NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

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STRESSES IN AND GENERAL INSTABILITY OF MONOCOQUE CYLINDERS WITH CUTOUTS

VI - CALCULATION OF THE BUCKLING LOAD OF CYLINDERS WITH SIDE CUTOUT SUBJECTED TO PURE BENDING

By N. J. Hoff, Bertram Klein, and Bruno A. Boley

Polytechnic Institute of Brooklyn

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SUMMARY

A strain-energy theory was developed for the calculation of the buckling load in general instability of circular reinforced monocoque cylinders having a side cutout and subjected to pure bending. The theory was applied to two series of specimens, each containing three cylinders, which were tested and reported in part IV in this program at the Polytechnic Institute of Brooklyn Aeronautical Laboratories. The average deviation between theoretical and experimental buckling load was 27.1 percent for the first series and 34.4 percent for the second.

INTRODUCTION

In the present report, the last report of a series of six (see references 1 to 5) dealing with monocoque cylinders with cutouts, the calculation of the buckling load of a cylinder which has a side cutout and fails in general instability when subjected to pure bending is undertaken. General instability is defined as the simultaneous buckling of the circumferential and longitudinal reinforcing elements of a monocoque cylinder together with the sheet attached to them. As the calculations given herein follow closely those presented in reference 3, it should be consulted for the development of some of the fundamental connections used in the present report.

The first step in the strain-energy calculation was the assumption of a buckled shape. This was done after examination of the deflection patterns observed in the tests described in reference 4. The stringer bordering the cutout on the compression side always showed the greatest deflections, and its distorted shape was very similar to a full sine wave. For the specimens having a symmetric cutout (reference 1) the wave length was only slightly greater than the length of the cutout, but for the specimens with side cutout the length of the wave varied and was observed to be definitely greater. For this reason, in the present report the wave length L was considered a parameter, the value of which had to be determined from the requirement that the buckling load be a minimum. This procedure differs from that adopted in the calculation of the general instability of the cylinders with symmetric cutout in which the wave length was assumed to be a known constant, namely the length of the cutout.

In the circumferential direction the deflected shape at buckling is represented by the first seven terms of a Fourier expansion. The circumferential coordinate is measured from the edge of the cutout, and the length of the interval in which the Fourier series is defined is considered as one of the parameters of the problem. The boundary conditions at the end of the interval determine four of the seven coefficients of the series, and one of them is indeterminate as in all buckling problems. The remaining two coefficients, as well as the wave-length parameter, are calculated from the requirement that the buckling load be a minimum.

The assumptions made regarding the buckling pattern in the circumferential direction are identical for the cylinders having a symmetric cutout and for those having a side cutout. For cylinders having a side cutout, however, additional considerations were necessary because the deflections extended into the complete portions of the cylinder. It was decided to assume that the expressions developed for the buckled shape in the cutout portion were also valid for the complete portions of the cylinder; this assumption, in effect, means that the restraint due to the continuity of rings and sheet in the complete portions is neglected. The justification for this assumption is the resulting comparative simplicity of the calculations, as well as the observation that the deflections always were considerably smaller in the complete portions than in the cutout portion.

The following strain-energy quantities were considered: radial and tangential bending, as well as torsion of the stringers; bending of the rings in their plane; and shear in the sheet. The extensional strain energy in the sheet was taken into account by adding an effective width of sheet to the stringers and the rings. In the calculation of the external work it was assumed that the applied moment caused a linear distribution of the strain in the cutout portion of the cylinder. The forces corresponding to these strains were applied to the stringers at the ends of the wave, and the sum of the products of these forces and the displacements of their points of application was taken as the external work.

The buckling load was calculated from the requirement that the strain energy corresponding to the transition from the unbuckled into the buckled shape be equal to the work done by the applied loads. The minimum value of the buckling load was found by assuming the circumferential wave length to be equal to the length of some integral number of stringer fields, and the axial wave length, some integral number of ring fields. The values of the two undetermined Fourier coefficients were calculated to make the buckling load a minimum. This minimum value of the buckling load then was determined and compared with other minimum values obtained on the basis of different choices of circumferential and axial wave lengths. Between 6 and 10 combinations were investigated for each of the six cylinders - three of which are shown in figure 1 - in order to find the absolute minimum value of the buckling load.

The investigation was conducted at the Polytechnic Institute of Brooklyn Aeronautical Laboratories under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics. For his substantial share in the numerical work the authors are indebted to Mr. John G. Pulos.

SYMBOLS

a, a_o, a_1, a_2, a_3	Fourier coefficients
A	cross-sectional area of stringer plus its effective width of sheet
b, b_1, b_2, b_3	Fourier coefficients
C	geometric factor in torsional rigidity GC
d	width of panel measured along circumference
E	Young's modulus
G	shear modulus
G_o	shear modulus of sheet covering at zero compressive load
G_{eff}	effective shear modulus
i	index indicating position along circumference
I	moment of inertia
I_r	moment of inertia of ring cross section and its effective width of sheet for bending in its own plane

I_{str_r}	moment of inertia of stringer cross section and its effective width of sheet for bending in radial direction (about a tangential axis)
I_{str_t}	moment of inertia of stringer cross section and its effective width of sheet for bending in tangential direction (about a radial axis)
j	index indicating position along axial direction
k_1,k_3,k_4	trigonometric functions of φ, n, a, and b
L	length of wave in axial direction
L_1	distance between adjacent rings
$m + 1$	number of ring fields involved in failure
M	applied bending moment; function of n, a, and b appearing in strain energy of bending in rings
n	parameter defining wave length in circumferential direction
p_1,p_2	polynomial functions of a and b
P_{cr}	maximum compressive force acting at buckling
P_i	force carried by the ith stringer at buckling
Q	function of x appearing in shear strain energy
r	radius of cylinder
R	function of φ, n, a, and b appearing in shear strain energy
s	number of stringer fields involved in failure
S	total number of stringers in cylinder
t	thickness of sheet covering
U	strain energy
U_r	bending strain energy stored in rings
U_{sh}	shear strain energy stored in sheet
U_{str_r}	radial bending strain energy stored in stringers
U_{str_t}	tangential bending strain energy stored in stringers

U_t	torsion strain energy stored in stringers,
$2w$	effective width of sheet
w_r	radial displacement of a point on a ring or a stringer
w_t	tangential displacement of a point on a ring or a stringer
W	work done by applied forces
x	axial coordinate
α	angle subtended by cutout
α_r, α_t	coefficients used in calculation of shear strain in a panel due to displacements of its corners
γ	shear strain
δ	shift of neutral axis from horizontal diameter
ϵ	normal strain in a stringer
ϵ_{max}	maximum compressive strain at buckling
φ	angular coordinate

THE DEFLECTED SHAPE

The shape of the bulge at buckling is determined mainly by the radial deflections. The following expression was chosen to represent the radial deflections:

$$\begin{aligned} w_r &= a_o k_1 \sin^2(\pi x/L) \\ &= \sin^2(\pi x/L)\Big(a_o + a_1 \cos n\varphi + a_2 \cos 2n\varphi + a_3 \cos 3n\varphi \\ &\qquad + b_1 \sin n\varphi + b_2 \sin 2n\varphi + b_3 \sin 3n\varphi\Big) \end{aligned} \tag{1}$$

provided that

$$0 \leq \varphi \leq (\pi/n) \tag{1a}$$

and

$$w_r = 0 \tag{1b}$$

when $\varphi > (\pi/n)$

The notation and the sign conventions are shown in figure 2.

