NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1786

RECOMMENDATIONS FOR NUMERICAL SOLUTION OF REINFORCED-PANEL AND FUSELAGE-RING PROBLEMS

By N. J. Hoff and Paul A. Libby

Polytechnic Institute of Brooklyn

Washington
December 1948

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 1786

RECOMMENDATIONS FOR NUMERICAL SOLUTION OF REINFORCED-PANEL AND FUSELAGE-RING PROBLEMS

By N. J. Hoff and Paul A. Libby

SUMMARY

Procedures are recommended for solving the equations of equilibrium of reinforced panels and isolated fuselage rings as represented by the external loads and the operations table established according to Southwell's method. From the solution of these equations the stress distribution can be easily determined. The recommendations are based on the experience of the past 4 years in applying numerical procedures to monocoque stress analysis at the Polytechnic Institute of Brooklyn Aeronautical Laboratories. The method of systematic relaxations, the matrix calculus method, and several other methods applicable in special cases are discussed.

Definite recommendations are made for obtaining the solution of reinforced-panel problems which are generally designated as shear lag problems. The procedures recommended are demonstrated in the analysis of a number of panels, several of which were discussed in previous PIBAL reports, whereas others are shown for the first time.

In the case of fuselage rings it is not possible to make definite recommendations for the solution of the equilibrium equations for all rings and loadings. However, suggestions based on the latest experience are made and demonstrated on several rings.

INTRODUCTION

The application of the indirect methods of Hardy Cross (reference 1) and R. V. Southwell (reference 2) to the analysis of monocoque structures has been shown in a series of investigations (references 3 to 8) carried out at the Polytechnic Institute of Brooklyn Aeronautical Laboratories. These indirect methods are likely to lead to solutions of problems in stress analysis that are intractable by direct analytical methods because the structure is tapered, has large cut-outs, its reinforcing elements are distributed irregularly, or the like.

The distorted shape corresponding to equilibrium under the applied loads is determined first in the indirect methods. From it the stresses, forces, and moments required can be calculated without difficulty. This approach is justified by the comparative ease with which the stresses in a complex structure can be determined for an individual displacement of one point and with which the final distorted shape of a complex structure can be represented by a summation of such individual displacements.

The complete structure is considered to be composed of appropriate elements and its degrees of freedom are the displacements of the several reference points on the boundary of each element. Each of these points is displaced in turn and the reactions at the reference points caused by the displacement are listed. If by suitable displacements of all points the reaction forces and moments are made equal and opposite to the external loads at each point, the whole structure is in equilibrium and its distorted shape is determined.

In applying the indirect methods to monocoque structures the terminology of Southwell (reference 2) has been retained. Thus, the elements which compose the complete structure are "units" and the determination of the forces and moments due to a displacement of a boundary point of such units is termed the "unit problem." The magnitudes of these forces and moments are given by "influence coefficients." The complete effect of a displacement is given in an "operations table," and the step-by-step process, which can be employed to determine the equilibrium distorted shape, is called the "method of systematic relaxations." At each step of this process forces and moments referred to as "residuals" remain unbalanced at each point in the structure. A running account of the residuals and of the displacements or "operations" undertaken is kept in the "relaxation table."

The operations table along with the external forces constitutes a system of linear equations, which are equal in number to the degrees of freedom of the structure and which have as variables the displacements. Each equation represents the condition of equilibrium for the force or moment associated with one degree of freedom. When the method of systematic relaxations is applied an approximate solution to this system of equations and accordingly an approximate equilibrium state of the structure are found.

The indirect method of analysis just outlined has been applied at PIBAL to the reinforced-panel and ring components of a monocoque structure as well as to complete circular cylinders with and without cut-outs. In references 3 and 4 the stress distribution in the sheet and stringers of a reinforced panel was determined under loads applied parallel to the stringers. Fuselage rings with and without internal bracing elements were investigated in reference 5. The determination of the influence coefficients for the ring unit problem was found to involve considerable computational work and therefore appropriate graphs and tables are given in reference 6 to facilitate their calculation. In references 7 and 8 the elements, namely, the reinforced panel and the ring, are combined into a circular cylinder and the stress distribution in the cylinder was investigated for the case when the loading is a pure bending moment.

In the application of the indirect-stress-analysis methods to the problems mentioned the major obstacle has been to find an approximate solution of the system of equations with a reasonable expenditure of effort. In each problem it has been readily possible to establish satisfactory units and to combine them to represent the complex structure. During the past 4 years considerable experience has been gained at PIBAL in overcoming this obstacle to the wider application of numerical procedures in the analysis of monocoque structures. On the basis of this experience some recommendations can be made as to the most expeditious method of solving reinforced-panel and fuselage-ring problems after the operations table has been established as described in references 3 to 5.

In many problems solution of the set of linear equations by means of matrix algebra was found easier and less time consuming than the determination of the displacements by systematic relaxations. In other cases special methods, such as the growing-unit method, proved to be most expeditious.

It is assumed that the reader is familiar with the terminology of Southwell's relaxation method and with the solution of the unit problem as well as the establishment of the operations table for both the reinforced-panel and fuselage-ring problems. Complete details of these are given in references 3 to 6.

This work, carried out at the Polytechnic Institute of Brooklyn, was sponsored by and conducted with the financial assistance of the National Advisory Committee for Aeronautics. Arnold O. Ostrand contributed the growing-unit method for reinforced panels. The authors also wish to acknowledge their indebtedness to the following members of the staff of the Polytechnic Institute of Brooklyn: Professors George B. Hoadley and William MacLean of the Department of Electrical Engineering for their work on the electric analogue, Burton Erickson for carrying out the major portion of the computations, and Bruno A. Boley for his editorial advice.

