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TECHNICAL NOTE

No. 1565

BUCKLING IN SHEAR OF CONTINUOUS FLAT PLATES

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SUMMARY

As basic information for the design of thin-web spars, the theoretical shear buckling stress is presented for an infinitely long, clamped plate divided into square panels by rigid intermediate supports. In addition, results are given for the shear buckling stress of a plate of infinite length and width having intermediate rigid supports that form an array of square panels. The results indicate the fallacy of the usually made assumption that each panel buckles in shear as if it were simply supported along the intermediate supports.

INTRODUCTION

The design of a thin-web beam requires a knowledge of the shear buckling stress of the web. In many cases the beam has fairly heavy flanges, and many equally spaced, intermediate stiffeners of high flexural rigidity and low torsional rigidity. The shear buckling of the web of such a beam may be analyzed by considering the web to be a long clamped plate, divided into equal panels by intermediate supports which completely restrain the plate from deflecting but offer no torsional restraint. (See fig. 1.)

The assumption is implicit in the work of many writers that each panel of the plate shown in figure 1 would buckle in shear as if it were simply supported along the intermediate supports. (See references 1, 2, and 3.) Although the assumption is valid for the compressive buckling of continuous plates having equal bays, it does not hold true for shear buckling. Adjacent panels buckling in shear restrain each other, so that the plate bending moments do not vanish along the intermediate supports, as they must for simple support. This continuity effect is probably a maximum for nearly square panels; from physical considerations it is evident that the continuity effect disappears for very small and very large values of the ratio b/a.

The present paper gives the theoretical shear buckling stress for an infinitely long, clamped plate divided into square bays by intermediate rigid supports. In addition, as further information on the effect of plate continuity on shear buckling stresses, results are given for the shear buckling stress of a plate of infinite length and width having intermediate rigid supports which form an array of square panels. (See fig. 2.) The theoretical analyses of both problems, performed by the Lagrangian multiplier method, are presented in appendixes.

SYMBOLS

a	length of panel
b	width of panel
t	plate thickness
E	Young's modulus of elasticity
μ	Poisson's ratio
D	plate stiffness in bending $\left(\frac{Et^3}{12(1 - \mu^2)}\right)$
τ	critical shear stress
k_s	critical-shear-stress coefficient in the formula $\tau = k_s\left(\frac{\pi^2 D}{b^2 t}\right)$
x	plate coordinate parallel to length
y	plate coordinate parallel to width
w	deflection normal to plane of the plate
$a_{mn},b_{mn},c_{mn},d_{mn}$	Fourier coefficients
$\alpha,\mu_i,\lambda_j,\eta_i$	Lagrangian multipliers
V	internal bending energy
T	external work of applied stress
m,n,i,j,p,q	integers
δ_{mn}	Kronecker delta: 1 if $m = n$; 0 if $m \neq n$

RESULTS AND DISCUSSION

Infinitely Long Clamped Plate with Square Bays

The critical shear stress of the plate shown in figure 1 is given by the formula

$$\tau = k_s\left(\frac{\pi^2 D}{b^2 t}\right)$$

The theoretical analysis given in appendix A yields the result

$$k_s = 13.14$$

for the case of square bays. This result, which represents the average of upper-limit and lower-limit solutions, is within 0.6 percent of the true buckling stress coefficient.

It is of interest to compare the present result for a continuous plate with the shear buckling stress coefficients of square plates with (a) two opposite edges clamped and the other two simply supported, and (b) all four edges clamped. This comparison is shown in the following table:

Boundary conditions	k_s
Two opposite edges clamped, two simply supported	[a]12.6 [b]12.28
Two opposite edges clamped, two continuous	13.14
All edges clamped	[c]14.71

[a]Upper limit (reference 1).
[b]Reference 4.
[c]Maximum error 0.6 percent (reference 5, appendix A).

Thus, for square bays, changing the boundary conditions along the transverse edges from simple support to continuity provides about 25 percent of the increase in shear buckling stress that would be obtained by changing the boundary conditions from simple support to clamped edges.

Doubly Infinite Array of Square Panels

The buckling in shear of the doubly infinite array of square panels continuous over rigid supports shown in figure 2 is analyzed in appendix B. The result for the critical-shear-stress coefficient, obtained to four-figure accuracy by means of upper-limit and lower-limit solutions, is

$$k_s = 11.10$$

Comparison of this result with those for square plates having other boundary conditions is shown in the following table:

Boundary conditions	k_s
All edges simply supported	[a]9.35
All edges continuous	11.10
All edges clamped	14.71

[a]Reference 6.