The deformations of the rings were assumed to be inextensional. The condition of inextensionality is

$$w_r = - dw_t/d\varphi \tag{2}$$

Equations (1) and (2) determine the tangential deflections as follows:

$$w_t = \sin^2(\pi x/L) \left[- a_o\varphi - (a_1/n) \sin n\varphi - (a_2/2n) \sin 2n\varphi - (a_3/3n) \sin 3n\varphi + (b_1/n) \cos n\varphi + (b_2/2n) \cos 2n\varphi + (b_3/3n) \cos 3n\varphi \right] \tag{3}$$

provided that

$$0 \leq \varphi \leq (\pi/n) \tag{3a}$$

Integration of the bracketed expression in the right-hand member of equation (1) yields an integration constant that was omitted from the bracketed expression in the right-hand member of equation (3). The physical meaning of this constant is a rigid-body rotation of the ring. Moreover,

$$w_t = 0 \tag{3b}$$

when $\varphi > (\pi/n)$

If it is required that there be a smooth transition between the bulge and the nondistorted part of the cylinder at $\varphi = (\pi/n)$, the following conditions must be satisfied:

The tangential displacement must vanish, that is,

$$w_t = 0 \tag{4a}$$

when $\varphi = (\pi/n)$

The radial displacement must vanish, that is,

$$w_r = 0 \tag{4b}$$

when $\varphi = (\pi/n)$

There must be no abrupt change in the direction of the tangent, that is,

$$\partial w_r/\partial\varphi = 0 \tag{4c}$$

when $\varphi = (\pi/n)$

There must be no abrupt change in the curvature, that is,

$$\partial^2 w_r/\partial\varphi^2 = 0 \tag{4d}$$

when $\varphi = (\pi/n)$

The four conditions (equations (4a) to (4d)) establish four relations between the Fourier coefficients and make it possible to express any four coefficients by means of the remaining three. If a_o, a_1, and b_1 are retained as the basic parameters, the following four equations are obtained:

$$\left.\begin{aligned}
a_2 &= (8/5)a_1 - (9/5)a_o \\
a_3 &= (3/5)a_1 - (4/5)a_o \\
b_2 &= (16/5)b_1 + (18/5)\ \pi a_o \\
b_3 &= (9/5)b_1 + (12/5)\pi a_o
\end{aligned}\right\} \tag{5}$$

With the notation

$$\left.\begin{aligned}
(a_1/a_o) &= a \\
\text{and} \qquad & \\
(b_1/a_o) &= b
\end{aligned}\right\} \tag{6}$$

and after substitution of equation (5) in equations (1) and (3),

$$w_r = a_o k_1 \sin^2(\pi x/L) \tag{7}$$

where

$$k_1 = \Big[1 + a \cos n\varphi + (1.6a - 1.8) \cos 2n\varphi + (0.6a - 0.8) \cos 3n\varphi + b \sin n\varphi + (3.2b + 3.6\pi) \sin 2n\varphi + (1.8b + 2.4\pi) \sin 3n\varphi\Big] \tag{7a}$$

and

$$w_t = a_o k_3 \sin^2(\pi x/L) \tag{8}$$

where

$$k_3 = (1/n)\Big[-n\varphi - a \sin n\varphi - (1/2)(1.6a - 1.8) \sin 2n\varphi - (1/3)(0.6a - 0.8)\sin 3n\varphi + b \cos n\varphi + (1/2)(3.2b + 3.6\pi) \cos 2n\varphi + (1/3)(1.8b + 2.4\pi)\cos 3n\varphi\Big] \tag{8a}$$

Equations (7) and (8) are valid, provided that

$$0 \leq \varphi \leq (\pi/n) \tag{8b}$$

When φ is greater than (π/n), the deflections are assumed to vanish. Typical examples of the deflection patterns in the plane of the rings are shown in figures 3 and 4

CALCULATION OF STRAIN ENERGY

Strain Energy Stored in Rings

The strain energy stored in any one ring is

$$U = (1/2)\left[(EI)_r/r^3\right] \int_0^{\pi/n} \left[w_r + \left(\partial^2 w_r/\partial\varphi^2\right)\right]^2 d\varphi \tag{9}$$

If the value of w_r is substituted from equation (7) and the strain energy is summed up over all the rings, the following expression is obtained:

$$U_r = (1/2)\left(a_o^2/r^3\right) \sum_{j=1}^{m} (EI)_r \sin^4(\pi x_j/L) \int_0^{\pi/n} \left[k_1 + \left(\partial^2 k_1/\partial\varphi^2\right)\right]^2 d\varphi \tag{10}$$

where m is the total number of rings included in the wave length. The integration yields a result in closed form. The same is true of the summation if all the rings have the same bending rigidity. In such a case the total strain energy U_r stored in all the rings becomes

$$U_r = (3/16)\left(a_o^2/r^3\right)(EI)_r(m + 1)M \tag{11}$$

when

$$m > 1 \tag{11a}$$

and

$$U_r = (1/2)\left(a_o^2/r^3\right)(EI)_r M \tag{12}$$

when

$$m = 1 \tag{12a}$$

where

$$\begin{aligned} nM = &\left[\pi + 10.053096\left(1 - 9n^2\right) + 206.01005\left(1 - 4n^2\right)^2 + 90.303387\left(1 - 9n^2\right)^2 \right.\\ &\left. - 18.095573\left(1 - 4n^2\right)\left(1 - 9n^2\right)\right] \\ &+ a\left[- 9.0477868\left(1 - 4n^2\right)^2 - 1.5079645\left(1 - 9n^2\right)^2 \right.\\ &\left. + 30.159289\left(1 - n^2\right)\left(1 - 4n^2\right) + 18.095573\left(1 - 4n^2\right)\left(1 - 9n^2\right)\right] \\ &+ b\left[4\left(1 - n^2\right) + 2.4\left(1 - 9n^2\right) + 113.69784\left(1 - 4n^2\right)^2 + 42.636690\left(1 - 9n^2\right)^2 \right.\\ &\left. + 2.4\left(1 - n^2\right)\left(1 - 4n^2\right) - 3.68\left(1 - 4n^2\right)\left(1 - 9n^2\right)\right] \\ &+ a^2\left[1.5707963\left(1 - n^2\right)^2 + 4.0212386\left(1 - 4n^2\right)^2 + 0.5654867\left(1 - 9n^2\right)^2\right] \\ &+ b^2\left[1.5707963\left(1 - n^2\right)^2 + 16.084954\left(1 - 4n^2\right)^2 + 5.08938\left(1 - 9n^2\right)^2\right] \\ &+ ab\left[6.4\left(1 - n^2\right)\left(1 - 4n^2\right) + 3.84\left(1 - 4n^2\right)\left(1 - 9n^2\right)\right] \end{aligned} \tag{13}$$

Strain Energy Stored in Stringers

The strain energy stored in the stringers because of bending in the radial direction is

$$U_{str_r} = \sum (1/2)(EI)_{str_r} \int_0^L \left(\partial^2 w_r/\partial x^2\right)^2 dx \tag{14}$$

where the summation is extended over all the stringers contained in the bulge. Substitution and integration yield

$$\begin{aligned} U_{str_r} &= \sum (1/2)(EI)_{str_r}\, a_o^2 \int_0^L k_1^2\left(2\pi^2/L^2\right)^2 \cos^2(2\pi x/L)\, dx \\ &= \left(\pi^4/L^3\right) a_o^2 \sum {k_1}^2\, (EI)_{str_r} \end{aligned} \tag{15}$$

Both k_1 and $(EI)_{str_r}$ vary from stringer to stringer, $(EI)_{str_r}$ because the effective width of sheet to be added to the stringer section changes. For this reason the summation has to be evaluated numerically.