SYMBOLS

A	cross-sectional area of stringer and effective sheet
A - Q	points on a ring or a reinforced panel; group operations
A*	effective shear area of ring section
a	distance between adjacent longitudinal stringers
b	distance between adjacent transverse stringers
C	electrical conductance

E	Young's modulus of elasticity
F	tensile force in stringer; applied external load
G	shear modulus of elasticity
H	horizontal direction
I	moment of inertia of cross section; electrical current
I-XX	group operations
L	length of straight bar or length of arc of curved bar
M	bending moment
N	moment acting on a joint
q	shear flow
R	radial force acting at a joint; electrical resistance
T	tangential force acting on a joint
t	sheet thickness
u	displacement of a joint in tangential direction
V	electrical potential; vertical direction
v	displacement of a joint in radial direction; displacement of a joint in vertical direction
v_{Block}	vertical block displacement
w	rotation of a joint
x	magnitude of group operation to be determined
x,y	rectangular coordinates
Y	force in y-axis direction
β	angle subtended by ring segment
γ	section-length parameter (AL^2/I)
ξ	ratio of effective shear area to tension area (A^*/A)
$\sum$	summation

REINFORCED PANELS

Introduction

In this section plane and slightly curved reinforced panels are discussed when the loads are applied in the plane of the flat panels or tangentially to the surface of the slightly curved panels.

In most airplane structures there is a predominant direction in which the major forces act and in which the major reinforcing elements lie. When the panel is symmetric and symmetrically loaded experience has shown that it suffices to consider displacements and force equilibrium in the predominant direction only. Even when the structure or the loads are nonsymmetric, the displacements and forces in the transverse direction are usually of secondary importance but they may be considered in a more refined analysis.

In references 3 and 4 numerical procedures for the determination of the stress distribution in reinforced panels subjected to axial stringer loads are developed and demonstrated on several flat and curved panels with and without cut-outs. The results obtained by means of these procedures are in good agreement with those of tests.

Solution of the system of equations represented by the operations table and the external forces can be found by several methods, five of which are described herein. The various conditions of loading and structure which suggest the use of one method rather than another are discussed.

Relaxation Method

For most reinforced-panel problems the relaxation method of solution is the most suitable. Simple group and block operations lead to a rapid elimination of the residuals and require little initiative on the part of the computer familiar with the sequence of step-by-step operations. The method, however, is not efficient in the case of panels with many bays in the direction of the stringer loads or panels with sheet covering of large shearing rigidity, since large forces are then introduced into adjacent stringers when one stringer is balanced. These forces in turn must be liquidated in successive operations with the consequence that the procedure becomes time consuming. Also in problems involving many loading conditions it may be expeditious to use the electric-analogy method described in the section entitled "The Electric Analogue," since in the relaxation method each new loading requires new step-by-step operations.

In this section panels are discussed which are not excluded from application of the relaxation method by the foregoing considerations. They may be classified according to the boundary conditions of the stringers into four groups. Recommendations for each group follow with a fifth subsection added containing suggestions for panels in which transverse forces and displacements are considered.

(a) Panels with boundary conditions at both ends of stringers specified in terms of force.- The following two procedures are recommended for liquidating the residuals on a panel of this group:

First procedure:

1. Consider each stringer isolated by cutting the sheet and the transverse reinforcing elements. Select the stringer for which the algebraic sum of the external forces is the largest. Displace the entire stringer as a rigid body (block displacement) until this sum vanishes.

2. Balance one end joint of the stringer by displacing the adjacent joint on the same stringer.

3. After step 2 is completed the end joint is balanced but the joint that was moved is unbalanced. Displace the third joint on the same stringer until the second joint is balanced.

4. Continue the procedure until the second end joint is moved. In this last step both the end joint and the adjacent one will be approximately balanced at the same time since the algebraic sum of all the forces acting upon the stringer was zero after completion of step 1 and this equilibrium has been disturbed only slightly by the shear forces transmitted by the sheet during the individual operations.

5. Stringer 1 is now approximately balanced. Carry out the same procedure with the other stringers of the panel successively.

6. When all the stringers are approximately balanced, return to the first stringer and balance it again by undertaking steps 1 to 4. Repeat the procedure with the other stringers until all the residual forces can be considered negligible for engineering purposes.

Second procedure:

1. Consider each stringer isolated by cutting the sheet and the transverse reinforcing elements. Select the stringer for which the algebraic sum of the external forces is the largest. Displace the entire stringer as a rigid body (block displacement) until this sum vanishes.

2. Displace one end point of this stringer so as to balance the residual thereon.

3. Displace by equal amounts the adjacent joint on the same stringer and the end joint which was balanced in step 2 so as to balance this second joint. The equilibrium of the end joint will be disturbed only by a small amount due to shear in the sheet.

4. Displace by equal amounts the third joint on the same stringer and the two joints that were placed in approximate balance by the operation described in step 3 so as to balance this third joint.

5. Continue this procedure until the joint next to the midjoint of the stringer is balanced by equal displacements of all the joints situated between it and the end joint first displaced.

6. Repeat the process described in steps 2 to 5, starting from the other end joint of the stringer and continuing to the midjoint from this direction. After this step is completed this stringer will be in approximate balance, the only residuals being those introduced by shear in the sheet.

7. Consider next the stringer on either side of the approximately balanced stringer. Undertake a block displacement so as to equilibrate externally the stringer under its residual forces.