Thus, continuity of the square panels at all edges provides over 30 percent of the increase in shear buckling stress that would be obtained by clamping all edges of a simply supported square plate.

CONCLUDING REMARKS

The theoretical results presented indicate the fallacy of the usually made assumption that continuous plates having equal, finite bays buckle in shear as if each bay were simply supported at the intermediate supports. The increase in buckling stress due to continuity at the supported edges of square panels is of the order of 25 percent of the increase that would be provided by clamping the edges. This continuity effect is probably a maximum for nearly square panels; from physical considerations it is evident that the continuity effect disappears for very small and very large values of the width-length ratio b/a.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., January 20, 1948

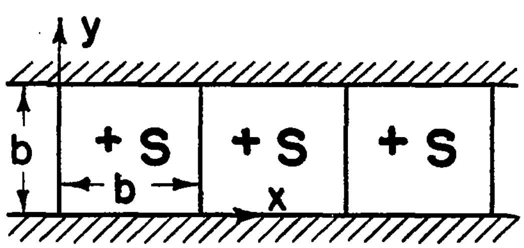
APPENDIX A

INFINITELY LONG CLAMPED PLATE WITH SQUARE BAYS

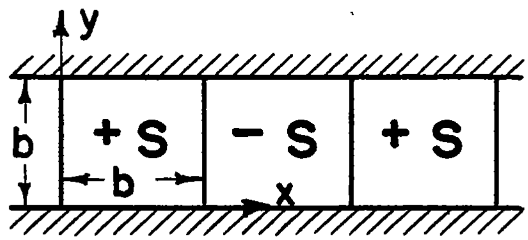
The shear buckling of the plate shown in figure 1 will be analyzed by the Lagrangian multiplier method (references 7 and 5) to obtain both upper and lower limits to the true buckling stress. Several possible types of buckling configurations will be investigated; the correct configuration is that which corresponds to the lowest buckling load.

Buckling configurations.— It is intuitively evident that, for square bays, each bay will have the same buckling configuration; however, it is not evident whether all bays will buckle in the same direction, or whether the directions will alternate. It is probable that the deflection of each square bay of the continuous plate is symmetrical about the bay midpoint since simply supported square plates (reference 6) and clamped square plates (reference 8) buckle in shear with symmetrical patterns. However, for completeness, the possibility of an antisymmetrical deflection in each bay will be investigated.

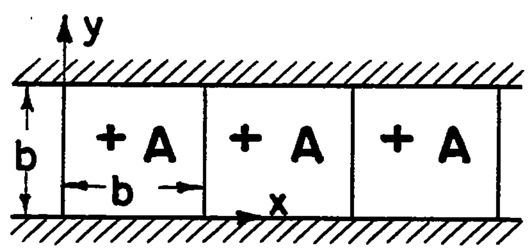
Boundary and continuity conditions, and deflection functions.— From the many possible types of Fourier series discussed in appendix B of reference 5, suitable series will be chosen to represent the various possible types of buckling patterns. Schematic representations of the three buckling configurations considered (where the notations S and A refer to buckling patterns that are symmetrical and antisymmetrical about the bay midpoint, respectively, and the signs preceding these notations indicate direction of buckling), with the series chosen for these configurations, and the conditions of continuity which they satisfy, are as follows:



$$\left.\begin{aligned} w &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{2m\pi x}{b}\sin\frac{2n\pi y}{b} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{mn}\cos\frac{2m\pi x}{b}\cos\frac{2n\pi y}{b} \\ \frac{\partial w}{\partial x}(0,y) &= \frac{\partial w}{\partial x}(b,y) \end{aligned}\right\} \quad \text{(A1a)}$$



$$\left.\begin{aligned} w = &\sum_{m=1,3,5,\ldots}^{\infty}\sum_{n=1}^{\infty} a_{mn}\cos\frac{m\pi x}{b}\sin\frac{2n\pi y}{b} \\ &+ \sum_{m=1,3,5,\ldots}^{\infty}\sum_{n=0}^{\infty} d_{mn}\sin\frac{m\pi x}{b}\cos\frac{2n\pi y}{b} \\ \frac{\partial w}{\partial x}(0,y) &= -\frac{\partial w}{\partial x}(b,y) \end{aligned}\right\} \quad \text{(A1b)}$$



$$\left.\begin{aligned} w = &\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} b_{mn}\sin\frac{2m\pi x}{b}\cos\frac{2n\pi y}{b} \\ &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{mn}\cos\frac{2m\pi x}{b}\sin\frac{2n\pi y}{b} \\ \frac{\partial w}{\partial x}(0,y) &= \frac{\partial w}{\partial x}(b,y) \end{aligned}\right\} \quad \text{(A1c)}$$