The strain energy stored in the stringers because of bending in the tangential direction is

$$U_{str_t} = \sum (1/2)(EI)_{str_t} \int_0^L \left(\partial^2 w_t/\partial x^2\right)^2 dx \tag{16}$$

where the summation is extended over all the stringers contained in the bulge. Substitution and integration yield

$$U_{str_t} = \left(\pi^4/L^3\right) a_o^2 \sum k_3^2 (EI)_{str_t} \tag{17}$$

Because both k_3 and $(EI)_{str_t}$ vary from stringer to stringer, the summation has to be evaluated numerically.

The strain energy stored in the stringer because of torsion is

$$U_t = \sum (1/2)GC \int_0^L (1/r)^2 \left[\left(\partial^2 w_r\right)\Big/(\partial x \partial \varphi)\right]^2 dx \tag{18}$$

In this equation $(1/r)\left(\partial^2 w_r\right)\Big/(\partial x \partial \varphi)$ is the unit angle of twist of the stringer, and the summation is extended over all the stringers contained in the bulge. In the expression for the Saint-Venant torsional rigidity,

$$C = 0.14a^4 \tag{18a}$$

because the test specimens were provided with square section stringers of edge length a. Differentiation gives

$$\left(\partial^2 w_r\right)\Big/(\partial x \partial \varphi) = a_o k_4 (\pi/L) \sin (2\pi x/L) \tag{19}$$

where

$$k_4 = n\Big[- a \sin n\varphi - (3.2a - 3.6) \sin 2n\varphi - (1.8a - 2.4) \sin 3n\varphi$$
$$+ b \cos n\varphi + (6.4b + 7.2\pi) \cos 2n\varphi + (5.4b + 7.2\pi) \cos 3n\varphi\Big] \tag{19a}$$

Hence the strain energy of torsion is

$$U_t = \left(\pi^2/4\right)a_o^2\left[(GC)/\left(Lr^2\right)\right]\sum k_4^2 \tag{20}$$

where the summation includes all the stringers contained in the bulge. Because the variation of the torsional rigidity with effective width is negligibly small, the factor GC was written before the summation sign. The summation was carried out numerically.

Strain Energy of Shear Stored in Sheet

The shear strain energy in the panel is taken as the average effective shear modulus G_{eff} multiplied by the square of the average shear strain γ in the panel. The value of γ is calculated from the displacements of the four corners of the panel. The total strain energy of shear stored in the sheet then is

$$U_{sh} = (1/2)\sum \gamma^2 G_{eff} L_1 td \tag{21}$$

The effective shear modulus depends upon the geometric and mechanical properties of the panel and the average strain therein. Its value was taken from the empirical curves established earlier at Polytechnic Institute of Brooklyn Aeronautical Laboratories and presented in figure 24 of reference 6.

The average angle of shear γ was calculated from the equation

$$\gamma = \left(\alpha_r/L_1\right)\left(w_{r_{i,j}} - w_{r_{i+1,j}} - w_{r_{i,j+1}} + w_{r_{i+1,j+1}}\right)$$
$$\left(-\alpha_t/L_1\right)\left(w_{t_{i,j}} + w_{t_{i+1,j}} - w_{t_{i,j+1}} - w_{t_{i+1,j+1}}\right) \tag{22}$$

where the first subscript refers to the circumferential location of the corner of the panel and the second to the axial location, as shown in figure 5. The values of the factors α_r and α_t were calculated from the equations

$$\left.\begin{aligned} \alpha_r &= (1/10)(d/r) = (1/10)(2\pi/S) \\ \alpha_t &= -1/2 \end{aligned}\right\} \tag{23}$$

Substitutions yield

$$U_{sh} = (1/2)(td/L_1)G_o \sum_{j=0}^{m} Q_j \sum_{i=0}^{s-1} (G_{eff}/G_o)_i R_i \tag{24}$$

where s is the number of stringer fields involved in the bulge, Q is a function of x only, and R is a function of φ only. The summation $\sum Q$ gives a result in closed form as follows:

$$\begin{aligned}\sum_{j=0}^{m} Q_j &= \sum_{j=0}^{m} \left\{\sin^2 \left[(\pi j)/(m+1)\right] - \sin^2 \left[\pi(j+1)/(m+1)\right]\right\}^2 \\ &= (1/4)(m+1)\left\{1 - \cos\left[(2\pi)/(m+1)\right]\right\}\end{aligned} \tag{25}$$

provided that

$$m > 1 \tag{25a}$$

When $m = 1$,

$$\sum Q = 2 \tag{25b}$$

The meaning of the symbol R is

$$R = \left[\alpha_r\left(k_{1,i} - k_{1,i+1}\right) - \alpha_t\left(k_{3,i} + k_{3,i+1}\right)\right]^2 \tag{26}$$

The values of $k_{1,i}$, $k_{1,i+1}$, $k_{3,i}$, and $k_{3,i+1}$ are obtained from those of k_1 and k_3 (equations (7a) and (8a)), respectively, by replacing the angle φ by $2\pi i/S$ or $2\pi(i+1)/S$. As examples, $k_{1,i}$ and $k_{3,i}$ are listed:

$$\begin{aligned}k_{1,i} &= 1 + a\cos(2\pi mi/S) + (1.6a - 1.8)\cos(4\pi mi/S) \\ &\quad + (0.6a - 0.8)\cos(6\pi mi/S) + b\sin(2\pi mi/S) \\ &\quad + (3.2b + 3.6\pi)\sin(4\pi mi/S) + (1.8b + 2.4\pi)\sin(6\pi mi/S)\end{aligned} \tag{27a}$$

$$\begin{aligned}nk_{3,i} &= -(2\pi mi/S) - a\sin(2\pi mi/S) - (1/2)(1.6a - 1.8)\sin(4\pi mi/S) \\ &\quad - (1/3)(0.6a - 0.8)\sin(6\pi mi/S) + b\cos(2\pi mi/S) \\ &\quad + (1/2)(3.2b + 3.6\pi)\cos(4\pi mi/S) + (1/3)(1.8b + 2.4\pi)\cos(6\pi mi/S)\end{aligned} \tag{27b}$$

the summation of the R quantities indicated in equation (24) was carried out numerically.

WORK DONE BY EXTERNAL FORCES

As was mentioned in the INTRODUCTION, the strain was assumed to be distributed linearly over the sections of the cutout portion of the cylinder. The force in each stringer was calculated as the product of the strain, the elastic modulus, and the cross-sectional area of stringer plus effective width of sheet. These forces were assumed to be the external forces applied to the stringers at the end of the axial wave length. Because of the reduction in the effective width of sheet on the compression side of the cylinder the neutral axis in bending is shifted toward the tension side. This shift was calculated and taken into account when the forces acting upon the ends of the stringers were determined.