8. Start at one end joint of this stringer and apply steps 2 to 6. This second stringer will be placed in approximate balance thereby, while the balance of the first stringer will be disturbed only through the shear in the sheet.

9. Either return to the first balanced stringer or proceed to the next stringer on the other side. Each newly considered stringer is first externally equilibrated under the external and residual forces by a block displacement. Then from each free end the residuals are balanced by group displacements involving equal displacements of all the joints situated between the one in question and the free end. Continue to balance individual stringers until all are balanced.

The relaxation tables for the panel shown in figure 1, for which table 1 is the operations table, are used to demonstrate the first and second procedures and are given as tables 2 and 3, respectively. It will be noticed that this operations table considers the displacements of only the joints on the left half of the panel. The panel is symmetrical and is symmetrically loaded. Therefore, the displacements in the balancing process are undertaken symmetrically and only those of the left side joints need be considered, those of the right being correspondingly equal. Since this panel has only three bays along each axially loaded stringer, the internal balancing process is undertaken from one end of the stringer only.

<u>(b) Panels with boundary conditions at one end of stringers specified in terms of force and at other in terms of displacement</u>.- This type of problem occurs, for instance, when one end of the panel is attached to a rigid body which is either held fixed in its position or is displaced a given amount. The recommended procedure for panels of this group is the same as the second procedure for panels in case (a) with two exceptions:

(1) No block displacements are needed (or possible) to equilibrate the stringers externally and (2) the internal balancing process can be started only from the one free end of each stringer.

The method is demonstrated on the panel shown in figure 2. It is identical with the panel used for case (a) with the exception of the fixed lower ends of the vertical stringers. The operations table is identical with that of the previous panel except that no block and no v_N and v_O displacements are admissible. The relaxation table is given as table 4.

(c) Panels with boundary conditions at both ends of stringer specified in terms of displacement.- Experience on panels of this type indicates that, although no systematic process of balancing the residuals can be recommended, the direct relaxation process is rapidly convergent. By starting from the midpoint joints on a stringer and by balancing successive joints toward the two fixed ends, the equilibrium position can be approximated rapidly. A further suggestion regarding this type of panel is contained in the later section "Niles Tables."

(d) Panels with irregularly specified boundary conditions.- For such panels a combination of the methods discussed under cases (a), (b), and (c) is recommended. By judicious use of block and group operations similar to those of cases (a) and (b) rapid convergence of the relaxation procedure will be obtained.

(e) Panels in which transverse displacements and forces are considered.- There are two general procedures for treating panels in which the transverse displacements and forces, usually considered negligible, are treated. These are described in the following paragraphs:

First procedure:

The procedure discussed under cases (a) and (b) can be applied to panels with cut-outs. The stringers are approximately balanced in the direction of the major axial forces by these procedures and then the residuals normal to this direction are considered. The same step-by-step operations can be applied in balancing transverse stiffeners under these transverse axial forces. The process of first balancing the stringers in one direction, then balancing the stiffeners in the normal direction, and then returning to the originally balanced stringers will be quite rapidly convergent for panels with sheet of low shearing rigidity.

Second procedure:

For panels with cut-outs requiring consideration of the transverse forces another procedure, which is demonstrated in reference 4, can be used. The panel is first considered to have continuous sheet and stringers, as if the cut-out did not exist, and the displacements for equilibrium of this panel under the external loads are determined by the usual methods. These displacements are then applied as a first approximation to the distorted shape of the actual panel with cut-outs. Displacements leading to a closer approximation are then undertaken. This procedure is found to be reasonably successful for the cases investigated in reference 4.

Matrix Calculus Method

The operations table together with the external forces can be considered as a system of linear equilibrium equations with the magnitudes of the displacements as the unknowns. Therefore, the methods of matrix calculus can be applied to find the solution of this system by direct mathematical means. The method described in reference 9 is recommended since a check on the calculations is maintained at each step in the process of solution.

Matrix methods of solution have several advantages. After the operations table is established by trained engineering personnel, the solution can be obtained by computing personnel familiar with the matrix calculus method. Under some conditions this economic advantage may be important. For reinforced panels with sheet of high shearing rigidity the relaxation procedures are slowly convergent even when the recommendations given in the preceding section are observed. The matrix calculus method is not affected by this physical characteristic of the structure.

When the number of equations is greater than 30 or 40, the work of computation becomes inconveniently large. Therefore, for panels having a sheet covering of small shearing rigidity relaxation methods are recommended. When the sheet covering is very rigid in shear the matrix method is likely to be more advantageous because the routine operations of the matrix method can always be carried out if enough time is allowed.

The equations of equilibrium for the panel shown in figure 1 are given by table 1 and are presented as follows to illustrate how the operations table and the external forces can be considered as a system of equilibrium equations:

$$\left.\begin{aligned}
-55.2v_B + 2.00v_E + 51.2v_F &= 0\\
2.00v_B - 101.6v_E + 4.00v_F + 46.8v_J + 2.00v_K &= 0\\
51.2v_B + 4.00v_E - 110.4v_F + 2.00v_J + 51.2v_K &= 0\\
46.8v_E + 2.00v_F - 101.6v_J + 4.00v_K + 46.8v_N + 2.00v_O &= 0\\
2.00v_E + 51.2v_F + 4.00v_J - 110.4v_K + 2.00v_N + 51.2v_O &= 0\\
46.8v_J + 2.00v_K - 50.8v_N + 2.00v_O + 60 \times 10^4 &= 0\\
2.00v_J + 51.2v_K + 2.00v_N - 55.2v_O + 60 \times 10^4 &= 0
\end{aligned}\right\} \qquad (1)$$

In considering the operations table and the external forces as a system of equilibrium equations, care must be taken to restrain enough joints so that the position of the panel as a rigid body is fixed. In the present case v_A and v_D are assumed to be zero, and since only displacements in the y-direction are considered in this problem, this restraint is sufficient.