The conditions of continuity are satisfied by these deflection functions term by term; but the boundary conditions at the clamped edges,

$$w(x,0) = w(x,b) = 0 \qquad \text{(A2)}$$

$$\frac{\partial w}{\partial y}(x,0) = \frac{\partial w}{\partial y}(x,b) = 0 \qquad \text{(A3)}$$

and the condition of zero deflection at the intermediate supports,

$$w(0,y) = w(b,y) = 0 \qquad \text{(A4)}$$

will be introduced by means of Lagrangian multipliers.

The remainder of the derivation will be performed for the first of the buckling configurations (A1a) (which, as will be shown, is actually the governing one); the details of analysis for the other two buckling patterns are analogous to those presented.

Energy expressions.- The internal energy of the plate V and the external work of the stresses T are given by the expressions

$$V = \frac{D}{2}\int_0^b \int_0^b \left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1 - \mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\} dx\,dy$$

$$T = -\tau t\int_0^b \int_0^b \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\,dx\,dy$$

Substituting the chosen expansion for w (equation (A1a)) into these energy integrals gives

$$V = \frac{2\pi^4 D}{b^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(m^2 + n^2\right)^2 \left[a_{mn}^2 \left(1 - \delta_{m0} - \delta_{0n}\right) + d_{mn}^2 \left(1 + \delta_{m0} + \delta_{0n}\right)\right]$$

$$T = 2\tau t \pi^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mna_{mn} d_{mn}$$

Then

$$(V - T)\frac{b^2}{2\pi^4 D} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{\frac{1}{2} A_{mn} \left[a_{mn}^2 \left(1 - \delta_{m0} - \delta_{0n}\right) + d_{mn}^2\right] - k_s mna_{mn} d_{mn}\right\} \tag{A5}$$

where

$$A_{mn} = 2\left(m^2 + n^2\right)\left(1 + \delta_{m0} + \delta_{0n}\right)$$

Note that $V - T$ is independent of d_{00}, since $A_{00} = 0$.

Constraining relationships.- In order to satisfy the boundary conditions of zero deflection (equations (A2) and (A4)), it is necessary to impose the following constraint relationships:

$$\sum_{m=0}^{\infty} d_{mj} = 0 \qquad (j = 0,1,2,\ldots) \tag{A6a}$$

$$\sum_{n=0}^{\infty} d_{in} = 0 \qquad (i = 0,1,2,\ldots) \tag{A6b}$$

Similarly, in order to satisfy the condition of zero slope along the clamped edges (equation (A3)), it must be true that

$$\sum_{n=1}^{\infty} na_{in} = 0 \qquad (i = 1,2,3,\ldots) \tag{A7}$$

The term d_{00}, which does not appear in the energy expression $V - T$ (equation (A5)), may be eliminated from these constraining relationships by subtracting the first of equations (A6b), the equation for $i = 0$, from the first of equations (A6a), the equation for $j = 0$. The necessary constraining relations finally become

$$\left.\begin{aligned} &\sum_{m=1}^{\infty} d_{m0} - \sum_{n=1}^{\infty} d_{0n} = 0 \\ &\sum_{m=0}^{\infty} d_{mj} = 0 \qquad (j = 1,2,3,\ldots) \\ &\sum_{n=0}^{\infty} d_{in} = 0 \qquad (i = 1,2,3,\ldots) \end{aligned}\right\} \tag{A8a}$$

$$\sum_{n=1}^{\infty} na_{in} = 0 \qquad (i = 1,2,3,\ldots) \tag{A8b}$$

Lower-limit solution.- A lower-limit solution requires that $V - T$ (equation (A5)) be minimized with respect to all the coefficients a_{mn} and d_{mn}, whereas the constraining relationships (equations (A8a) and (A8b)) are to be satisfied only as far as $i = p$ and $j = q$. As required by the Lagrangian multiplier method, the expression to be minimized is

$$G = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{1}{2} A_{mn} \left[a_{mn}^2 \left(1 - \delta_{m0} - \delta_{0n}\right) + d_{mn}^2 \right] - k_s mn a_{mn} d_{mn} \right\}$$