The distance between the points of application of the forces shortens when the stringers bend during the buckling process. This shortening multiplied by the force is the work done by the force. The total external work is the sum of all the work quantities calculated for the individual stringers:

$$W = (1/2)\sum P_i\int_0^L \left[\left(\partial w_r/\partial x\right)^2 + \left(\partial w_t/\partial x\right)^2\right] dx \tag{28}$$

where P_i is the external force acting upon the ith stringer and the summation is extended over all the stringers contained in the wave length. Substitutions and integration yield

$$\begin{aligned} W &= (1/2)a_o^2\sum P_i \ (\pi/L)^2\left(k_1^2 + k_3^2\right)\int_0^L \sin^2 \ (2\pi x/L) \ dx \\ &= (1/4)a_o^2(\pi^2/L)P_{cr}\sum_{i=0}^{s}\left(P_i/P_{cr}\right)\left({k_{1,i}}^2 + {k_{3,i}}^2\right) \end{aligned} \tag{29}$$

where P_{cr} is the force acting in the most highly compressed stringer. The summation was carried out numerically.

CALCULATION OF BUCKLING LOAD

The buckling condition is

$$U_r + U_{str_r} + U_{str_t} + U_t + U_{sh} = W \tag{30}$$

where the values of the quantities must be taken from equations (11) or (12), (15), (17), (20), (24), and (29), respectively. Equation (30) was solved for P_{cr}, which is a multiplying factor in the expression for W, and minimized by means of the following procedure:

A value of n corresponding to a circumferential wave length extending over an integral number of stringer fields was first assumed, and an integral number was chosen for $m + 1$, the number of ring fields included in the axial wave length. On the basis of these tentative values, M, k_1, k_3, and k_4 were computed. Next, P_{cr} was assumed. This assumption permitted the calculation of the effective width of sheet and consequently of the moments of inertia of the stringers and made possible the determination of the values of G_{eff}/G_o from the graph. The summations were then carried out. Substitution of the results in equation (30) resulted in a polynomial p_1 of the second degree in a and b in the left-hand member and another polynomial p_2 of the second degree in the right-hand member, p_2 multiplied by P_{cr}. Solution for P_{cr} gave the fraction

$$P_{cr} = \frac{p_1(a,b)}{p_2(a,b)} \tag{31}$$

This expression for P_{cr} may be minimized with respect to a and b by setting

$$P_{cr} = \frac{p_1}{p_2} = \frac{\partial p_1/\partial a}{\partial p_2/\partial a} = \frac{\partial p_1/\partial b}{\partial p_2/\partial b} \tag{32}$$

The partial differential coefficients of p_1 and p_2 are linear functions of a and b. Equation (32) represents three connections between P_{cr}, a, and b. They were solved by a rapidly converging trial-and-error method. First, a value was assumed for P_{cr}, and a and b were determined from the two linear equations. Then the values of a and b were substituted into the quadratic expression for P_{cr}. The procedure was repeated with the aid of new assumptions for P_{cr} until the value obtained was close enough to the assumed value.

When the value of P_{cr} obtained in these calculations differed substantially from the value assumed at the outset, the moments of inertia and the effective shear modulus had to be recalculated and the entire procedure repeated. All the calculations were carried out for a number of different choices of n and m. The buckling loads corresponding to these different values were compared, and the smallest one was considered as the true buckling load. Details of the procedure may be seen from the numerical example in the appendix.

COMPARISON OF THEORY AND EXPERIMENT

Numerical calculations were carried out for the two arrangements of stringers and three circumferential sizes of cutouts investigated in the experiments described in reference 4. Typical buckling patterns obtained in the calculations are shown in figures 3 and 4, and details of the numerical results are presented in table 1. Theoretical and experimental bending moments at buckling are compared in figure 6.

The theory predicted bending moments at buckling which were, with one exception, consistently higher than those obtained in the experiments. Moreover, the deviations between theory and experiment increased systematically with decreasing circumferential length of the cutout. Strain-energy calculations are known to yield too high buckling loads when the deflected shape assumed differs from the actual shape of distortions.

The circumferential wave length was predicted by theory with satisfactory accuracy. In the axial direction the theoretical wave length is greater than the one observed. The deviation is slight in the case of the 16-stringer specimens and large in the case of the 8-stringer specimens. Small changes in the axial wave length, however, have little effect upon the buckling load.

CONCLUSIONS

A strain-energy theory has been developed for the calculation of the buckling load in general instability of circular reinforced monocoque cylinders which have a side cutout and are subjected to pure bending. When the theory was applied to the test cylinders of part IV of the present series of investigations, the following percentage deviations from the experimental values were obtained: 54.8, 32.4, and 16.1 percent for the 45°, 90°, and 135° cutouts, respectively, of the 8-stringer series; and 47.4, 30.8, and −3.2 percent for the 45°, 90°, and 135° cutouts, respectively, of the 16-stringer series.

The authors believe it would be desirable to continue the theoretical investigations on the basis of more refined assumptions for the deflected shape at buckling. Strain-energy calculations are known to yield too high buckling loads when the deflected shape assumed differs from the actual shape of distortions. More experimental work is also needed for a better understanding of the general instability phenomenon of reinforced monocoque cylinders having a side cutout.

Polytechnic Institute of Brooklyn
Brooklyn, N. Y., December 26, 1946

APPENDIX

NUMERICAL EXAMPLE

Details of the numerical work performed in connection with the determination of the buckling load for a test specimen are shown. The cylinder considered is cylinder 39 of reference 4. Pertinent data to be referred to are listed as follows:

Radius, r, in. 10
Distance between adjacent rings, L_1, in. 6.429
Width of panel measured along circumference, d, in.. . . . . 7.854
Angle subtended by cutout, α, deg 45
Total number of stringers in cylinder, S 8
Stringer cross section, in.. . . . . $3/8 \times 3/8$
Ring cross section, in.. . . . . $3/8 \times 3/8$
Young's modulus, E, psi . . . . 10.5×10^6
Shear modulus, G, psi . . . . 3.9×10^6
Shear modulus of sheet covering at zero compressive load, G_o, psi . . . . 3.9×10^6
Thickness of sheet covering, t, in.. 0.012
Moment of inertia of ring cross section and its effective width of sheet for bending in its own plane, I_r $\left((1/2)(3/8)\left[(1/8) + (0.012)\right]^3\right)$, in^4. . . . . 80.35×10^{-6}

Once a value of s, the number of stringer fields involved in failure, is chosen, it is possible to reduce the expressions denoted by M, k_1, k_3, and k_4 defined by equations (13), (7a), (8a), and (19a), respectively, to arithmetic polynomials in a and b. The one for M is obtained by simply inserting the values of $n = (S/2s)$ and its powers into equation (13). For $s = 3$, the result is:

$$M \times 10^{-5} = 0.0020877a^2 + 0.0131007b^2 + 0.19654065 + 0.00286815ab + 0.0084369a + 0.1010576b \qquad (33)$$