Growing-Unit Method

For reinforced panels with sheet of high shearing rigidity or with a large number of bays in the direction of the axial forces, the relaxation procedure is not rapidly convergent. In such problems either the matrix calculus or the growing-unit method is recommended. The latter can be applied only to panels the boundary conditions of which are specified in terms of force at least at one end of the stringers.

The growing-unit method applied to reinforced panels is as follows. The joint at the free end of an arbitrarily selected unbalanced stringer, called hereinafter the principal joint and the principal stringer, respectively, is displaced so as to liquidate the residual on this joint. At the same time the joints lying on adjacent parallel stringers and the same transverse stiffener are displaced so that the residuals that would be otherwise introduced by shear from the balancing of the principal joint as well as any external forces applied to these joints are likewise liquidated. In the second operation the next joint on the principal stringer is relaxed while the previously balanced joints on the first transverse stiffener and the joints on the second transverse stiffener are kept in balance by suitable displacements. After this second operation no residuals remain at the joints of the first two transverse stiffeners. After a sufficient number of repetitions of the procedure all residuals will be confined to reaction points or will be liquidated; the panel will then be in equilibrium.

This procedure is demonstrated on the panel shown in figure 3. The physical properties of the panel are the same as those of the previously discussed panels except for the additional bay in the direction of the axial forces. Actually the convergence of the relaxation method for this panel would be quite rapid, but for convenience the growing-unit method, applicable when this convergence is slow, is demonstrated thereon. Table 5 is the operations table for this panel and contains not only the individual operations but also the group operations of the growing-unit method. Table 6 is the relaxation table in which these group operations are used.

The group operations given in table 5 require some explanation. In order to avoid introducing a Y_B residual when joint A is relaxed by application of operation (1), a v_B displacement is applied, the magnitude of which can be calculated from the equation

$$-55.2v_B + 2.00 = 0 \tag{2}$$

Thus operation (9) is $v_B = (2/55.2) = 0.0362$ and (10) is a group operation equal to the sum of operations (1) and (9), which liquidates the residual Y_A without introducing a Y_B unbalance.

After operation (10) is used, unbalances exist at joints E and F, that is, on the second transverse stiffener. In order to balance these without disturbing the recently established balance at A and B, two group operations are developed: one permitting the balancing of E and one permitting the balancing of F. The magnitudes of v_A and v_B required to maintain the balance of A and B when a displacement of $v_E = 1$ is undertaken are given by the following equations:

$$\left.\begin{aligned} -50.8v_A + 2.00v_B + 46.8 &= 0 \\ 2.00v_A - 55.2v_B + 2.00 &= 0 \end{aligned}\right\} \qquad (3)$$

These are satisfied by $v_A = 0.921$, operation (11), and $v_B = 0.0695$, operation (12). Operation (13) is therefore established as the sum of operations (3), (11), and (12). The magnitudes of v_A and v_B required to maintain the balance of A and B when a displacement of $v_F = 1$ is undertaken are given by the following equations:

$$\left.\begin{aligned} -50.8v_A + 2.00v_B + 2.00 &= 0 \\ 2.00v_A - 55.2v_B + 51.2 &= 0 \end{aligned}\right\} \qquad (4)$$

These are satisfied by $v_A = 0.0758$, operation (14), and $v_B = 0.923$, operation (15). Operation (16) is the sum of operations (4), (14), and (15). Since group operations (13) and (16) both introduce Y_E and Y_F forces, the magnitudes x_{13} and x_{16} of these groups required to liquidate the -111-pound and -9-pound residuals at E and F, respectively, are given by the following equations:

$$\left.\begin{aligned} -58.3x_{13} + 9.4x_{16} - 111 &= 0 \\ 9.4x_{13} - 62.6x_{16} - 9 &= 0 \end{aligned}\right\} \qquad (5)$$

Thus $x_{13} = -1.975$ and $x_{16} = -0.444$. Joints E and F are balanced without disturbing the balance of A and B by the use of these multiples of operations (13) and (16).

In eliminating the residuals at joints J and K multiples of operations (13) and (16) are applied since these operations permit displacements of E and F to be undertaken while the balance at A and B is left undisturbed. When joint J is displaced a unit amount, multiples of operations (13) and (16), defined by the following equations, are used so that the balance at A, B, E, and F is maintained:

$$\left.\begin{aligned} -58.3x_{13} + 9.4x_{16} + 46.8 &= 0 \\ 9.4x_{13} - 62.6x_{16} + 2.00 &= 0 \end{aligned}\right\} \qquad (6)$$

The solution to these equations is $x_{13} = 0.828$, operation (17), and $x_{16} = 0.158$, operation (18). Operation (19) is the sum of operations (5), (17), and (18).

In a similar manner all the individual and group displacements described in table 4 are found. It may be mentioned that in the present example no shearing stresses were set up in the middle bays because of the symmetry of structure and loading. The original operations table was already established in a manner which complied with these requirements of symmetry. When such is not the case or when there is a greater number of stringers in the panel, displacements of principal stringer joints will, in general, cause residuals to appear at more joints so that three or more, rather than two, simultaneous equations have to be solved at each step.

