$$w'' + w - qw\left[\frac{r^3}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2}\right] + \frac{q}{S}\left[\frac{r^2}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2}\right]\oint w\left[\frac{r^2}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2}\right]d\theta$$

$$= \frac{q}{2}\left[\frac{r^4}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2}\right] - \frac{q}{2}\frac{J}{S}\left[\frac{r^2}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2}\right] \tag{37}$$

In equation (37), let F_1, F_2, and F_3 stand for the following functions:

$$\left.\begin{aligned} F_1 &= \frac{r^2}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2} \\ F_2 &= \frac{r^3}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2} \\ F_3 &= \frac{r^4}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2} \end{aligned}\right\} \tag{38}$$

The functions F_1, F_2, and F_3 are known for any given ring. The equation becomes

$$w'' + w - wqF_2 + \frac{q}{S}F_1 \oint F_1 w \, d\theta = \frac{q}{2}\left(F_3 - \frac{J}{S}F_1\right) \qquad (39)$$

Since F_1, F_2, and F_3 are known functions, they can be expanded into Fourier series with known coefficients. Therefore,

$$\left.\begin{aligned} F_1 &= \sum_{n=0,2,4\ldots} a_n \cos n\theta \\ F_2 &= \sum_{n=0,2,4\ldots} b_n \cos n\theta \\ F_3 &= \sum_{n=0,2,4\ldots} c_n \cos n\theta \end{aligned}\right\} \qquad (40)$$

where a_n, b_n, and c_n are known Fourier coefficients. For double symmetry, the case under study herein, n is always an even number.

Now let w be represented by a Fourier series with unknown coefficients:

$$w = \sum_{m=0,2,4\ldots} A_m \cos m\theta \qquad (41)$$

where A_m's are to be determined. From the fact that the ring is double-symmetrical and the external load system is also double-symmetrical, the radial deflection w also is a double-symmetrical function. Therefore all m's are even numbers.

Differentiating equation (41) twice gives

$$w'' = -\sum_{m=0,2,4\ldots} m^2 A_m \cos m\theta \tag{42}$$

Therefore,

$$w'' + w = \sum_{m=0,2,4\ldots} \left(1 - m^2\right) A_m \cos m\theta \tag{43}$$

The substitution of these relations into equation (37) yields

$$\sum_{m=0,2,4\ldots} \left(1 - m^2\right) A_m \cos m\theta - q \sum_{n=0,2,4\ldots} b_n \cos n\theta \sum_{m=0,2,4\ldots} A_m \cos m\theta$$

$$+ \frac{q}{S} \sum_{n=0,2,4\ldots} a_n \cos n\theta \oint \sum_{m=0,2,4\ldots} A_m \cos m\theta \sum_{n=0,2,4\ldots} a_n \cos n\theta \, d\theta$$

$$= \frac{q}{2}\left(\sum_{n=0,2,4\ldots} c_n \cos n\theta - \frac{J}{S} \sum_{n=0,2,4\ldots} a_n \cos n\theta\right) \tag{44}$$

In order to facilitate the comparison of the coefficients on both sides, equation (44) is simplified. First, consider terms involving the integral sign in equation (44) as follows:

$$\oint \sum_{m=0,2,4\ldots} A_m \cos m\theta \sum_{n=0,2,4\ldots} a_n \cos n\theta \, d\theta$$

$$= \sum_{m=0,2,4\ldots} A_m \int_0^{2\pi} \cos m\theta \sum_{n=0,2,4\ldots} a_n \cos n\theta \, d\theta$$

$$= 2A_0 a_0 \pi + \sum_{m=2,4\ldots} \pi A_m a_m \tag{45}$$

and equation (44) becomes

$$\sum_{m=0,2,4\ldots}\left(1 - m^2\right)A_m \cos m\theta - q \sum_{n=0,2,4\ldots} b_n \cos n\theta \sum_{m=0,2,4\ldots} A_m \cos m\theta$$

$$+ \frac{q}{S} \sum_{n=0,2,4\ldots} a_n \cos n\theta \left(2\pi A_0 a_0 + \sum_{k=2,4\ldots} \pi A_k a_k\right)$$

$$= \frac{q}{2}\left(\sum_{n=0,2,4\ldots} c_n \cos n\theta - \frac{J}{S} \sum_{n=0,2,4\ldots} a_n \cos n\theta\right) \qquad (46)$$

The second step is to simplify the second term of the equation which contains the multiplication of two Fourier series. By actually carrying out the multiplication,

$$\sum_{n=0,2,4\ldots} b_n \cos n\theta \sum_{m=0,2,4\ldots} A_m \cos m\theta$$

$$= d_0 + d_2 \cos 2\theta + d_4 \cos 4\theta + \ldots$$

$$= \sum_{m=0,2,4\ldots} d_m \cos m\theta \qquad (47)$$

where

$$\left.\begin{aligned}
d_0 &= \frac{1}{2}A_0 b_0 + \frac{1}{2}\sum_{n=0,2,4\ldots} A_n b_n \\
d_2 &= \frac{1}{2}\left(A_0 b_2 + A_2 b_0\right) + \frac{1}{2}\sum_{n=0,2,4\ldots}\left(A_n b_{n+2} + A_{n+2} b_n\right) \\
d_4 &= \frac{1}{2}\left(A_0 b_4 + A_2 b_2 + A_4 b_0\right) + \frac{1}{2}\sum_{n=0,2,4\ldots}\left(A_n b_{n+4} + A_{n+4} b_n\right) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
d_m &= \frac{1}{2}\left(A_0 b_m + A_2 b_{m-2} + \ldots + A_{m-2} b_2 + A_m b_0\right) \\
&\quad + \frac{1}{2}\sum_{n=0,2,4\ldots}\left(A_n b_{n+m} + A_{n+m} b_n\right)
\end{aligned}\right\} \tag{48}$$

and equation (46) then becomes

$$\sum_{m=0,2,4\ldots}\left(1 - m^2\right) A_m \cos m\theta - q \sum_{m=0,2,4\ldots} d_m \cos m\theta$$

$$+ \frac{q}{S}\left(2\pi A_0 a_0 + \sum_{k=2,4\ldots} \pi A_k a_k\right) \sum_{m=0,2,4\ldots} a_m \cos m\theta$$

$$= \frac{q}{2} \sum_{m=0,2,4\ldots}\left(c_m - \frac{J}{S} a_m\right) \cos m\theta \tag{49}$$

or

$$\sum_{m=0,2,4\ldots}\left[\left(1 - m^2\right)A_m - qd_m + \frac{q}{S}\left(2\pi A_0 a_0 + \sum_{k=2,4\ldots}\pi A_k a_k\right)a_m\right]\cos m\theta$$

$$= \frac{q}{2}\sum_{m=0,2,4\ldots}\left(c_m - \frac{J}{S}a_m\right)\cos m\theta \tag{50}$$

Equation (44) has been reduced to a more usable form in equation (50).

Now the Fourier coefficients on both sides of equation (50) can be compared.

For $m = 0$,

$$A_0 - qd_0 + \frac{q}{S}\left(2A_0 a_0\pi + \sum_{k=2,4\ldots}\pi A_k a_k\right)a_0 = \frac{q}{2}\left(c_0 - \frac{J}{S}a_0\right)$$

For $m = 2$,

$$-3A_2 - qd_2 + \frac{q}{S}\left(2\pi A_0 a_0 + \sum_{k=2,4\ldots}\pi A_k a_k\right)a_2 = \frac{q}{2}\left(c_2 - \frac{J}{S}a_2\right)$$

For $m = 4$,

$$-15A_4 - qd_4 + \frac{q}{S}\left(2\pi A_0 a_0 + \sum_{k=2,4\ldots}\pi A_k a_k\right)a_4 = \frac{q}{2}\left(c_4 - \frac{J}{S}a_4\right)$$

. . . . (51)

As many simultaneous equations as wanted can be formed. The solution of m simultaneous equations gives m unknowns, A_0, A_2, A_4 . . . .

In actual cases, as shown in the examples in appendix E, three or four simultaneous equations, which give three or four A_m terms, are sufficiently accurate for ordinary rings. The following system is the system of four simultaneous equations which give four Fourier coefficients, A_0, A_2, A_4, and A_6. The following equations are obtained from equations (51) and (48) after collecting the terms:

$$
\left.
\begin{aligned}
&A_0\left(1 - qb_0 + 2\frac{q}{S}\pi a_0 a_0\right) + A_2\left(\frac{q}{2}b_2 + \pi\frac{q}{S}a_0 a_2\right) + A_4\left(-\frac{q}{2}b_4 + \pi\frac{q}{S}a_0 a_4\right) \\
&\quad + A_6\left(-\frac{q}{2}b_6 + \frac{q}{S}\pi a_0 a_6\right) = \frac{q}{2}\left(c_0 - \frac{J}{S}a_0\right) \\
&A_0\left(-qb_2 + 2\pi\frac{q}{S}a_0 a_2\right) + A_2\left(-3 - qb_0 - \frac{q}{2}b_4 + \pi\frac{q}{S}a_2 a_2\right) \\
&\quad + A_4\left(-\frac{q}{2}b_2 - \frac{q}{2}b_6 + \pi\frac{q}{S}a_2 a_4\right) + A_6\left(-\frac{q}{2}b_4 - \frac{q}{2}b_8 + \pi\frac{q}{S}a_2 a_6\right) = \frac{q}{2}\left(c_2 - \frac{J}{S}a_2\right) \\
&A_0\left(-qb_4 + 2\frac{q}{S}\pi a_0 a_4\right) + A_2\left(-\frac{q}{2}b_2 - \frac{q}{2}b_6 + \pi\frac{q}{S}a_4 a_2\right) \\
&\quad + A_4\left(-15 - qb_0 - \frac{q}{2}b_8 + \pi\frac{q}{S}a_4 a_4\right) + A_6\left(-\frac{q}{2}b_2 - \frac{q}{2}b_{10} + \pi\frac{q}{S}a_4 a_6\right) \\
&\quad = \frac{q}{2}\left(c_4 - \frac{J}{S}a_4\right) \\
&A_0\left(-qb_6 + 2\pi\frac{q}{S}a_0 a_6\right) + A_2\left(-\frac{q}{2}b_4 - \frac{q}{2}b_8 + \pi\frac{q}{S}a_6 a_2\right) \\
&\quad + A_4\left(-\frac{q}{2}b_2 - \frac{q}{2}b_{10} + \pi\frac{q}{S}a_6 a_4\right) + A_6\left(-35 - qb_0 - \frac{q}{2}b_{12} + \pi\frac{q}{S}a_6 a_6\right) \\
&\quad = \frac{q}{2}\left(c_6 - \frac{J}{S}a_6\right)
\end{aligned}
\right\} \tag{52}
$$

With A_m determined, the radial deflections can be obtained from equation (41) and the bending-moment distributions, from equation (14). The angular displacements ϕ can be determined by equations from the nonextension theory (appendix A).

Angular Displacements

When the radial deflections w are known, the angular displacements ϕ can be found from the following equation, obtained from the nonextension theory (appendix A).

$$\phi' = -\frac{w}{r} - \frac{w'}{r}\frac{r'}{r} \tag{53}$$

The integration of equation (53) gives the angular displacements ϕ.

$$\phi = -\int \frac{w}{r}\,d\theta - \int \frac{w'}{r}\frac{r'}{r}\,d\theta \tag{54}$$

Before evaluating the foregoing integration, the Fourier coefficients e_n for the function $1/r$ must be determined:

$$\frac{1}{r} = \sum_{n=0,2,4\ldots} e_n \cos n\theta \tag{55}$$

Since $1/r$ is also a double-symmetrical function, the n's are even numbers only. Therefore,

$$\frac{w}{r} = \sum_{m=0,2,4\ldots} A_m \cos m\theta \sum_{n=0,2,4\ldots} e_n \cos n\theta$$

$$= \sum_{n=0,2,4\ldots} f_n \cos n\theta \tag{56}$$

where

$$\left.\begin{aligned}
f_0 &= \frac{1}{2}A_0e_0 + \frac{1}{2}\sum_{n=0,2,4\ldots} A_ne_n \\
f_2 &= \frac{1}{2}\left(A_0e_2 + A_2e_0\right) + \frac{1}{2}\sum_{n=0,2,4\ldots}\left(A_ne_{n+2} + A_{n+2}e_n\right) \\
f_4 &= \frac{1}{2}\left(A_0e_4 + A_2e_2 + A_4e_0\right) + \frac{1}{2}\sum_{n=0,2,4\ldots}\left(A_ne_{n+4} + A_{n+4}e_n\right) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{57}$$

and

$$\int \frac{w}{r}\,d\theta = \int\left(f_0 + f_2\cos 2\theta + f_4\cos 4\theta + \ldots\right)d\theta$$

$$= f_0\theta + \sum_{n=2,4\ldots}\frac{f_n}{n}\sin n\theta \tag{58}$$

It is also known that

$$\frac{r'}{r^2} = -\left(\frac{1}{r}\right)' = \sum_{n=2,4\ldots} ne_n\sin n\theta \tag{59}$$

and

$$\frac{w'}{r}\frac{r'}{r} = -\sum_{m=0,2,4\ldots} mA_m\sin m\theta\sum_{n=2,4\ldots} ne_n\sin n\theta$$

$$= -\sum_{n=0,2,4\ldots} g_n\cos n\theta \tag{60}$$

where

$$\left.\begin{aligned}
g_0 &= \frac{1}{2}\sum_{n=2,4\ldots} n^2 A_n e_n \\
g_2 &= \frac{1}{2}\sum_{n=2,4\ldots} n(n + 2)\left(A_n e_{n+2} + A_{n+2}e_n\right) \\
g_4 &= \frac{1}{2}\sum_{n=2,4\ldots} n(n + 4)\left(A_n e_{n+4} + A_{n+4}e_n\right) - \frac{1}{2}\left(2A_2\right)\left(2e_2\right) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{61}$$

Therefore,

$$\int \frac{w'}{r}\frac{r'}{r}\,d\theta = -\int \sum_{n=0,2,4\ldots} g_n \cos n\theta\, d\theta$$

$$= -g_0\theta - \sum_{n=2,4\ldots} \frac{g_n}{n}\sin n\theta \tag{62}$$

Substituting equations (58) and (62) into equation (54), the following form for ϕ is obtained:

$$\phi = -\left(f_0\theta + \sum_{n=2,4\ldots}\frac{f_n}{n}\sin n\theta - g_0\theta - \sum_{n=2,4\ldots}\frac{g_n}{n}\sin n\theta\right) + C$$

$$= -\left[\left(f_0 - g_0\right)\theta + \sum_{n=2,4\ldots}\frac{f_n - g_n}{n}\sin n\theta\right] + C \tag{63}$$

where C is the constant of integration.

The present boundary conditions require that:

$$\left.\begin{aligned} &(1)\text{ at } \theta = 0, \quad \phi = 0 \\ &\text{and } (2)\text{ at } \theta = \tfrac{\pi}{2}, \quad \phi = 0 \end{aligned}\right\} \tag{64}$$

For condition (1), equation (63) becomes

$$\phi(0) = 0 = 0 + \sum_{n=2,4\ldots} (0) + C$$

$$C = 0$$

For condition (2), equation (63) becomes

$$\phi\left(\frac{\pi}{2}\right) = 0 = \left(f_0 - g_0\right)\frac{\pi}{2} - \sum_{n=2,4\ldots} (0)$$

$$\left(f_0 - g_0\right)\frac{\pi}{2} = 0$$

$$f_0 - g_0 = 0$$

And equation (63) becomes

$$\phi = - \sum_{n=2,4\ldots} \frac{f_n - g_n}{n} \sin n\theta \tag{65}$$

which is the final form for the angular displacement ϕ.

The fact that $f_0 - g_0$ must be zero gives a good check on the values of A_0, A_2, A_6 . . . . This condition should always be satisfied. A proof that this condition $f_0 - g_0 = 0$ is always satisfied automatically is given as follows: From equation (5),

$$\oint \frac{M}{EI} ds = 0$$

whereas from equation (28)

$$\frac{M}{EI} = \frac{w + w''}{r^2\sqrt{1 + \left(\frac{r'}{r}\right)^2}}$$

Therefore,

$$\oint \frac{M}{EI} ds = \oint \frac{w + w''}{r^2\sqrt{1 + \left(\frac{r'}{r}\right)^2}} ds$$

$$= \int_0^{2\pi} \frac{w + w''}{r} d\theta = 0 \qquad (66)$$

Since

$$\frac{1}{r} = \sum_{n=0,2,4\ldots} e_n \cos n\theta$$

$$w + w'' = \sum_{m=0,2,4\ldots} \left(1 - m^2\right) A_m \cos m\theta$$

therefore

$$\frac{w + w''}{r} = \sum_{m=0,2,4\ldots} \left(1 - m^2\right) A_m \cos m\theta \sum_{n=0,2,4\ldots} e_n \cos n\theta$$

$$= \sum_{n=0,2,4\ldots} h_n \cos n\theta$$

where

$$\left.\begin{aligned} h_0 &= e_0A_0 + \sum_{n=2,4\ldots} \frac{1}{2}\left(1 - n^2\right) e_nA_n \\ h_2 &= \ldots \end{aligned}\right\} \quad (67)$$

Substituting into equation (66) gives

$$\int_0^{2\pi} \frac{w + w''}{r}\, d\theta = \int_0^{2\pi} \left(h_0 + \sum_{n=2,4\ldots} h_n \cos n\theta\right) d\theta$$

$$= 2\pi h_0 = 0$$

or

$$h_0 = 0$$

Comparing equations (57), (61), and (67), it can be seen that

$$h_0 = f_0 - g_0 = 0$$

Thus $f_0 - g_0 = 0$ is a condition which must be satisfied automatically.

Summary of Procedure

In the foregoing sections, there has been developed an analytic method of finding the bending-moment distribution in double-symmetrical rings of arbitrary shapes, acted upon by internal pressure loads. The ring must have a regular and smooth contour, with its thickness negligible compared to the radius. For such rings the procedures to solve this problem can be summarized in the following steps:

(1) Determine the Fourier coefficients of the following given functions:

$$\left.\begin{aligned}
F_1 &= \frac{r^2}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2} = \sum_{n=0,2,4\ldots} a_n \cos n\theta \\
F_2 &= \frac{r^3}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2} = \sum_{n=0,2,4\ldots} b_n \cos n\theta \\
F_3 &= \frac{r^4}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2} = \sum_{n=0,2,4\ldots} c_n \cos n\theta
\end{aligned}\right\} \qquad (68)$$

If the contour of the ring is expressed analytically in simple functions, the Fourier coefficients can be obtained by the usual method of integration. Otherwise, the coefficients must be obtained by graphical integration, which is explained in appendix D.

(2) Perform the following integrations:

$$J = \oint \frac{r^2}{EI}\,ds = \int_0^{2\pi} \frac{r^3}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2}\,d\theta$$

$$S = \oint \frac{ds}{EI} = \int_0^{2\pi} \frac{r}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2}\,d\theta$$

(3) Calculate the following constants by substituting the results obtained from steps (1) and (2):

$$K_{00} = -b_0 + \frac{2\pi}{S}a_0 a_0$$

$$K_{02} = -\frac{b_2}{2} + \frac{\pi}{S}a_0 a_2$$

$$K_{04} = -\frac{b_4}{2} + \frac{\pi}{S}a_0 a_4$$

$$K_{06} = -\frac{b_6}{2} + \frac{\pi}{S}a_0 a_6$$

$$K_{20} = -b_2 + \frac{2\pi}{S}a_2a_0$$

$$K_{22} = -b_0 - \frac{b_4}{2} + \frac{\pi}{S}a_2a_2$$

$$K_{24} = -\frac{b_2}{2} - \frac{b_6}{2} + \frac{\pi}{S}a_2a_4$$

$$K_{26} = -\frac{b_4}{2} - \frac{b_8}{2} + \frac{\pi}{S}a_2a_6$$

$$K_{40} = -b_4 + \frac{2\pi}{S}a_4a_0$$

$$K_{42} = -\frac{b_2}{2} - \frac{b_6}{2} + \frac{\pi}{S}a_4a_2$$

$$K_{44} = -b_0 - \frac{b_8}{2} + \frac{\pi}{S}a_4a_4$$

$$K_{46} = -\frac{b_2}{2} - \frac{b_{10}}{2} + \frac{\pi}{S}a_4a_6$$

$$K_{60} = -b_6 + \frac{2\pi}{S}a_6a_0$$

$$K_{62} = -\frac{b_4}{2} - \frac{b_8}{2} + \frac{\pi}{S}a_6a_2$$

$$K_{64} = -\frac{b_2}{2} - \frac{b_{10}}{2} + \frac{\pi}{S}a_6a_4$$

$$K_{66} = -b_0 - \frac{b_{12}}{2} + \frac{\pi}{S}a_6a_6$$

$$Q_0 = \frac{1}{2}\left(c_0 - \frac{J}{S}a_0\right)$$

$$Q_2 = \frac{1}{2}\left(c_2 - \frac{J}{S}a_2\right)$$

$$Q_4 = \frac{1}{2}\left(c_4 - \frac{J}{S}a_4\right)$$

$$Q_6 = \frac{1}{2}\left(c_6 - \frac{J}{S}a_6\right) \qquad (69)$$

(4) Substitute the foregoing constant terms into the following system of simultaneous equations:

$$\left.\begin{aligned}
A_0\left(\frac{1}{q} + K_{00}\right) + A_2K_{02} + A_4K_{04} + A_6K_{06} &= Q_0 \\
A_0K_{20} + A_2\left(-\frac{3}{q} + K_{22}\right) + A_4K_{24} + A_6K_{26} &= Q_2 \\
A_0K_{40} + A_2K_{42} + A_4\left(-\frac{15}{q} + K_{44}\right) + A_6K_{46} &= Q_4 \\
A_0K_{60} + A_2K_{62} + A_4K_{64} + A_6\left(-\frac{35}{q} + K_{66}\right) &= Q_6
\end{aligned}\right\} \qquad (70)$$

(5) For a given pressure load q, the foregoing simultaneous equations can be solved for A_0, A_2, A_4, and A_6.

(6) The radial deflections w are given by the following equation:

$$w = A_0 + A_2 \cos 2\theta + A_4 \cos 4\theta + A_6 \cos 6\theta \qquad (71)$$

(7) In order to find the angular displacements, the Fourier coefficient e_n of the following functions should be first determined:

$$\frac{1}{r} = \sum_{n=0,2,4\ldots} e_n \cos n\theta \qquad (72)$$

Then values of A_n and e_n are substituted into the following equation to see if the condition $f_0 - g_0 = 0$ is satisfied.

$$f_0 - g_0 = A_0e_0 + \frac{1}{2}\sum_{n=2,4\ldots}\left(1 - n^2\right)A_ne_n \qquad (73)$$

(8) Knowing e_n and A_n, determine f_n and g_n as given in the following equation:

$$\left.\begin{aligned}
f_2 &= \frac{1}{2}\left(A_0 e_2 + A_2 e_0\right) + \frac{1}{2} \sum_{n=0,2,4\ldots} \left(A_n e_{n+2} + A_{n+2} e_n\right) \\
f_4 &= \frac{1}{2}\left(A_0 e_4 + A_2 e_2 + A_4 e_0\right) + \frac{1}{2} \sum_{n=0,2,4\ldots} \left(A_n e_{n+4} + A_{n+4} e_n\right) \\
f_6 &= \frac{1}{2}\left(A_0 e_6 + A_2 e_4 + A_4 e_2 + A_6 e_0\right) \\
&\quad + \frac{1}{2} \sum_{n=0,2,4\ldots} \left(A_n e_{n+6} + A_{n+6} e_n\right) \\
g_2 &= \frac{1}{2} \sum_{n=2,4\ldots} n(n + 2)\left(A_n e_{n+2} + A_{n+2} e_n\right) \\
g_4 &= \frac{1}{2} \sum_{n=2,4\ldots} n(n + 4)\left(A_n e_{n+4} + A_{n+4} e_n\right) - 2A_2 e_2 \\
g_6 &= \frac{1}{2} \sum_{n=2,4\ldots} n(n + 6)\left(A_n e_{n+6} + A_{n+6} e_n\right) - 4\left(A_2 e_4 + A_4 e_2\right)
\end{aligned}\right\} \qquad (74)$$

(9) The angular displacements ϕ can be obtained from the following equation:

$$\phi = - \sum_{n=2,4\ldots} \frac{f_n - g_n}{n} \sin n\theta \qquad (75)$$

(10) The moment distribution M is given by the following equation:

$$M = \frac{q}{2}\left(r_1^2 - \frac{J_1}{S}\right)$$

$$= \frac{q}{2}\left(r^2 - \frac{J}{S} + 2rw - \frac{2}{S}\oint rw \frac{ds}{EI}\right)$$

$$= \frac{q}{2}\left[r^2 - \frac{J}{S} + 2rw - \frac{2}{S}\left(2\pi A_0 a_0 + \sum_{n=2,4\ldots} \pi A_n a_n\right)\right] \tag{76}$$

From the foregoing procedures it can be seen that the main part of the solution of the problem is the determination of the Fourier coefficients of several known functions. The rest of the procedures are simply algebraic. On the average, it takes 20 hours to solve completely a problem.

Preparation of Charts

Following the foregoing procedures, solutions have been obtained for two specific families of rings. The first family consists of elliptical rings (fig. 5(a)) with various eccentricities. The second family consists of rings formed by two semicircles and two straight lines. (See fig. 5(b).) The ratio of the radius of the semicircle to the height of the straight-line portion is variable. All rings are assumed to be of constant EI.

Charts and tables are provided for both families of rings. The Fourier coefficients a_n, b_n, c_n, and e_n are given in tables 1 and 2 and are also plotted in figures 6 to 13. Since both families of rings are of constant EI, the Fourier coefficients are obtained in terms of EI and the dimensions of the rings.

Values of J and S are given in table 3 and also plotted in figures 14 and 15. When these values are substituted into equation (70) the simultaneous equations are obtained. For a given value of q and EI, the simultaneous equations can be solved for A_0, A_2, A_4, A_6 . . . . Figures 16 to 21 give the values of A_n for various ratios of q/EI.

Knowing A_n, w, ϕ, and M are obtained from equations (71), (75), and (76), respectively. The results are plotted in figures 22 to 42.

Two examples are given in appendix E to illustrate the foregoing procedures in detail.

TEST

Test Specimens

Two steel rings were tested, one elliptical and the other made up of two semicircles joined by two straight lines. (See fig. 5.)

Test Apparatus and Procedures

Figures 43 and 44 show the test setup. The ring was tested in a horizontal plane. Its circumference was divided into a number of segments with equal arc lengths. Wires, loaded equally, were attached to these points. Each wire was led through a pulley, connected to a fixed horizontal ring in such a manner that the pull on the test ring was normally outward at each point. (See fig. 45.) The concentrated loads were sufficiently close so that they could be assumed as simulating pressure.

In order to register the deflections of the ring, a wooden board was placed on top of the ring. The contour of the ring was traced on the board, with marks denoting the division points, before and after the ring was loaded, as shown in figure 46. The radial and angular displacements can be measured directly on the board.

Bending moments were found by electrical strain-gage readings at three points on the ring. (See fig. 47.) The value of the effective EI used was determined from bending tests on a specimen cut from the ring.

It might be mentioned that the test was at first tried with the aid of a pressure bag. The ring was laid around the bag which was blown up with compressed air. Special devices were used to adjust the pressure between the bag and ring so that the ring would be uniformly loaded. The test was unsuccessful, however, since it was found that uniform loading could be obtained only through long and tedious work. (See fig. 48.)

Test Results and Discussion

The test results are plotted in figures 49 to 54, together with curves obtained by the method developed in this paper. The agreement between test data and calculated results is good.

In figures 51 and 54 where bending-moment distributions along the circumferences of the rings are given, an additional curve is shown in each figure. These added curves represent the bending-moment distribution when changes of geometric shapes of the rings caused by loading are neglected. The symbol M_0 is used to designate the bending moment calculated without considering the deflections of the ring. The value of M_0 can be found from the following equation:

$$M_0 = \frac{q}{2}\left(r^2 - \frac{J}{S}\right)$$

It can be seen from figures 51 and 54 that the bending moment obtained without considering the change of geometric shape is too conservative. At the point of maximum bending moment the difference is quite large.

CONCLUSIONS

From an analytical method derived for the determination of bending-moment distribution and radial and angular displacements of flexible rings of arbitrary shape under internal pressure load, with the change of geometric shapes caused by the load being considered, the following conclusions can be made:

1. Results obtained by the present method showed good agreement with test data and proved that results obtained when change of geometric shape was not considered are inadequate for flexible rings.

2. Although only rings with double symmetry under internal pressure loads are discussed in the present work, the method can be extended to include rings of any shape under any system of external loads.

University of Michigan
Ann Arbor, Mich., October 17, 1946

APPENDIX A

NONEXTENSION THEORY

The deflections of a loaded ring can be completely defined by the radial deflections w and angular displacements ϕ, both of which are functions of θ. (See fig. 3.) The deflections w and ϕ are independent of each other if there are no additional conditions imposed on the ring. They are definitely related, however, if the ring is assumed to be inextensible. The nonextension theory gives the relation between w and ϕ for such rings.

Timoshenko (reference 3, p. 208) has given the relation between the radial deflections w and the tangential deflections v (which correspond to angular displacements ϕ in the present case) for circular rings as follows:

$$\frac{dv}{d\theta} + w = 0 \qquad \text{(A1)}$$

For circular rings this can be written as

$$r\phi' + w = 0 \qquad \text{(A2)}$$

where

$$\phi' = \frac{d\phi}{d\theta}$$

This relation, however, is not quite accurate for rings of noncircular shapes. A relationship between w and ϕ which is more general and can be applied to rings of any shape is derived by means of several steps as follows.

Arc Length in Polar Coordinates

Given a small element $AB = ds$ on an arc C.

$$OA = r$$

$$OB = r + dr$$

With O as the center, OA as the radius, swing a circular arc which will cut OB at D. Then

$$OD = OA = r$$

$$BD = dr$$

and

$$\widehat{AB} = (ds)_{AB} = \sqrt{(AD)^2 + (BD)^2}$$

$$= \sqrt{(r\,d\theta)^2 + (dr)^2}$$

$$= r\,d\theta\sqrt{1 + \left(\frac{r'}{r}\right)^2} \qquad (A3)$$

where the prime means derivative with respect to θ.

Increase of Arc Length Due to Angular Displacements

When point A (fig. 55) is allowed an angular displacement ϕ and point B an angular displacement $\phi + d\phi$, with no radial deflections, the new position of the element AB becomes EF. Since there is no change of radius,

$$OE = OA = r$$

$$OF = OB = r + dr$$

and the length of the arc EF is

$$\widehat{EF} = (ds)_{EF} = \sqrt{r^2(d\theta + d\phi)^2 + (dr)^2}$$

$$= r\,d\theta\sqrt{(1 + \phi')^2 + \left(\frac{r'}{r}\right)^2}$$

The increase of arc length due to the angular displacement is $(\Delta ds)_\phi$ where

$$(\Delta ds)_\phi = (ds)_{EF} - (ds)_{AB}$$

$$= r\,d\theta\sqrt{(1 + \phi')^2 + \left(\frac{r'}{r}\right)^2} - r\,d\theta\sqrt{1 + \left(\frac{r'}{r}\right)^2} \qquad \text{(A5)}$$

Increase of Arc Length Due to Radial Deflections

The element of arc EF is now allowed to have radial deflections. Point E moves radially to G and F, to H. (See fig. 56.)

$$GE = w$$

$$FH = w + dw$$

The arc length of GH is then

$$\widehat{GH} = (ds)_{GH} = \sqrt{(GI)^2 + (IH)^2}$$

$$= \sqrt{(w + r)(d\theta + d\phi)^2 + (dw + dr)^2}$$

$$= (r + w)d\theta\sqrt{(1 + \phi')^2 + \left(\frac{r' + w'}{r + w}\right)^2} \qquad \text{(A6)}$$

and the increase of length of the arc due to radial deflections alone is $(\Delta ds)_w$ where

$$(\Delta ds)_w = (r + w)d\theta\sqrt{(1 + \phi')^2 + \left(\frac{r' + w'}{r + w}\right)^2}$$

$$-r\,d\theta\sqrt{(1 + \phi')^2 + \left(\frac{r'}{r}\right)^2} \qquad \text{(A7)}$$

Equation for Deflections of an Inextensible Ring

The nonextension theory requires that the total increase of the arc length due to both radial deflections and angular displacements should be zero. In other words,

$$(\Delta ds)_{\phi} + (\Delta ds)_{w} = 0$$

From equations (A5) and (A7),

$$(r + w)d\theta\sqrt{(1 + \phi')^2 + \left(\frac{r' + w'}{r + w}\right)^2} - r\ d\theta\sqrt{1 + \left(\frac{r'}{r}\right)^2}$$

or

$$(r + w)\sqrt{(1 + \phi')^2 + \left(\frac{r' + w'}{r + w}\right)^2} = r\sqrt{1 + \left(\frac{r'}{r}\right)^2} \qquad \text{(A8)}$$

Squaring both sides and dividing by r^2 yields

$$\left(1 + \frac{w}{r}\right)^2\left[(1 + \phi')^2 + \left(\frac{r' + w'}{r + w}\right)^2\right] = 1 + \left(\frac{r'}{r}\right)^2$$

or

$$\left(1 + \frac{w}{r}\right)^2(1 + \phi')^2 + \left(\frac{r'}{r} + \frac{w'}{r}\right)^2 = 1 + \left(\frac{r'}{r}\right)^2$$

or

$$\left(1 + \frac{w}{r}\right)^2(1 + \phi')^2 = 1 - 2\frac{r'}{r}\frac{w'}{r} - \left(\frac{w'}{r}\right)^2$$

or

$$\phi' = \frac{\sqrt{1 - 2\frac{r'}{r}\frac{w'}{r} - \left(\frac{w'}{r}\right)^2}}{1 + \frac{w}{r}} - 1 \qquad \text{(A9)}$$

Equation (A9) gives the relation between the angular displacements and the radial deflections for an inextensible ring. Equation (A9) is very general and can be used for rings of any shape.