For the evaluation of k_1, k_3, and k_4 it is convenient to set up a tabular arrangement. For $s = 3$ this arrangement takes the following form:

i	Constant	cos $(\pi i/3)$	cos $(2\pi i/3)$	cos $(3\pi i/3)$	$2\pi i/8$	sin $(\pi i/3)$	sin $(2\pi i/3)$	sin $(3\pi i/3)$
0	1	1	1	1	0	0	0	0
1	1	.5	-.5	-1	.785398	.8660255	.8660255	0
2	1	-.5	-.5	1	1.570796	.8660255	-.8660255	0
Multipliers for k_1	1	a	1.6a-1.8	0.6a-0.8		b	3.2b+11.3097336	1.8b+7.5398224
Multipliers for k_3		0.75b	1.2b+4.2411501	0.45b+1.8849556	-1	-0.75a	-0.6a+0.675	-0.15a+0.2
Multipliers for k_4		1.333...b	8.5333...b+30.1592894	7.2b+30.1592894		-1.333...a	-4.2666...a+4.8	-2.4a+3.2

For the value of the index $i = 0$ denoting the position of the stringer at the edge of the cutout, the polynomial for k_1 in a and b is found by multiplying the expressions appearing in the first row below the double line (labeled multipliers for k_1) by the numbers in the corresponding column listed in the first row (labeled 0) and adding like quantities of the results of the products. In a similar manner the polynomials for $i = 1$ and $i = 2$ for k_1 are obtained, as well as the three values each for k_3 and k_4. The results are presented in tabular form as follows:

TABLE 2

i	k_1	k_3	k_4
0	3.2a-1.6	2.4b + 6.1261056	17.0666...b + 60.3185787
1	- 0.9a + 3.673071b + 12.4945177	- 1.1691344a - 0.675b - 4.2063614	- 4.849728a - 10.8b - 41.0820117
2	- 0.7a - 1.9052561b - 8.6945177	- 0.1299038a - 0.525b - 2.3909827	2.5403415a + 2.2666..b + 10.9227221

The functions $(k_{1,i} - k_{1,i+1})$ and $(k_{3,i+1} + k_{3,i})$ can be determined with the aid of table 2 by simply subtracting or adding the polynomials of adjacent rows in the first two columns of the table. It must be remembered that $k_1 = 0$ and $k_3 = 0$ when $i = 3$ since the deflections have been assumed to vanish at the third stringer, where $i = 3$. If α_r is taken as 0.0785 and α_t as -0.5, the function $R = (k_{1,i} - k_{1,i+1})\alpha_r - (k_{3,i+1} + k_{3,i})\alpha_t$ becomes for each field:

TABLE 3

ith field	R
1	-0.2625532a + 0.576826b - 0.147112
2	-0.6652272a - 0.164686b - 1.6344854
3	-0.1199299a - 0.4121398b - 1.878358

Before the stringer bending strain energy, shear strain energy, and external work can be evaluated, it is necessary to assume first a value of the buckling load so that the numerical values of the stringer moments of inertia, effective shear modulus, and effective area may be determined. Since the last three quantities are functions of the normal strain acting in the axial fibers at buckling, a value of the critical strain is assumed. Also in order to locate the position of the neutral axis at failure, the critical strain must be known. In the present example, the critical strain is guessed to be 21×10^{-4}. Then the shift of the neutral axis, calculated by taking first moments of area, is found to be 0.07, expressed in percentage of the radius. The following table contains the afore-mentioned items for $\epsilon_{max} = 21 \times 10^{-4}$ and $\delta/r = 0.07$

TABLE 4

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
i	ϵ	2w	I_{str_r}	I_{str_t}	$\frac{\epsilon}{3.3}$	G_{eff}/G_o	A_{eff}	$A_{eff}(\epsilon/\epsilon_{max})$
0	8.884×10^{-4}	4.290	22.15×10^{-4}	435×10^{-4}	2.69×10^{4}	0.633	0.1664	0.0704
1	19.506	2.648	23.85	200	5.91	.419	.1724	.16014
2	19.506	2.648	23.85	200	5.91	.419	.1724	.16014

Column (1) refers to the stringer station. Column (2), the strain at these locations, is directly proportional to the distance from the neutral axis, since a linear strain distribution is assumed. Column (3) is the effective width of curved sheet calculated from equation (30) of reference 7. Columns (4) and (5) give the moments of inertia of the stringers plus their effective width of curved sheet. These are calculated from equations (34) and (36) of reference 7, with special considerations for the edge stringer because it has effective width on only one side. Column (6) indicates the ratio of the actual strain in the sheet when the monocoque cylinder buckles to the buckling strain of a panel of sheet of 3.30×10^{-4} as given in reference 6. These values are needed to obtain the percentage reduction in the value of the shear modulus recorded in figure 24 of reference 6 and presented in column (7). Adding to the cross-sectional area of the stringer, 0.140625 square inch, the area of the effective width of sheet which is 0.012 times a value of column (3) yields a value of effective area shown in column (8). The entries in column (9) can be computed with the aid of columns (8) and (3).

The next step in the calculations is to assume a value of $m + 1$ the number of ring fields involved in failure. It is then possible to write the equation for the buckling condition in terms of the parameters a and b. It is convenient to multiply the numerator and denominator terms of equation (31) by $\left[(m+1)/E\right] \times 10^4$ and to solve for $\epsilon_{max} \times 10^4$ instead of ϵ_{max}; this procedure requires that each strain energy be multiplied by $\left[(m+1)/E\right] \times 10^4$ and that the external work be multiplied by $\left[(m+1)/E\right] \times 10^4$. When $m+1 = 11$, from equations (11), (15), (17), (20), (24), and (29), there results:

$$U_r\left[(m+1)/E\right] \times 10^4 = a_o^2\left[(3/16)(11)^2\left(80.35 \times 10^{-6}\right)/10^3\right]M \times 10^4$$

$$= a_o^2\ 1822.94\ (M \times 10^{-5}) \qquad (34)$$

where $M \times 10^{-5}$ is given in equation (33).

$$U_{str}\left[(m+1)/E\right] \times 10^4 = a_o^2\left[\pi^4/(6.429)^3(11)^2\right]\left[\sum_i k_1^2 I_{str_r} + \sum_i k_3^2 I_{str_t}\right] \times 10^4$$

$$= a_o^2 \sum_i \left[\left(30.298\ I_{str_r}\right)k_1^2 + \left(30.298\ I_{str_t}\right)k_3^2\right] \qquad (35)$$

where the indicated summation can be evaluated by taking the sum of the products of the squares of the polynomials for both k_1 and k_3 appearing in table 2 and 30.298 times the entries for corresponding values of i listed in columns (4) and (5) of table 4. For this purpose it is helpful first to calculate and record the quadratic polynomials in a and b

which represent the squares of k_1 and k_3 for each value of i. These expressions are not shown here.

$$U_t\left[(m + 1)/E\right] \times 10^4 = a_o^2\left[\pi^2/4(10)^2(6.429)\right](3.9/10.5)\ (0.002769)\left(\sum_i k_4^2\right) \times 10^4$$

$$= a_o^2(0.03947)\left(\sum_i k_4^2\right) \tag{36}$$

where the summation is performed by squaring and adding the values of k_4 given in table 2 for each value of i.