$$- \alpha \left(\sum_{m=1}^{\infty} d_{m0} - \sum_{n=1}^{\infty} d_{0n} \right) - \sum_{j=1}^{q} \lambda_j \sum_{m=0}^{\infty} d_{mj}$$

$$- \sum_{i=1}^{p} \mu_i \sum_{n=0}^{\infty} d_{in} - \sum_{i=1}^{p} \eta_i \sum_{n=1}^{\infty} n a_{in} \qquad \text{(A9)}$$

with $\eta_i = 0$ for $i > p$

The quantities α, λ_j, μ_i, and η_i are Lagrangian multipliers. The equations for minimizing $V - T$ while satisfying the constraining relationships on the a's and d's (equations (A8a) and (A8b) up to $i = p$ and $j = q$) become

$$\left.\begin{aligned} &\frac{\partial G}{\partial a_{mn}} = 0 \qquad (m,n = 1,2,3,\ldots) \\ &\frac{\partial G}{\partial d_{mn}} = 0 \qquad (m,n = 0,1,2,\ldots) \\ &\text{Equations (A8a) and (A8b) up to } i = p \text{ and } j = q \end{aligned}\right\} \qquad \text{(A10)}$$

By evaluation,

$$\frac{\partial G}{\partial a_{mn}} = A_{mn} a_{mn} - k_s mn d_{mn} - n \eta_m = 0 \qquad \text{(A11)}$$

$$\frac{\partial G}{\partial d_{mn}} = A_{mn} d_{mn} - k_s mn a_{mn} - \lambda_n - \mu_m = 0 \qquad (m,n \neq 0) \qquad \text{(A12)}$$

$$\frac{\partial G}{\partial d_{m0}} = A_{m0}d_{m0} - \alpha - \mu_m = 0 \tag{A13}$$

$$\frac{\partial G}{\partial d_{0n}} = A_{0n}d_{0n} + \alpha - \lambda_n = 0 \tag{A14}$$

In equations (A11) to (A14) λ_n does not appear for $n > q$, nor do η_m and μ_m for $m > p$. When both $m > p$ and $n > q$, the multipliers λ_n, η_m, and μ_m vanish from equations (A11) and (A12). Then one of two conditions is possible: either

$$A_{mn}^2 - k_s^2 m^2 n^2 = 0$$

or

$$a_{mn} = d_{mn} = 0$$

The first alternative, however, for given values of $m > p$ and $n > q$, will ordinarily lead to a very high value of the buckling coefficient k_s. For the lowest buckling load, therefore, when $m > p$ and $n > q$,

$$a_{mn} = d_{mn} = 0$$

For the remaining a's and d's, solving equations (A11) and (A12) gives

$$a_{mn} = nB_{mn}\eta_m + C_{mn}\left(\lambda_n + \mu_m\right) \tag{A15a}$$

$$d_{mn} = B_{mn}\left(\lambda_n + \mu_m\right) + nC_{mn}\eta_m \tag{A15b}$$

where

$$B_{mn} = \frac{A_{mn}}{A_{mn}^2 - D_{mn}^2}$$

$$C_{mn} = \frac{D_{mn}}{A_{mn}^2 - D_{mn}^2}$$

in which

$$D_{mn} = k_s mn$$

From equations (A13) and (A14),

$$d_{m0} = \frac{1}{A_{m0}}\left(\alpha + \mu_m\right) \qquad \text{(A16)}$$

$$d_{0n} = \frac{1}{A_{0n}}\left(-\alpha + \lambda_n\right) \qquad \text{(A17)}$$

It is to be emphasized that, in equations (A15a) to (A17),

$$\eta_m = \mu_m = 0 \qquad \text{for} \quad m > p$$

and

$$\lambda_n = 0 \qquad \text{for} \quad n > q$$

Substituting the values of a_{mn} and d_{mn} given by equations (A15a) to (A17) in constraining relationships (A8a) and (A8b) up to $j = q$ and $i = p$ gives

$$\alpha\left(\sum_{m=1}^{\infty} B_{m0} + \sum_{n=1}^{\infty} B_{0n}\right) + \sum_{m=1}^{p} B_{m0}\mu_m - \sum_{n=1}^{q} B_{0n}\lambda_n = 0 \qquad \text{(A18a)}$$