Simplifications

Equation (A9) can be greatly simplified for rings, the deflections w and ϕ of which are not very large when compared with the radius of the ring. In other words, $\frac{w}{r} \ll 1$ and $\phi \ll 1$. Also, it must be assumed that $\frac{w'}{r} \ll 1$ and $\phi' \ll 1$. Then any term of power higher than two or products of the foregoing items can be neglected. Equation (A9) becomes

$$\left(1 + \frac{w}{r}\right)(1 + 2\phi') = 1 - 2\frac{r'}{r}\frac{w'}{r}$$

or

$$1 + 2\phi' + 2\frac{w}{r} = 1 - 2\frac{r'}{r}\frac{w'}{r}$$

or

$$\phi' = -\frac{w}{r} - \frac{r'}{r}\frac{w'}{r} \qquad \text{(A10)}$$

Differentiate equation (A10) once as follows:

$$\phi'' = \frac{w}{r}\frac{r'}{r} + \frac{w'}{r}\left[-1 + 2\left(\frac{r'}{r}\right)^2 - \frac{r''}{r}\right] - \frac{w''}{r}\frac{r'}{r} \qquad \text{(A11)}$$

Equations (A10) and (A11) are the relationship between ϕ and w for inextensible rings.

For circular rings for which $r = \text{Constant}$ and $r' = 0$, equations (A10) and (A11) reduce to

$$\phi' = -\frac{w}{r}$$

$$\phi'' = -\frac{w'}{r}$$

which are exactly the relations given by Timoshenko in reference 3.

APPENDIX B

EQUIVALENT RINGS OF ELASTIC WEIGHT

The term "elastic weight" has been used by Mohr (reference 4). It is designated by "dW" and is defined as

$$dW = \frac{ds}{EI} \tag{B1}$$

The equivalent ring of elastic weight is then defined as the ring, the median of which has a shape which is exactly the same as the shape of the original ring, but the thickness t_1 of the equivalent ring is $1/EI$, where EI is the bending flexibility of the original ring.

The cross-sectional area of the equivalent ring then is

$$A = \oint t_1 \, ds = \oint \frac{ds}{EI} \tag{B2}$$

and the polar moment of inertia of the equivalent ring is

$$I = \oint r^2 t_1 \, ds = \oint \frac{ds}{EI} r^2 \tag{B3}$$

When equations (B2) and (B3) are compared with equation (11), it is seen that

$$S = A$$

$$J = I$$

of the equivalent ring of elastic weight.

APPENDIX C

DETAILED DERIVATION OF EQUATION (27)

The details of deriving equation (27) are as follows. Start with equation (22), where

$$\frac{1}{\rho_1} = \frac{(r + w)^2 - (r + w)\left[\frac{r'' + w''}{(1 + \phi')^2} - \frac{r' + w'}{(1 + \phi')^3}\phi''\right] + 2\left(\frac{r' + w'}{1 + \phi'}\right)^2}{\left[(r + w)^2 + \left(\frac{r' + w'}{1 + \phi'}\right)^2\right]^{3/2}}$$

$$= \frac{\left(1 + \frac{w}{r}\right)^2 - \left(1 + \frac{w}{r}\right)\left[\frac{\frac{r''}{r} + \frac{w''}{r}}{(1 + \phi')^2} - \frac{\frac{r'}{r} + \frac{w'}{r}}{(1 + \phi')^3}\phi''\right] + 2\left(\frac{\frac{r'}{r} + \frac{w'}{r}}{1 + \phi'}\right)^2}{r\left(1 + \frac{w}{r}\right)^3\left[1 + \frac{1}{\left(1 + \frac{w}{r}\right)^2}\left(\frac{\frac{r'}{r} + \frac{w'}{r}}{1 + \phi'}\right)^2\right]^{3/2}} \qquad \text{(C1)}$$

By neglecting all terms containing second or higher power or products of the deflection items and using the binomial theorem, the following relations are obtained:

$$\frac{\frac{r''}{r} + \frac{w''}{r}}{(1 + \phi')^2} = \frac{r''}{r} + \frac{w''}{r} - 2\phi'\frac{r''}{r}$$

$$\frac{\frac{r'}{r} + \frac{w'}{r}}{(1 + \phi')^3}\phi'' = \frac{r'}{r}\phi''$$

$$\left(\frac{\frac{r'}{r}+\frac{w'}{r}}{1+\phi'}\right)^2 = \left(\frac{r'}{r}\right)^2 + 2\frac{w'}{r}\frac{r'}{r} - 2\phi'\left(\frac{r'}{r}\right)^2$$

$$\frac{1}{\left(1+\frac{w}{r}\right)^2}\left(\frac{\frac{r'}{r}+\frac{w'}{r}}{1+\phi'}\right)^2 = \left(\frac{r'}{r}\right)^2 - 2\frac{w}{r}\left(\frac{r'}{r}\right)^2 + 2\frac{w'}{r}\frac{r'}{r} - 2\phi'\left(\frac{r'}{r}\right)^2 \qquad \text{(C2)}$$

Substitution of equation (C2) into equation (C1) yields

$$\frac{1}{\rho_1} = \frac{\left(1+2\frac{w}{r}\right) - \left(1+\frac{w}{r}\right)\left(\frac{r''}{r} + \frac{w''}{r} - 2\phi'\frac{r''}{r} - \phi''\frac{r'}{r}\right) + 2\left[\left(\frac{r'}{r}\right)^2 + 2\frac{w'}{r}\frac{r'}{r} - 2\phi'\left(\frac{r'}{r}\right)^2\right]}{r\left(1+\frac{w}{r}\right)^3\left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{w}{r}\left(\frac{r'}{r}\right)^2 + 2\frac{w'}{r}\frac{r'}{r} - 2\phi'\left(\frac{r'}{r}\right)^2\right]^{3/2}}$$

The foregoing equation can be further simplified by using equations (A10) and (A11). The numerator of equation (C2) becomes

$$\begin{aligned}
\text{Numerator} = {} & \left[1 - \frac{r''}{r} + 2\left(\frac{r'}{r}\right)^2\right] + \frac{w}{r}\left(2 - \frac{r''}{r}\right) + 4\frac{w'}{r}\frac{r'}{r} - \frac{w''}{r} \\
& - \frac{w}{r}\left[2\frac{r''}{r} - 4\left(\frac{r'}{r}\right)^2\right] - \frac{w'}{r}\left[2\frac{r'}{r}\frac{r''}{r} - 4\left(\frac{r'}{r}\right)^3\right] \\
& + \frac{w}{r}\left(\frac{r'}{r}\right)^2 + \frac{w'}{r}\left[-\frac{r'}{r} - \frac{r'}{r}\frac{r''}{r} + 2\left(\frac{r'}{r}\right)^3\right] - \frac{w''}{r}\left(\frac{r'}{r}\right)^2 \\
= {} & \left[1 - \frac{r''}{r} + 2\left(\frac{r'}{r}\right)^2\right] + \frac{w}{r}\left[2 - 3\frac{r''}{r} + 5\left(\frac{r'}{r}\right)^2\right] \\
& + \frac{w'}{r}\left[3\frac{r'}{r} - 3\frac{r'}{r}\frac{r''}{r} + 6\left(\frac{r'}{r}\right)^3\right] + \frac{w''}{r}\left[-1 - \left(\frac{r'}{r}\right)^2\right]
\end{aligned} \tag{C3}$$

and the denominator becomes

$$\begin{aligned}
\text{Denominator} = {} & r\left(1 + \frac{w}{r}\right)^3\left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{w}{r}\left(\frac{r'}{r}\right)^2 + 2\frac{w'}{r}\frac{r'}{r}\right. \\
& \left. + 2\frac{w}{r}\left(\frac{r'}{r}\right)^2 + 2\frac{w'}{r}\left(\frac{r'}{r}\right)^3\right]^{3/2} \\
= {} & r\left(1 + \frac{w}{r}\right)^3\left[1 + \left(\frac{r'}{r}\right)^2\right]^{3/2}\left(1 + 2\frac{w'}{r}\frac{r'}{r}\right)^{3/2}
\end{aligned}$$

Therefore

$$\frac{1}{\rho_1} = \frac{\text{Numerator}}{r\left(1 + \frac{w}{r}\right)^3\left[1 + \left(\frac{r'}{r}\right)^2\right]^{3/2}\left(1 + 2\frac{w'}{r}\frac{r'}{r}\right)^{3/2}} \tag{C4}$$

or

$$\frac{r\left[1 + \left(\frac{r'}{r}\right)^2\right]^{3/2}}{\rho_1} = \text{Numerator}\left(1 - 3\frac{w}{r} - \ldots\right)\left(1 - 3\frac{w'}{r}\frac{r'}{r} - \ldots\right)$$

$$= \left[1 - \frac{r''}{r} + 2\left(\frac{r'}{r}\right)^2\right] - \frac{w}{r}\left[1 + \left(\frac{r'}{r}\right)^2\right] - \frac{w''}{r}\left[1 + \left(\frac{r'}{r}\right)^2\right] \quad \text{(C5)}$$

But from equation (18)

$$\frac{1}{\rho_0} = \frac{1 - \frac{r''}{r} + 2\left(\frac{r'}{r}\right)^2}{r\left[1 + \left(\frac{r'}{r}\right)^2\right]^{3/2}}$$

or

$$\frac{r\left[1 + \left(\frac{r'}{r}\right)^2\right]^{3/2}}{\rho_0} = 1 - \frac{r''}{r} + 2\left(\frac{r'}{r}\right)^2 \quad \text{(C6)}$$

Therefore

$$r\left[1 + \left(\frac{r'}{r}\right)^2\right]^{3/2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right) = -\left(\frac{w}{r} + \frac{w''}{r}\right)\left[1 + \left(\frac{r'}{r}\right)^2\right]$$

or

$$r\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)\sqrt{1 + \left(\frac{r'}{r}\right)^2} = -\left(\frac{w}{r} + \frac{w''}{r}\right)$$

or

$$\frac{1}{\rho_1} - \frac{1}{\rho_0} = -\frac{w + w''}{r^2\sqrt{1 + \left(\frac{r'}{r}\right)^2}}$$

APPENDIX D

GRAPHICAL METHOD TO DETERMINE THE FOURIER COEFFICIENTS a_n, b_n, c_n, AND e_n

For rings, the contours of which cannot be expressed analytically as simple functions, the following procedure is a graphical method for quick evaluation of the Fourier coefficients a_n, b_n, c_n, and e_n of the following functions:

$$F_1 = \frac{r^2}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2} = \sum_{n=0,2,4\ldots} a_n \cos n\theta$$

$$F_2 = \frac{r^3}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2} = \sum_{n=0,2,4\ldots} b_n \cos n\theta$$

$$F_3 = \frac{r^4}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2} = \sum_{n=0,2,4\ldots} c_n \cos n\theta$$

$$\frac{1}{r} = \sum_{n=0,2,4\ldots} e_n \cos n\theta$$

Table 4 is filled in first. Then rows 7 to 18 are plotted against the arc length s, and rows 19 to 22 are plotted against θ. Let E_7, E_8, . . . E_{22} represent the area under the corresponding curves. The Fourier coefficients are

$$a_0 = \frac{2}{\pi}E_7 \qquad b_0 = \frac{2}{\pi}E_{11} \qquad c_0 = \frac{2}{\pi}E_{15} \qquad e_0 = \frac{2}{\pi}E_{19}$$

$$a_2 = \frac{4}{\pi}E_8 \qquad b_2 = \frac{4}{\pi}E_{12} \qquad c_2 = \frac{4}{\pi}E_{16} \qquad e_2 = \frac{4}{\pi}E_{20}$$

$$a_4 = \frac{4}{\pi}E_9 \qquad b_4 = \frac{4}{\pi}E_{13} \qquad c_4 = \frac{4}{\pi}E_{17} \qquad e_4 = \frac{4}{\pi}E_{21}$$

$$a_6 = \frac{4}{\pi}E_{10} \qquad b_6 = \frac{4}{\pi}E_{14} \qquad c_6 = \frac{4}{\pi}E_{18} \qquad e_6 = \frac{4}{\pi}E_{22}$$

APPENDIX E

EXAMPLES

Two examples are given in this appendix. A method of checking the final results is also presented at the end of this appendix.

Example 1

Given an elliptical ring with the following data (see fig. 5(a)): $a = 30$, $b = 25.11$, $EI = 81{,}000$, and $q = 10$.

With the coordinate system shown in figure 5(a), the equation of the ellipse can be expressed either by

$$r = b\sqrt{1 - k^2 \sin^2\theta} \tag{E1}$$

or by

$$r = a\sqrt{1 - k^2 \cos^2\beta} \tag{E2}$$

where

$$k^2 = 1 - \left(\frac{b}{a}\right)^2 \tag{E3}$$

The general procedures given under Summary of Procedure are followed.

(1) In order to determine the Fourier coefficients a_n, b_n, and c_n of the following functions

$$F_1 = \frac{r^2}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2} = \sum_{0,2,4} a_n \cos n\theta$$

$$F_2 = \frac{r^3}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2} = \sum_{0,2,4} b_n \cos n\theta$$

$$F_3 = \frac{r^4}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2} = \sum_{0,2,4} c_n \cos n\theta$$

the following formulas can be used:

$$a_n = \frac{4}{\pi\left(1 + \delta_{0n}\right)} \int_0^{\pi/2} \frac{r^2}{EI} \sqrt{1 + \left(\frac{r'}{r}\right)^2} \cos n\theta \, d\theta$$

$$b_n = \frac{4}{\pi\left(1 + \delta_{0n}\right)} \int_0^{\pi/2} \frac{r^3}{EI} \sqrt{1 + \left(\frac{r'}{r}\right)^2} \cos n\theta \, d\theta$$

$$c_n = \frac{4}{\pi\left(1 + \delta_{0n}\right)} \int_0^{\pi/2} \frac{r^4}{EI} \sqrt{1 + \left(\frac{r'}{r}\right)^2} \cos n\theta \, d\theta$$

where

$$\delta_{0n} = 1 \qquad \text{if} \quad n = 0$$

$$\delta_{0n} = 0 \qquad \text{if} \quad n \neq 0$$

Performing the integration gives

$$a_0 = 0.84334\left(\frac{a^2}{EI}\right)$$

$$a_2 = -0.15059\left(\frac{a^2}{EI}\right)$$

$$a_4 = 0.00689\left(\frac{a^2}{EI}\right)$$

$$b_0 = 0.77600\left(\frac{a^3}{EI}\right)$$

$$b_2 = -0.20669\left(\frac{a^3}{EI}\right)$$

$$b_4 = 0.01703\left(\frac{a^3}{EI}\right)$$

$$c_0 = 0.71684\left(\frac{a^4}{EI}\right)$$

$$c_2 = -0.25300\left(\frac{a^4}{EI}\right)$$

$$c_4 = 0.02817\left(\frac{a^4}{EI}\right)$$

(2) In order to evaluate J and S the following integration should be performed:

$$J = \int_0^{2\pi} \frac{r^3}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2}\, d\theta = 2\pi b_0$$

$$S = \int_0^{2\pi} \frac{r}{EI}\sqrt{1 + \left(\frac{r'}{r}\right)^2}\, d\theta$$

Substituting in these equations gives

$$J = 4.87580\left(\frac{a^3}{EI}\right)$$

$$S = 5.78147\left(\frac{a}{EI}\right)$$

Therefore

$$\frac{J}{S} = 0.84335a^2$$

(3) When the values of a_n, b_n, c_n, J, and S are substituted into equation (69), the following constants are obtained:

$$K_{00} = -0.00306\left(\frac{a^3}{EI}\right)$$

$$K_{02} = 0.03433\left(\frac{a^3}{EI}\right)$$

$$K_{04} = -0.00536\left(\frac{a^3}{EI}\right)$$

$$K_{20} = 0.06867\left(\frac{a^3}{EI}\right)$$

$$K_{22} = -0.77220\left(\frac{a^3}{EI}\right)$$

$$K_{24} = 0.10278\left(\frac{a^3}{EI}\right)$$

$$K_{40} = -0.01067\left(\frac{a^3}{EI}\right)$$

$$K_{42} = 0.10278\left(\frac{a^3}{EI}\right)$$

$$K_{44} = 0.10338\left(\frac{a^3}{EI}\right)$$

$$Q_0 = 0.00280\left(\frac{a^4}{EI}\right)$$

$$Q_2 = -0.06300\left(\frac{a^4}{EI}\right)$$

$$Q_4 = 0.01118\left(\frac{a^4}{EI}\right)$$

(4) When these constants are substituted into equation (70), the following equations are obtained:

$$A_0(t - 0.00306) + A_2(0.03433) + A_4(-0.00536) = 0.00280a$$

$$A_0(0.06867) + A_2(-3t - 0.77221) + A_4(0.10278) = -0.06300a$$

$$A_0(-0.01067) + A_2(0.10278) + A_4(-15t + 0.10338) = 0.01118a \qquad \text{(E4)}$$

where

$$t = \frac{EI}{qa^3}$$

The parameter t is nondimensional and is a measure of the flexibility of the ring. In the present example $t = 0.3$.

(5) Solving the preceding simultaneous equations gives

$$A_0 = 0.00503a$$

$$A_2 = 0.03778a$$

$$A_4 = -0.00167a$$

(6) The radial deflection w is then given by the following equation:

$$w = A_0 + A_2 \cos 2\theta + A_4 \cos 4\theta$$

$$= (0.00503 + 0.03778 \cos 2\theta - 0.00167 \cos 4\theta)a \qquad \text{(E5)}$$

The values of w/a for various values of the angle θ are given in table 5 and also plotted in figure 24.

(7) The Fourier coefficient e_n can be obtained from the following formula:

$$e_n = \frac{4}{\pi\left(1 + \delta_{0n}\right)} \int_0^{\pi/2} \frac{1}{r} \cos n\theta \, d\theta$$

Performing the integration gives

$$e_0 = \frac{1.09979}{a}$$

$$e_2 = \frac{0.09752}{a}$$

$$e_4 = \frac{-0.00217}{a}$$

Substituting A_n and e_n into equation (73) gives

$$f_0 - g_0 = A_0e_0 - 1.5A_2e_2 - 7.5A_4e_4 = -0.00001 = 0$$

Therefore the results are checked.

(8) The coefficients f_n and g_n are obtained from equation (74) as follows:

$$f_2 = 0.04188$$

$$f_4 = -0.00001$$

$$g_2 = -0.00100$$

$$g_4 = -0.00737$$

(9) The angular displacement ϕ is, therefore,

$$\phi = -\frac{1}{2}\left(f_2 - g_2\right) \sin 2\theta - \frac{1}{4}\left(f_4 - g_4\right) \sin 4\theta$$

$$= -0.02144 \sin 2\theta - 0.00184 \sin 4\theta \qquad (E6)$$

Values of ϕ are given in table 6 and are also plotted in figure 31.

(10) Finally equation (76) is used to calculate the bending-moment distribution.

$$M = \frac{q}{2}\left[r^2 - \frac{J}{S} + 2rw - \frac{2}{S}\left(2\pi A_0 a_0 + \sum_{2,4} \pi A_k a_k\right)\right]$$

Since

$$\frac{2}{S}\left(2\pi A_0 a_0 + \sum_{2,4} \pi A_k a_k\right) = 0.00302a^2$$

and the bending-moment distribution if the change of geometric shape of ring is neglected is

$$M_0 = \frac{q}{2}\left(r^2 - \frac{J}{S}\right)$$

the bending-moment distribution M can be expressed as

$$M = M_0 + qrw - 0.00151qa^2$$

or

$$\frac{M}{qa^2} = \frac{M_0}{qa^2} + \left(\frac{r}{a}\frac{w}{a} - 0.00151\right) \qquad (E7)$$

Values of M_0/qa^2 and M/qa^2 are given in table 7; M/qa^2 is also plotted in figure 38. The last term of equation (E7) is evidently the correction needed when the change of geometric shape is considered. In this particular case the correction term amounts to about 37 percent at the point of maximum bending moment.

Example 2

Given a ring as shown in figure 5(b), with the following data: R = 15, H = 9, q = 10, and EI = 84,375.

With the coordinate system as shown in figure 5(b), the equations for the ring outline are

$$\left.\begin{aligned} \frac{r}{R} &= k' \cos\theta + \sqrt{1 - k'^2 \sin^2\theta} \qquad && 0 \leqslant \theta \leqslant \alpha \\ \frac{r}{R} &= \frac{1}{\sin\theta} && \alpha \leqslant \theta \leqslant \frac{\pi}{2} \end{aligned}\right\} \qquad \text{(E8)}$$

where

$$k' = \frac{H}{R}$$

The procedure as followed for example 1 is also used for example 2 by means of the following steps:

(1) The Fourier coefficients are

$$a_0 = 1.85526\left(\frac{R^2}{EI}\right)$$

$$a_2 = 0.78339\left(\frac{R^2}{EI}\right)$$

$$a_4 = -0.09307\left(\frac{R^2}{EI}\right)$$

$$b_0 = 2.55170\left(\frac{R^3}{EI}\right)$$

$$b_2 = 1.57561\left(\frac{R^3}{EI}\right)$$

$$b_4 = -0.02956\left(\frac{R^3}{EI}\right)$$

$$c_0 = 3.58542\left(\frac{R^4}{EI}\right)$$

$$c_2 = 2.83147\left(\frac{R^4}{EI}\right)$$

$$c_4 = 0.15947\left(\frac{R^4}{EI}\right)$$

(2) The values of J and S are

$$J = 16.0332\left(\frac{R^3}{EI}\right)$$

$$S = 8.6832\left(\frac{R}{EI}\right)$$

$$\frac{J}{S} = 1.8464R^2$$

(3) The constants are

$$K_{00} = -0.06107\left(\frac{R^3}{EI}\right)$$

$$K_{02} = -0.26192\left(\frac{R^3}{EI}\right)$$

$$K_{04} = -0.04769\left(\frac{R^3}{EI}\right)$$

$$K_{20} = -0.52399\left(\frac{R^3}{EI}\right)$$

$$K_{22} = -2.31488\left(\frac{R^3}{EI}\right)$$

$$K_{24} = -0.81417\left(\frac{R^3}{EI}\right)$$

$$K_{40} = -0.09538\left(\frac{R^3}{EI}\right)$$

$$K_{42} = -0.81417\left(\frac{R^3}{EI}\right)$$

$$K_{44} = -2.54857\left(\frac{R^3}{EI}\right)$$

$$Q_0 = 0.07993\left(\frac{R^4}{EI}\right)$$

$$Q_2 = 0.69251\left(\frac{R^4}{EI}\right)$$

$$Q_4 = 0.16565\left(\frac{R^4}{EI}\right)$$

(4) Introduce the nondimensional parameter $t' = \frac{EI}{q} \frac{1}{R^3}$ which is equal to 2.5 for the present example. The following simultaneous equations are obtained:

$$2.43893A_0 - 0.26196A_2 - 0.04769A_4 = 0.07993R$$

$$-0.52394A_0 - 9.81488A_2 - 0.81417A_4 = 0.69251R$$

$$-0.09538A_0 - 0.81417A_2 - 40.04857A_4 = 0.16565R$$

(5) The solution of the foregoing equations is

$$A_0 = 0.02503R$$

$$A_2 = -0.07166R$$

$$A_4 = -0.00274R$$

(6) The radial deflection w is

$$w = (0.02503 - 0.07166 \cos 2\theta - 0.00274 \cos 4\theta)R \qquad \text{(E9)}$$

Values of w/R are given in table 8 and also plotted against θ in figure 28.

(7) The Fourier coefficients e_n are obtained as

$$e_0 = \frac{0.78538}{R}$$

$$e_2 = \frac{-0.18829}{R}$$

$$e_4 = \frac{0.03070}{R}$$

In order to check the results, equation (73) is used.

$$f_0 - g_0 = -0.00005 = 0$$

(8) The coefficients f_n and g_n are then evaluated.

$$f_2 = -0.06173$$

$$f_4 = 0.00537$$

$$g_2 = -0.00672$$

$$g_4 = -0.02698$$

(9) The angular displacements can be expressed by the following equation:

$$\phi = 0.02751 \sin 2\theta - 0.00809 \sin 4\theta \qquad \text{(E9)}$$

Values of ϕ are given in table 9 and also plotted in figure 35.

(10) The bending-moment distribution is then given by the following equation:

$$\frac{M}{qR^2} = \frac{M_0}{qR^2} + \frac{r}{R}\frac{w}{R} - 0.01339 \qquad \text{(E10)}$$

Values of M/qR^2 and M_0/qR^2 are given in table 10 and also plotted in figure 57 for comparison. The maximum error introduced by neglecting the change of geometric shape of the ring is about 35 percent.

Method of Checking

A method of checking the final bending moment and displacements is given. Example 2 is used for illustration.

Assume the results obtained from example 2 to be correct. Then

$$w = (0.02503 - 0.07166 \cos 2\theta - 0.00274 \cos 4\theta)R$$

$$\phi = 0.02751 \sin 2\theta - 0.00809 \sin 4\theta$$

$$\frac{M}{qR^2} = \frac{M_0}{qR^2} + \frac{r}{R}\frac{w}{R} - 0.01339$$

Using the bending-moment expression, the deflections u and v, parallel to the x- and y-axis, respectively, can be determined from the original shape of the ring. The deflected ring defined by the displacements u and v should agree with the deflected ring defined by w and ϕ.

The equations for the determination of u and v are

$$u_\theta = \int_0^\theta \frac{M_\alpha}{EI} r_\alpha \cos \alpha \, ds + r_\theta \cos \theta \int_0^\theta \frac{M_\alpha}{EI} \, ds \qquad \text{(E11)}$$

$$v_\theta = \int_{\pi/2}^\theta \frac{M_\alpha}{EI} r_\alpha \sin \alpha \, ds - r_\theta \sin \theta \int_{\pi/2}^\theta \frac{M_\alpha}{EI} \, ds \qquad \text{(E12)}$$

Values of u and v, obtained from equations (E11) and (E12), are given in table 11 and are also plotted in figure 58, together with the deflections defined by w and ϕ. The agreement is very good.

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3. Timoshenko, S.: Theory of Elastic Stability. First ed., McGraw-Hill Book Co., Inc., 1936, pp. 206, 208.

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TABLE 1.- FOURIER COEFFICIENTS a_n, b_n, c_n, AND e_n FOR VARIOUS VALUES OF PARAMETER k

[Family I, elliptical rings.]

$k^2 = 1 - \left(\frac{b}{a}\right)^2$	0	0.1	0.2	0.3	0.4
$a_0\left(EI/a^2\right)$	1.00000	0.94934	0.89722	0.84334	0.78735
$a_2\left(EI/a^2\right)$	0	-0.05002	-0.10016	-0.15059	-0.20160
$a_4\left(EI/a^2\right)$	0	0.00066	0.00282	0.00689	0.01354
$b_0\left(EI/a^3\right)$	1.00000	0.92514	0.85052	0.77601	0.70146
$b_2\left(EI/a^3\right)$	0	-0.07308	-0.14210	-0.20669	-0.26638
$b_4\left(EI/a^3\right)$	0	0.00177	0.00731	0.01703	0.03151
$c_0\left(EI/a^4\right)$	1.00000	0.90187	0.80749	0.71684	0.62988
$c_2\left(EI/a^4\right)$	0	-0.09493	-0.17944	-0.25300	-0.31494
$c_4\left(EI/a^4\right)$	0	0.00313	0.01251	0.02817	0.05014
$e_0(a)$	1.00000	1.02722	1.05984	1.09979	1.15013
$e_2(a)$	0	0.02705	0.05899	0.09752	0.14519
$e_4(a)$	0	-0.00018	-0.00082	-0.00217	-0.00470

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TABLE 2.- FOURIER COEFFICIENTS a_n, b_n, c_n, AND e_n FOR VARIOUS VALUES OF PARAMETER k'

[Family II, rings formed by two semicircles and two straight lines.]

$k' = \frac{H}{R}$	0	0.2	0.4	0.6
$a_0\left(EI/R^2\right)$	1.00000	1.26456	1.54962	1.85526
$a_2\left(EI/R^2\right)$	0	0.19985	0.46057	0.78339
$a_4\left(EI/R^2\right)$	0	-0.03398	-0.06703	-0.09307
$b_0\left(EI/R^3\right)$	1.00000	1.42356	1.93754	2.55170
$b_2\left(EI/R^3\right)$	0	0.33803	0.85649	1.57561
$b_4\left(EI/R^3\right)$	0	-0.04495	-0.06725	-0.02956
$c_0\left(EI/R^4\right)$	1.00000	1.60789	2.45013	3.58542
$c_2\left(EI/R^4\right)$	0	0.50563	1.41205	2.83147
$c_4\left(EI/R^4\right)$	0	-0.05622	-0.03567	0.15947
$e_0(R)$	1.00000	0.89756	0.83083	0.78538
$e_2(R)$	0	-0.07617	-0.13812	-0.18829
$e_4(R)$	0	0.01605	0.02677	0.03070

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TABLE 3.- VALUES OF J AND S

Family I

$k^2 = 1 - \left(\frac{b}{a}\right)^2$	0	0.1	0.2	0.3	0.4
$J\left(EI/a^3\right)$	6.2832	5.81282	5.34397	4.87580	4.40741
$S(EI/a)$	6.2832	6.12302	5.95614	5.78147	5.59762
$\frac{J}{S}\left(\frac{1}{a^2}\right)$	1.0000	0.94933	0.89722	0.84335	0.78737

Family II

$k' = \frac{H}{R}$	0	0.2	0.4	0.6
$J\left(EI/R^3\right)$	6.2832	8.9452	12.1736	16.0332
$S(EI/R)$	6.2832	7.0832	7.8832	8.6832
$\frac{J}{S}\left(\frac{1}{R^2}\right)$	1.0000	1.2628	1.5442	1.8464

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TABLE 4.- GRAPHICAL METHOD TO DETERMINE THE FOURIER COEFFICIENTS a_n, b_n, c_n, AND e_n

[See appendix D.]

Row	Item	Procedure	Results at -									
1	θ		0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
2	r											
3	EI											
4	cos 2θ											
5	cos 4θ											
6	cos 6θ											
7	r/EI	(2)/(3)										
8	(r/EI) cos 2θ	(7) × (4)										
9	(r/EI) cos 4θ	(7) × (5)										
10	(r/EI) cos 6θ	(7) × (6)										
11	r^2/EI	(2) × (7)										
12	(r^2/EI) cos 2θ	(11) × (4)										
13	(r^2/EI) cos 4θ	(11) × (5)										
14	(r^2/EI) cos 6θ	(11) × (6)										
15	r^3/EI	(2) × (11)										
16	(r^3/EI) cos 2θ	(15) × (4)										
17	(r^3/EI) cos 4θ	(15) × (5)										
18	(r^3/EI) cos 6θ	(15) × (6)										
19	1/r	1/(2)										
20	(1/r) cos 2θ	(19) × (4)										
21	(1/r) cos 4θ	(19) × (5)										
22	(1/r) cos 6θ	(19) × (6)										
23	Arc length, S											

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TABLE 5.- RADIAL DEFLECTIONS w/a AT VARIOUS VALUES OF θ

[Family I. $k^2 = 0.3$; $t = 0.3$.]

1	2	3	4	5	6	7
θ (deg)	cos 2θ	cos 4θ	0.03778 × (2)	-0.00167 × (3)	0.00503	w/a (4) + (5) + (6)
0	1.00000	1.00000	0.03778	-0.00167	0.00503	0.04114
10	.93969	.76604	.03550	-.00128	.00503	.03925
20	.76604	.17365	.02894	-.00029	.00503	.03368
30	.50000	-.50000	.01889	.00084	.00503	.02476
40	.17365	-.93969	.00656	.00157	.00503	.01316
50	-.17365	-.93969	-.00656	.00157	.00503	.00004
60	-.50000	-.50000	-.01889	.00084	.00503	-.01302
70	-.76604	.17365	-.02894	-.00029	.00503	-.02420
80	-.93969	.76604	-.03550	-.00128	.00503	-.03175
90	-1.00000	1.00000	-.03778	-.00167	.00503	-.03442

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TABLE 6.- ANGULAR DISPLACEMENTS ϕ AT VARIOUS VALUES OF θ

[Family I. $k^2 = 0.3$; $t = 0.3$.]

1	2	3	4	5	6	7
θ (deg)	sin 2θ	sin 4θ	$-0.02144 \times (2)$	$-0.00184 \times (3)$	$\phi = (4) + (5)$ (rad.)	ϕ (deg)
0	0	0	0	0	0	0
10	.34202	.64279	-.00733	-.00118	-.00851	-.49
20	.64279	.98481	-.01378	-.00181	-.01559	-.89
30	.86603	.86603	-.01857	-.00159	-.02016	-1.16
40	.98481	.34202	-.02111	-.00063	-.02174	-1.25
50	.98481	-.34202	-.02111	.00063	-.02048	-1.17
60	.86603	-.86603	-.01857	.00159	-.01698	-.97
70	.64279	-.98481	-.01378	.00181	-.01197	-.69
80	.34202	-.64279	-.00733	.00118	-.00615	-.35
90	0	0	0	0	0	0

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TABLE 7.- BENDING-MOMENT DISTRIBUTION

[Family I. $k^2 = 0.3$; $t = 0.3$; $\frac{J}{S}\frac{1}{a^2} = 0.84335$.]

1	2	3	4	5	6	7	8
θ (deg)	$\frac{r}{a}$	$\left(\frac{r}{a}\right)^2$	$\left(\frac{r}{a}\right)^2 - \frac{J}{S}\frac{1}{a^2}$ = (3) - 0.84335	$\frac{M_0}{q}\frac{1}{a^2}$ = $\frac{1}{2}$(4)	$\frac{w}{a}$ from table 5	$\frac{w}{a}\frac{r}{a}$	$\frac{M}{q}\frac{1}{a^2}$ = (5) + (7) - 0.00151
0	0.83666	0.70000	-0.14335	-0.07168	0.04114	0.03442	-0.03877
10	.84040	.70639	-.13696	-.06848	.03925	.03299	-.03750
20	.85173	.72545	-.11790	-.05895	.03368	.02869	-.03177
30	.86989	.75675	-.08660	-.04330	.02476	.02154	-.02330
40	.89394	.79904	-.04431	-.02216	.01316	.01176	-.01189
50	.92172	.84956	.00621	.00310	.00004	.00004	-.00163
60	.95038	.90322	.05987	.02994	-.01302	-.01237	.01606
70	.97584	.95226	.10891	.05446	-.02420	-.02362	.02933
80	.99365	.98725	.14390	.07195	-.03175	-.03155	.03889
90	1.00000	1.00000	.15665	.07833	-.03442	-.03442	.04240

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TABLE 8.- RADIAL DEFLECTIONS w/R AT VARIOUS VALUES OF θ

[Family II. k' = 0.6; t' = 2.5.]