$$U_{sh}\left[(m + 1)/E\right] \times 10^4 = a_o^2\left[(0.012)(7.854)/(8)(6.429)\right](3.9/10.5)\left[(11)^2\left(1 - \cos\frac{2\pi}{11}\right)\right] \times$$

$$\left[\sum_i\left(G_{eff}/G_o\right)R_i^2\right] \times 10^4 = a_o^2\sum_i\left[130.57\left(G_{eff}/G_o\right)\right]R_i^2 \tag{37}$$

where the summation can be deduced by squaring each polynomial of table 3, multiplying the result by 130.57 times the proper value of $\left(G_{eff}/G_o\right)$ to be found in column (7) of table 4, and adding all such products. A table containing the terms in R_i^2 is needed. It is not given here. And finally,

$$W\left[(m + 1)/E\right] \times 10^4 = a_o^2\left[\pi^2/(4)(6.429)\right]\epsilon_{max} \times 10^4\left[\sum_i A_{eff}\left(\epsilon/\epsilon_{max}\right)\left(k_1^2 + k_3^2\right)\right]$$

$$= a_o^2\ \epsilon_{max} \times 10^4\left\{\sum_i\left[(0.38379)A_{eff}\left(\epsilon/\epsilon_{max}\right)\right]\left(k_1^2 + k_3^2\right)\right\} \tag{38}$$

where the summation is carried out by finding for each value of i the sum of the squares of k_1 and k_3. These quantities are evaluated in connection with the determination of the stringer bending strain energy, multiplying this sum by 0.38379 times the corresponding value appearing in column (9) of table 4, and adding all such products.

The value of $\epsilon_{max} \times 10^4$ is obtained by adding equations (34), (35), (36), and (37) and dividing this sum by equation (38) divided by $\epsilon_{max} \times 10^4$. If the foregoing procedure is followed, the numerator of the ratio of the two quadratic polynomials in a and b to which $\epsilon_{max} \times 10^4$ is equal can be computed immediately without writing the individual strain-energy terms of which it is composed. The result for the critical strain is:

$$\epsilon_{max} \times 10^4 = \frac{36.2426a^2 + 85.6323b^2 + 886.8543 + 1.23321ab + 180.5144a + 419.21505b}{0.441621a^2 + 1.236806b^2 + 16.762969 + (-0.133067)ab - 0.268132a + 8.920413b} \tag{39}$$

For buckling, a minimum value of ϵ_{max} is desired, that is, ϵ_{max} must be minimized with respect to both a and b. The two conditions for satisfying this requirement are:

$$\epsilon_{max} \times 10^4 = \frac{2(36.2426)a + 1.23321b + 180.5144}{2(0.441621)a - 0.133067b - 0.268132} \tag{40}$$

and

$$\epsilon_{max} \times 10^4 = \frac{1.23321a + 2(85.6323)b + 419.21505}{-0.133067a + 2(1.236806)b + 8.920413} \tag{41}$$

Equations (39), (40), and (41) represent three equations in the three unknowns ϵ_{max}, a, and b. If a value of ϵ_{max} is assumed, equations (40) and (41) are reduced to a pair of simultaneous equations in a and b. After these equations are solved, substitution of the results into equation (39) yields a value of ϵ_{max}. This value may be used as a second approximation and the process repeated if necessary.

In the present numerical example $\epsilon_{max} \times 10^4$ has been assumed to be 21 for the calculation of the position of the neutral axis, effective widths of sheet, moments of inertia, and effective areas. The same value is substituted into equations (40) and (41). The values of a and b are found to be

$$\left.\begin{aligned} a &= -3.4349 \\ b &= -1.74254 \end{aligned}\right\} \tag{42}$$

When these results are inserted into equation (39), $\epsilon_{max} \times 10^4$ becomes 22.44. Experience has shown that the difference between the value $\epsilon_{max} = 22.44 \times 10^{-4}$ obtained and the value $\epsilon_{max} = 21 \times 10^{-4}$ assumed is small enough to make a repetition of the calculations unnecessary. The changes in the values of the effective widths, moments of inertia, location of neutral axis, and effective shear modulus resulting from the increase in ϵ_{max} from 21×10^{-4} to 22.44×10^{-4} would have a negligible effect upon the calculated buckling strain. Hence 22.44×10^{-4} may be taken as the critical buckling strain.

The critical moment is computed from the relation

$$M_{cr} = \sum Pd$$

where P is the force in a stringer at buckling, d is the distance of the stringer from the neutral axis, and the summation must be taken over the entire cylinder. Experessed in terms of strain, this relation is

$$M_{cr} = \sum \left(EA_{eff}\epsilon\right)d = \left[E\epsilon_{max}/(r + \delta)\right]\sum A_{eff}d^2 \tag{43}$$

since

$$\epsilon = \epsilon_{max}d/(r + \delta)$$

For a critical strain of 22.44×10^{-4}, M_{cr} becomes 174,960 inch-pounds.

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TABLE 1

TABULATION OF RESULTS

S	α (deg)	M_{theor} (in.-lb)	M_{exp} (in.-lb)	Percent Difference	δ/r	ϵ_{max}	s	(m + 1)	-a	-b
8	45	174,960	113,000	54.8	0.07	22.44×10^{-4}	3	7	3.4349	1.7425
	90	142,890	107,900	32.4	.07	18.67	4	10	2.7724	2.7464
	135	116,070	100,000	16.1	.05	16.44	4	10	2.8213	2.8887
16	45	342,680	232,500	47.4	0.05	24.83	6	7	2.3523	3.1193
	90	225,690	172,600	30.8	.04	16.76	6	7	2.3449	3.1744
	135	164,470	169,900	-3.2	.04	14.13	6	7	2.3471	3.2175