Niles Tables

In reference 10, A. S. Niles demonstrates for the solution of reinforced-panel problems a method which essentially parallels the previously described relaxation method. The Niles method is a procedure for balancing a stringer by the use of tables which give the displacements of each joint on the stringer required to liquidate a residual on a given joint of the stringer. The tables are worked out for various end conditions and sheet shearing rigidities.

Since reference 10 contains tables only for sheet of relatively low shearing rigidity, the Niles method is limited in this respect in the same way as the relaxation method. However, the tables can be employed on stringers with the boundary conditions at both ends specified in terms of displacement; for such problems no step-by-step routine relaxation method has been recommended. Also by use of the tables exact balance of a stringer is gained after a single displacement of each joint, whereas in the relaxation method, because of the shear, small unbalances remain after each joint is moved.

On the other hand, the relaxation method can be applied to stringers with irregularly spaced joints for which no tables were set up by Niles.

Since in reference 10 several examples of the procedure are given, no application of the Niles method is shown herein.

Electric Analogue

Another convenient method of solving the problem of force distribution in a reinforced panel is that in which the voltages are measured in an electric network which is so constructed as to make it a complete analogue of the reinforced panel. When suitable electric equipment is available, an analogous network can be hooked up and tested with very little work. A particularly attractive property of the stress-analysis procedure by means of electric measurement is the ease with which the effect upon the stress distribution of changes in loading and in dimensions of the various structural elements of the reinforced panel can be investigated. This permits the development of an efficient design with little analytic work.

The analogy between the forces transmitted through the different structural elements of the reinforced panel and the currents flowing through the various branches of the direct-current network can be explained with the aid of figures 4 and 5. The problem investigated is the so-called "one-dimensional shear lag." It is assumed that the transverse stiffeners are infinitely rigid so that the vertical, or longitudinal, displacements v alone need to be determined. The portion of the sheet covering considered effective in tension or compression is added to the cross-sectional area of each stringer and the panels of sheet are assumed to carry shear stresses only. A consequence of these assumptions is that the shearing stress must be constant in each panel.

The analogous direct-current network contains as many binding posts as the number of joints in the reinforced panel. Adjacent binding posts are connected by conductors having prescribed resistances R. Predetermined electric currents I, which correspond to the forces F applied to joints A and B of the reinforced panel, are introduced into the network at points A and B.

It is now recalled that in the relaxation method the joints of the panel are first assumed to be rigidly fixed to a rigid wall behind the panel. The external loads are first applied to these rigid pegs, referred to as the "constraints." The panel is obviously in equilibrium under these conditions but this artificial equilibrium is entirely different from that prevailing in the actual panel, which is not attached to any rigid wall. The actual state of equilibrium is approached by the step-by-step procedure of the relaxation method, in each step of which one single constraint is removed and the corresponding joint is displaced until it reaches its equilibrium position in the system in which all the other joints are still rigidly fixed.

For instance when joint 1 of the reinforced panel is moved through a distance v in the positive direction, this displacement imposes forces upon all the adjacent joints numbered from 2 to 9. Three typical forces are given by the equations:

$$F_{81} = v\left(\frac{EA}{b} - \frac{Gbt}{2a}\right) \tag{7}$$

$$F_{91} = v\,\frac{Gbt}{4a} \tag{8}$$

$$F_{61} = v\,\frac{Gbt}{2a} \tag{9}$$

where

F_{81}, F_{91}, F_{61}	the forces acting upon joints 8, 9, and 6, respectively, because of the displacement of joint 1
E	modulus of elasticity of stringer
G	shear modulus of sheet
t	thickness of sheet
v	displacement of joint 1

In the case of the analogous network it can be assumed that the potential of each binding post is zero at the outset. If there is no potential difference, no current flows between the posts. It can be imagined that the currents introduced at points A and B are taken out of the system by means of some imaginary conductors. However, the actual distribution of currents in the network prevails without the aid of the imaginary conductors. This actual state can be approached also by means of a step-by-step, approximation-type calculation. For instance it can be assumed first that the potential of binding post 1 is elevated to the value V. After this change there is a potential difference between binding posts 1 and 8 and consequently a current will flow from post 1 to post 8. The magnitude of this current can be calculated from the equation

$$I_{81} = V/R_{81} = C_{81}V \tag{10}$$

where R_{81} is the resistance and $C_{81} = 1/R_{81}$ is the conductance of the conductor between posts 1 and 8. Similarly the current flowing from post 1 to post 9 is

$$I_{91} = C_{91}V \tag{11}$$

The current flowing from post 1 to post 6 is

$$I_{61} = C_{61}V \tag{12}$$

Comparison of equations (7) to (9) with equations (10) to (12) reveals an analogy between the effects of a displacement v of joint 1 and the raising of the voltage of binding post 1 by an amount V. The current caused by the change in potential corresponds to the force caused by the displacement, provided that the conductance of each conductor is made equal to the influence coefficient in the corresponding force equation. Hence

$$C_{81} = \frac{EA}{b} - \frac{Gbt}{2a} \tag{13}$$

$$C_{91} = \frac{Gbt}{4a} \tag{14}$$

$$C_{61} = \frac{Gbt}{2a} \tag{15}$$

In the relaxation procedure the equilibrium state is approached by displacing individually the joints and summing the effects of each displacement. In exactly the same way the actual distribution of the currents in the network can be determined by changing individually the voltages of each binding post and summing the effects of these changes. In the reinforced panel equilibrium is obtained when at each joint the sum of the external forces and of all the internal forces caused by the displacements is zero. The forces are considered positive if they are directed as the positive displacements. In the form of an equation,