1	2	3	4	5	6	7
θ (deg)	cos 2θ	cos 4θ	-0.07166 × (2)	-0.00274 × (3)	0.02503	$\frac{w}{R}$ = (4) + (5) + (6)
0	1.00000	1.00000	-0.07166	-0.00274	0.02503	-0.04937
10	.93969	.76604	-.06734	-.00210	02503	-.04441
20	.76604	.17365	-.05489	-.00048	02503	-.03034
30	.50000	-.50000	-.03583	.00137	.02503	-.01217
40	.17365	-.93969	-.01244	.00257	.02503	.01516
50	-.17365	-.93969	.01244	.00257	.02503	.04004
60	-.50000	-.50000	.03583	.00137	.02503	.06223
70	-.76604	.17365	.05489	-.00048	.02503	.07944
80	-.93969	.76604	.06734	-.00210	.02503	.09027
90	-1.00000	1.00000	.07166	-.00274	.02503	.09395

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TABLE 9.- ANGULAR DISPLACEMENTS ϕ AT VARIOUS VALUES OF θ

[Family II. k' = 0.6; t' = 2.5.]

1	2	3	4	5	6	7
θ (deg)	sin 2θ	sin 4θ	0.02751 × (2)	-0.00809 × (3)	ϕ = (4) + (5) (rad.)	ϕ (deg)
0	0	0	0	0	0	0
10	.34202	.64279	.00941	-.00520	.00421	.24
20	.64279	.98481	.01768	-.00797	.00971	.56
30	.86603	.86603	.02382	-.00701	.01681	.96
40	.98481	.34202	.02709	-.00277	.02432	1.39
50	.98481	-.34202	.02709	.00277	.02986	1.71
60	.86603	-.86603	.02382	.00701	.03083	1.77
70	.64279	-.98481	.01768	.00797	.02565	1.47
80	.34202	-.64279	.00941	.00520	.01461	.84
90	0	0	0	0	0	0

TABLE 10.- BENDING-MOMENT DISTRIBUTION

[Family II. k' = 0.6; t' = 2.5; $\frac{J}{S}\frac{1}{R^2}$ = 1.8464.]

1	2	3	4	5	6	7	8
θ (deg)	$\frac{r}{R}$	$\left(\frac{r}{R}\right)^2$	$\left(\frac{r}{R}\right)^2 - \frac{J}{S}\frac{1}{R^2}$ = (3) - 1.8464	$\frac{M_0}{q}\frac{1}{R^2}$ = (4)/2	$\frac{w}{R}$ from table 8	$\frac{w}{R}\frac{r}{R}$ = (2) × (6)	$\frac{M}{q}\frac{1}{R^2}$ = (5) + (7) - 0.01339
0	1.60000	2.56000	0.71360	0.35680	-0.04937	-0.07899	0.26442
10	1.58545	2.51365	.66725	.33363	-.04441	-.07041	.24983
20	1.54252	2.37937	.53297	.26649	-.03034	-.04680	.20630
30	1.47356	2.17138	.32498	.16249	-.01217	-.01793	.13117
40	1.38226	1.91064	.06424	.03212	.01516	.02096	.03969
50	1.27376	1.62246	-.22394	-.11197	.04004	.05100	-.07436
60	1.15469	1.33331	-.51309	-.25655	.06223	.07186	-.19808
70	1.06418	1.13248	-.71392	-.35696	.07944	.08454	-.28581
80	1.01542	1.03108	-.81532	-.40766	.09027	.09166	-.32939
90	1.00000	1.00000	-.84640	-.42320	.09395	.09395	-.34264

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TABLE 11.- DEFLECTIONS u AND v AT

VARIOUS VALUES OF θ

[Family II. $k' = 0.6$; $t' = 2.5$.]

1	2	3
θ (deg)	u/R	v/R
0	0	-0.05064
10	.00001	-.04649
20	.00446	-.03553
30	.01549	-.02158
40	.03270	-.00920
50	.05300	-.00182
60	.07165	0
70	.08424	0
80	.08726	0
90	.08939	0

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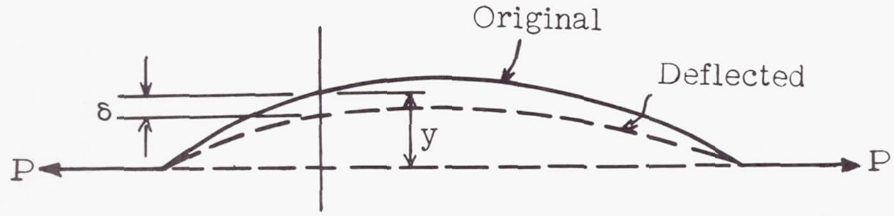


Figure 1.- Initially curved bar in original and in deflected shape.

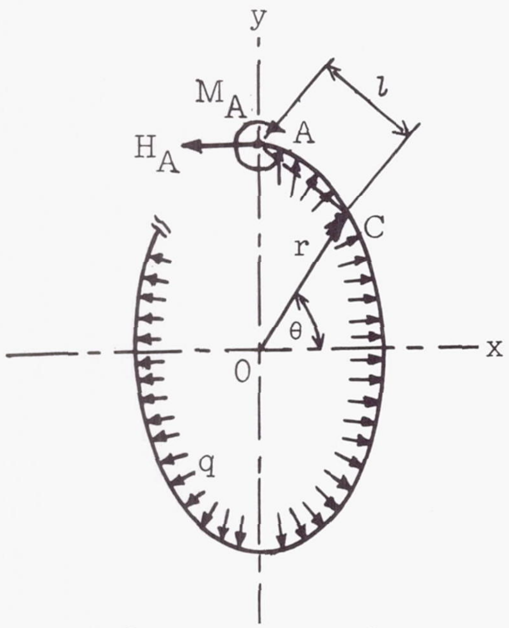


Figure 2.- Forces and moments on a pressure-loaded ring.

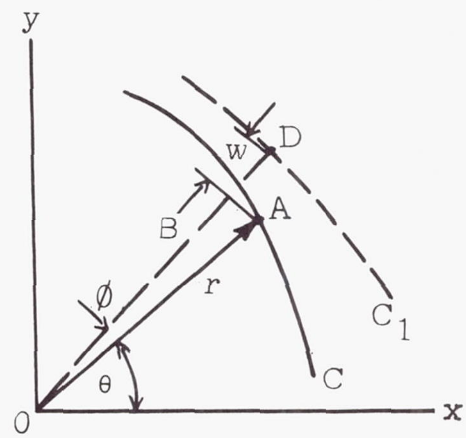


Figure 3.- Angular rotation and radial deflection of a point on a ring after loading.

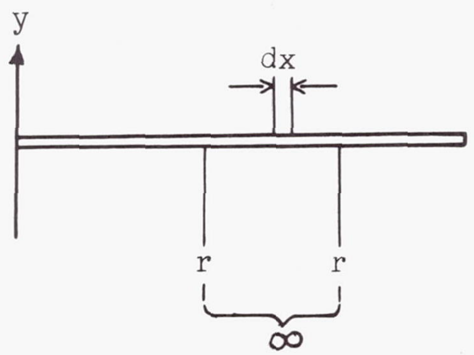
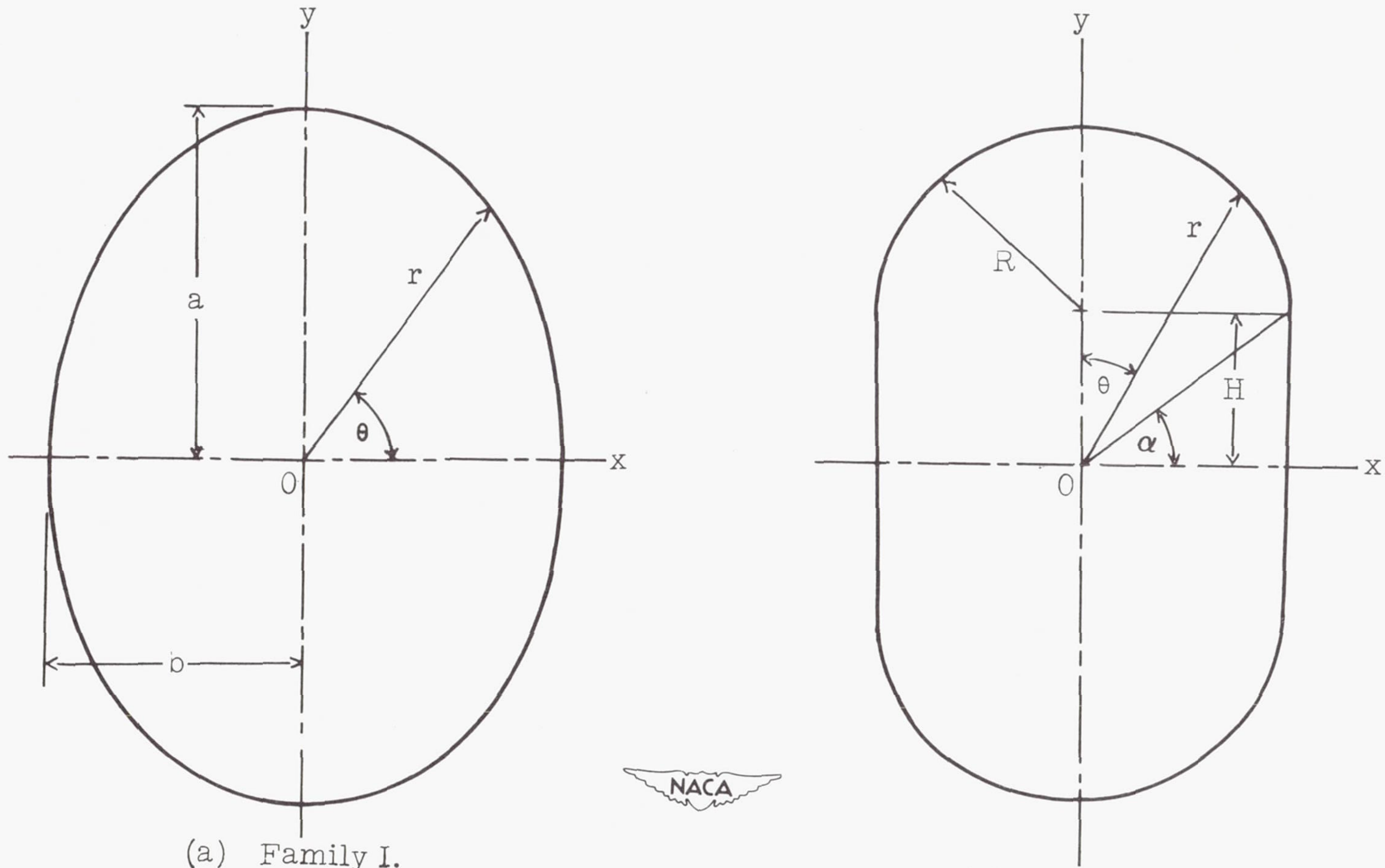


Figure 4.- Coordinate system used in beam deflection.



(a) Family I.

(b) Family II.

Figure 5.- Shapes of rings used in numerical examples.

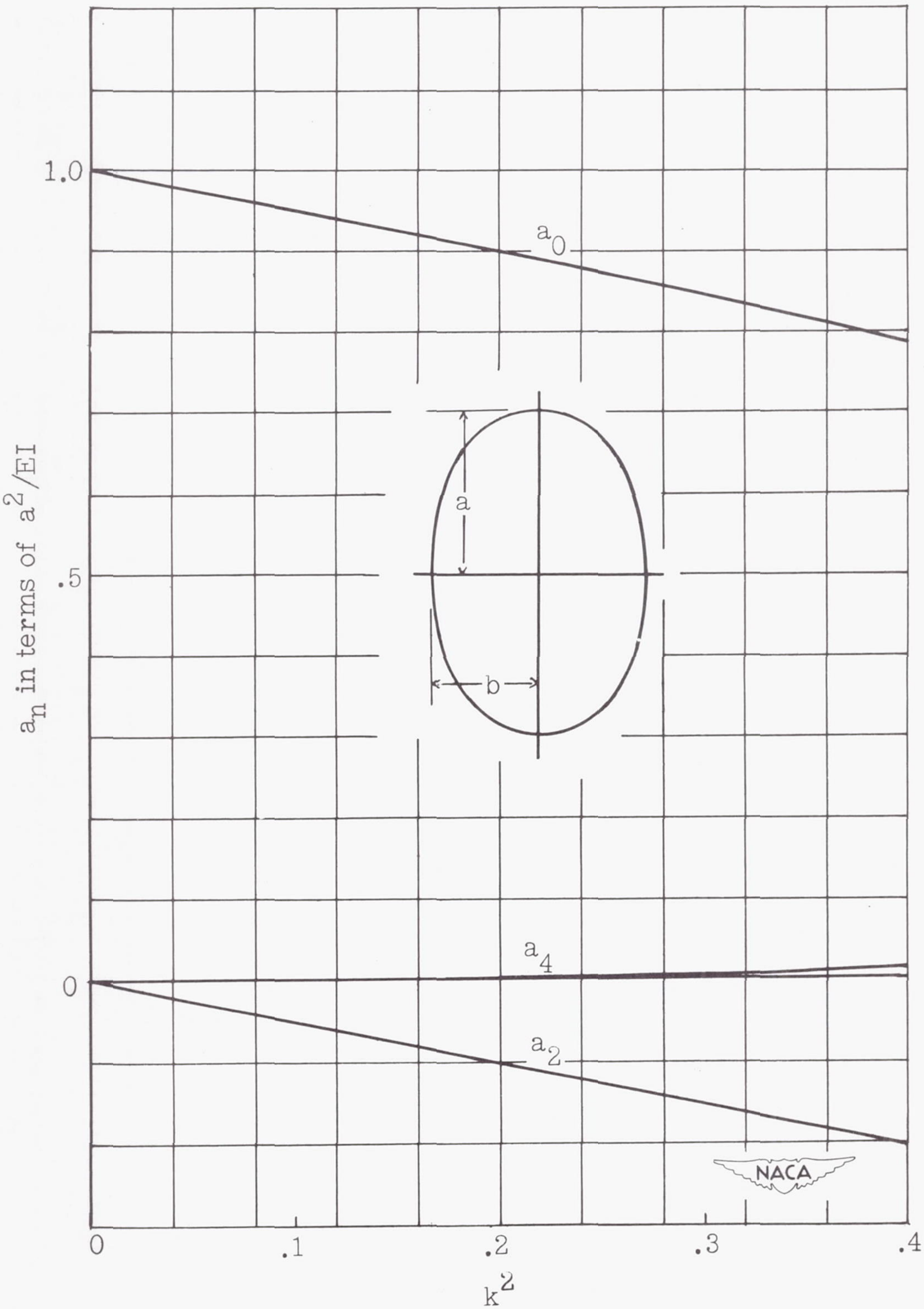


Figure 6.- Fourier coefficients a_n for various values of parameter k. $k^2 = 1 - \left(\frac{b}{a}\right)^2$.

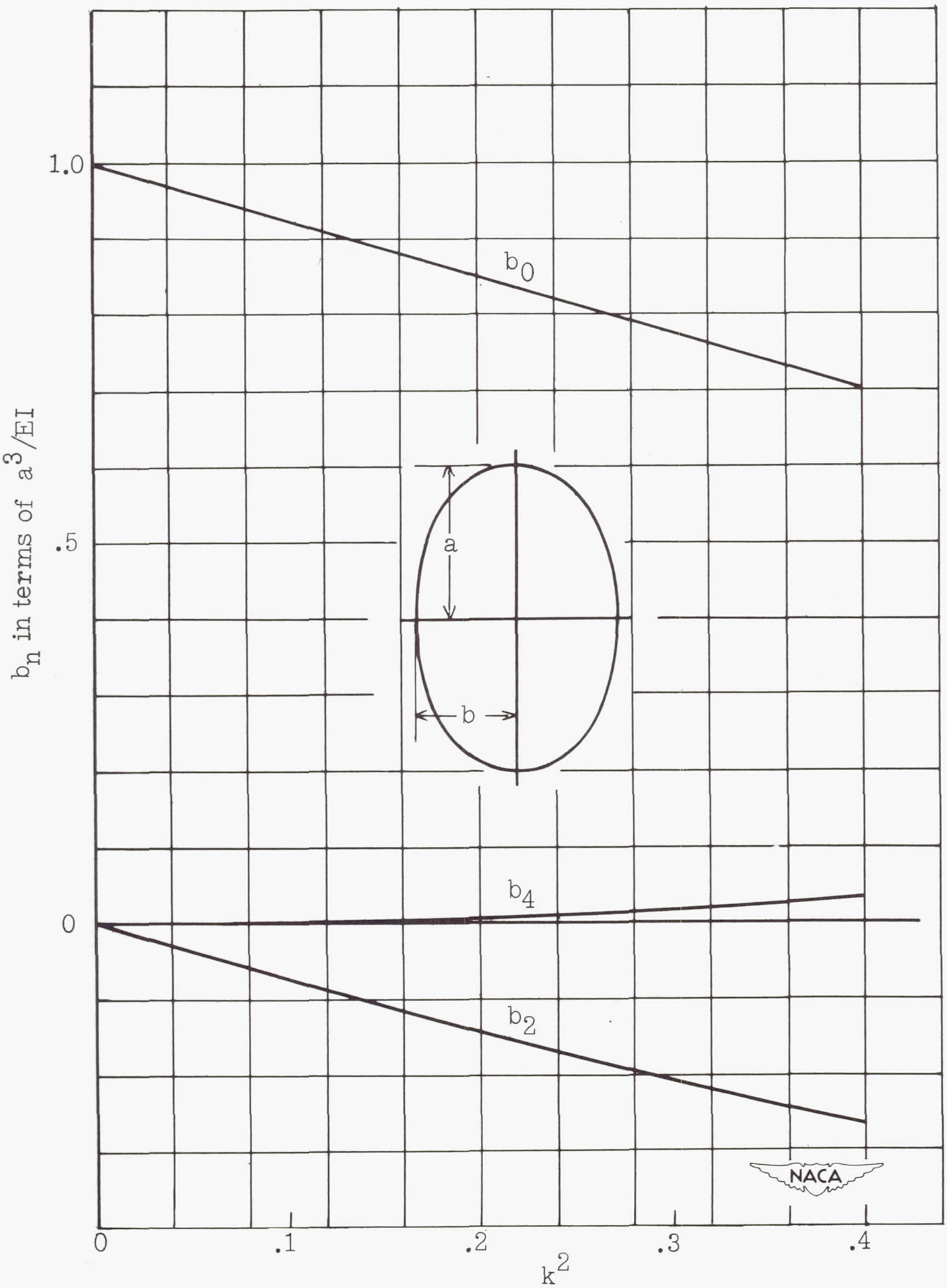


Figure 7.- Fourier coefficients b_n for various values of parameter k. $k^2 = 1 - \left(\frac{b}{a}\right)^2$.

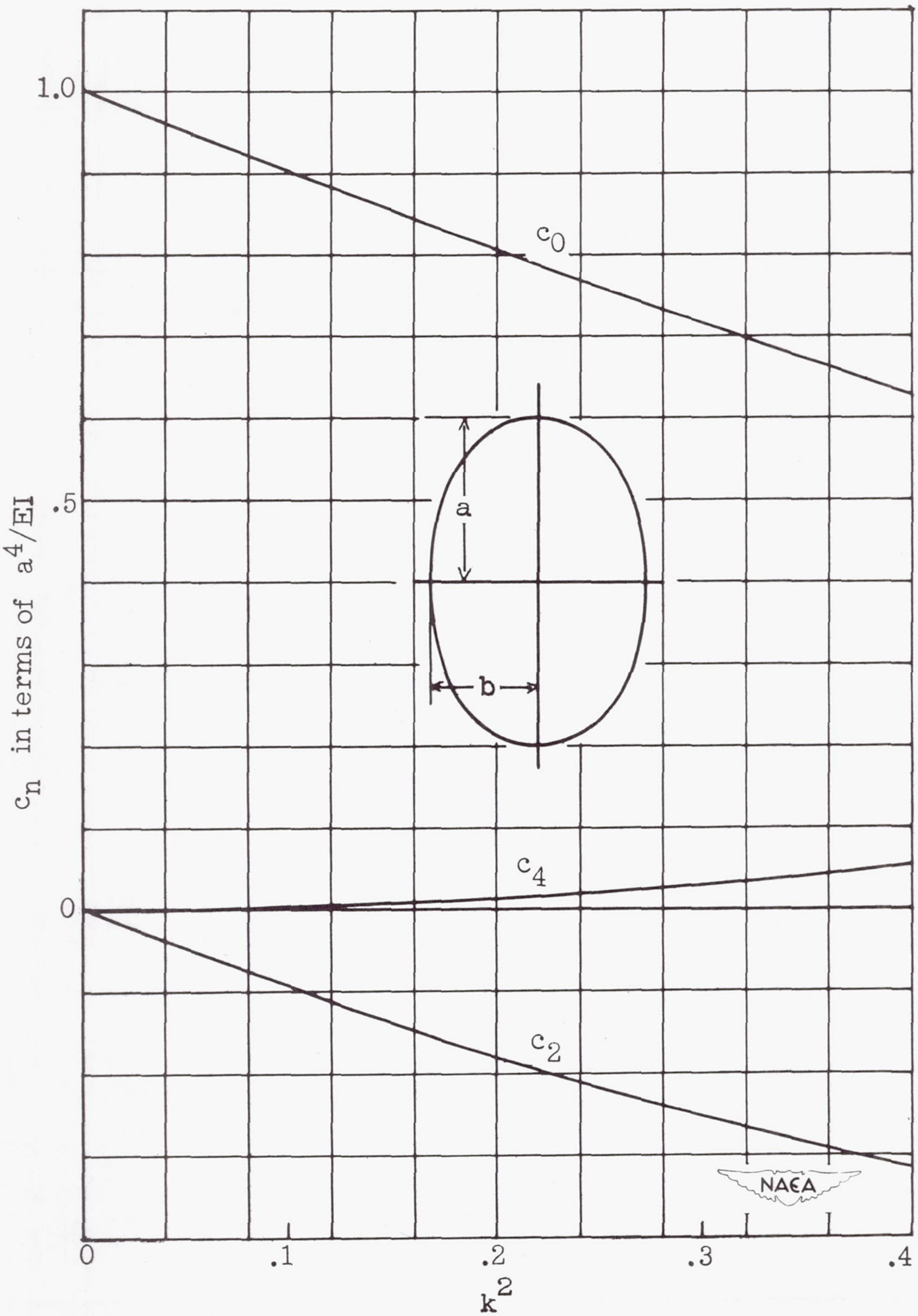


Figure 8.- Fourier coefficients c_n for various values of parameter k. $k^2 = 1 - \left(\frac{b}{a}\right)^2$.

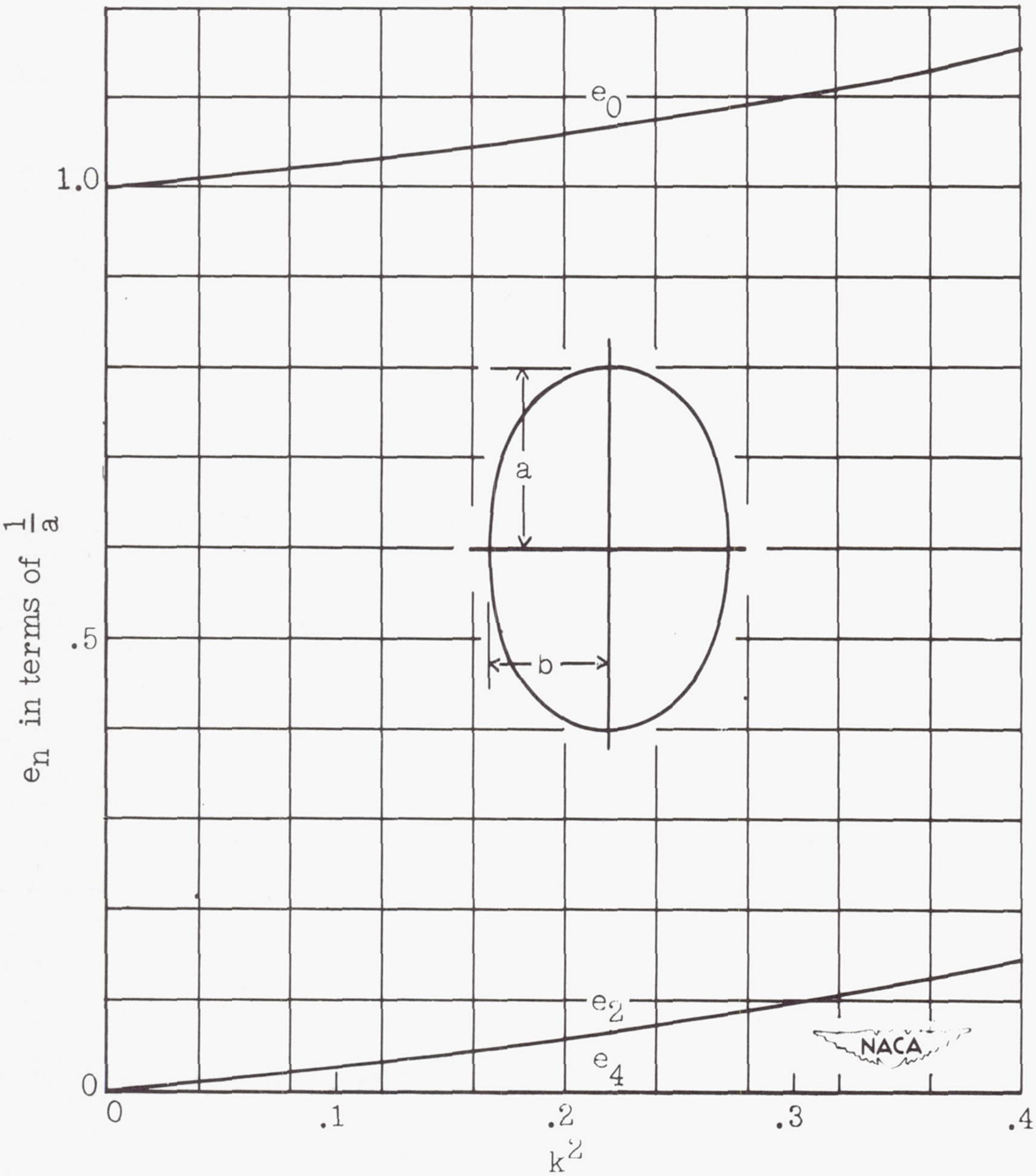


Figure 9.- Fourier coefficients e_n for various values of parameter k. $k^2 = \left(\frac{b}{a}\right)^2$.

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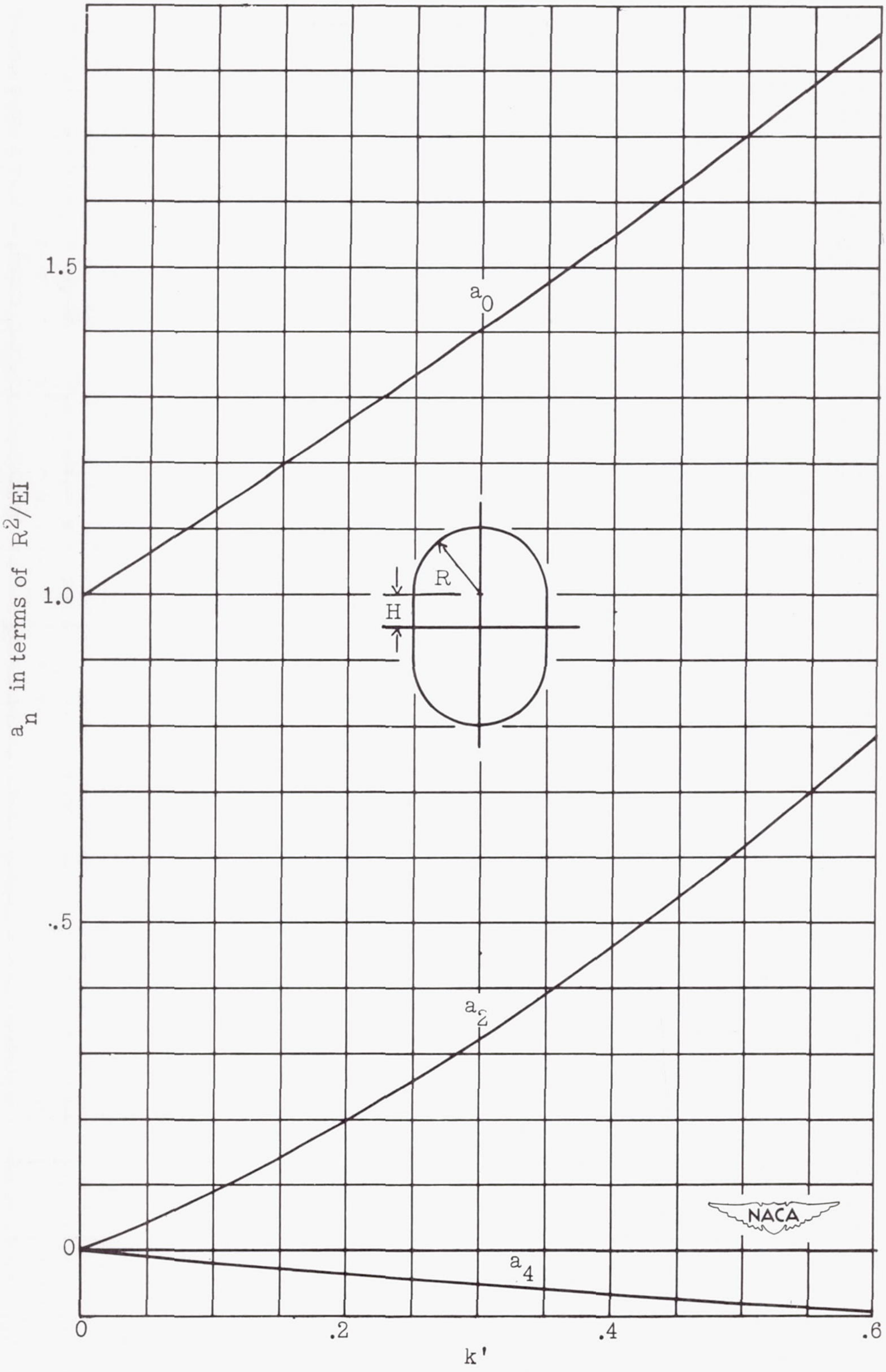


Figure 10.- Fourier coefficients a_n for various values of parameter k'. k' = H/R.

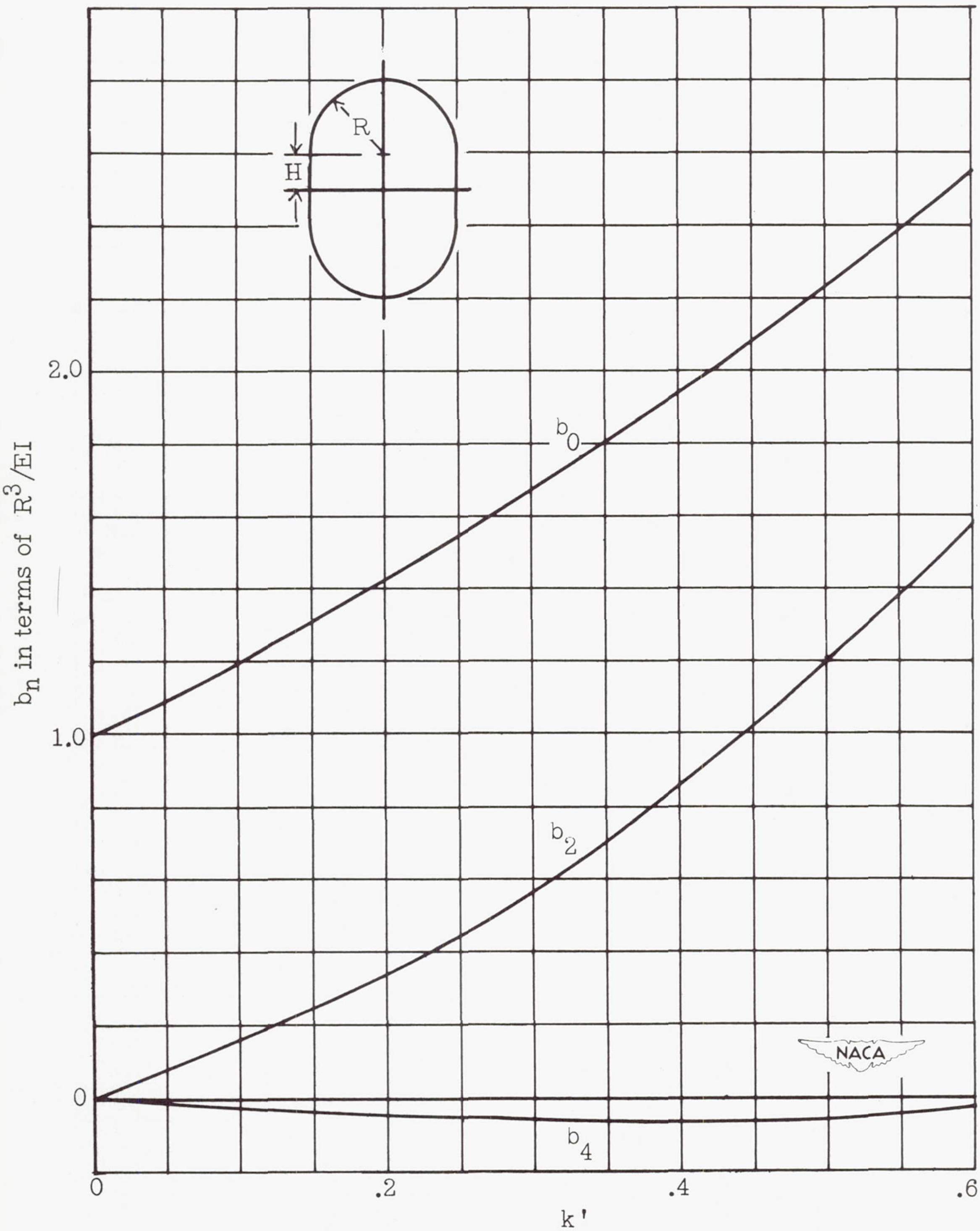


Figure 11.- Fourier coefficients b_n for various values of parameter k'. $k' = H/R$.