$$\lambda_j \sum_{m=0}^{\infty} B_{mj} - \alpha B_{0j} + \sum_{m=1}^{p} B_{mj}\mu_m + \sum_{m=1}^{p} jC_{mj}\eta_m = 0 \qquad (j = 1,2,3,\ldots q) \qquad \text{(A18b)}$$

$$\mu_i \sum_{n=0}^{\infty} B_{in} + \alpha B_{i0} + \sum_{n=1}^{q} B_{in}\lambda_n + \eta_i \sum_{n=1}^{\infty} nC_{in} = 0 \qquad (i = 1,2,3,\ldots p) \qquad \text{(A18c)}$$

$$\eta_i \sum_{n=1}^{\infty} n^2 B_{in} + \sum_{n=1}^{q} nC_{in}\lambda_n + \mu_i \sum_{n=1}^{\infty} nC_{in} = 0 \qquad (i = 1,2,3,\ldots p) \qquad \text{(A18d)}$$

In order for these $2p + q + 1$ equations to be compatible, the determinant of the coefficients of the Lagrangian multipliers must vanish. From this determinantal equation the critical value of the buckling coefficient may be found. Those terms in the stability determinant involving infinite summations may be evaluated by means of computation aids similar to those described in the appendix of reference 8.

Upper-limit solution.— The theory of the upper-limit solution in the Lagrangian multiplier method (reference 7) requires that some a's and d's arbitrarily be set equal to zero, that equation (A5) be minimized with respect to all the remaining a's and d's, and that all the constraining relationships of equations (A8a) and (A8b) be satisfied.

As a result of the necessity for satisfying all the constraining relationships, a redundancy exists among equations (A8a) (see reference 7); this redundancy can be removed by discarding the first of equations (A8a). The necessary constraint relationships now become

$$\sum_{m=0}^{\infty} d_{mj} = 0 \qquad (j = 1,2,3,\ldots) \tag{A19a}$$

$$\sum_{n=0}^{\infty} d_{in} = 0 \qquad (i = 1,2,3,\ldots) \tag{A19b}$$

$$\sum_{n=1}^{\infty} na_{in} = 0 \qquad (i = 1,2,3,\ldots) \tag{A19c}$$

The arrays of Fourier coefficients to be retained in an upper-limit solution are chosen with the intention of combining accuracy with ease of solution. The procedure followed in the analysis of the shear buckling of a clamped plate (reference 5, appendix A) suggests that the restriction on the existence of the coefficients can be as follows:

$$d_{mn} = 0 \qquad (\text{when either } m > p \text{ or } n > q)$$

$$a_{mn} = 0 \qquad (\text{when } m > p)$$

When these limits are imposed on the existence of the coefficients, the constraint relationships (equations (A19)) take the form

$$\sum_{m=0}^{p} d_{mj} = 0 \qquad (j = 1,2,3,\ldots q) \tag{A20a}$$

$$\sum_{n=0}^{q} d_{in} = 0 \qquad (i = 1,2,3,\ldots p) \tag{A20b}$$

$$\sum_{n=1}^{\infty} na_{in} = 0 \qquad (i = 1,2,3,\ldots p) \tag{A20c}$$

The function to be minimized is

$$G = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{1}{2}A_{mn}\left[a_{mn}^2\left(1 - \delta_{m0} - \delta_{0n}\right) + d_{mn}^2\right] - k_s mna_{mn}d_{mn}\right\}$$

$$-\sum_{j=1}^{q}\lambda_j\sum_{m=0}^{p}d_{mj} - \sum_{i=1}^{p}\mu_i\sum_{n=0}^{q}d_{in} - \sum_{i=1}^{p}\eta_i\sum_{n=1}^{\infty}na_{in} \tag{A21}$$

Setting $\dfrac{\partial G}{\partial a_{mn}} = \dfrac{\partial G}{\partial d_{mn}} = 0$ then gives

$$a_{mn} = nB_{mn}\eta_m + C_{mn}\left(\lambda_n + \mu_m\right) \qquad (\text{for} \quad d_{mn} \neq 0)$$

$$a_{mn} = \frac{n}{A_{mn}}\eta_m \qquad (\text{for} \quad d_{mn} = 0)$$

$$d_{mn} = B_{mn}\left(\lambda_n + \mu_m\right) + nC_{mn}\eta_m \qquad (\text{for} \quad m,n \neq 0)$$