$$\sum F = 0 \tag{16}$$

An analogous equation in the direct-current network is furnished by Kirchhoff's first law, according to which the sum of the currents flowing into any binding post must be zero. Currents in the direction of any binding post are considered as positive. In the form of an equation,

$$\sum I = 0 \tag{17}$$

Comparison of the last two equations reveals that the conditions of equilibrium for the reinforced panel and Kirchhoff's first law in the case of the direct-current network complete the analogy of the two systems considered. It is possible therefore to construct an electric network with the same configuration of binding posts as that of the joints of the reinforced panel. The conductances of the conductors connecting the binding posts must be so chosen as to make them proportional to the corresponding influence coefficients in the operations table of the reinforced panel. If then currents are introduced at the binding posts which correspond to the joints at which external loads are applied, the distribution of the currents in the network will be the same as the distribution of the forces between the various structural elements of the reinforced panel.

In the first applications of the relaxation process to reinforced panels each joint was displaced until equilibrium was established. It was noted in the section dealing with the solution of the problem by matrix methods that this procedure permitted rigid body displacements of the structure. Rigid body displacements can be eliminated if one or more joints are considered as rigidly fixed. In the case of the reinforced panel of figure 4 the degree of freedom of motion of each joint is one, because the problem is considered as a one-dimensional shear lag problem. Consequently it suffices to fix one single joint so that it is prevented from displacing vertically. However, if joint C, for instance, is fixed, the symmetry of the structure and loading requires the simultaneous fixation of joint D.

In the analogous network binding posts C and D are given predetermined values of the potentials by connecting them to the ground. It is customary to attribute the value zero to the potential of the ground. Consequently V_C and V_D are zero just as in the reinforced panel v_C and v_D are zero.

It will be noticed that in figure 4 the direction of F at joints A and B is upward, whereas the direction of I at binding posts A and B in figure 5 is downward. This corresponds to the difference in the sign convention in the two systems. In the panel upward forces were considered positive and in the network currents flowing toward the binding posts were given the positive sign. The directions of the forces and the currents at points C and D are the same. This again corresponds to the correct signs required by the sign convention since the downward forces at these points are negative just as the currents which flow away from the binding posts are negative. Hence the reinforced panel is under the action of external tensile forces, whereas through the network currents are flowing in the downward direction.

In the case under discussion it is easy enough to introduce the two equal currents at posts A and B and to regulate their magnitude by means of an adjustable rheostat. However, when there are a number of impressed currents of different magnitude stipulated, their adjustment may become a lengthy trial-and-error procedure. In such cases it is advantageous to employ a number of commercially available electronic devices, known as constant-current generators, which have the property of maintaining a constant current independently of the properties of the network.

When the construction of the network is completed and the required external currents are introduced, the deflection of any joint of the reinforced panel can be obtained by measuring the potential of the corresponding post in the network with respect to the ground. This quantity multiplied by the conversion factor is the relative displacement of the corresponding joint of the reinforced panel with respect to the fixed points C and B. In most cases, however, the displacement quantities are of interest only indirectly and the main quantities sought are the forces in the stringers and the shear stresses in the sheet. These quantities can be obtained in a simple manner by multiplying potential differences by the appropriate conductances and by the conversion factor.

For instance when the force in stringer segment 1-8 is sought, the voltage drop between posts 1 and 8 must be measured and multiplied by the conductance C_{81} and the conversion factor. This is a consequence of equations (7) and (10). Similarly when the shear stress in panel 1689 is required, the voltage drops in conductors 1-6 and 8-9 have to be measured. From figure 4 the average displacement of stringer segment 6-9 is $(v_6 + v_9)/2$ and the average displacement of stringer segment 1-8 is $(v_1 + v_8)/2$. The difference of these two average displacements multiplied by Gtb/a is the shear force transmitted from the panel to stringer segment 6-9. Consequently the sum of the displacement differences $v_6 - v_1$ and $v_9 - v_8$ multiplied by the influence coefficient 1-6 is the shear force sought. In other words the sum of the voltage drops from post 1 to post 6 and from post 8 to post 9 multiplied by the conductance C_{61} and the conversion factor is the shear force in question. This shear force divided by the length b gives the average shear flow in panel 1689 and this shear flow divided by the thickness of the sheet is the average shear stress.

With the cooperation of the Department of Electrical Engineering a network was constructed at the Polytechnic Institute of Brooklyn which was the analogue of the reinforced panel investigated earlier at PIBAL both experimentally and by relaxation methods. The results of these investigations are described in reference 3. The constant currents were introduced by means of constant-current generators. In the electrical system the unit of the potential was chosen as 1 volt and that of the current as 100 milliamperes. Then the unit of the conductance had to be a millimho and that of the resistance, a kilohm. In the mechanical system the unit displacement was 10^{-4} inch and the unit force, 1 pound. Consequently in this problem the voltage differences had to be multiplied by the conversion factor 10^{-4} inch per volt in order to obtain displacements. The factor converting currents into forces was 10 pounds per ampere. The results of the measurements were in excellent agreement with the results quoted in reference 3.

Similar experiments were carried out by R. E. Newton and M. E. Engle at the Curtiss-Wright Corporation, Airplane Division, in St. Louis and are described in two reports listed as references 11 and 12. Newton's approach to the problem is fundamentally the same as the argument given herein. However, his electric network is slightly simpler since it does not contain the conductors arranged diagonally in the system shown in figure 5. The network of figure 5 was chosen in this report in preference to Newton's simpler network since by this presentation the identity of the conductances of the network and the influence coefficients used in the other parts of this report could be established.