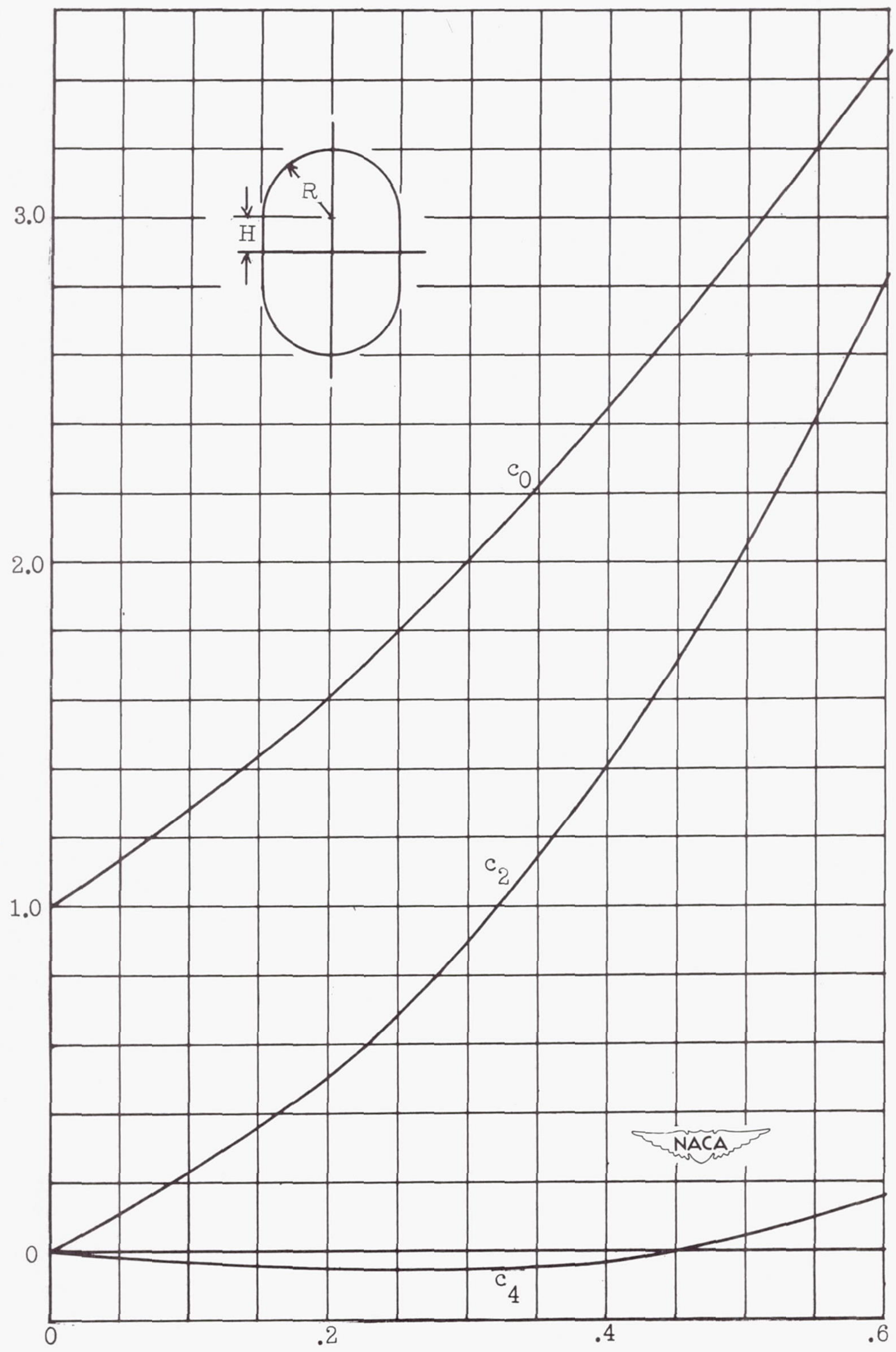


Figure 12.- Fourier coefficients c_n for various values of parameter k'. k' = H/R.

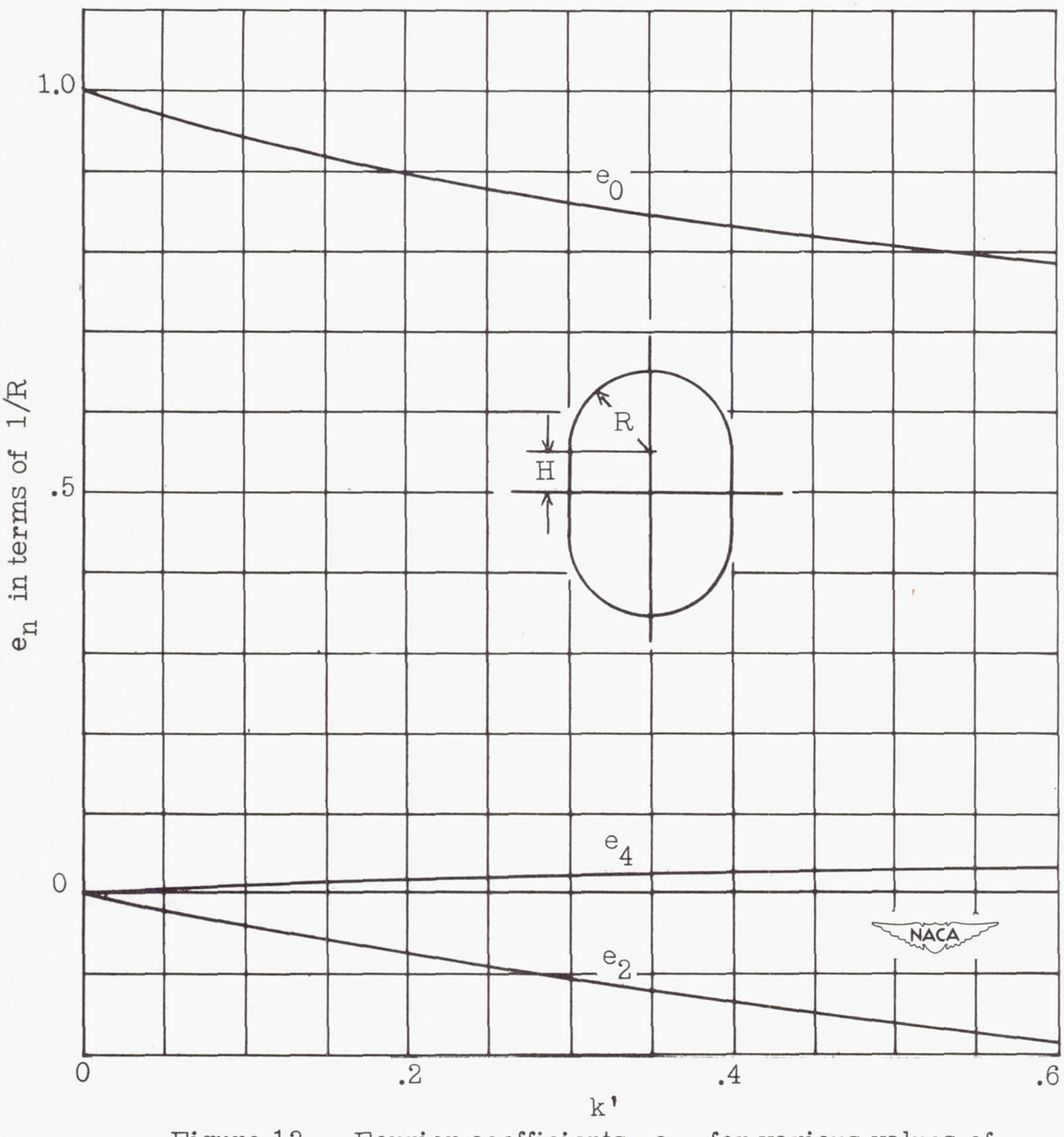


Figure 13.- Fourier coefficients e_n for various values of parameter k'. k' = H/R.

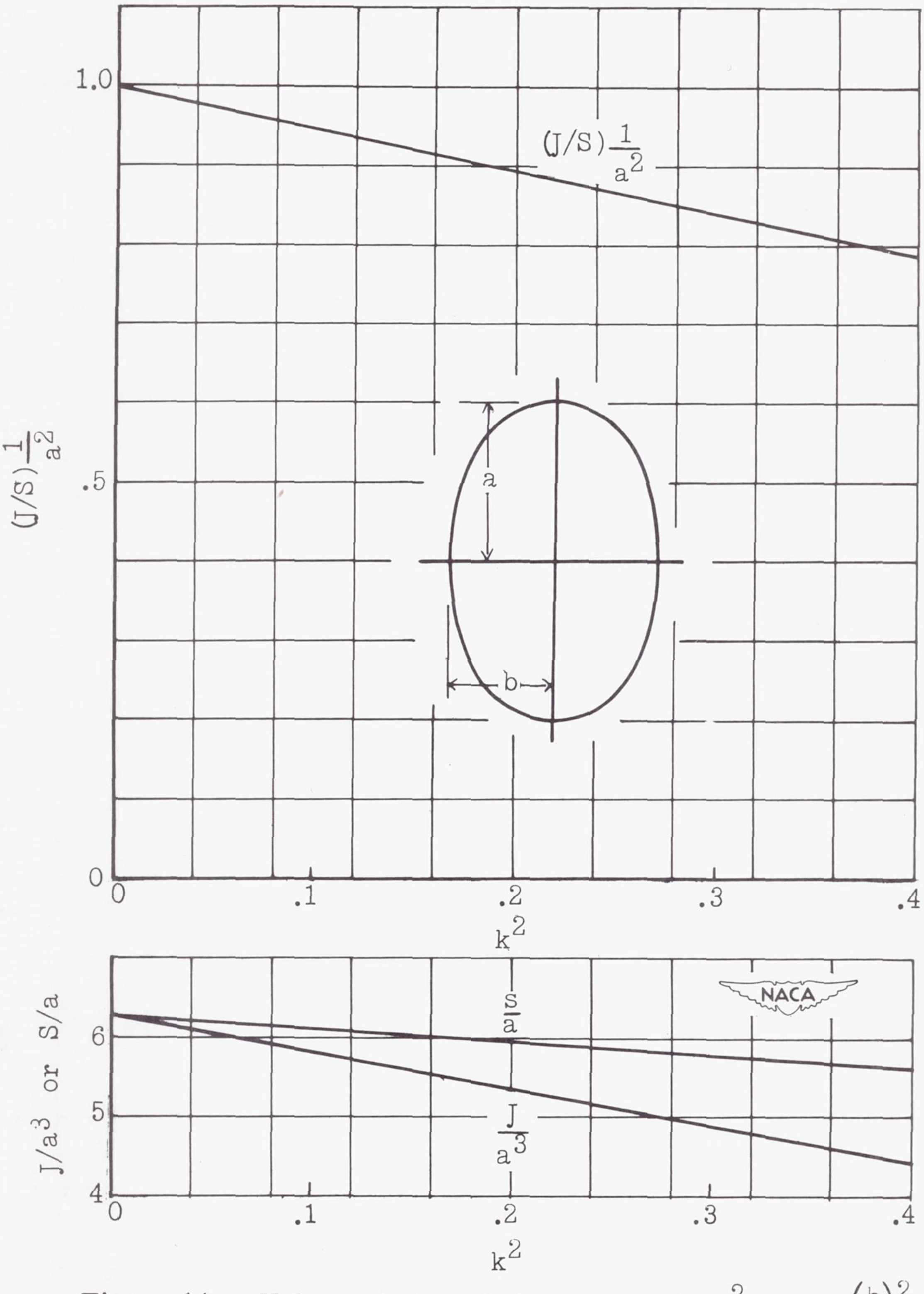


Figure 14.- Values of J and S. Family I. $k^2 = 1 - \left(\frac{b}{a}\right)^2$.

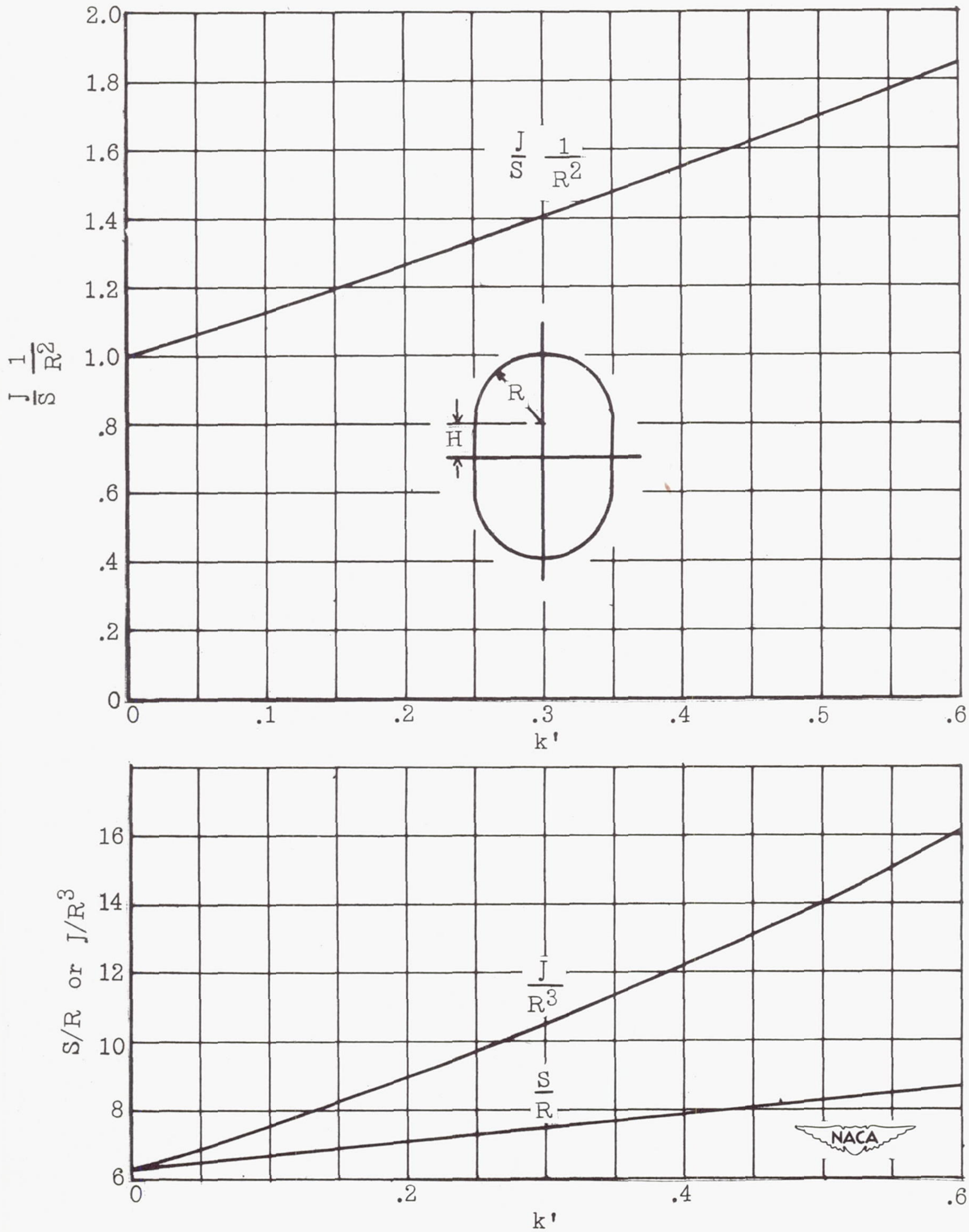


Figure 15.- Values of J and S. Family II. $k' = H/R$.

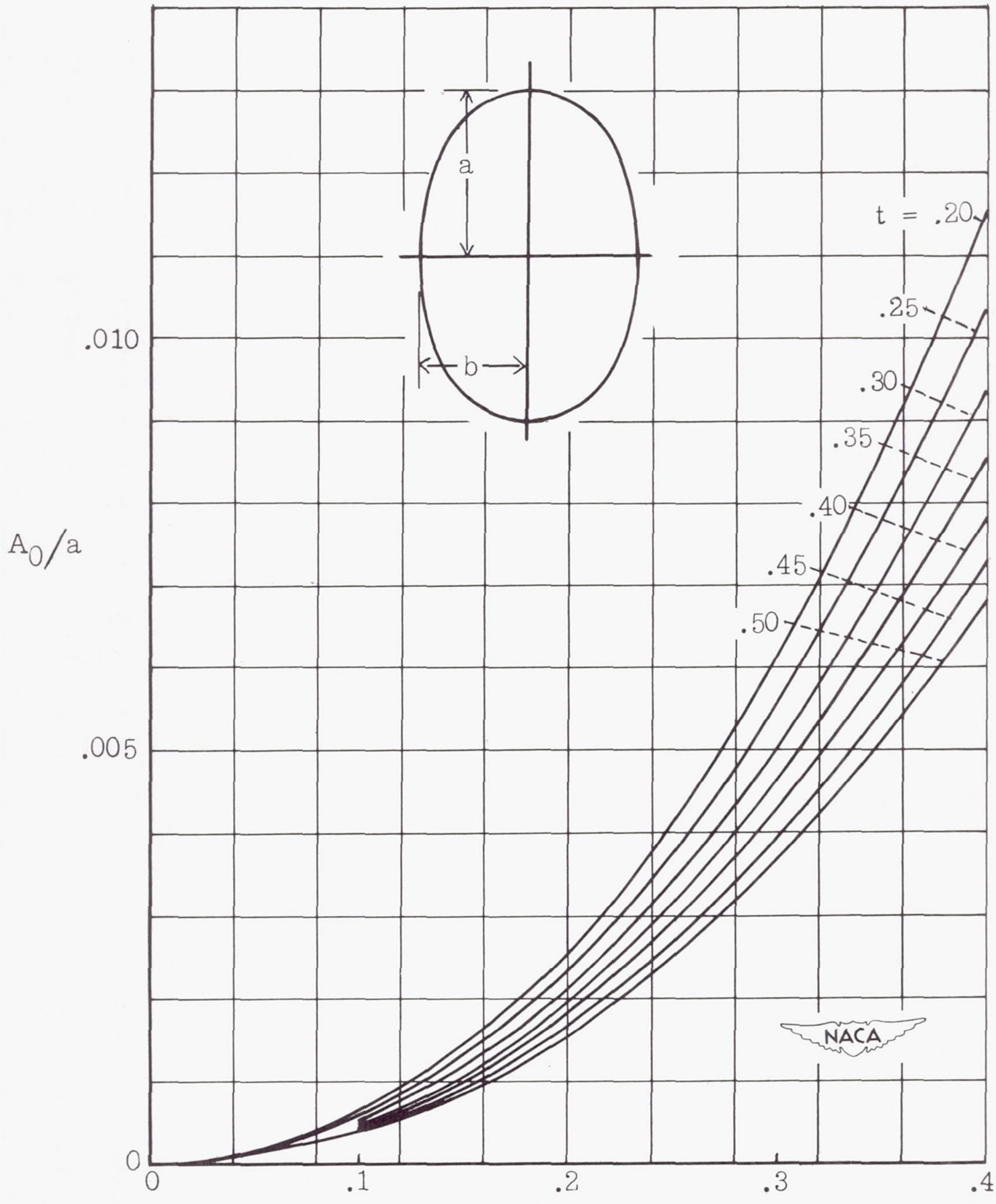


Figure 16.- Values of A_0/a for various ratios of $\frac{q}{EI}$. $t = \frac{EI}{q}\frac{1}{a^3}$; $k^2 = 1 - \left(\frac{b}{a}\right)^2$.

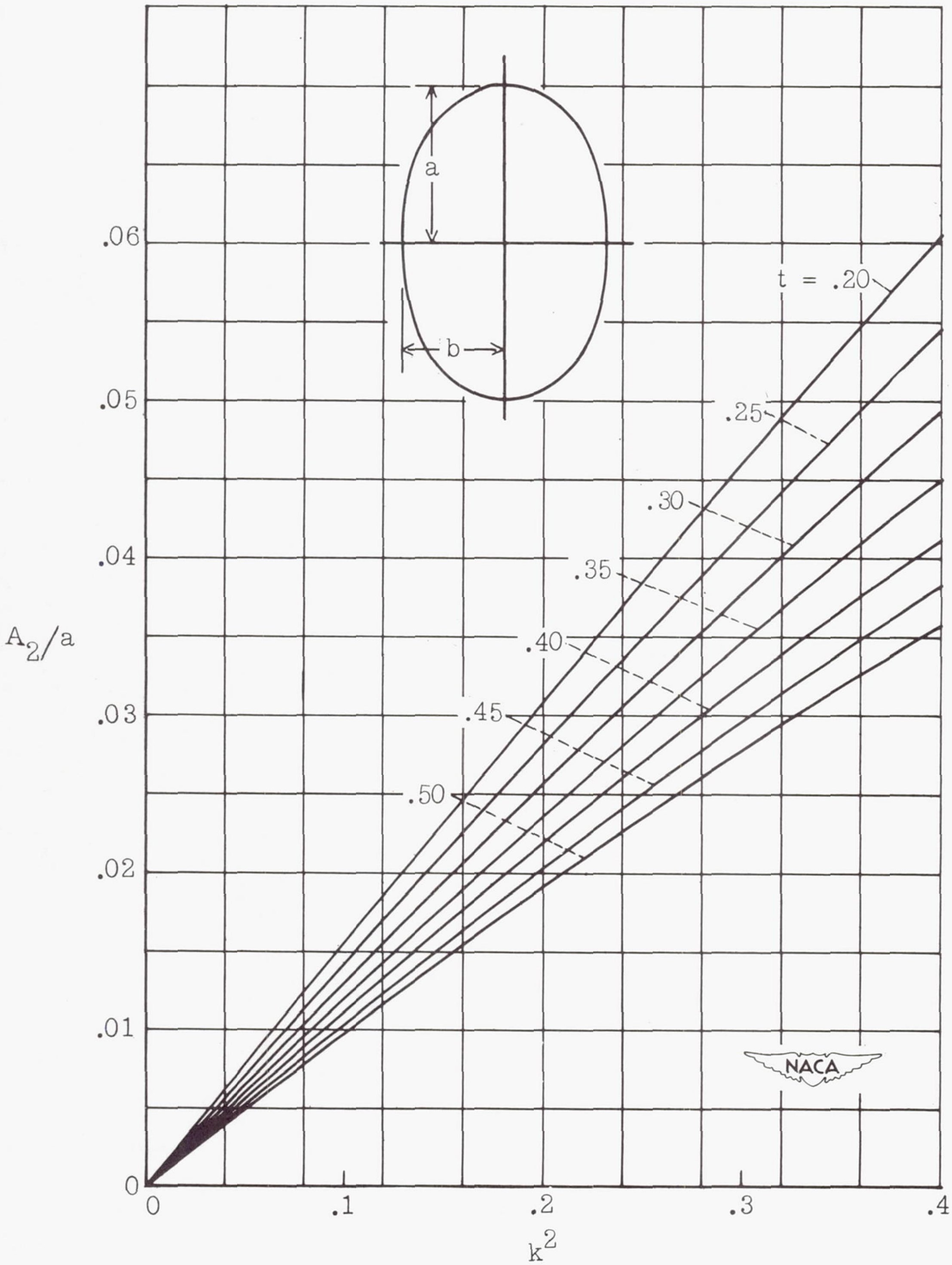


Figure 17.- Values of A_2/a for various ratios of $\frac{q}{EI}$. $t = \frac{EI}{q}\frac{1}{a^3}$; $k^2 = 1 - \left(\frac{b}{a}\right)^2$

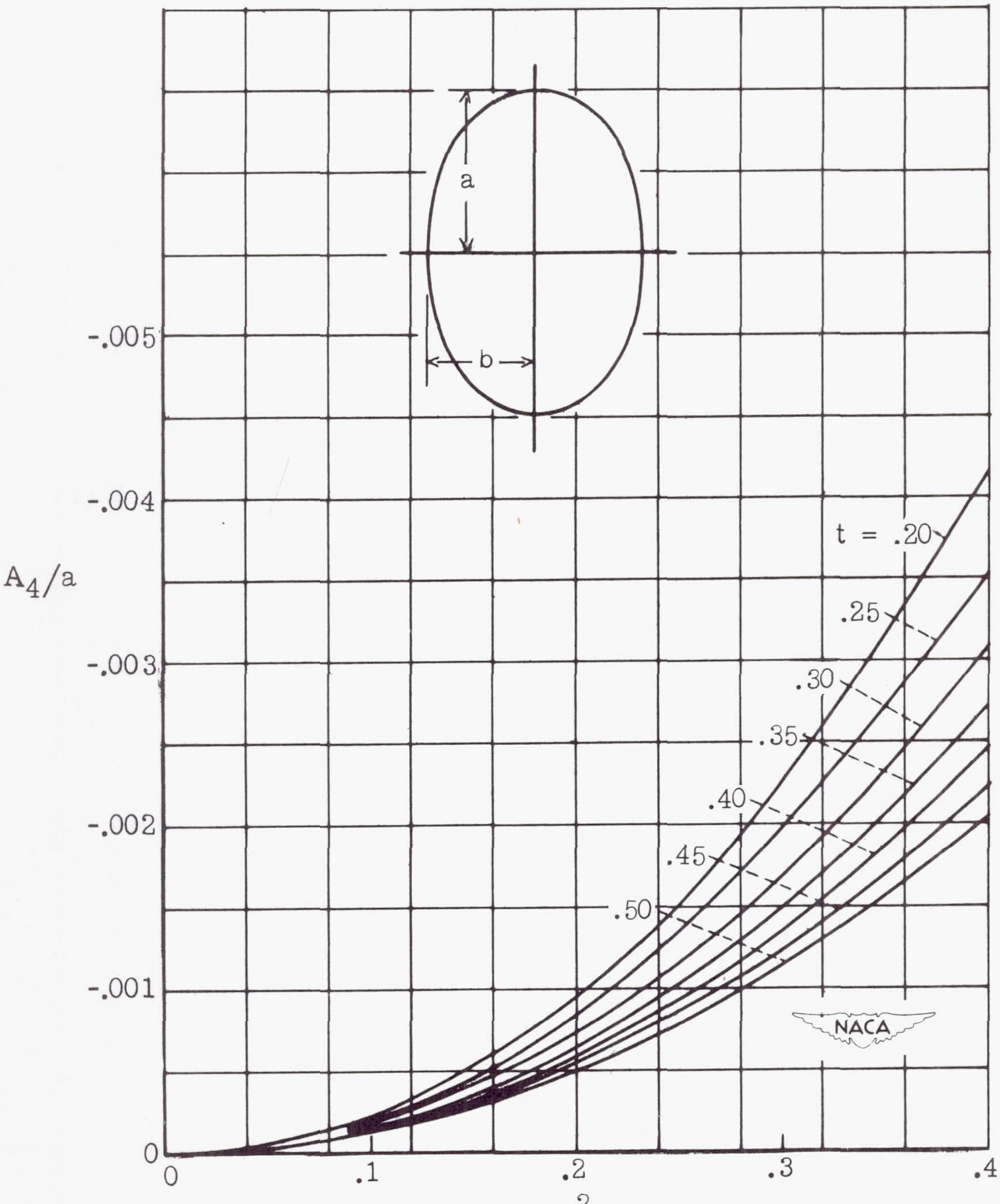


Figure 18.- Values of A_4/a for various ratios of $\frac{q}{EI}$. $t = \frac{EI}{q}\frac{1}{a^3}$. $k^2 = 1 - \left(\frac{b}{a}\right)^2$.

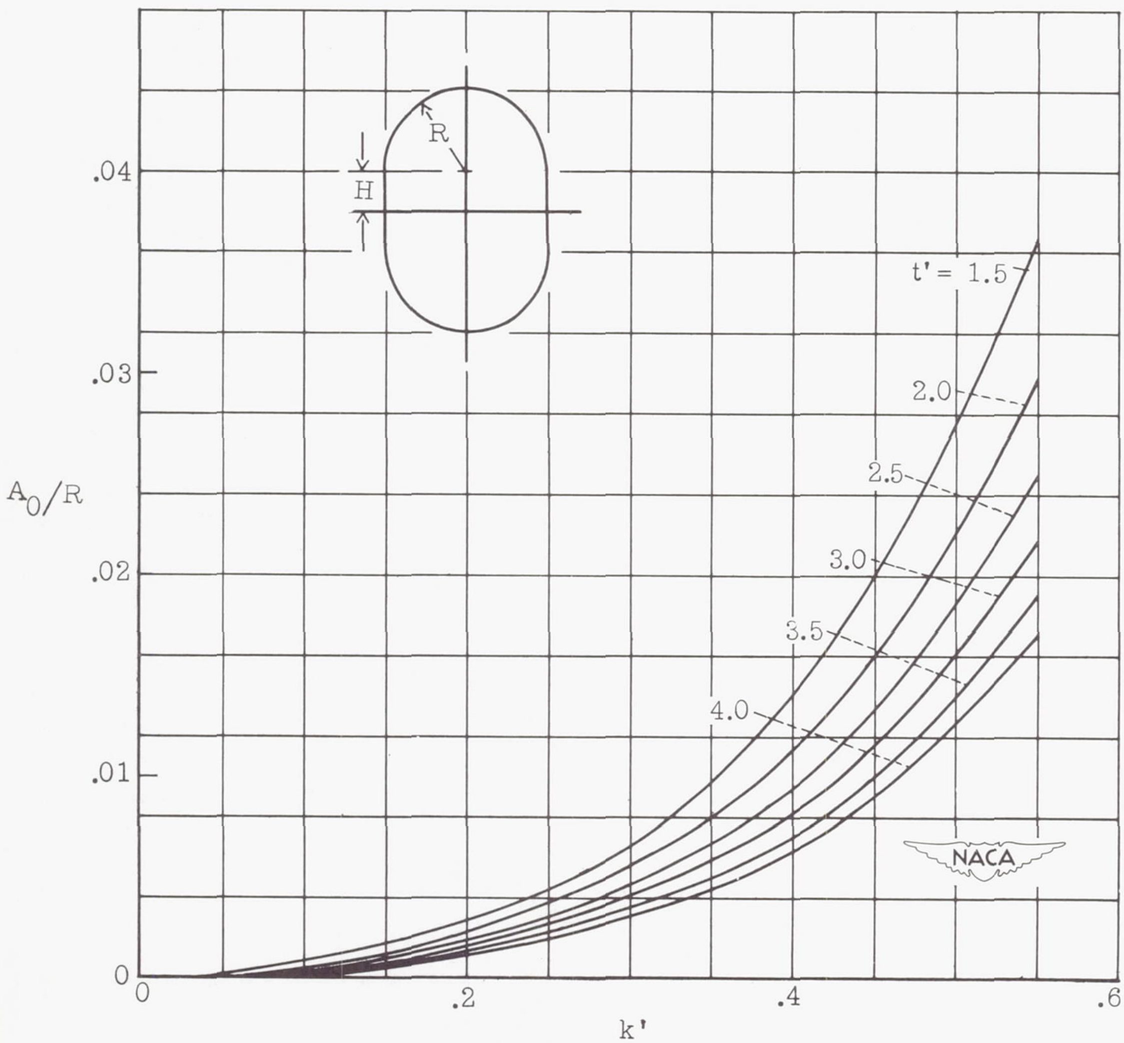


Figure 19.- Values of A_0/R for various ratios of $\frac{q}{EI}$. $t' = \frac{EI}{q}\frac{1}{R^3}$. $k' = H/R$.