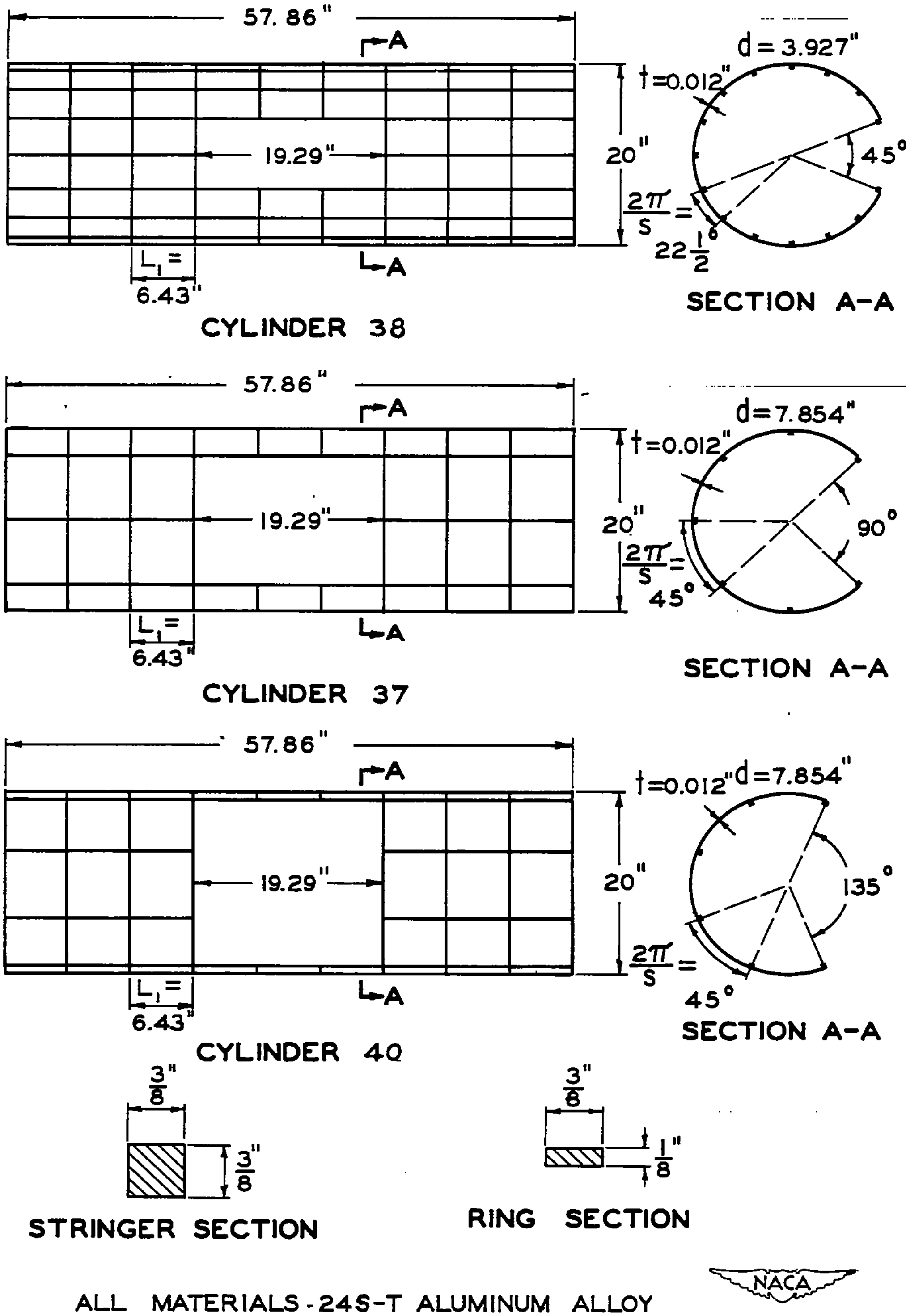

Figure 1.- Typical monocoque cylinders.

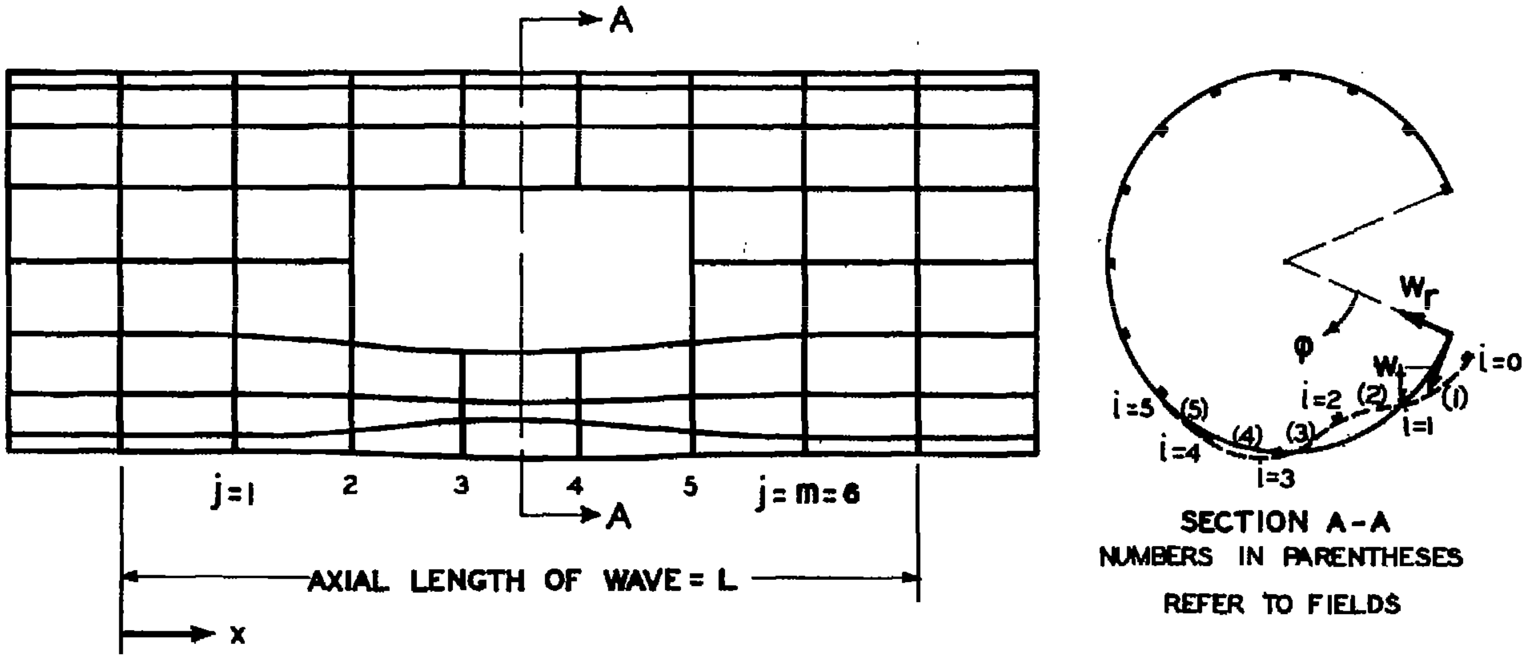

Figure 2.- Deflected shape.

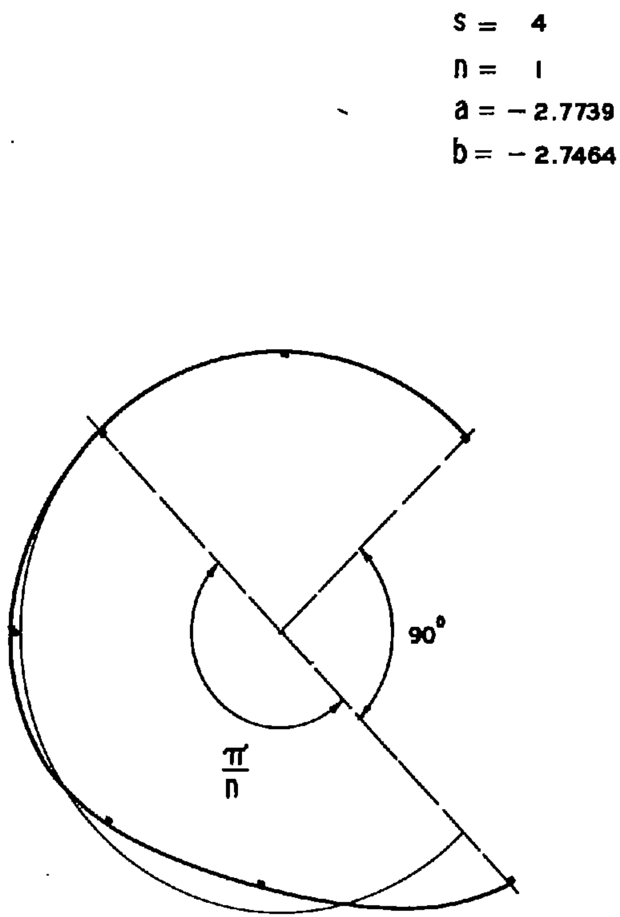

Figure 3.- Deflected shape of ring in its own plane (according to theory, exaggerated). Cylinder 37; 8 stringers; 90° cutout.