It should be mentioned that in many cases it is possible to construct a dual type of network in which the currents correspond to the displacements of the joints of the reinforced panel and the potential differences correspond to the forces in the stringers and in the sheet covering of the panels. In this type of network the external loads can be introduced more easily as impressed potential differences. However, the network described herein is more advantageous since it can always be constructed directly from the geometry of the reinforced panel.

The usefulness of the analogue with the direct-current network breaks down when the influence coefficient in equation (7) becomes negative. In such a case the conductance and consequently the resistance of the corresponding branch of the network should be negative; this is obviously impossible. However, the situation can be usually remedied in the case of one-dimensional shear lag problems. The fundamental assumptions of the problem are not changed if a number of additional horizontal bracing elements are introduced in the panel since all of them are assumed to be infinitely rigid. If, however, the panel length b is reduced to one-half its original value, then the negative term in the influence coefficient appearing in equation (7) is halved and the positive term is doubled. In most cases this will suffice to change the sign of the influence coefficient. When such is not the case distance b can be reduced in any other suitable ratio.

Negative influence coefficients can be realized if the analogous network is fed by an alternating current. The quantity corresponding in an alternating-current circuit to the resistance of the direct-current circuit is the impedance. In the impedance the inductance retards the phase of the current and the capacitance advances it so that the two have opposite effects. If one is designated as positive, the other is negative. However, no inductance is entirely free of resistance and for this reason the accuracy of a complicated alternating-current network may not be sufficient for the solution of some of the problems encountered in practice.

The use of the electric analogue for solution of shear lag problems is recommended when several similar panels with many loading conditions are to be analyzed. For such a case the construction of the analogous network, the variation of the loading by varying the impressed currents, and the determination of the potentials at the binding posts would be simpler than any analytic method of solution.

FUSELAGE RINGS

Introduction

In reference 5 numerical procedures for the determination of the bending-moment distribution in fuselage rings are developed and demonstrated on several simple and internally braced fuselage rings. The number of redundant internal bracing elements increases little the work involved in establishing the operations table for the ring and affects not at all the amount of numerical work in the solution of the operations table. This nonsensitivity to the number of redundances constitutes the advantage of this method in the analysis of fuselage rings.

The methods suggested for the solution of the system of equations represented by the operations table and the external forces are three: relaxation, matrix calculus, and growing-unit. The latter two may be considered as direct mathematical methods and as in reinforced-panel problems require only computing personnel. For the analysis of isolated fuselage rings of complex shape the use of these direct methods is recommended since an accurate solution is assured in a reasonable length of time, whereas the relaxation method may not lead to sufficiently accurate results even after considerable effort has been expended. However, for simply shaped rings and for problems of stress distribution in sheet, stringer, and ring combinations, application of the relaxation method to fuselage rings is advantageous. For this reason the relaxation method for fuselage-ring problems is presented and new, more rapidly convergent procedures are developed.

It has not been found possible to make concrete reommmendations for relaxation procedures which are rapidly convergent for all types of ring and loading. However, satisfactory procedures for several distinct types of ring and loading are demonstrated and explained in some detail. It is felt that consideration of these examples will suggest to the analyst means of solving more rapidly other ring and cylinder problems which are not efficiently attacked by direct mathematical means. The procedures, which involve essentially appropriate combined operations, are demonstrated on two rings solved in reference 5 by the usual relaxation methods and on a new internally braced ring. Application of the growing-unit and matrix calculus methods to the latter problem is made to demonstrate these methods and to verify the results of the relaxation procedure.

Torsion of a Circular Ring

In reference 5 the bending-moment distribution for a simple circular ring with antisymmetric loading consisting of concentrated forces and distributed and constant shear flow is determined by application of numerical methods. The dimensions and loading for this ring are shown in figure 6 and the operations table is given as table 7. Relaxation methods are applied to the solution of this ring problem in reference 5. By a process of increasing all the residuals in such a proportion that one key operation would liquidate them all to within the desired degree of accuracy, the residuals were reduced to within 2 percent of the maximum applied load in 12 operations.

In the present report combined operations which increase the rate of convergence are demonstrated. Tangential and angular displacements of A and C balance these points in four operations and place all remaining residuals at B. Since no tangential forces exist at A and C, the force residual at B must be vertical and the moment residual, equal to the couple of the vertical forces. Suppose the residual moment at B is liquidated by a rotation of that joint while the balance of A and C is preserved by suitable displacements of A and C. Then from equilibrium considerations the residual forces at B must also be liquidated. Thus in five operations balance will be obtained. This procedure is used and proves to be satisfactory.