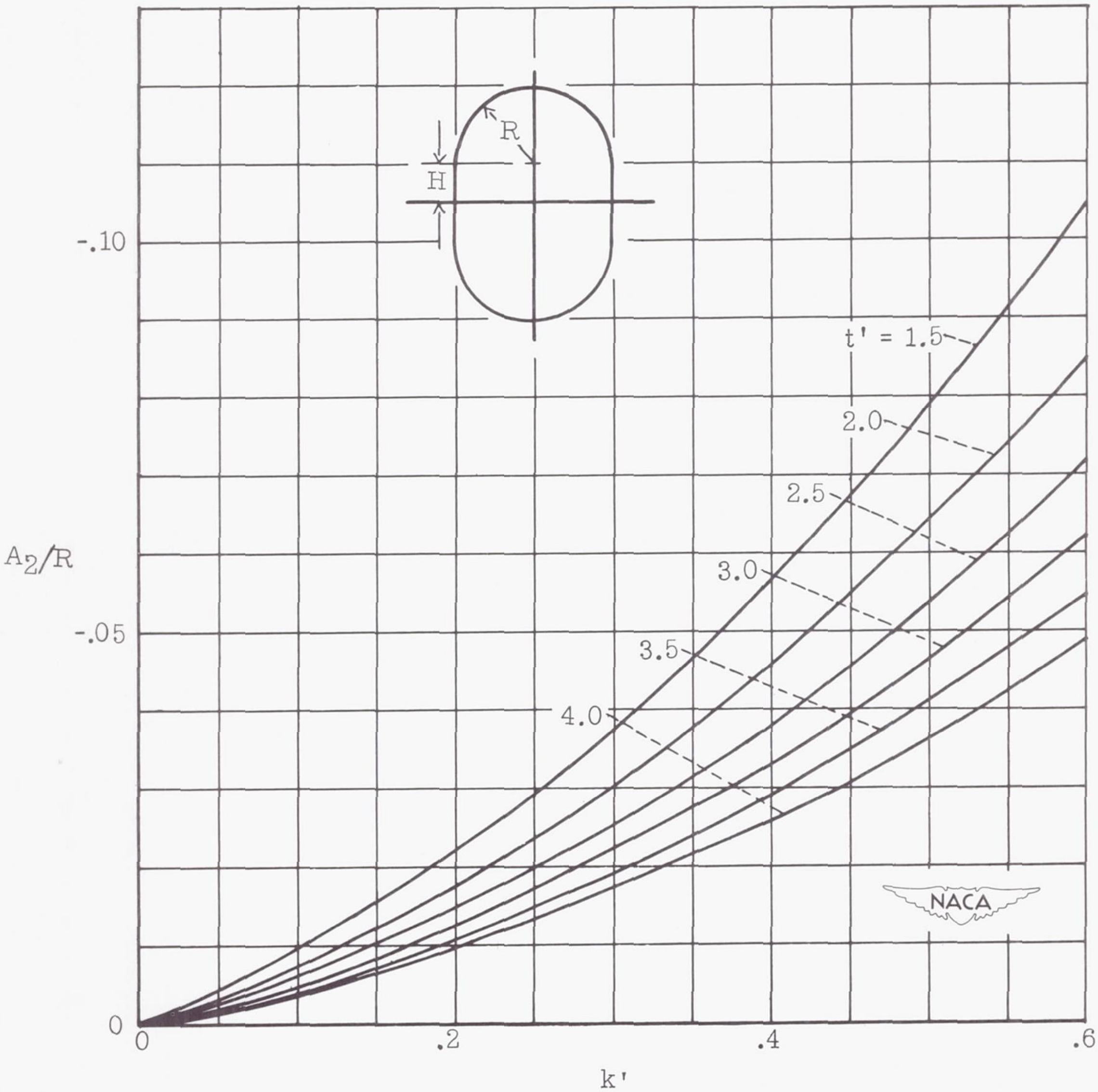


Figure 20.- Values of A_2/R for various ratios of $\frac{q}{EI}$. $t' = \frac{EI}{q} \frac{1}{R^3}$. $k' = H/R$.

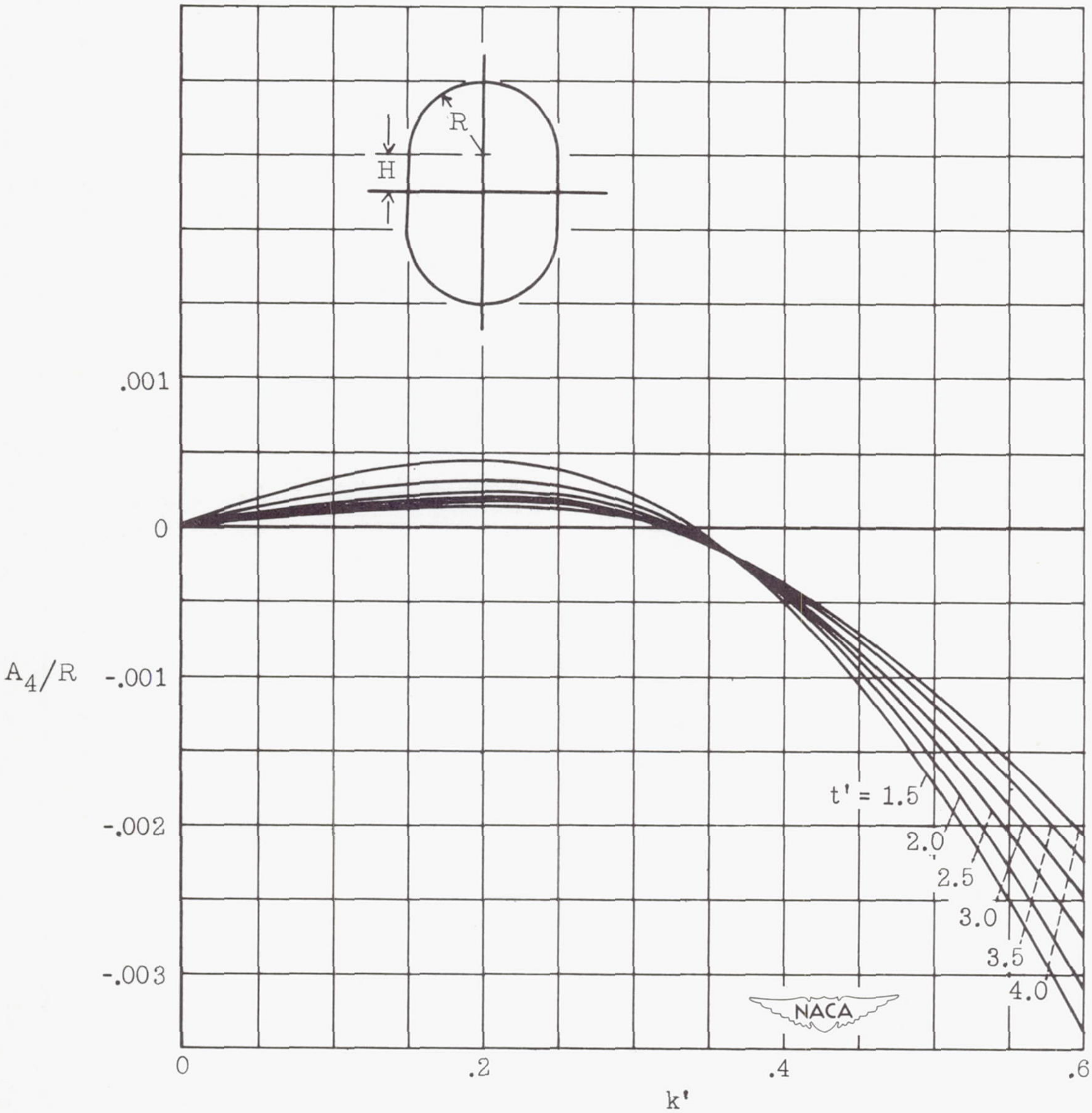


Figure 21.- Values of A_4/R for various ratios of $\frac{q}{EI}$. $t' = \frac{EI}{q}\,\frac{1}{R^3}$. $k' = H/R$.

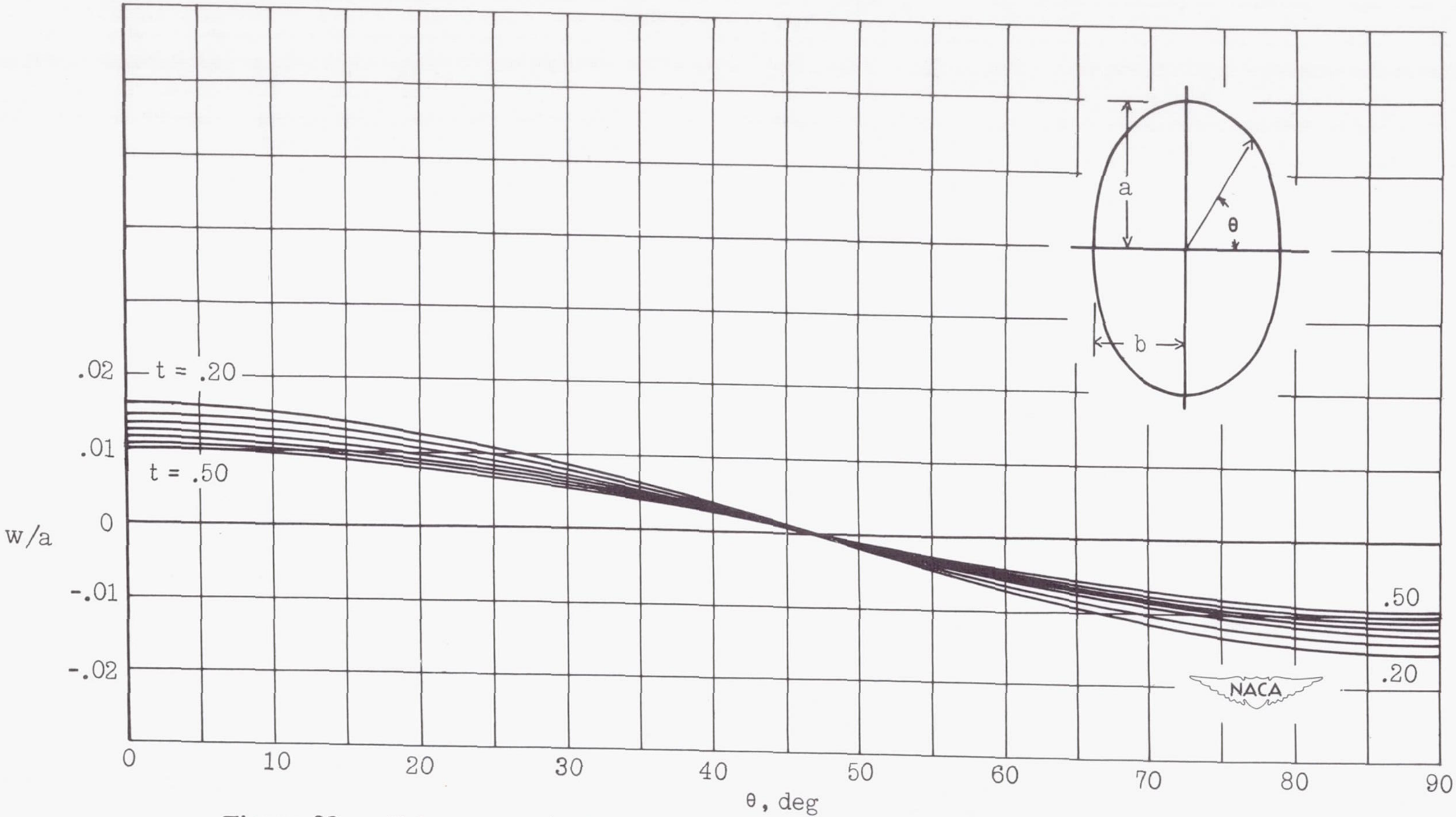


Figure 22.- Values of w/a for various positions around ring from t = 0.20 to t = 0.50.

$$t = \frac{EI}{q}\,\frac{1}{a^3};\; k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.1.$$

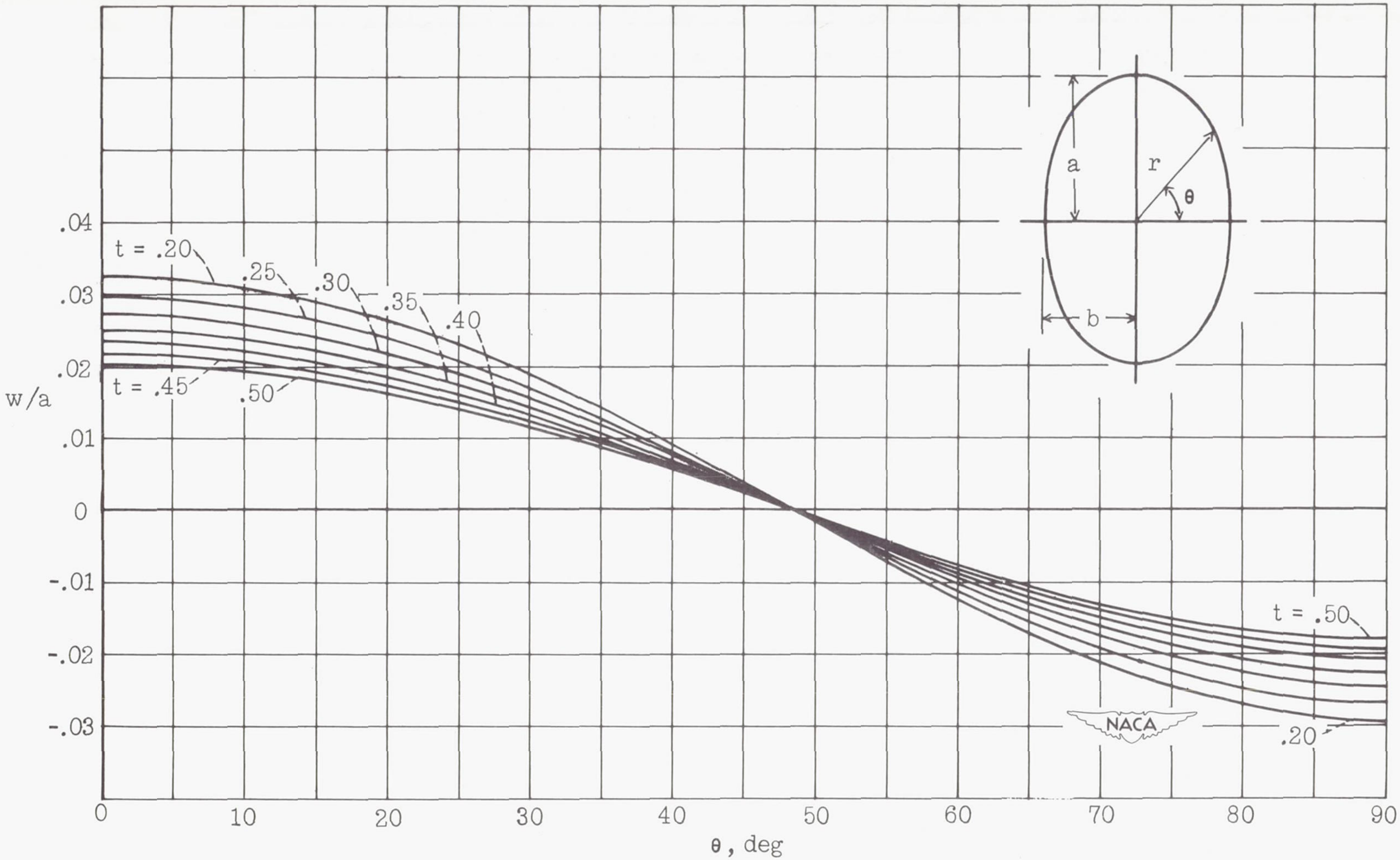


Figure 23.- Values of w/a for various positions around ring from t = 0.20 to t = 0.50.

$$t = \frac{EI}{q}\frac{1}{a^3};\ k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.2.$$

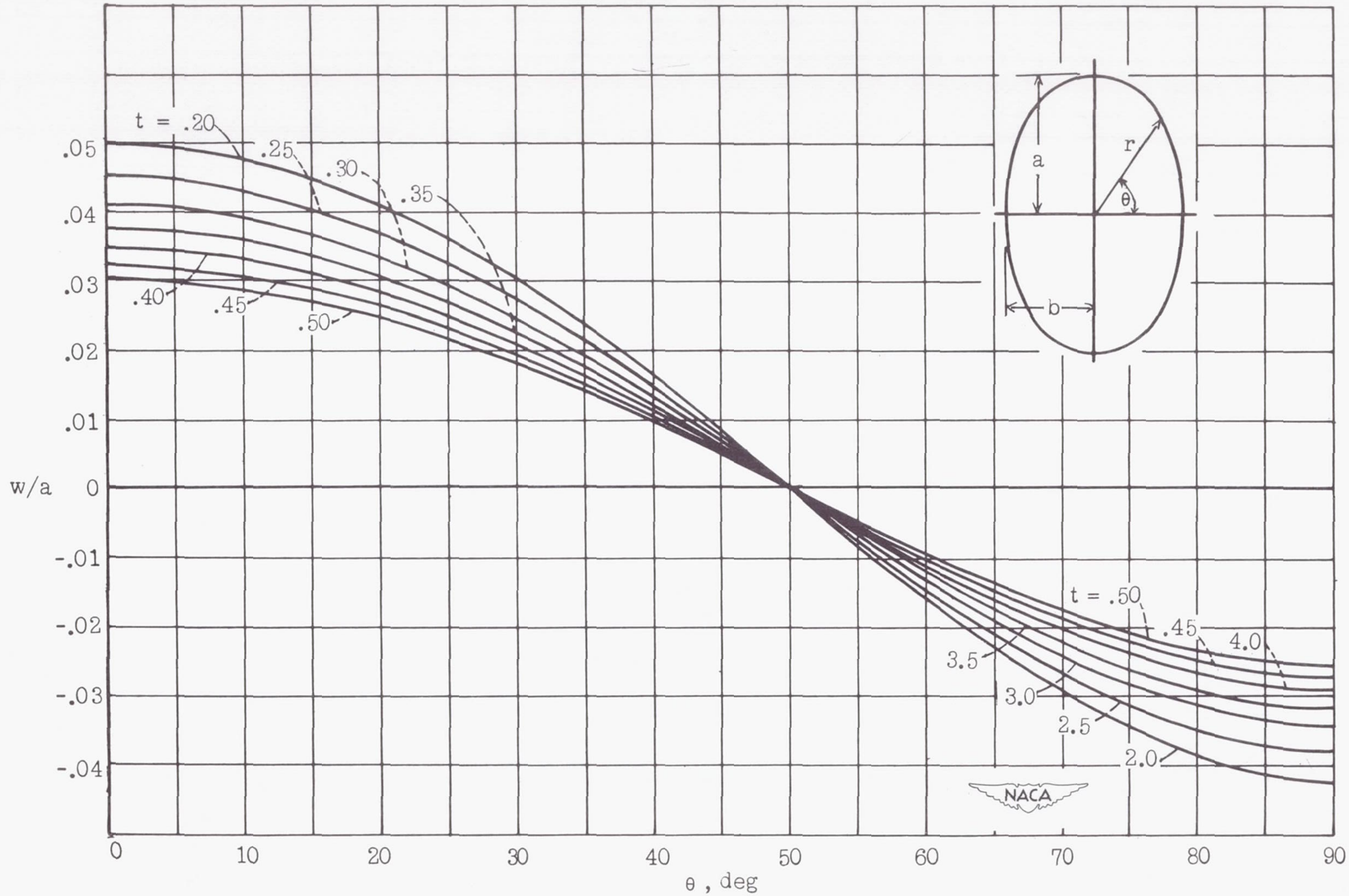


Figure 24.- Values of w/a for various positions around ring from t = 0.20 to t = 0.50.

$t = \frac{EI}{q}\frac{1}{a^3}$; $k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.3.$

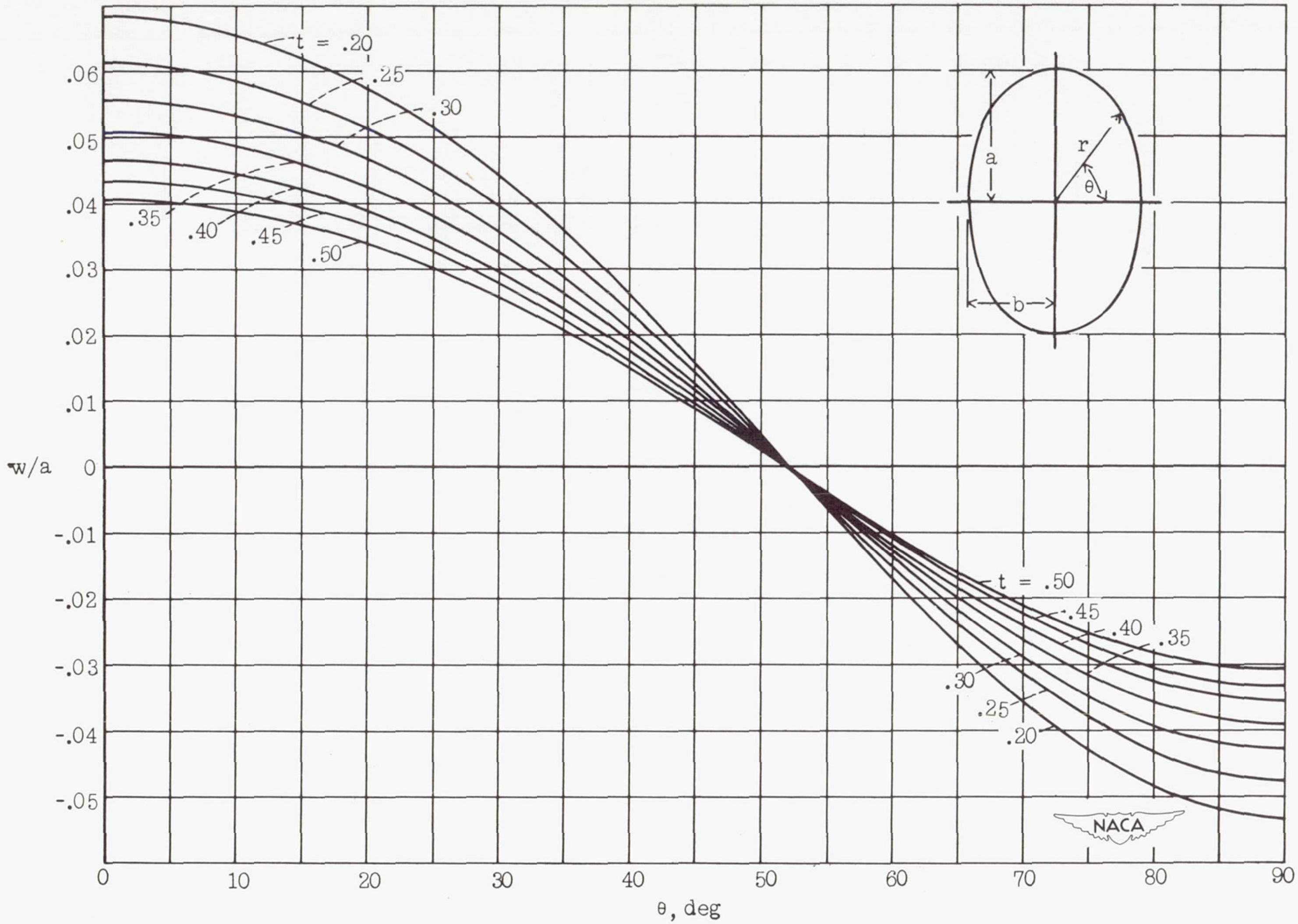


Figure 25.- Values of w/a for various positions around ring from $t = 0.20$ to $t = 0.50$.

$$t = \frac{EI}{q}\frac{1}{a^3};\ k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.4.$$

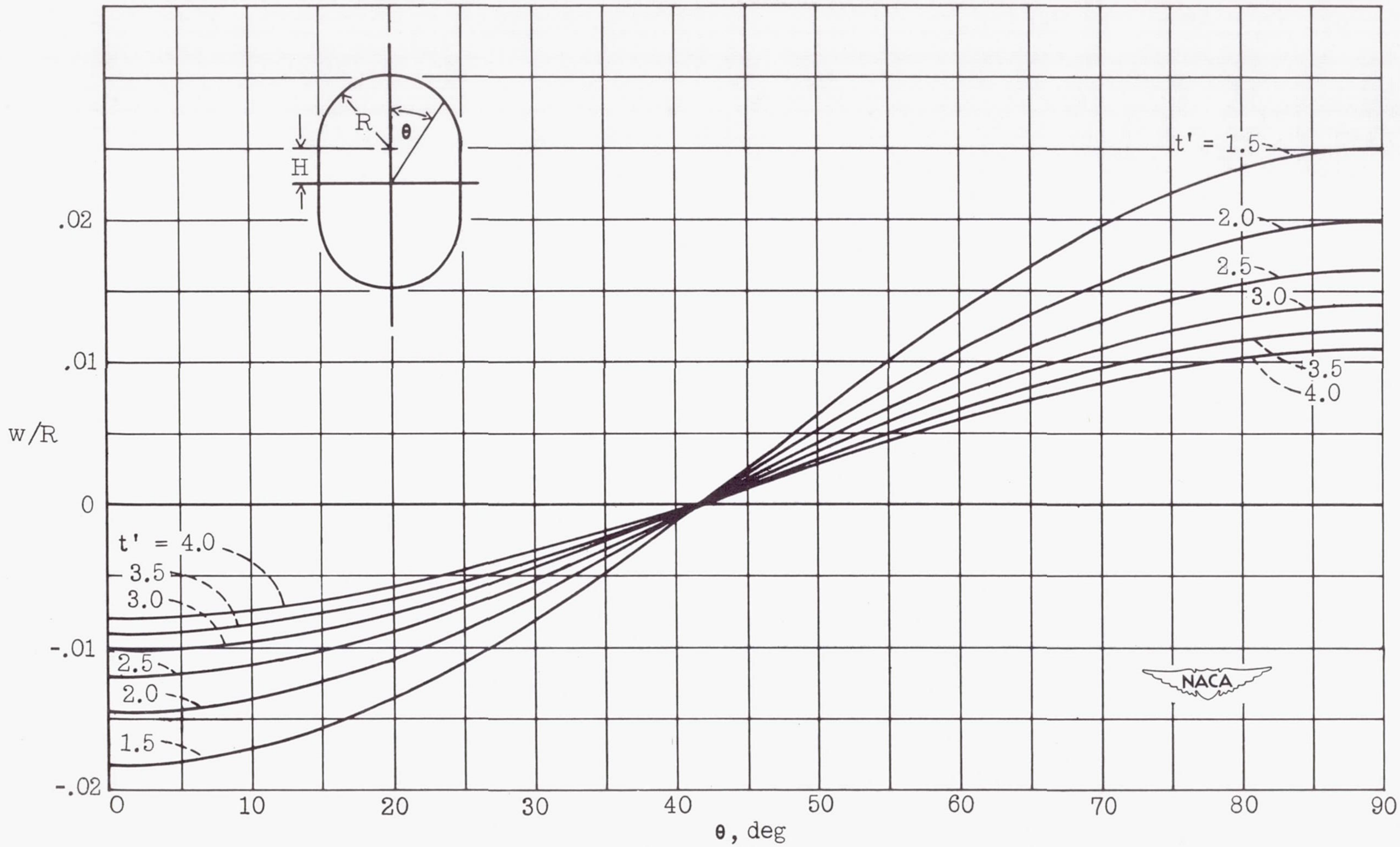


Figure 26.- Values of w/R for various positions around ring from t' = 1.5 to t' = 4.0.

$$t' = \frac{EI}{q}\frac{1}{R^3};\ k' = \frac{H}{R} = 0.2.$$

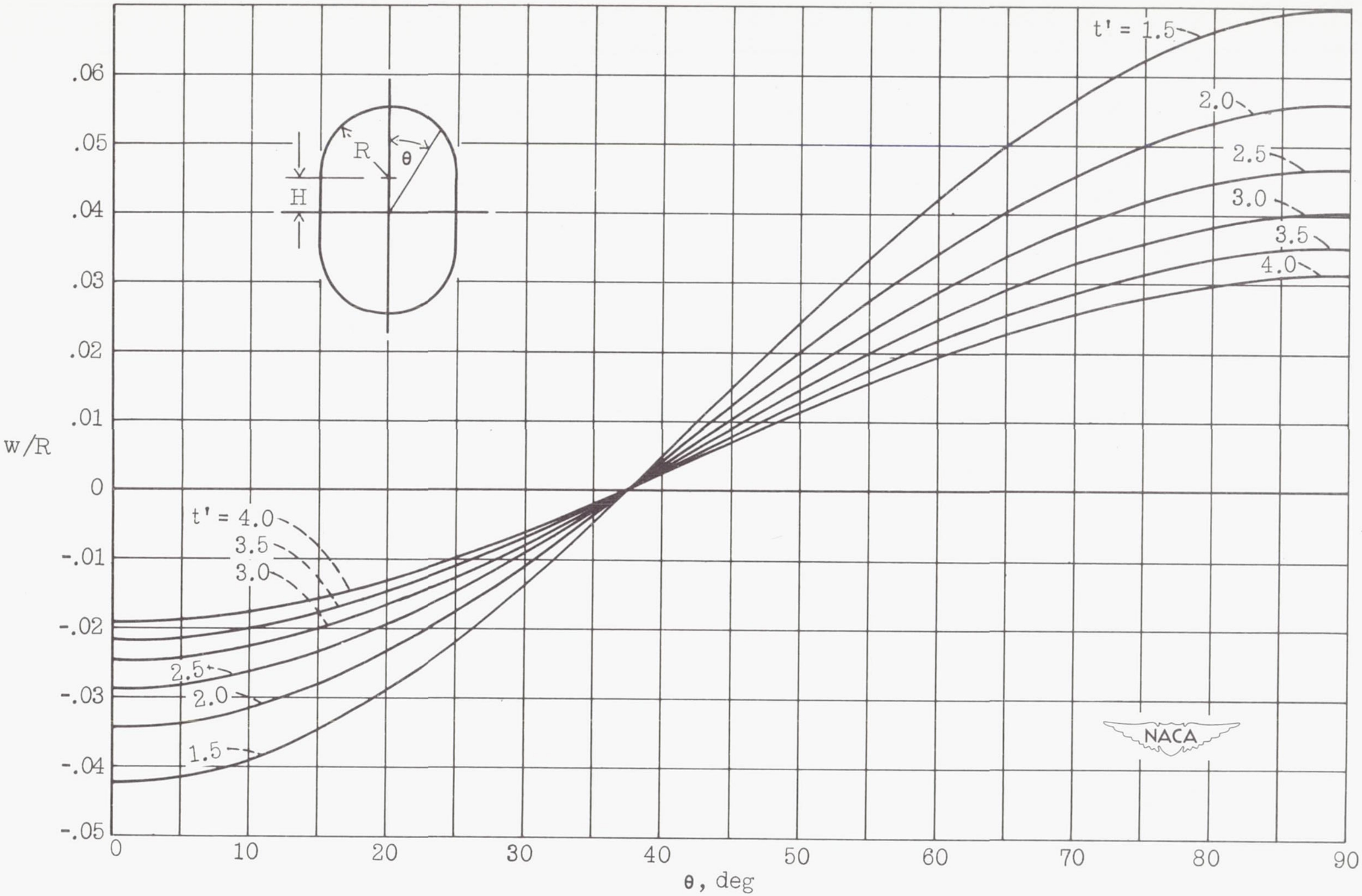


Figure 27.- Values of w/R for various positions around ring from t' = 1.5 to t' = 4.0.

$$t' = \frac{EI}{q}\frac{1}{R^3}; \quad k' = \frac{H}{R} = 0.4.$$

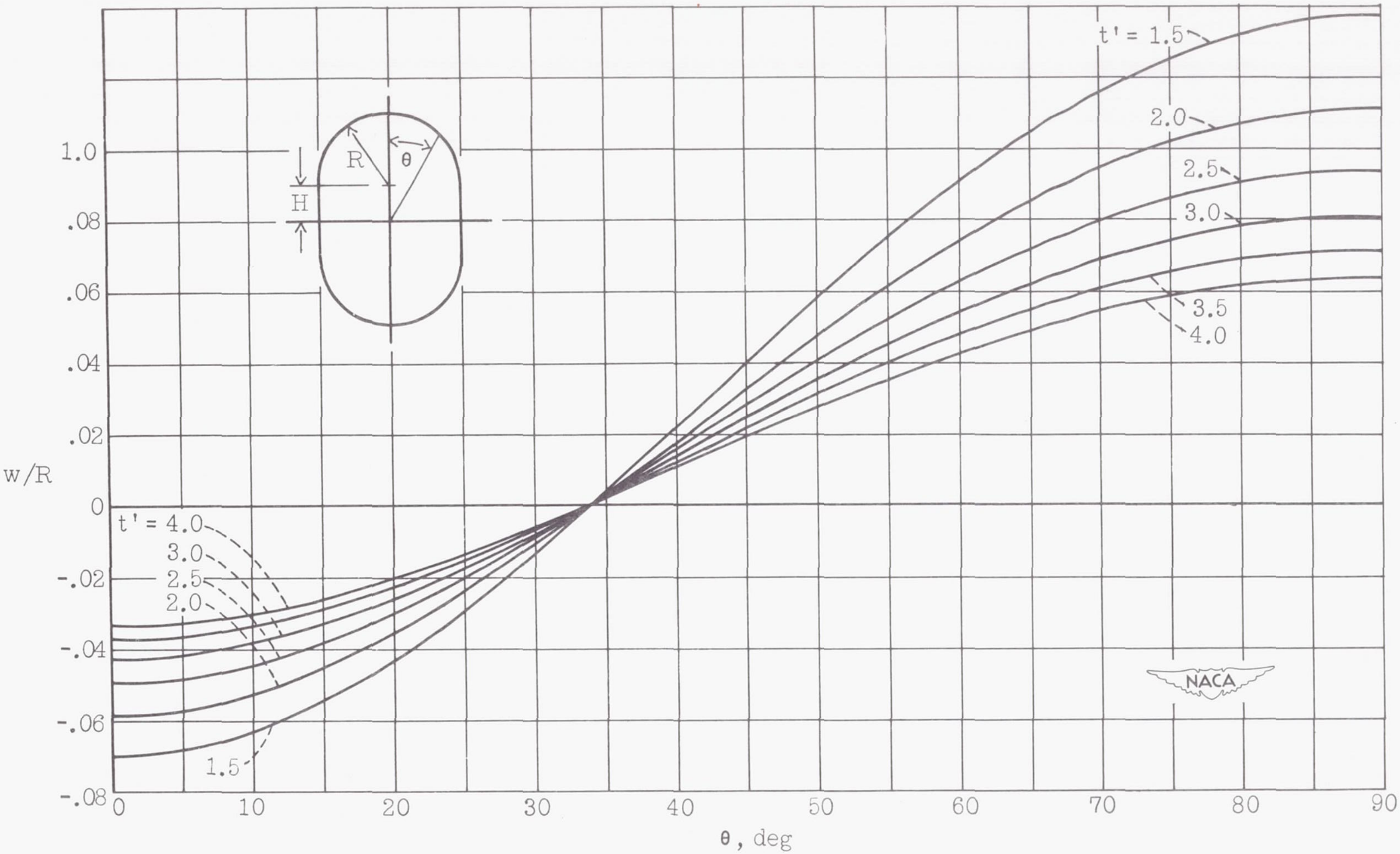


Figure 28.- Values of w/R for various positions around ring from t' = 1.5 to t' = 4.0.