$$d_{m0} = \frac{1}{A_{m0}}\mu_m$$

$$d_{0n} = \frac{1}{A_{0n}}\lambda_n$$

Substituting these values in the constraint relationships (equations (A20)) gives the final stability equations:

$$\lambda_j \sum_{m=0}^{p} B_{mj} + \sum_{m=1}^{p} B_{mj}\mu_m + \sum_{m=1}^{p} jC_{mj}\eta_m = 0 \qquad (j = 1,2,3,\ldots q) \qquad \text{(A22a)}$$

$$\sum_{n=1}^{q} B_{in}\lambda_n + \mu_i \sum_{n=0}^{q} B_{in} + \eta_i \sum_{n=1}^{q} nC_{in} = 0 \qquad (i = 1,2,3,\ldots p) \qquad \text{(A22b)}$$

$$\eta_i\left(\sum_{n=1}^{q} n^2 B_{in} + \sum_{n=q+1}^{\infty} \frac{n^2}{A_{in}}\right) + \mu_i \sum_{n=1}^{q} nC_{in} + \sum_{n=1}^{q} nC_{in}\lambda_n = 0 \qquad (i = 1,2,3,\ldots p) \qquad \text{(A22c)}$$

The stability criterion is the determinant of the coefficients of the Lagrangian multipliers in equations (A22).

Numerical results.- Numerical results for the critical-shear-stress coefficient obtained in the present case are as follows:

Approximation	Lower limit	Upper limit
First; $p = q = 1$	12.50	13.56
Second; $p = q = 2$	13.06	13.22

Thus, the average of the upper-limit and lower-limit results of the second approximation must be within 0.6 percent of the true buckling stress coefficient.

Lower-limit results for the other types of buckling configurations were sufficient to indicate that the buckling configuration just analyzed corresponds to the lowest buckling stress. Analysis of the second configuration (equation (A1b)) provided the lower-limit result $k_s = 14.29$ with a sixth-order determinant; calculations for the third buckling pattern (equation (A1c)) gave the lower-limit value $k_s = 14.1$ with a fourth-order determinant. Since both of these values are higher than the upper-limit result for the first buckling configuration (equation (A1a)), it is evident that this first configuration is the governing one for the case of square bays.

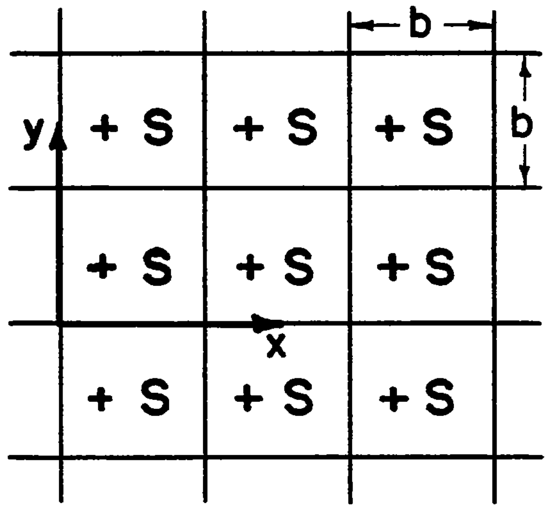
Although the same general method of analysis outlined in this appendix may be applied to the problem of the buckling of a continuous plate where the bays are not square, care must be taken to investigate all the possible buckling configurations; it may no longer be true that each bay will exhibit the same configuration as every other bay. Also, as the depth of the bays increases, the effect of continuity decreases, until, when b/a is infinite (see fig. 1), adjacent bays act as simply supported strips.

APPENDIX B

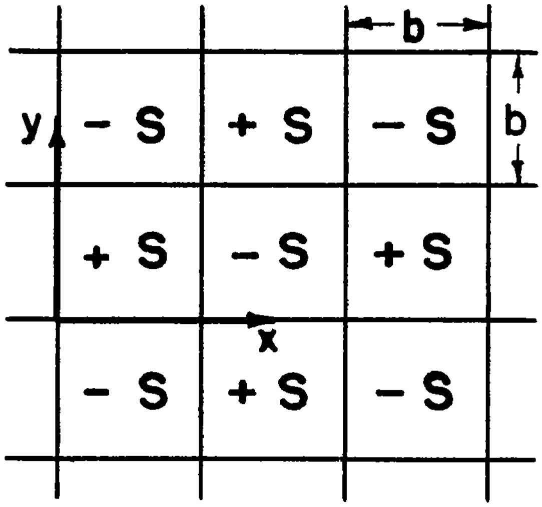
DOUBLY INFINITE ARRAY OF SQUARE PANELS

The shear buckling of the plate shown in figure 2 will be analyzed by a method closely analogous in general principle and specific detail to that of appendix A. Two possible buckling configurations will be investigated; as before, the governing configuration is that which provides the lowest buckling load.