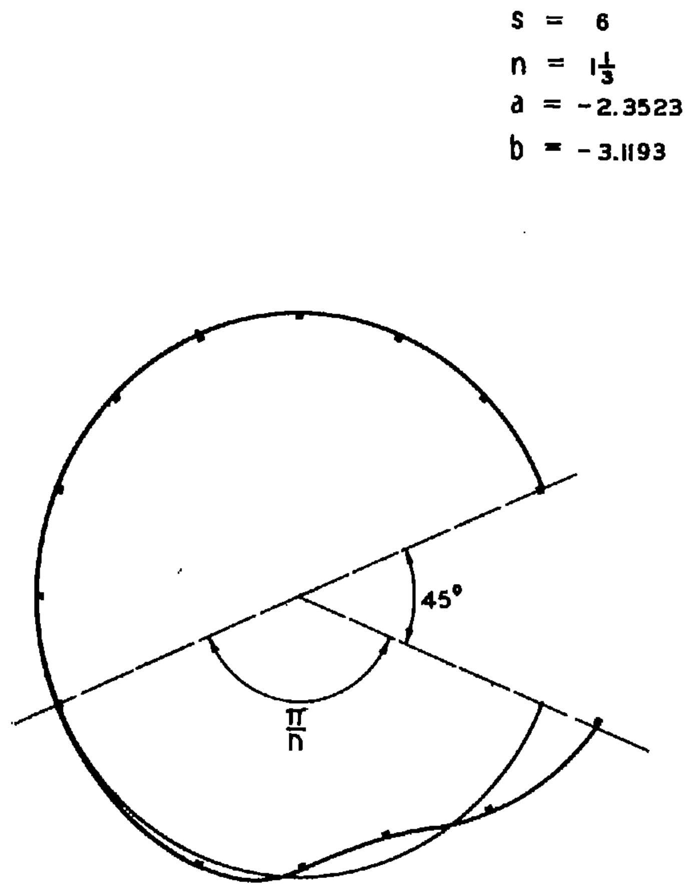

Figure 4.- Deflected shape of ring in its own plane (according to theory, exaggerated). Cylinder 38; 16 stringers; 45° cutout.

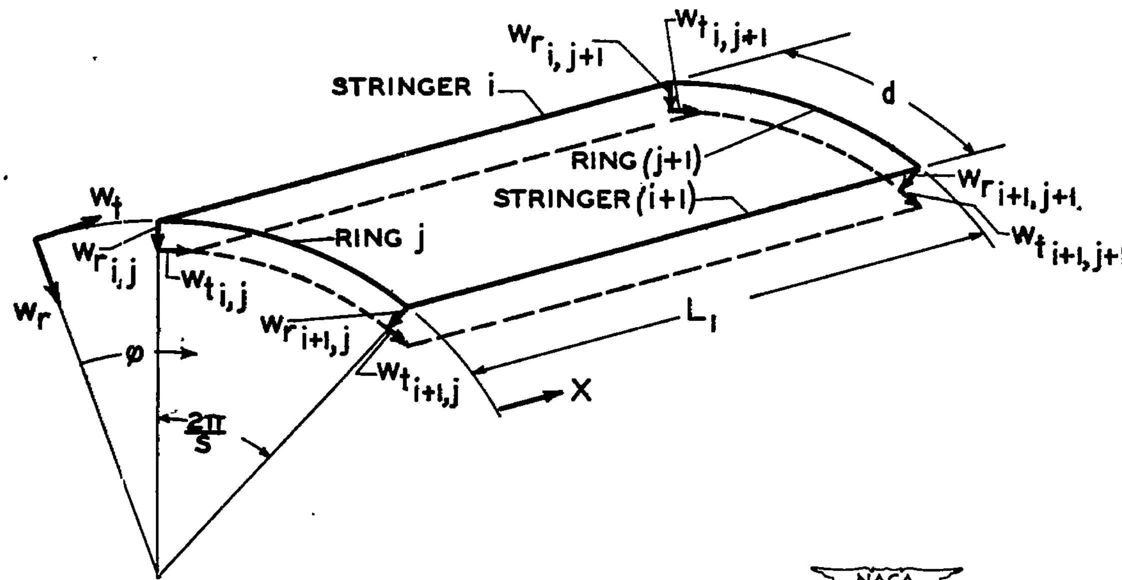

Figure 5.- Deformations of panel corners with notation for shear strain calculations.

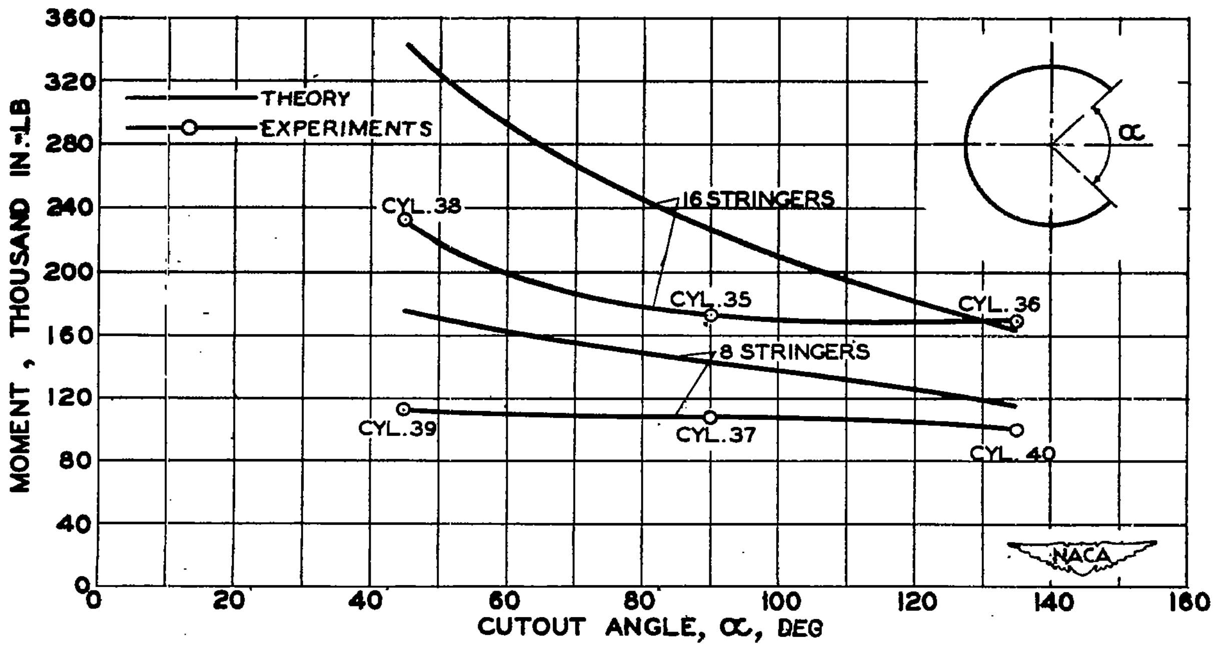

Figure 6.- Comparison of calculated and experimental critical moments.