$$t' = \frac{EI}{q}\frac{1}{R^3};\ k' = \frac{H}{R} = 0.6.$$

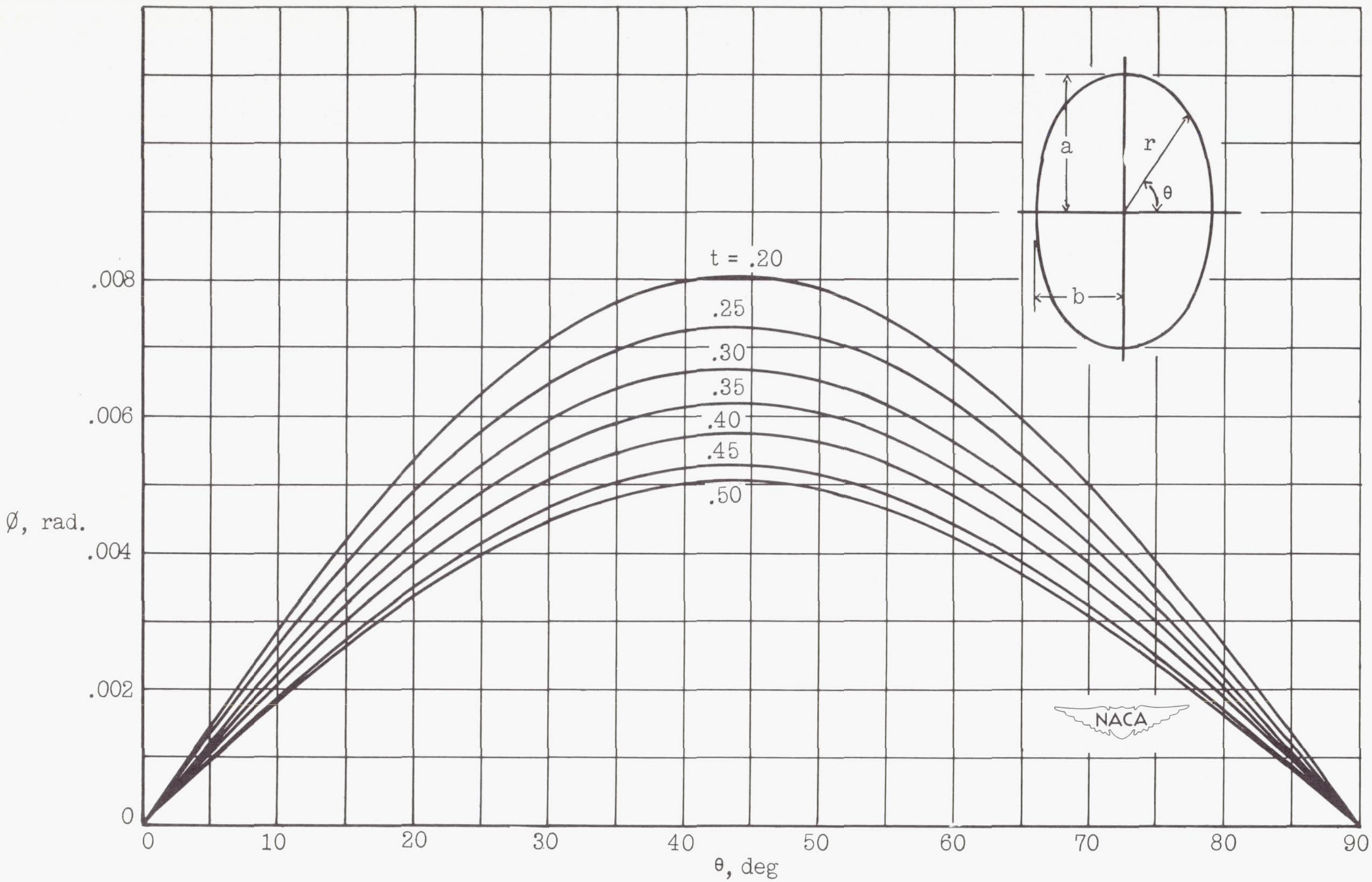


Figure 29.- Values of Ø for various positions around ring from t = 0.20 to t = 0.50.

$$t = \frac{EI}{q}\frac{1}{a^3};\ k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.1.$$

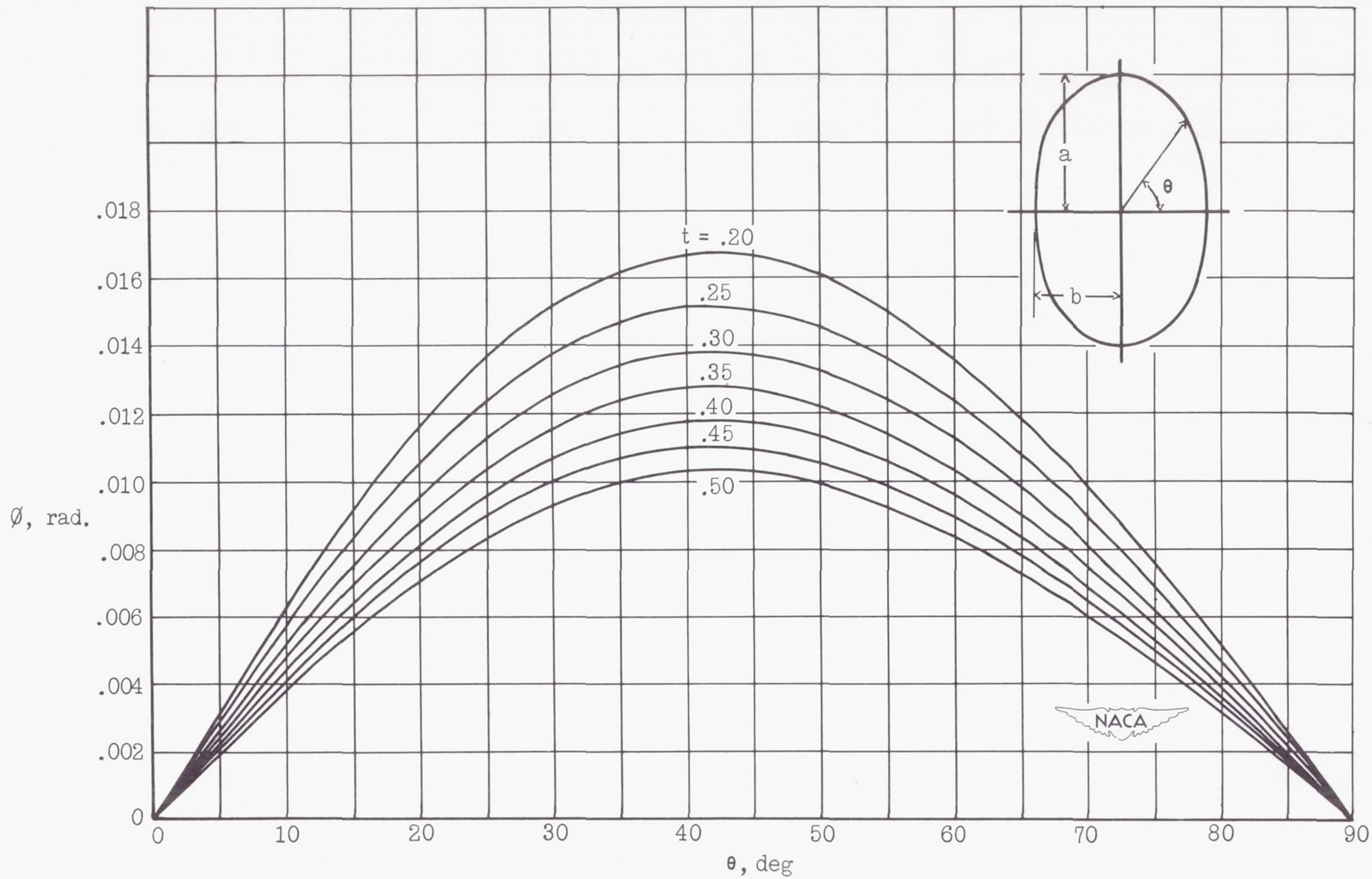


Figure 30.- Values of Ø for various positions around ring from t = 0.20 to t = 0.50.

$$t = \frac{EI}{q}\frac{1}{a^3};\ k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.2.$$

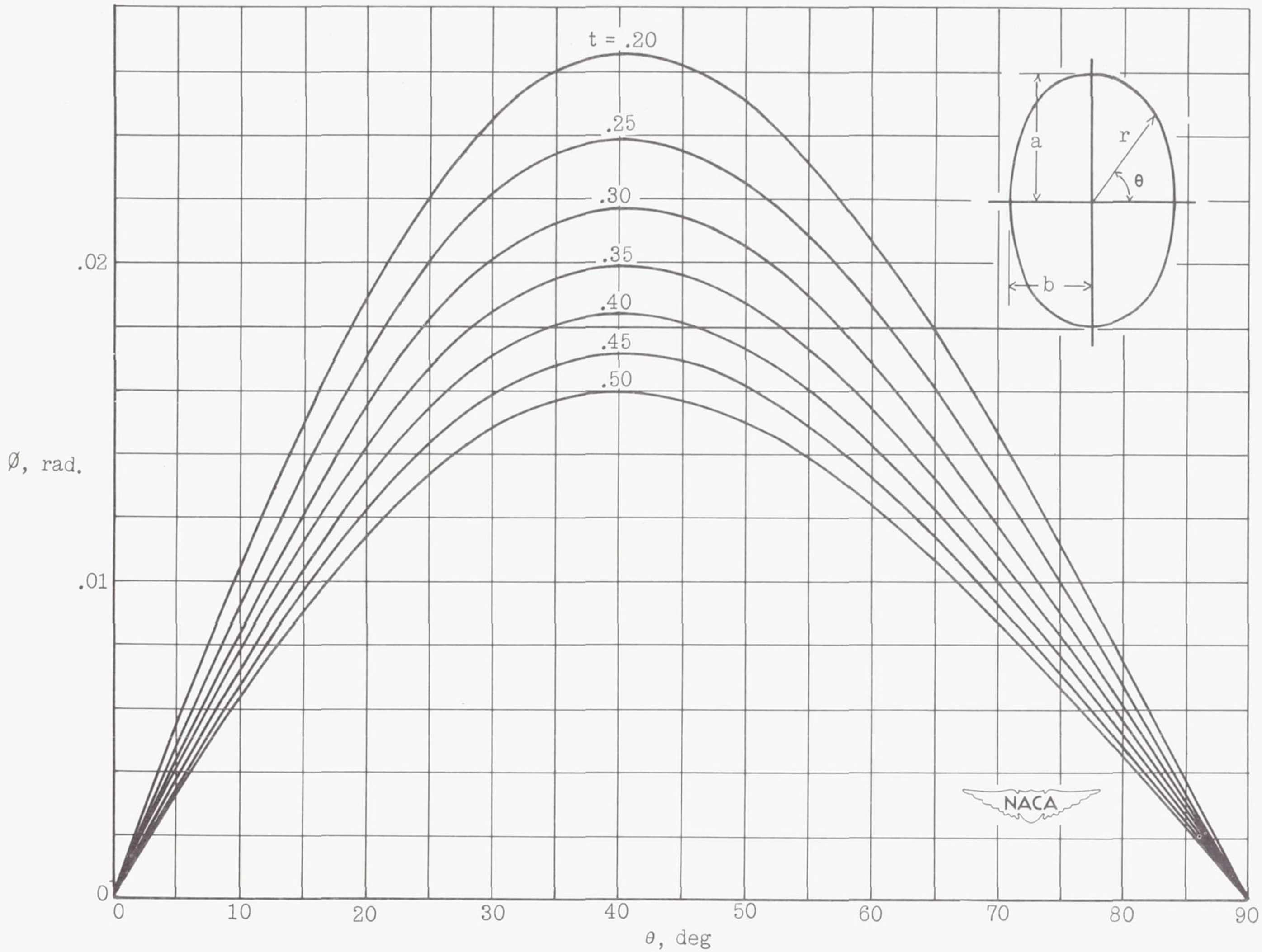


Figure 31.- Values of Ø for various positions around ring from t = 0.20 to t = 0.50.

$$t = \frac{EI}{q}\frac{1}{a^3};\ k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.3.$$

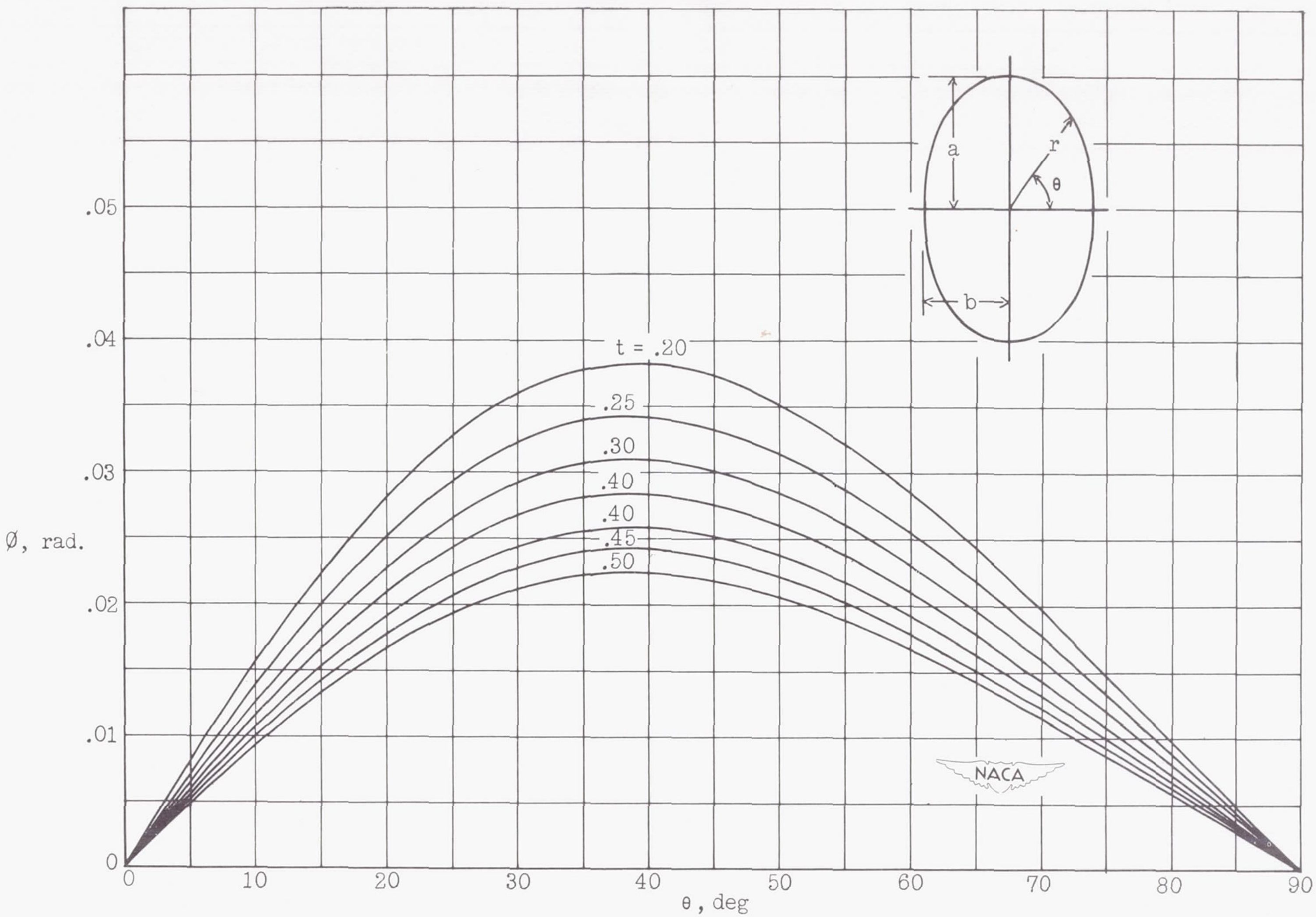


Figure 32.- Values of Ø for various positions around ring from t = 0.20 to t = 0.50.

$$t = \frac{EI}{q}\,\frac{1}{a^3};\; k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.4.$$

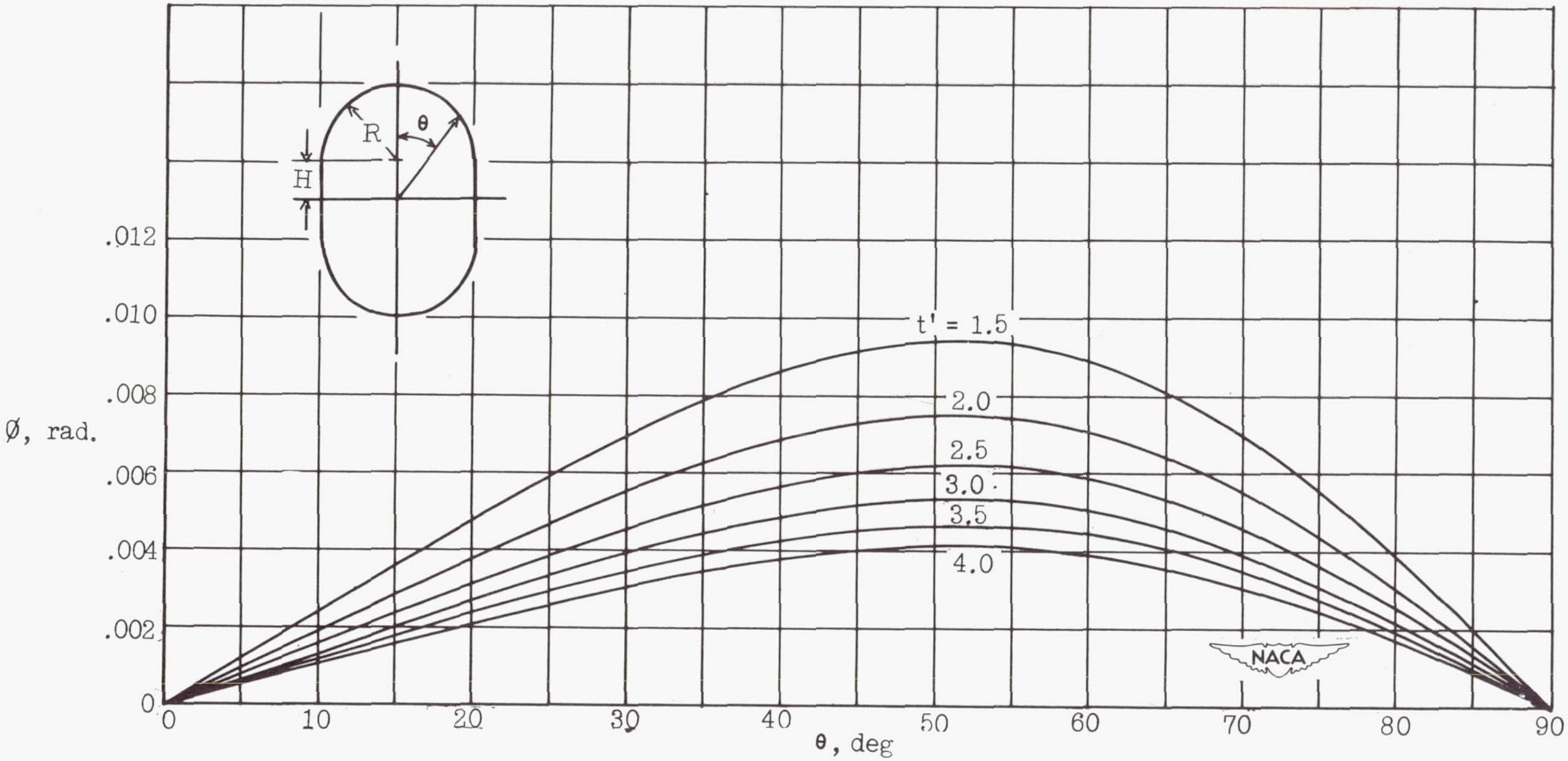


Figure 33.- Values of ϕ for various positions around ring from $t' = 1.5$ to $t' = 4.0$.

$$t' = \frac{EI}{q}\,\frac{1}{R^3};\; k' = \frac{H}{R} = 0.2.$$

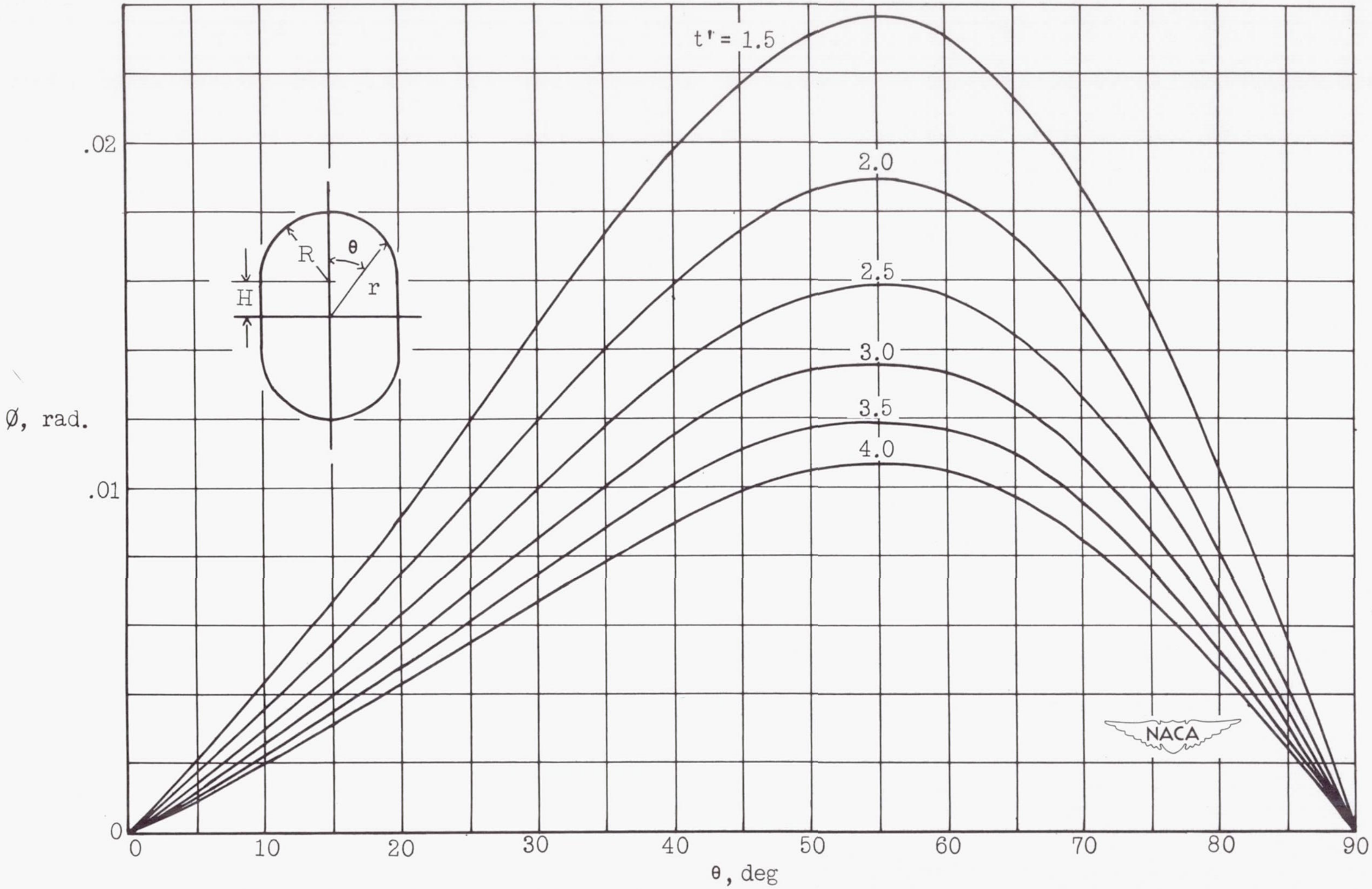


Figure 34.- Values of Ø for various positions around ring from t' = 1.5 to t' = 4.0.

$$t' = \frac{EI}{q}\,\frac{1}{R^3};\quad k' = \frac{H}{R} = 0.4.$$

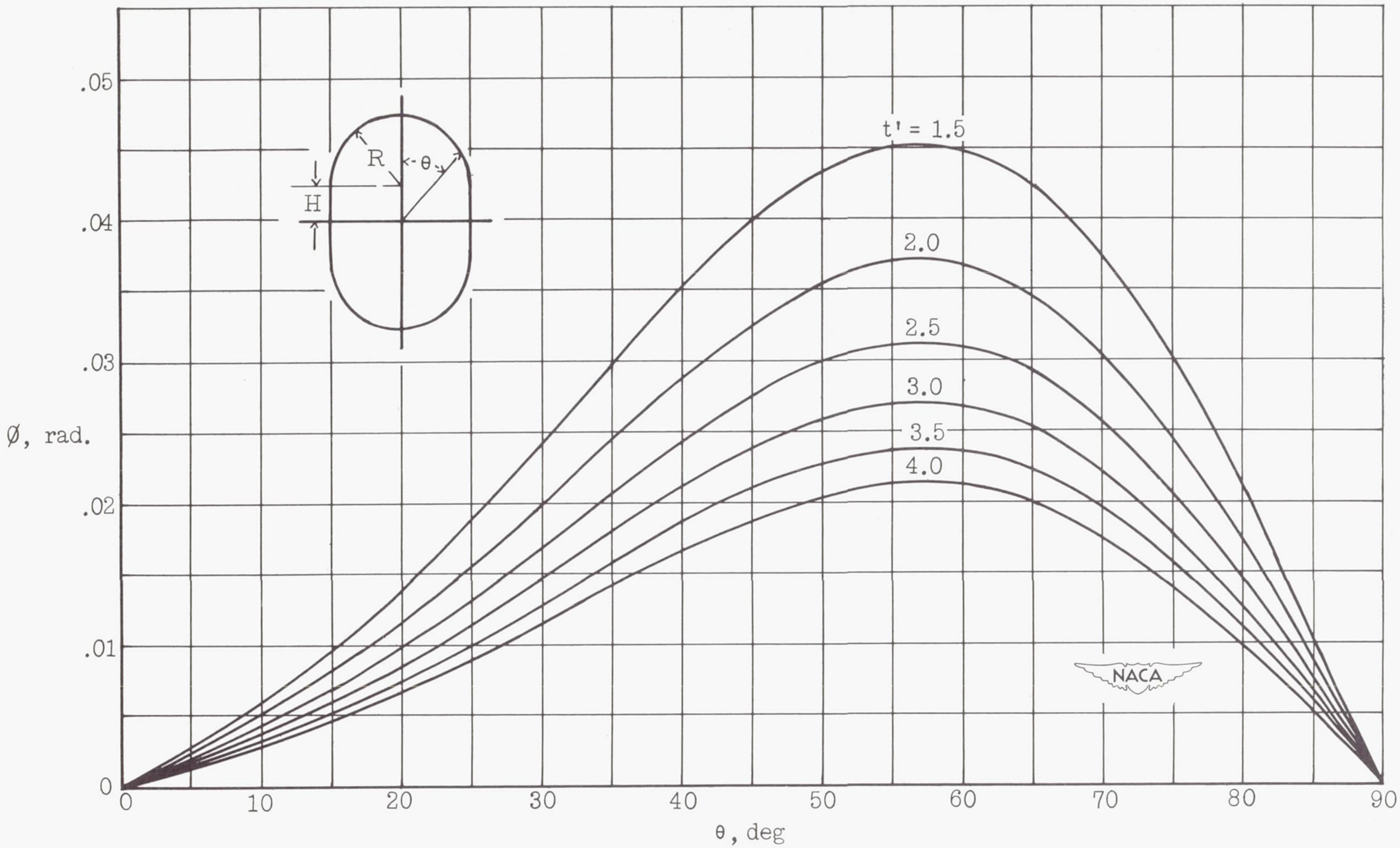


Figure 35.- Values of Ø for various positions around ring from t' = 1.5 to t' = 4.0.

$t' = \frac{EI}{q} \frac{1}{R^3}$; $k' = \frac{H}{R} = 0.6$.

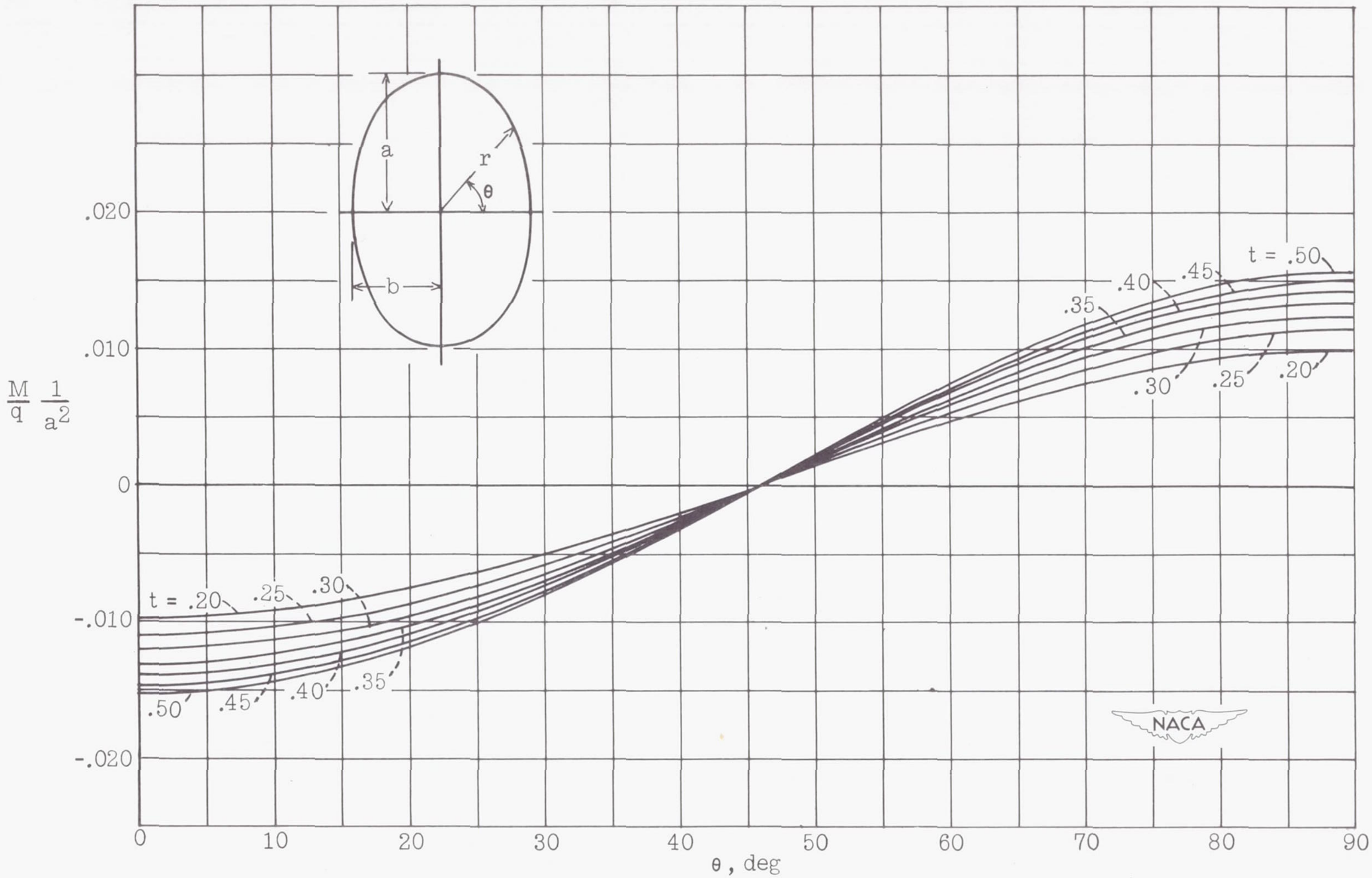


Figure 36.- Values of $\frac{M}{q}\frac{1}{a^2}$ for various positions around ring from t = 0.20 to t = 0.50.

$t = \frac{EI}{q}\frac{1}{a^3}$; $k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.1$.