<u>Boundary and continuity conditions, and deflection functions</u>.– In this problem it is quite evident that the buckling configuration will be symmetrical about the midpoint of each bay; it is only necessary, therefore, to investigate the question of whether the direction of buckling is the same or alternating from bay to bay. The two possible configurations may be represented by the following series, which satisfy the required continuity conditions term by term:



$$\left.\begin{aligned} w &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{2m\pi x}{b} \sin\frac{2n\pi y}{b} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{mn} \cos\frac{2m\pi x}{b} \cos\frac{2n\pi y}{b} \\ \frac{\partial w}{\partial x}(0,y) &= \frac{\partial w}{\partial x}(b,y) \\ \frac{\partial w}{\partial y}(x,0) &= \frac{\partial w}{\partial y}(x,b) \end{aligned}\right\} \quad \text{(B1a)}$$



$$\left.\begin{aligned} w = & \sum_{m=1,3,5,\ldots}^{\infty} \sum_{n=1,3,5,\ldots}^{\infty} a_{mn} \cos\frac{m\pi x}{b} \cos\frac{n\pi y}{b} \\ & + \sum_{m=1,3,5,\ldots}^{\infty} \sum_{n=1,3,5,\ldots}^{\infty} d_{mn} \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{b} \\ & \frac{\partial w}{\partial x}(0,y) = -\frac{\partial w}{\partial x}(b,y) \\ & \frac{\partial w}{\partial y}(x,0) = -\frac{\partial w}{\partial y}(x,b) \end{aligned}\right\} \qquad \text{(B1b)}$$

The boundary conditions of zero deflection at the supports must be introduced by means of Lagrangian multipliers.

Stability criterions.- The first of the aforementioned buckling configurations (equations (B1a)) was the governing one; the remainder of this analysis will be performed for that case.

It may be seen that the deflection function chosen to represent the buckling configuration in the present problem is identical with that employed in the detailed analysis of appendix A. As a result of the close similarity of the two problems, much of the analysis of appendix A may be applied to the present problem with only simple revisions. The constraining relationships are the same for the two cases except that in the present problem no condition of zero slope at the boundaries exists; therefore, the condition of equation (A7) is omitted, and, in equations (A9) and (A21), the multiplier η_i is set equal to zero.

A further simplification occurs in the present problem because of the symmetry about both diagonals of the buckling pattern within each bay. As a result it can be proved that, among the terms of deflection function w,

$$a_{mn} = a_{nm}$$

$$d_{mn} = d_{nm}$$

Consequently, in the application of equations (A9) and (A21) to the present case,

$$\alpha = 0$$

and

$$\lambda_j = \mu_i \quad \text{for} \quad i = j$$

Hence the stability equations may be written directly from equations (A18b) and (A22a) by employing the following relations:

$$\eta_i = 0$$

$$\alpha = 0$$

$$\lambda_j = \mu_i \quad \text{for} \quad i = j$$

First, for the lower-limit result, from equation (A18b), the stability equations are

$$\lambda_j \sum_{m=0}^{\infty} B_{mj} + \sum_{m=1}^{p} B_{mj}\lambda_m = 0 \qquad (j = 1,2,3,\ldots p) \tag{B2}$$

For the upper-limit result, from equation (A23a), the stability equations become

$$\lambda_j \sum_{m=0}^{p} B_{mj} + \sum_{m=1}^{p} B_{mj}\lambda_m = 0 \qquad (j = 1,2,3,\ldots p) \qquad \text{(B3)}$$

Numerical results.- The computed results for k_s in the present problem are tabulated below:

Approximation	Lower limit	Upper limit
First; p = 2	11.07	11.18
Second; p = 3	11.10	11.10

The lower-limit result for k_s obtained from the other buckling configuration (equation (B1b)) is $k_s = 13.10$; this result indicates that the first pattern is the governing one.

Extension of this analysis to bays other than square is subject to the same qualification concerning the introduction of other possible buckling configurations as that stated in appendix A.