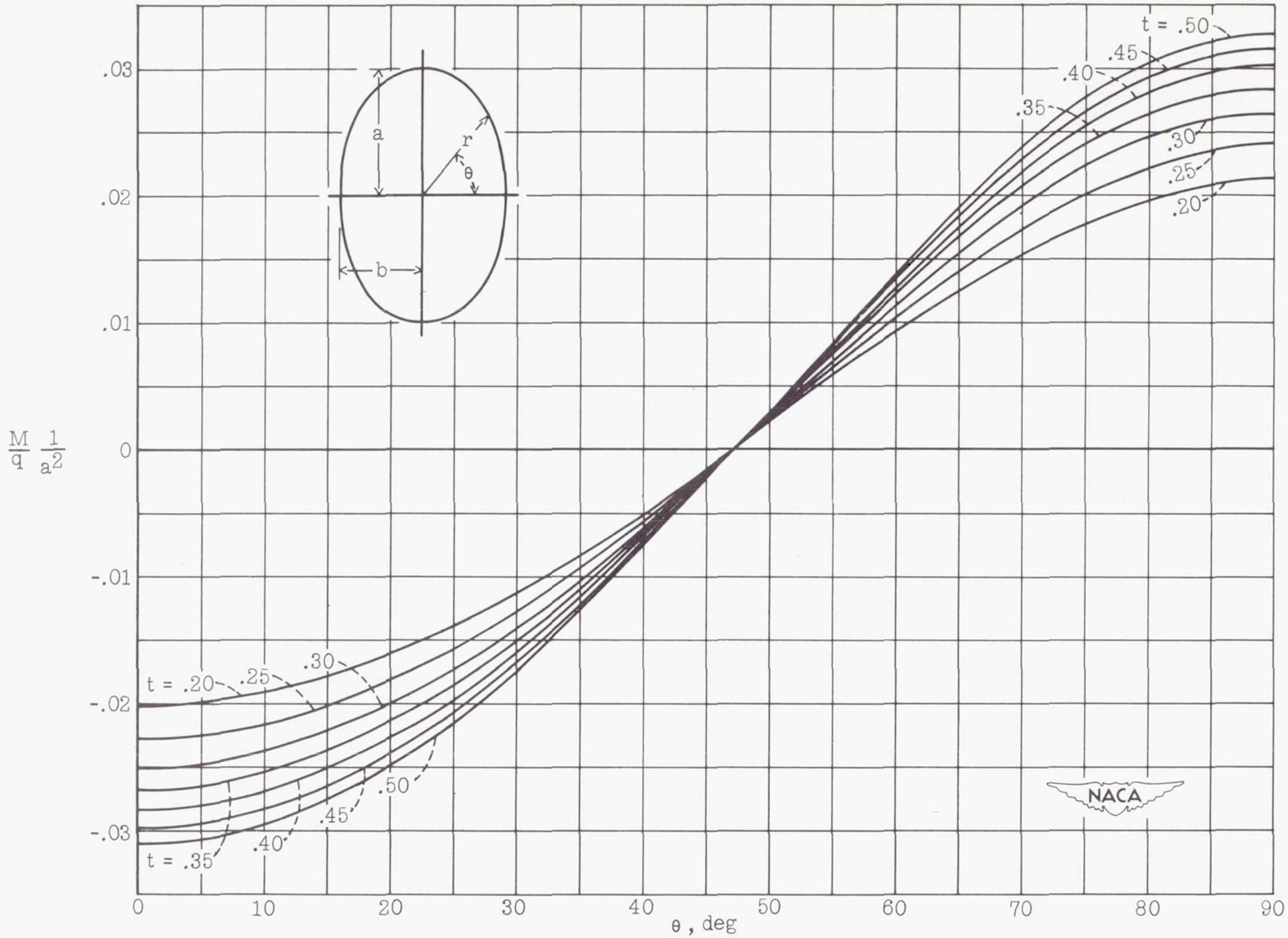


Figure 37.- Values of $\frac{M}{q}\frac{1}{a^2}$ for various positions around ring from t = 0.20 to 0.50.

$$t = \frac{EI}{q}\frac{1}{a^3};\ k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.2.$$

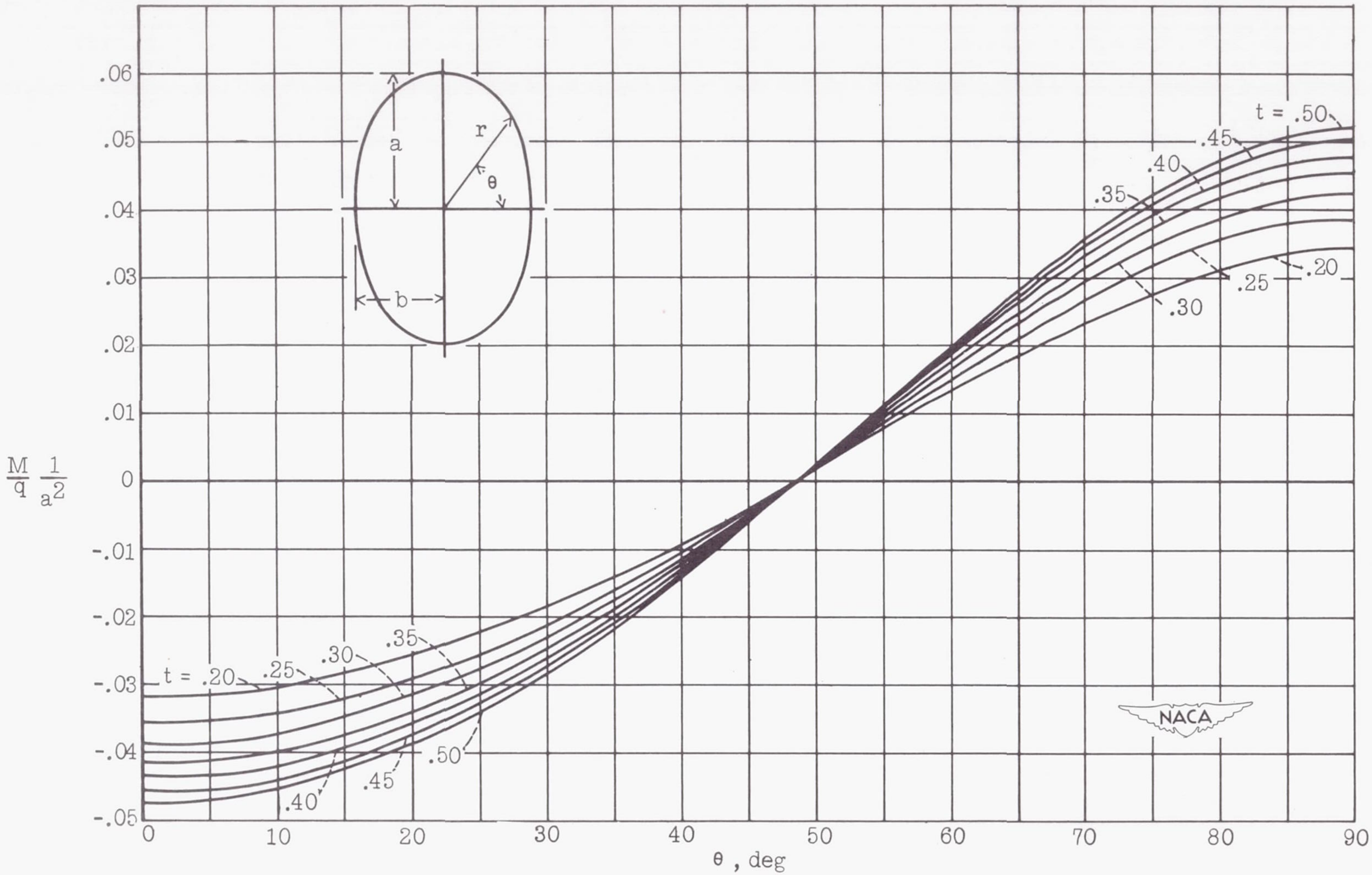


Figure 38.- Values of $\frac{M}{q}\frac{1}{a^2}$ for various positions around ring from $t = 0.20$ to $t = 0.50$.

$$t = \frac{EI}{q}\frac{1}{a^3};\ k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.3.$$

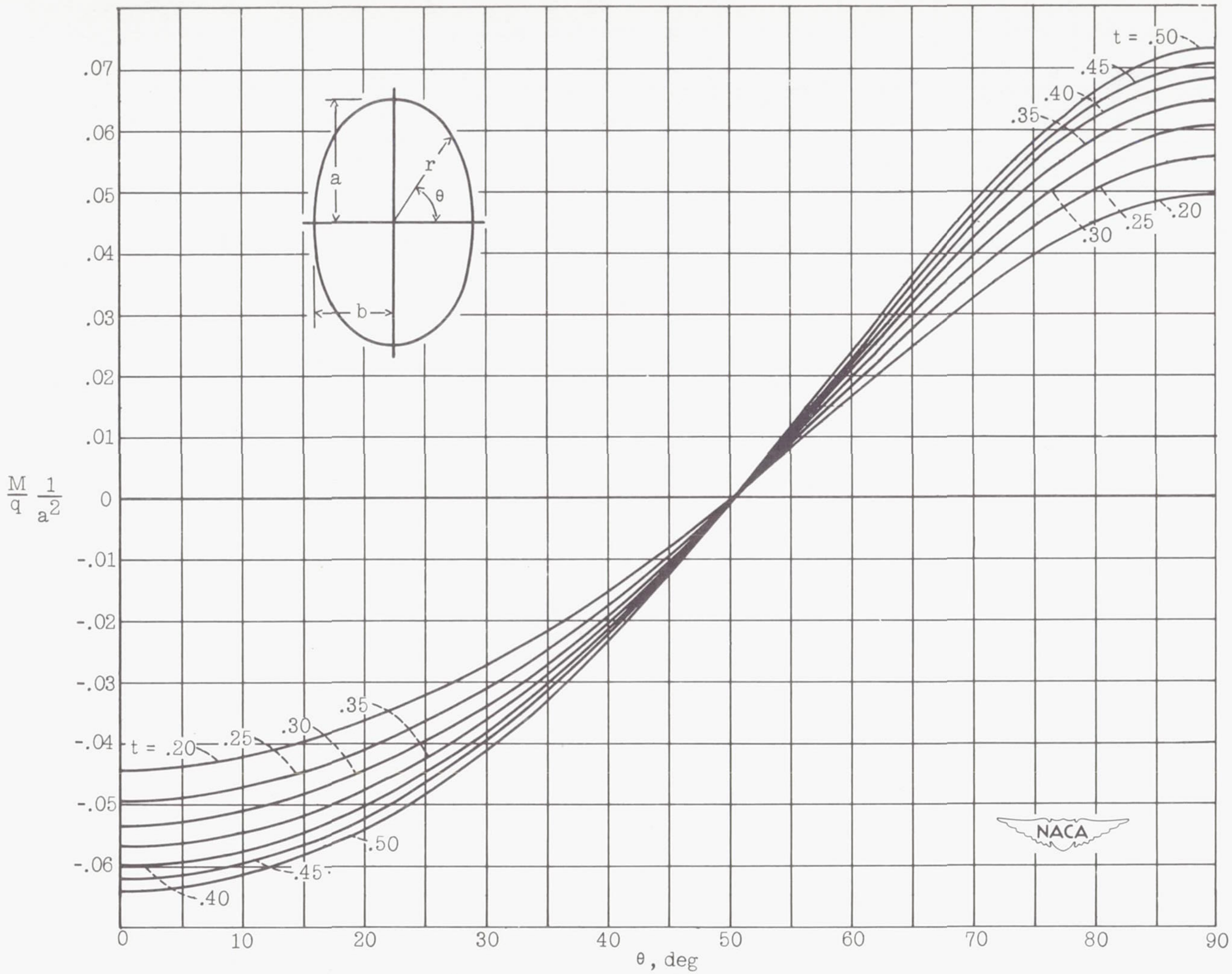


Figure 39.- Values of $\frac{M}{q}\frac{1}{a^2}$ for various positions around ring from t = 0.20 to t = 0.50.

$t = \frac{EI}{q}\frac{1}{a^3}$; $k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.4$.

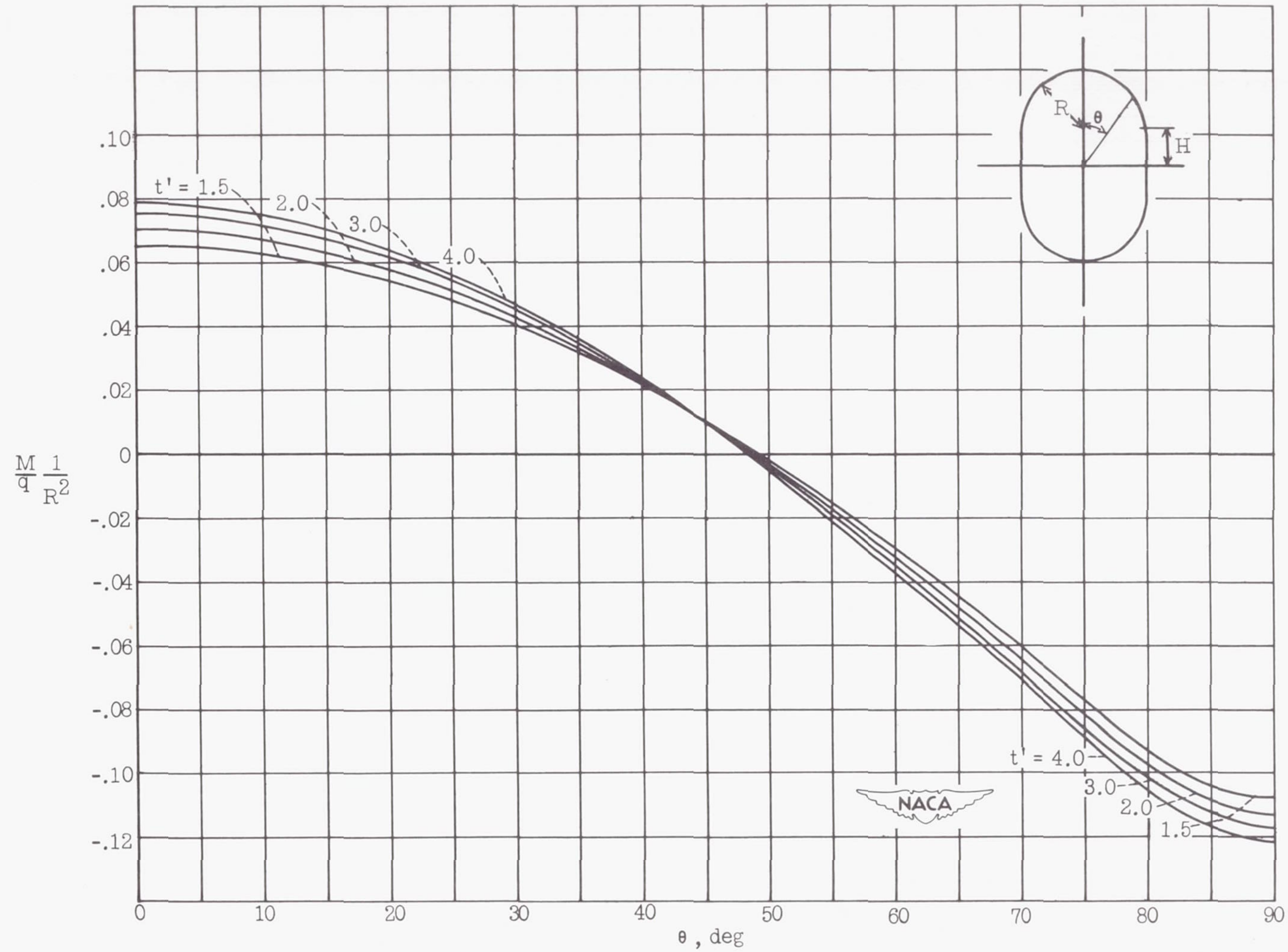


Figure 40.- Values of $\frac{M}{q}\frac{1}{R^2}$ for various positions around ring from $t' = 1.5$ to $t' = 4.0$.

$t' = \frac{EI}{q}\frac{1}{R^3}$; $k' = \frac{H}{R} = 0.2$.

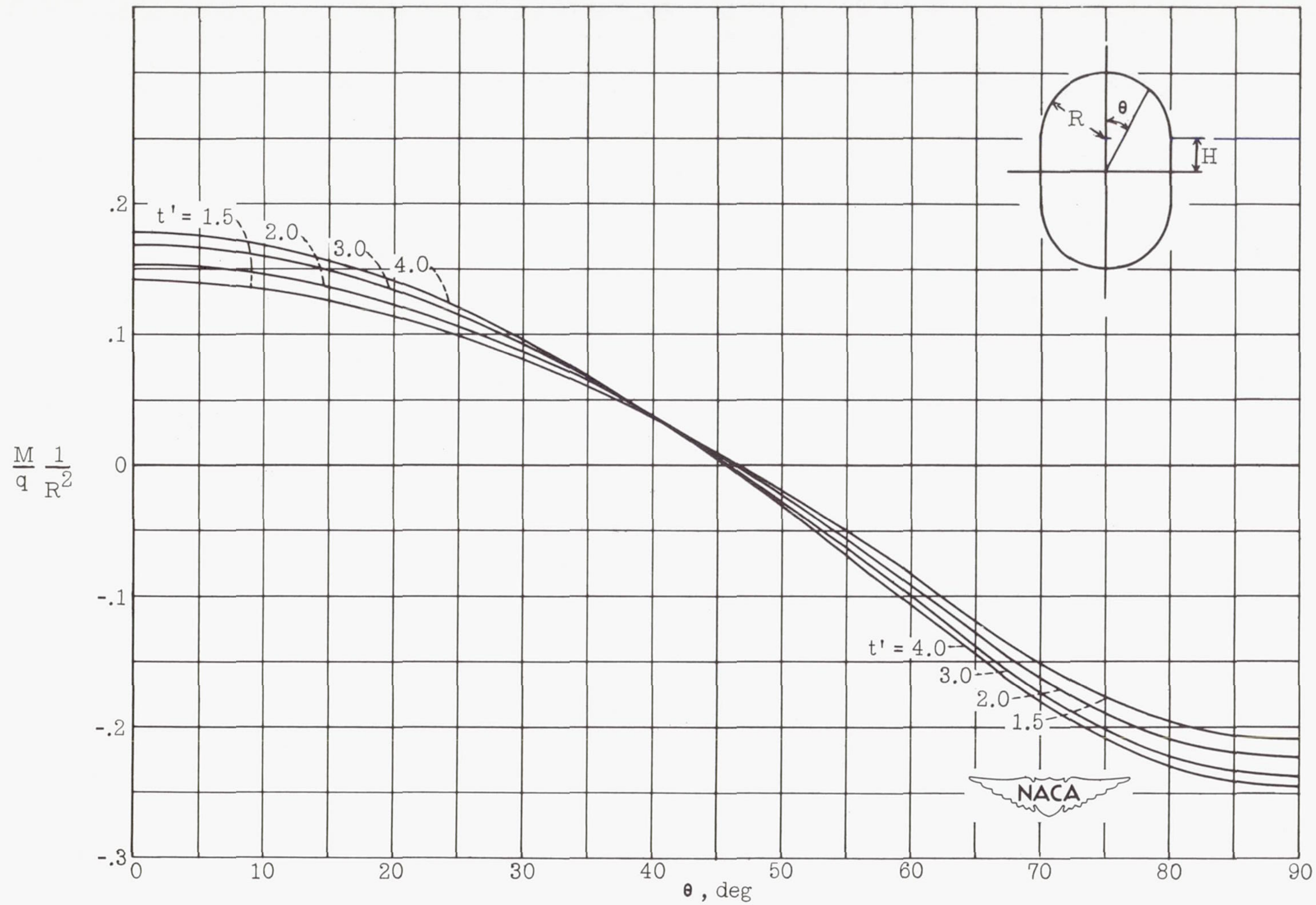


Figure 41.- Values of $\frac{M}{q}\frac{1}{R^2}$ for various positions around ring from t' = 1.5 to t' = 4.0.

$t' = \frac{EI}{q}\frac{1}{R^3}$; $k' = \frac{H}{R} = 0.4$.

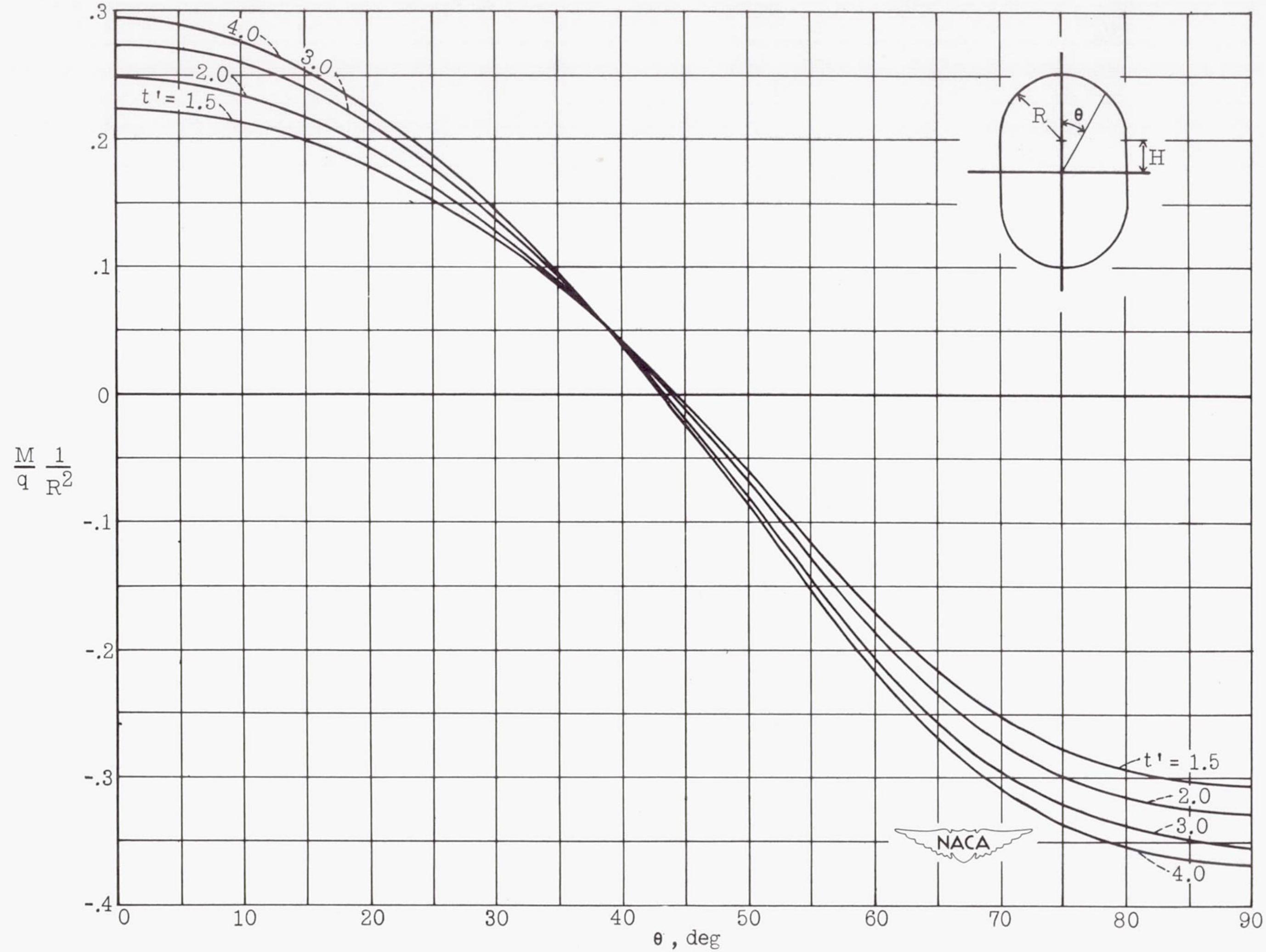


Figure 42.- Values of $\frac{M}{q} \frac{1}{R^2}$ for various positions around ring from t' = 1.5 to t' = 4.0.

$t' = \frac{EI}{q} \frac{1}{R^3}$; $k' = \frac{H}{R} = 0.6$.

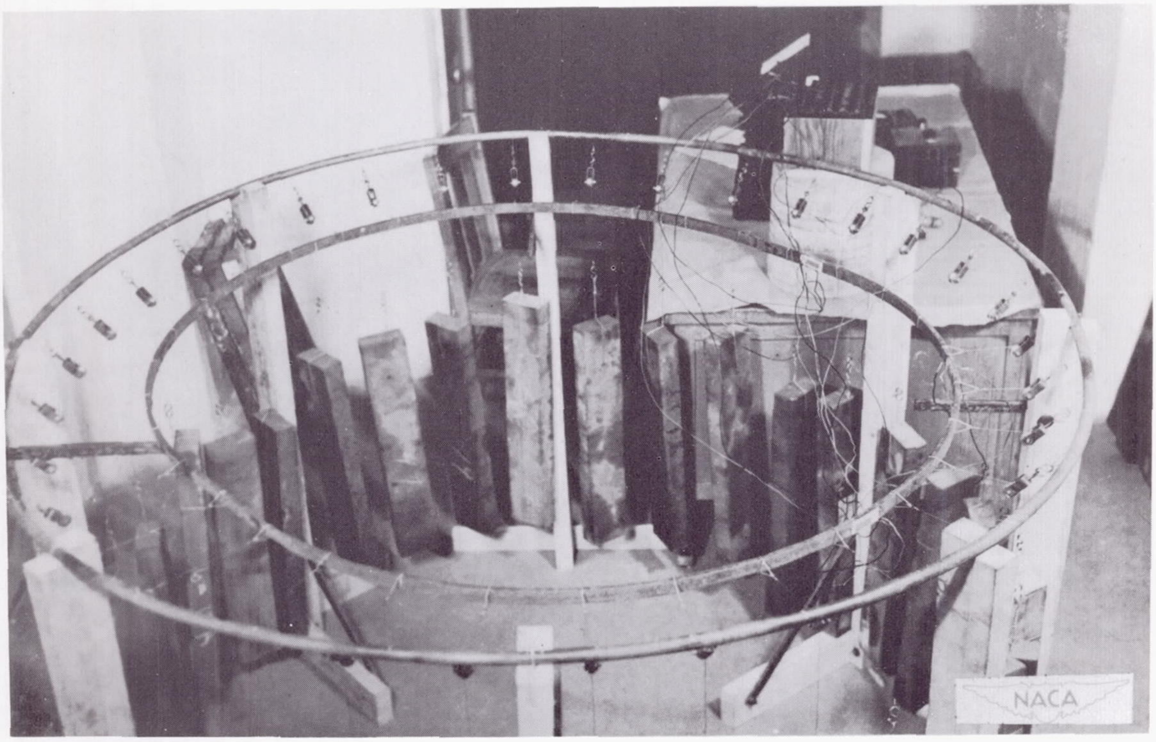


Figure 43.- Top view of test setup.

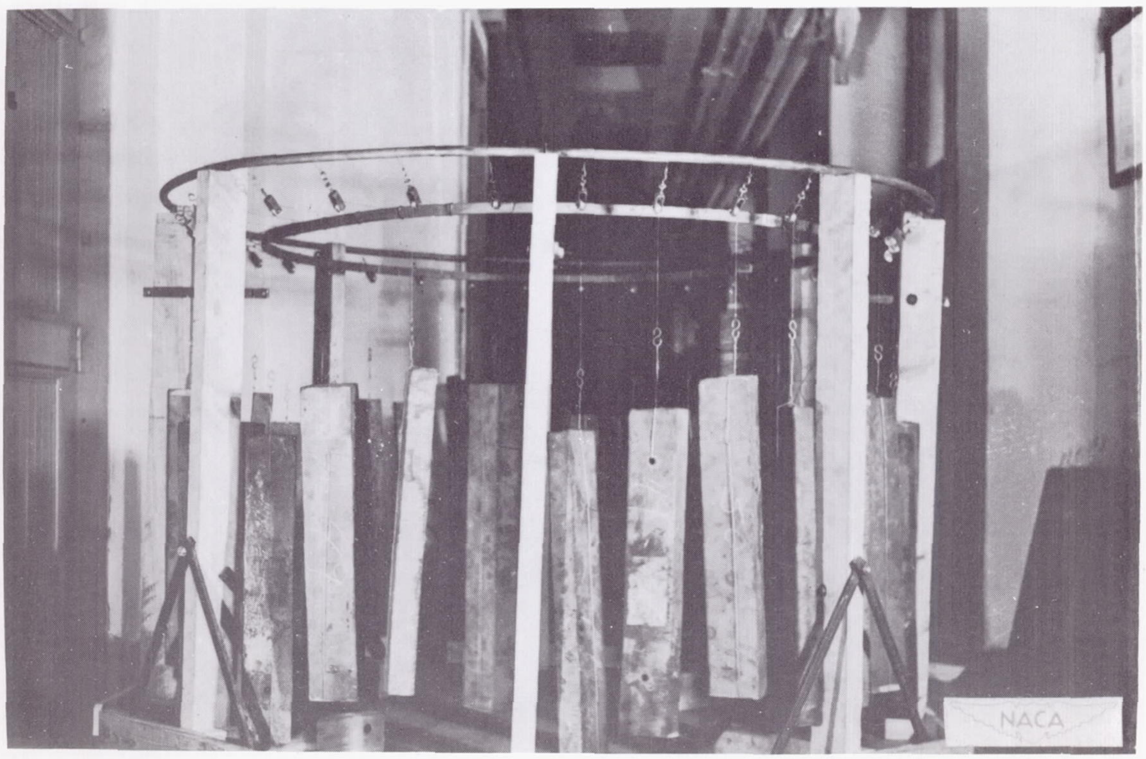


Figure 44.- General view of test setup.

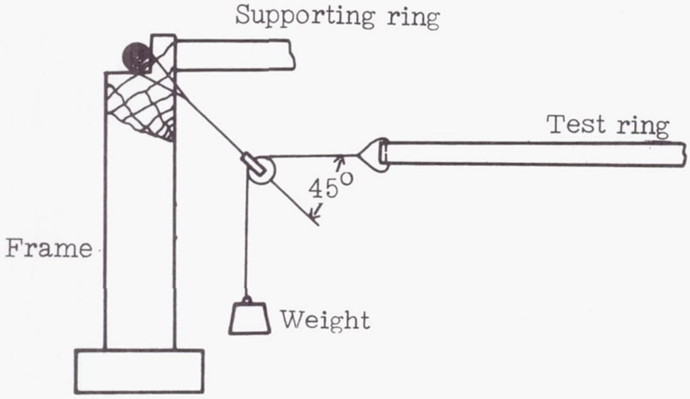
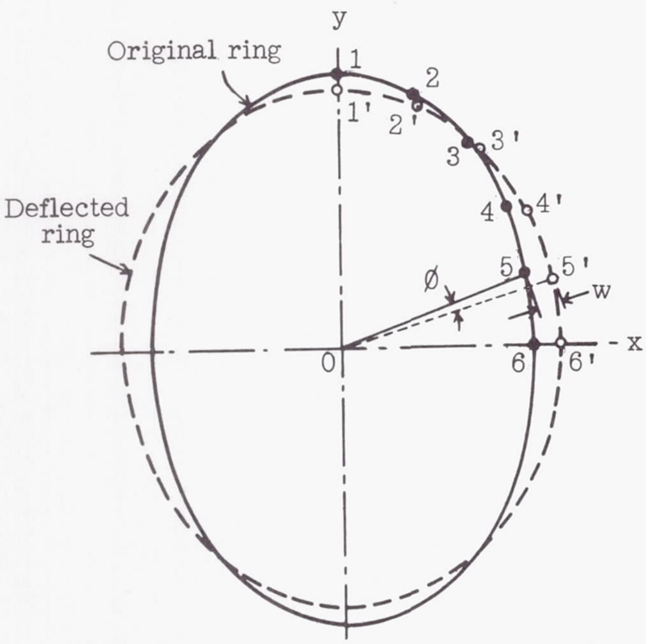


Figure 45.- Schematic drawing of test setup.



1,2,3,4... Division points before loading
1',2',3',4'...Division points after loading

Figure 46.- Wooden board to register deflections.

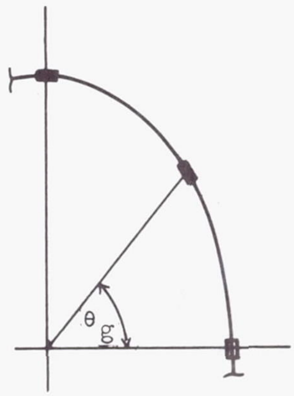


Figure 47.- Strain-gage locations.

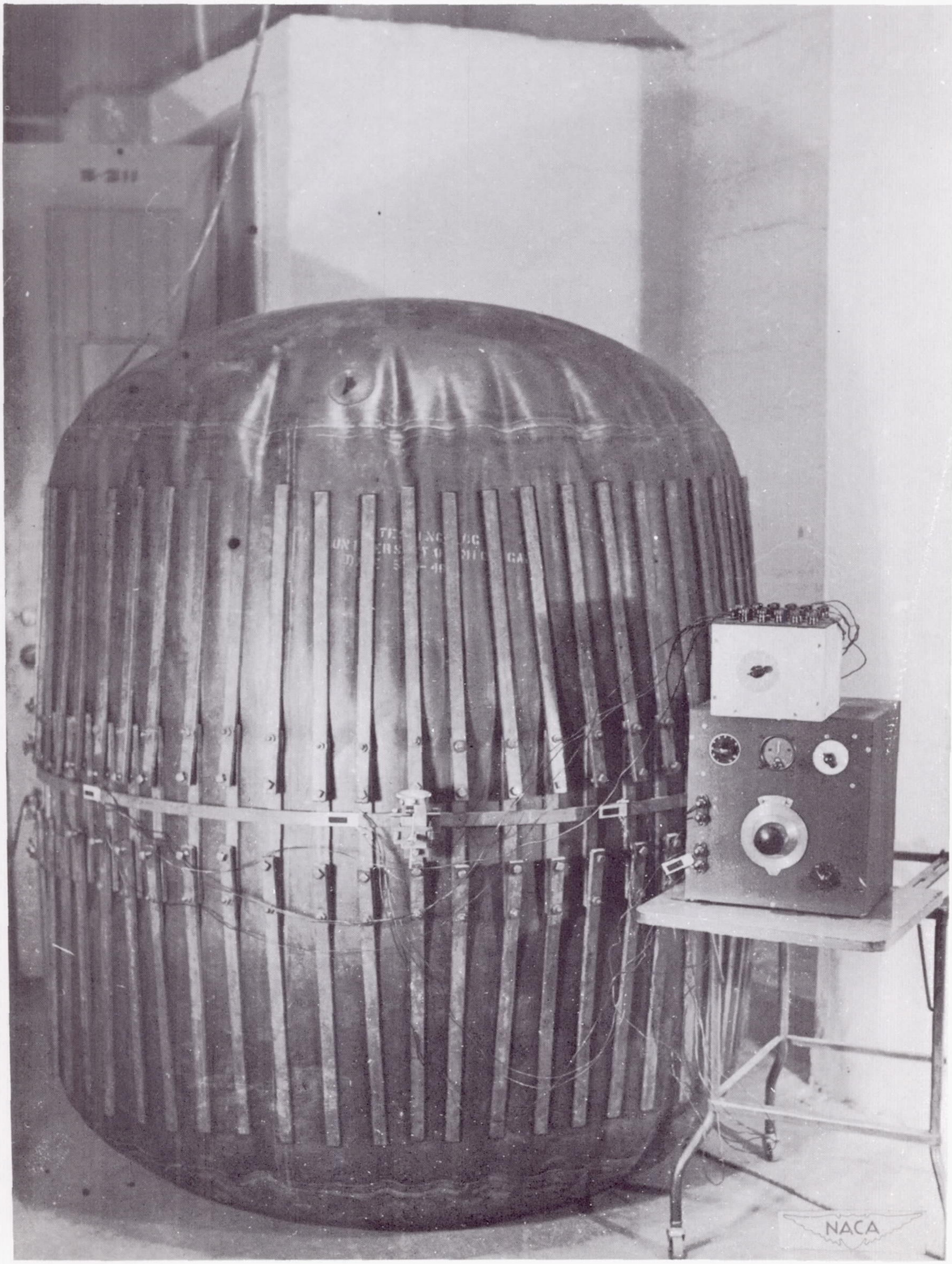


Figure 48.- Pressure bag used to load ring. (Uniform load could not be obtained.)

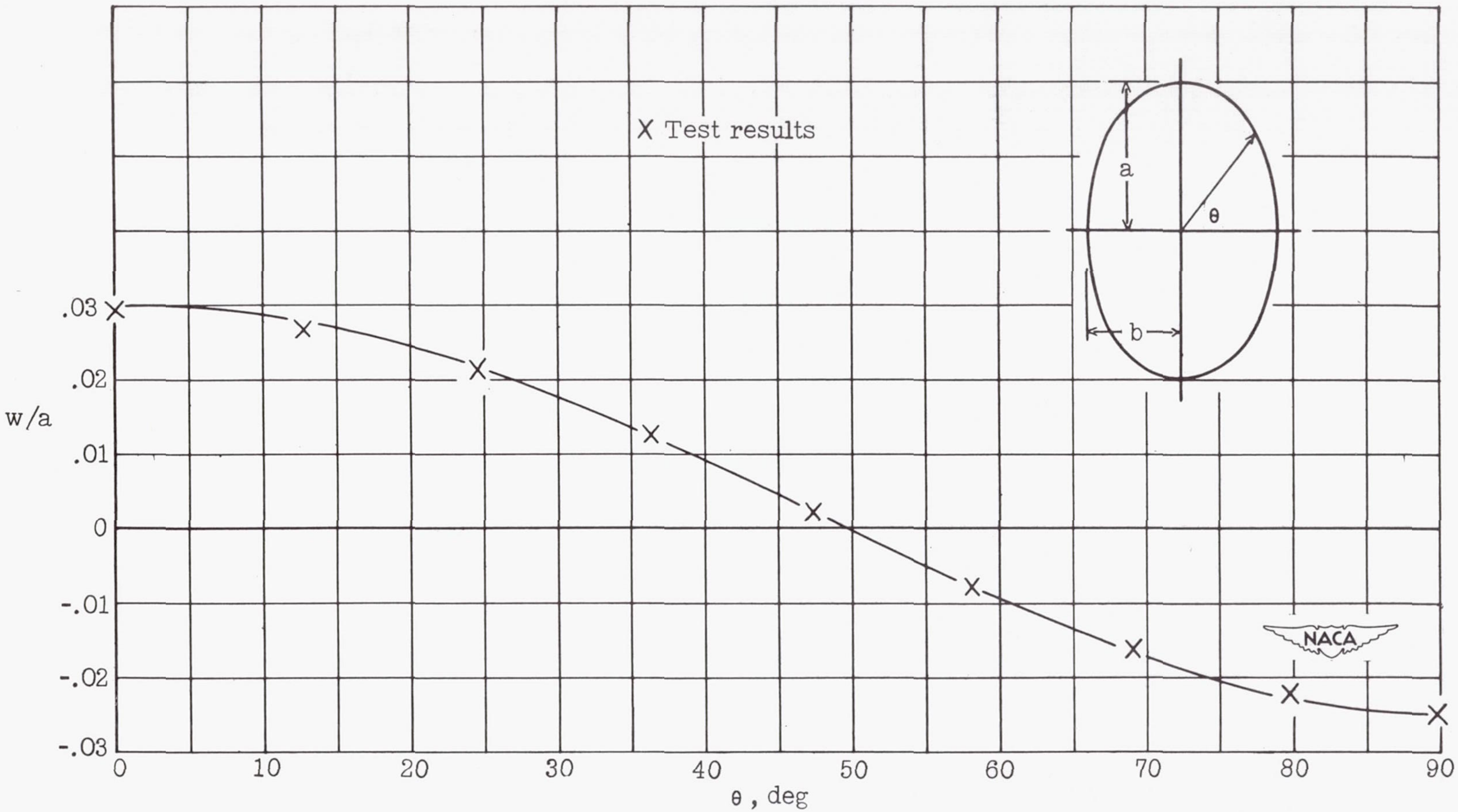


Figure 49.- Comparison of experimental values of w/a and values of w/a computed by present method. a = 27.74 in.; b = 23.24 in.; EI = 105,410; q = 10.12 lb/in.;

$k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.296; \quad t = \frac{EI}{q}\frac{1}{a^3} = 0.488.$

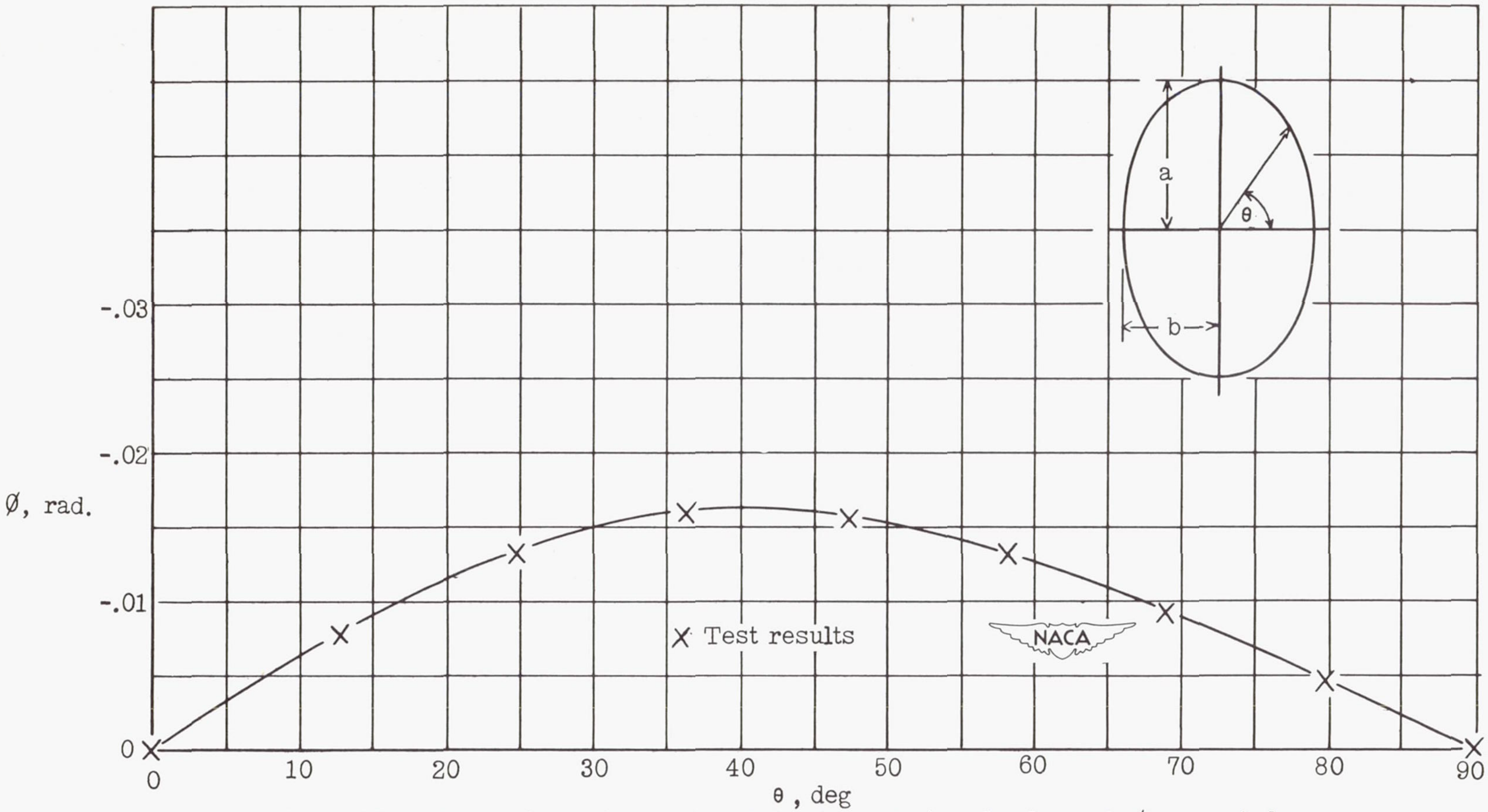


Figure 50.- Comparison of experimental values of ϕ and values of ϕ computed by the present method. a = 27.74 in.; b = 23.24 in.; EI = 105,410; q = 10.12 lb/in.;

$$k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.296; \quad t = \frac{EI}{q}\,\frac{1}{a^3} = 0.488.$$

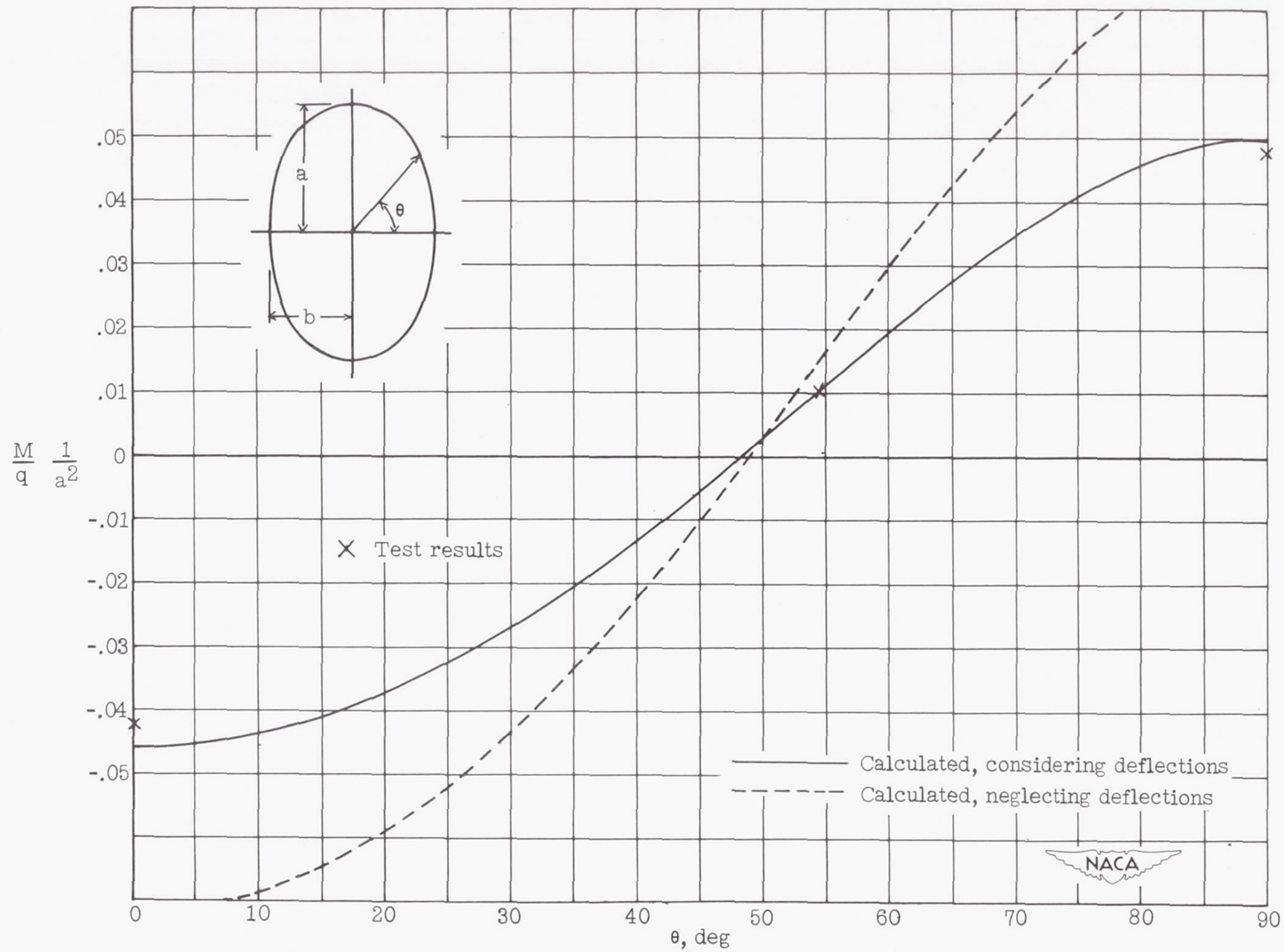


Figure 51.- Comparison of experimental values of $\frac{M}{q}\frac{1}{a^2}$ and values of $\frac{M}{q}\frac{1}{a^2}$ computed by present method. $a = 27.74$ in.; $b = 23.24$ in.; $EI = 105,410$; $q = 10.12$ lb/in.; $k^2 = 1 - \left(\frac{b}{a}\right)^2 = 0.296$; $t = \frac{EI}{q}\frac{1}{a^3} = 0.488$.

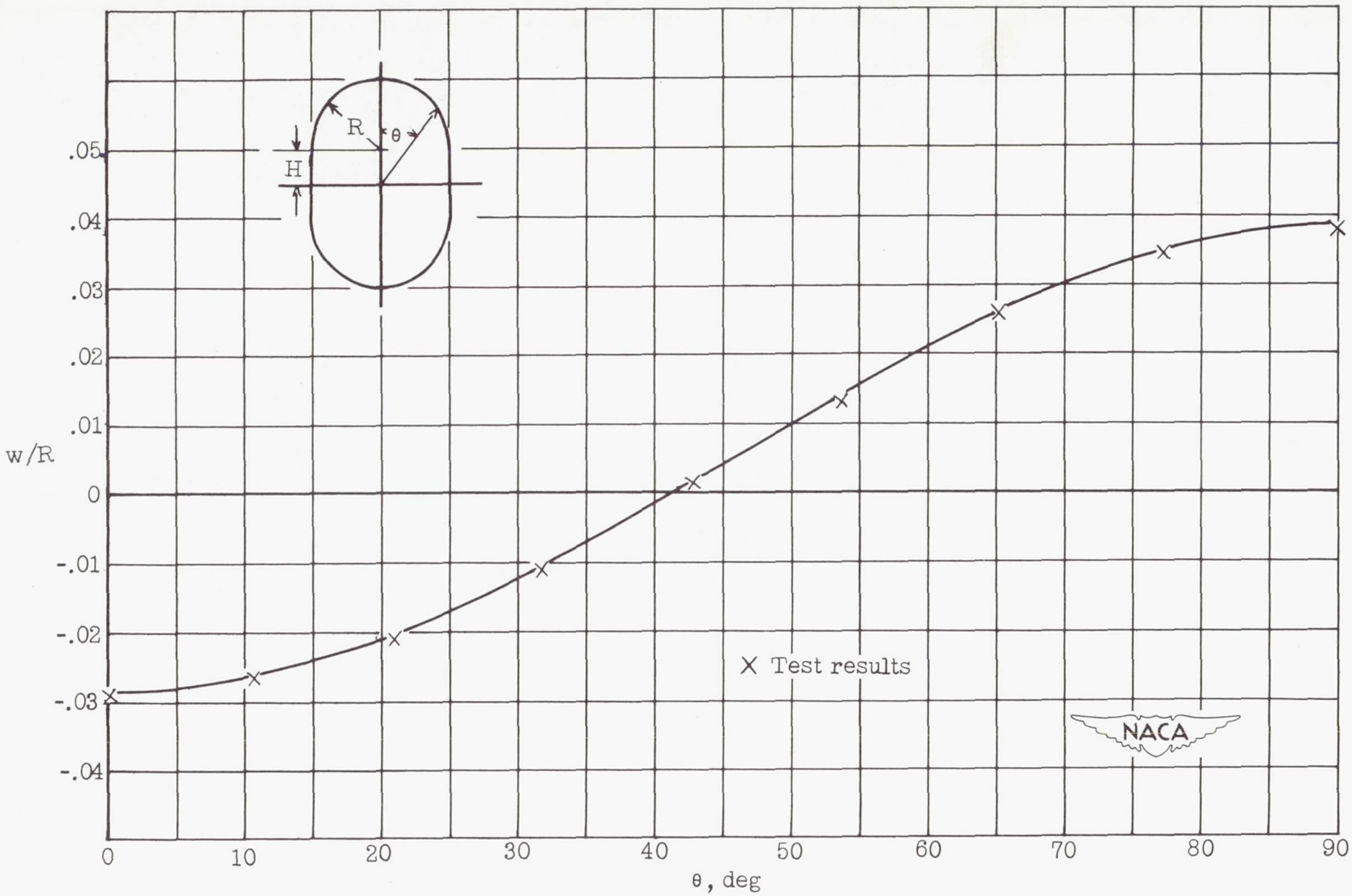


Figure 52.- Comparison of experimental values of w/R and values of w/R computed by present method. H = 4.77 in.; R = 23.85 in.; EI = 105,410; q = 10 lb/in.;

$k' = \frac{H}{R} = 0.2$; $t' = \frac{EI}{q}\frac{1}{R^3} = 0.794$.

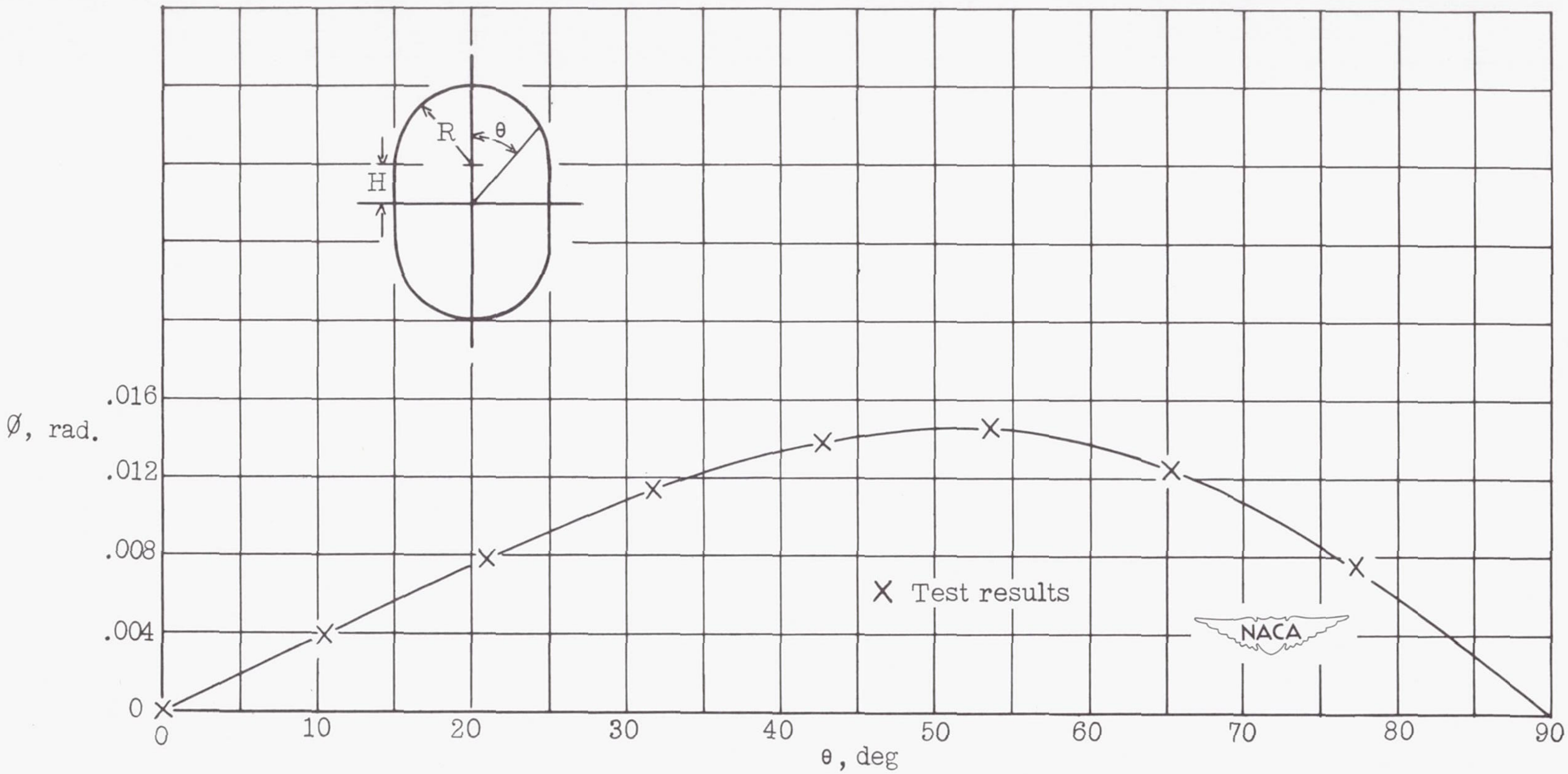


Figure 53.- Comparison of experimental values of Ø and values of Ø computed by present method. H = 4.77 in.; R = 23.85 in.; EI = 105,410; q = 10 lb/in.; $k' = \frac{H}{R} = 0.2$; $t' = \frac{EI}{q}\frac{1}{R^3} = 0.794$.

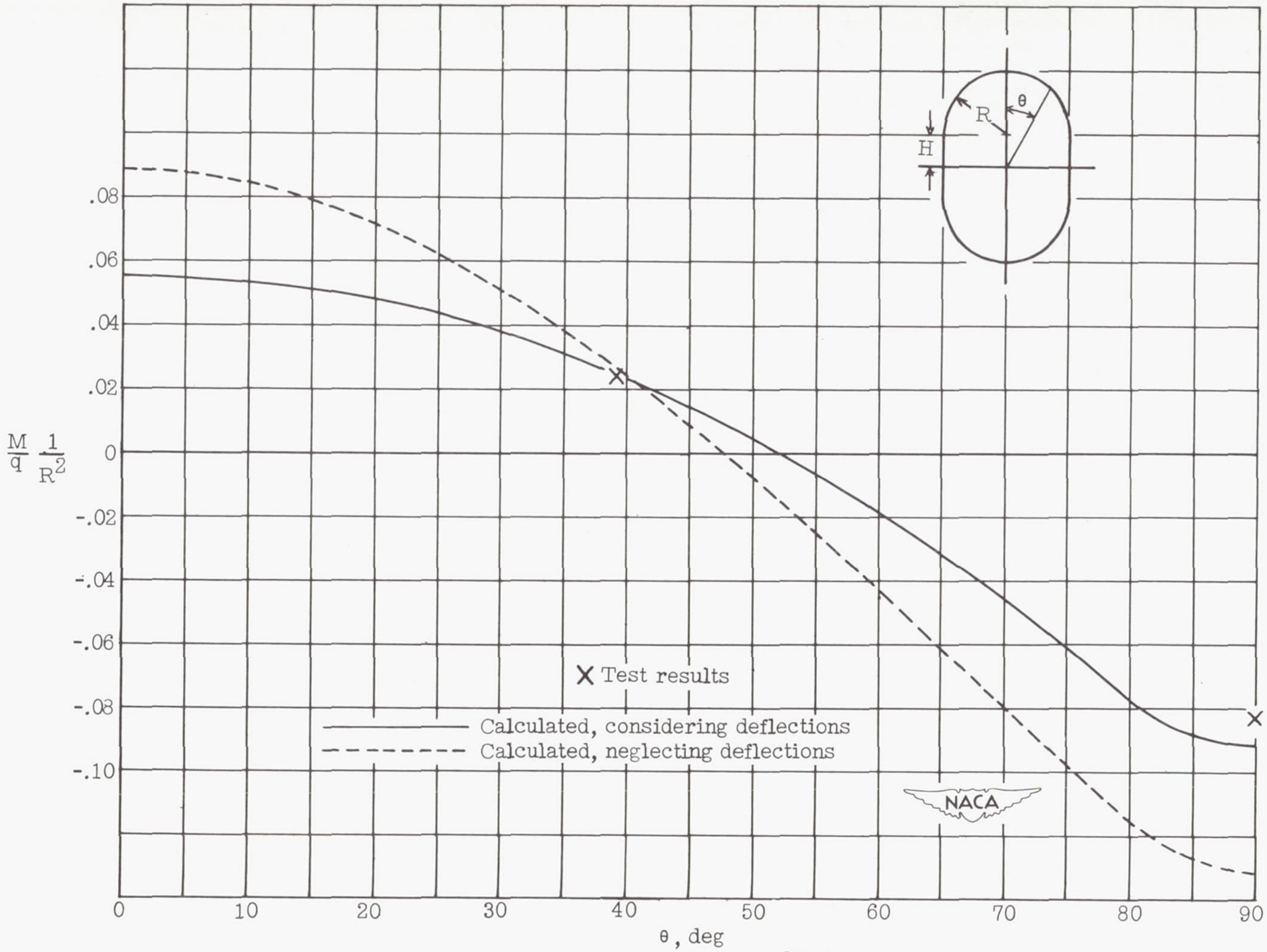


Figure 54.- Comparison of experimental values of $\frac{M}{q}\frac{1}{R^2}$ and values of $\frac{M}{q}\frac{1}{R^2}$ computed by present method. H = 4.77 in.; R = 23.85 in.; EI = 105,410; q = 10 lb/in.; $k' = \frac{H}{R} = 0.2$; $t' = \frac{EI}{q}\frac{1}{R^3} = 0.794$.

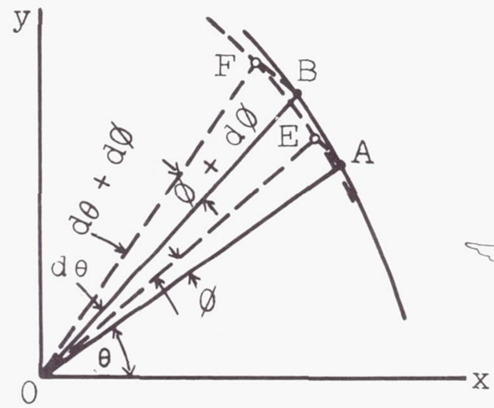


Figure 55.- Increase of arc length due to angular displacements.

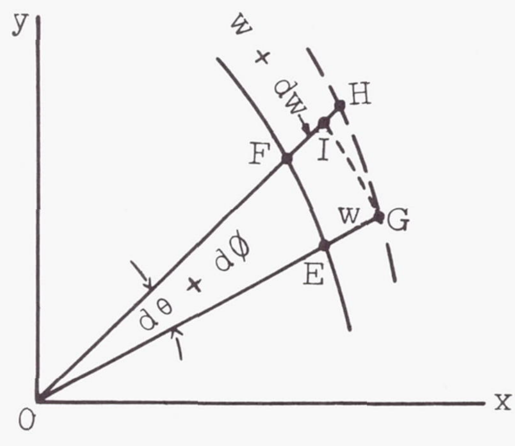


Figure 56.- Increase of arc length due to radial deflections.

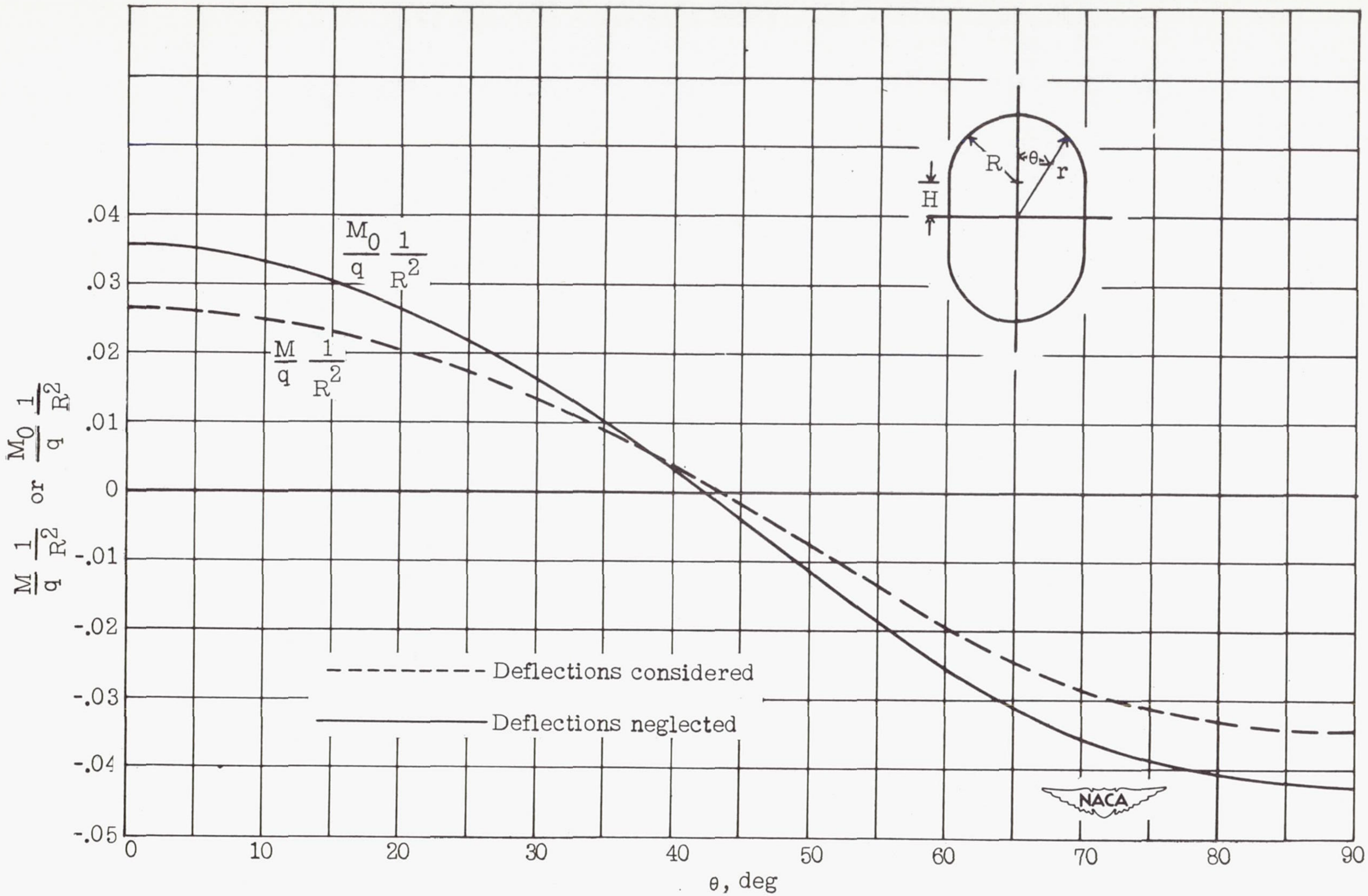


Figure 57.- Bending-moment distribution. $k' = \frac{H}{R} = 0.6$; $t' = \frac{EI}{q}\frac{1}{R^3} = 2.5$.

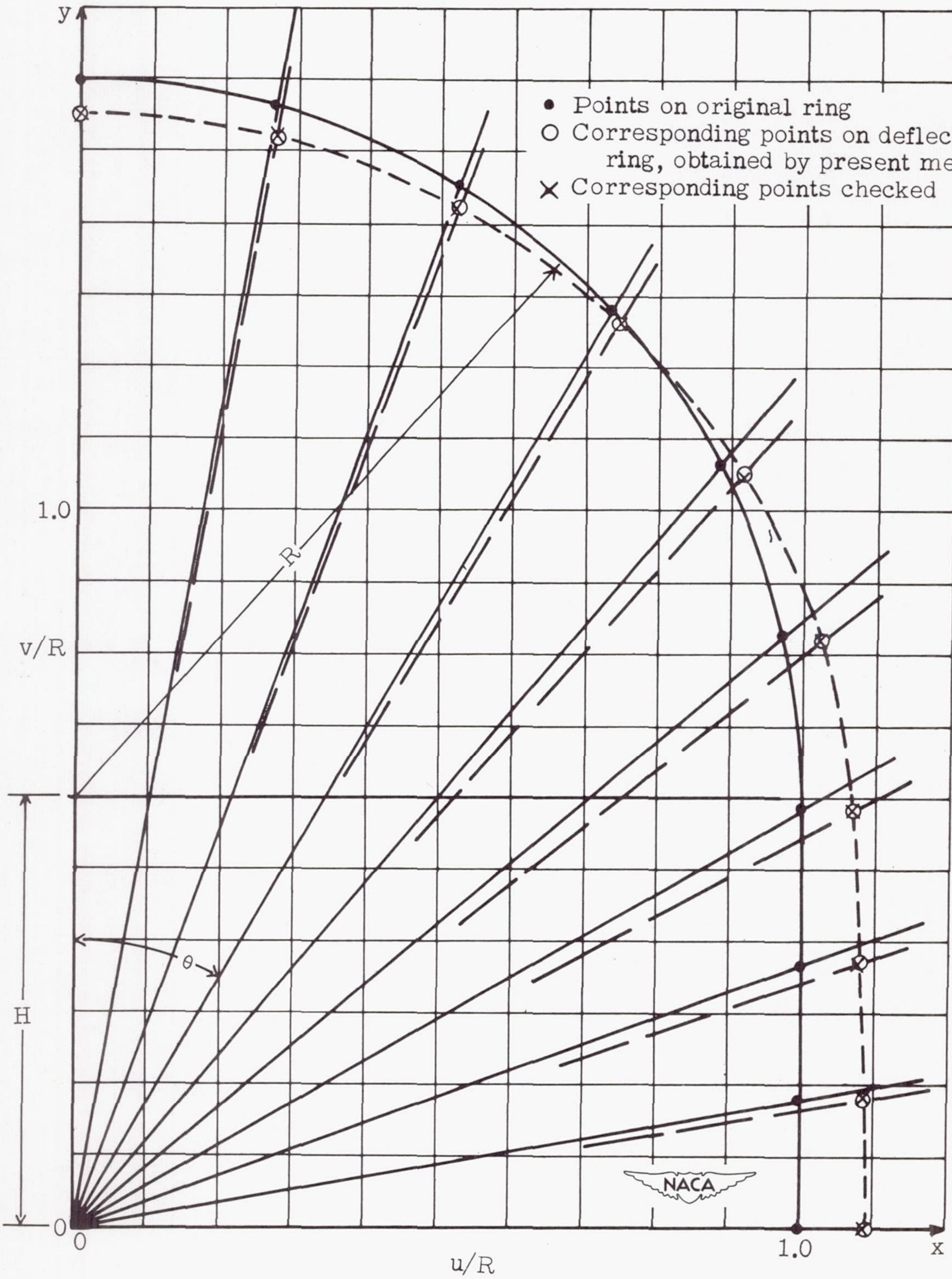


Figure 58.- Deflections u and v at various values of θ. $\frac{H}{R} = 0.6$; $t' = \frac{EI}{q} \frac{1}{R^3} = 2.5$.

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