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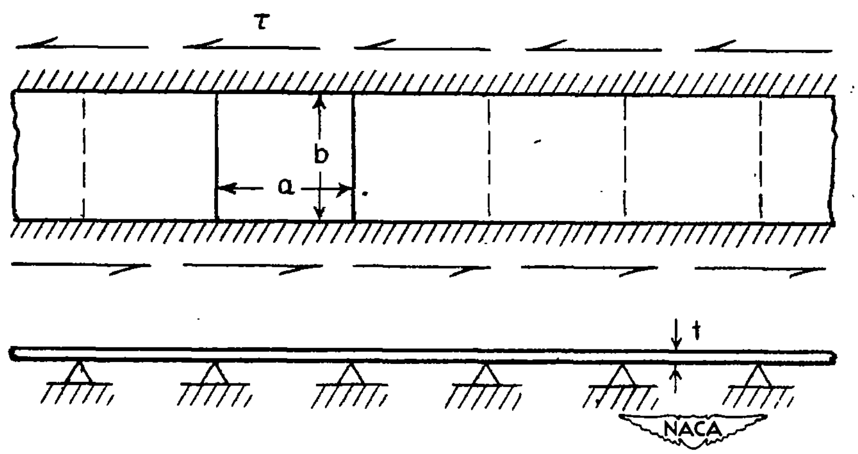

Figure 1.- Infinitely long clamped plate with equally spaced intermediate rigid supports.

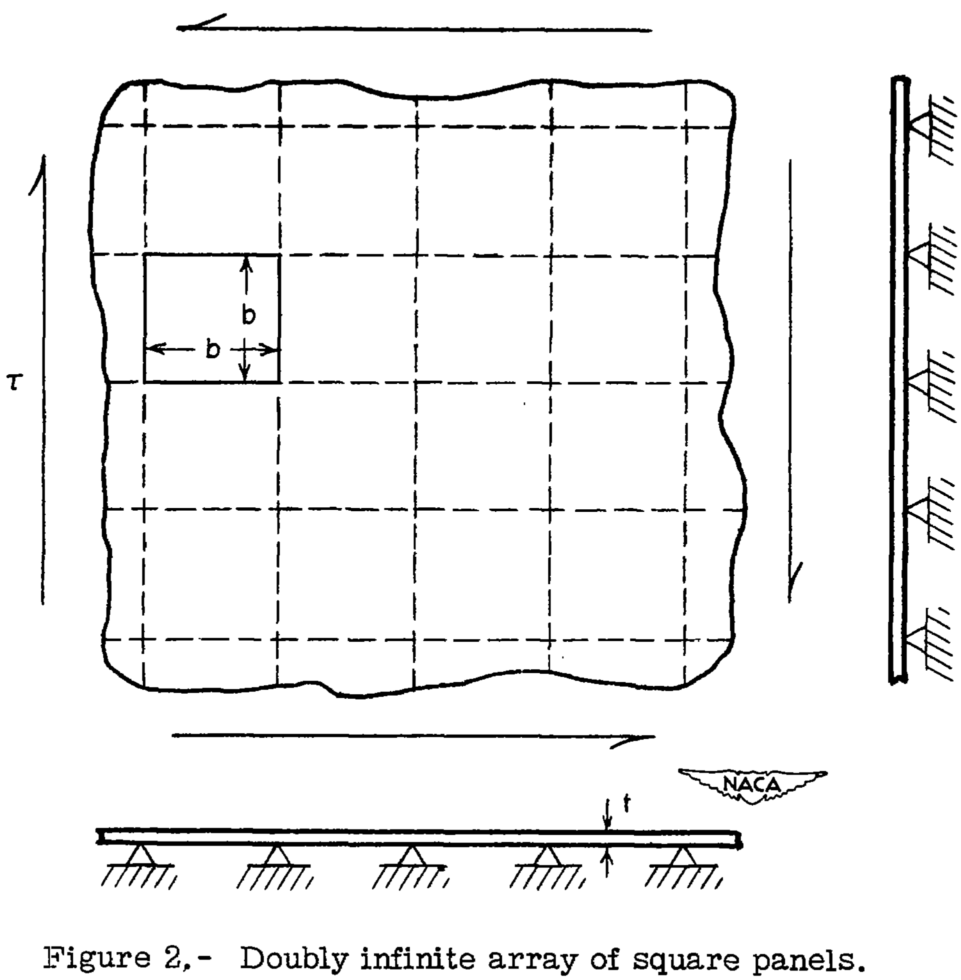

Figure 2.- Doubly infinite array of square panels.