In order to balance the residuals at A two combined operations are developed. The first combines a unit angular displacement w_A with a tangential displacement u_A such that no tangential force at A results when the two individual operations are simultaneously applied. The forces and moments introduced by the individual operations as well as by the combination are given in the following table:

Operation \ Forces and moments	N_A	T_A	N_B	R_B	T_B	N_C	T_C
$w_A = 10^{-3}$ radian	-281.95	-49.079	-29.966	-4.733	64.675	0	0
$u_A = -0.93848 \times 10^{-3}$ in.	46.060	49.079	-60.696	21.060	-48.347	0	0
$\sum \rightarrow$ Operation A = 1	-235.89	0	-90.662	16.327	16.328	0	0

The second operation combines a unit tangential displacement u_A with an angular rotation w_A such that at A no moment arises from the combined operation. The forces and moments introduced by the individual operations as well as by the combined operation are given in the following table:

Operation \ Forces and moments	N_A	T_A	N_B	R_B	T_B	N_C	T_C
$u_A = 10^{-3}$ in.	-49.079	-52.296	64.675	-22.441	51.516	0	0
$w_A = -0.17407 \times 10^{-3}$ radian	49.079	8.5432	5.2162	0.82387	-11.258	0	0
$\sum \rightarrow$ Operation B = 1	0	-43.753	69.891	-21.617	40.258	0	0

Thus by using the necessary amounts of the combined operations A and B joint A is balanced in two operations. Two similar operations are found for joint C and are given as follows without explanation:

Operation \ Forces and moments	N_A	T_A	N_B	R_B	T_B	N_C	T_C
$w_C = 10^{-3}$ radian	0	0	56.512	8.842	6.632	-157.899	-1.563
$u_C = -4.8540 \times 10^{-3}$ in.	0	0	-32.192	-2.5435	-0.33250	7.5868	1.563
$\sum \rightarrow$ Operation C = 1	0	0	24.320	6.2985	6.2995	-150.31	0

Operation \ Forces and moments	N_A	T_A	N_B	R_B	T_B	N_C	T_C
$u_C = 10^{-3}$ in.	0	0	6.632	0.524	0.0685	-1.563	-0.322
$w_C = -0.0098987 \times 10^{-3}$ radian	0	0	-0.55939	-0.08752	-0.0656	1.563	0.01547
$\sum \rightarrow$ Operation D = 1	0	0	6.0726	0.43648	0.0029	0	-0.30653

In order to balance the residuals at B without disturbing the balance at A and C obtained by use of operations A to D, combined operations involving tangential and angular displacements of A and C and a unit rotation of B are developed. If joint A is to remain in balance when a rotation of B is undertaken, joint A must be rotated and displaced in such a manner that the tangential force and the moment introduced at A by this rotation of B are equilibrated. Since the angular displacement introduces tangential forces at A, and the tangential displacement introduces moments, two simultaneous equations must be solved for the unknown tangential and angular displacements. The equations for A are:

$$\left.\begin{aligned} -281.95w_A - 49.079u_A - 29.966 \times 10^{-3} &= 0 \\ -49.079w_A - 52.296u_A + 64.675 \times 10^{-3} &= 0 \end{aligned}\right\} \qquad (18)$$

The solution to these equations is $w_A = -0.38434 \times 10^{-3}$ radian and $u_A = 1.5974 \times 10^{-3}$ inch. A unit rotation of B and tangential and angular displacements of C are combined in equations (19) so that the tangential force and moment introduced at C by the combined operations are zero.

$$\left.\begin{aligned} -157.899w_C - 1.563u_C + 56.5117 \times 10^{-3} &= 0 \\ -1.563w_C - 0.322u_C + 6.632 \times 10^{-3} &= 0 \end{aligned}\right\} \qquad (19)$$

The solution to these equations is $w_C = 0.16180 \times 10^{-3}$ radian and $u_C = 19.811 \times 10^{-3}$ inch.

If the forces and moments introduced by the three sets of displacements (unit rotation of B, the tangential and angular displacements of A, and the tangential and angular displacements of C) are combined, a combined operation is obtained such that only forces and moments at B and radial forces at A and C are introduced. These latter forces are of no interest in the relaxation procedure since they are equilibrated automatically by the other half of the ring. The combined operation from these three sets of displacements is given in the following table:

Operation \ Forces and moments	N_A	T_A	N_B	R_B	T_B	N_C	T_C
$w_B = 10^{-3}$ radian	-29.966	64.675	-439.849	31.443	-50.642	56.5117	6.632
$w_A = -0.3843 \times 10^{-3}$ radian	108.37	18.863	11.517	1.8191	-24.857	0	0
$u_A = 1.5974 \times 10^{-3}$ in.	-78.399	-83.538	103.31	-35.847	82.292	0	0
$w_C = 0.16180 \times 10^{-3}$ radian	0	0	9.1436	1.4306	1.0730	-25.548	-0.25289
$u_C = 19.811 \times 10^{-3}$ in.	0	0	131.39	10.381	1.3570	-30.964	-6.3791
$\sum \rightarrow$ Operation E = 1	0	0	-184.49	9.2267	9.2230	0	0

The relaxation table using these five combined operations, A to E, is given as table 8. The balancing process was carried out on a slide rule and after five operations all the residuals were reduced to negligible quantities. From the magnitudes of these group operations the total individual displacements of A, B, and C can be found and the unknown radial forces at A and C calculated.

The procedure just described involves essentially the development of group operations so that full advantage of the symmetry properties of the ring may be realized. This method is applicable to other rings. The internally braced circular ring subjected to antisymmetric loads and analyzed in reference 5 can be treated in the same way as this simple ring. If these rings had been symmetrically loaded, the force residuals at B, after A and C had been balanced by simple radial displacements, would have a horizontal resultant. By combining radial and tangential displacements of A and C such that the resultant force introduced at B is horizontal and such that A and C remain in balance, the horizontal resultant at B could be liquidated by application of such a combined operation. The moment residual at B is not necessarily eliminated when the force residual at B is balanced. Joint B must be rotated while A and C are displaced radially so that the moment at B is liquidated and joints A and C are kept in balance. If the process of liquidating first the residual force and then the moment at B, preserving in each operation the balance at A and C, is not rapidly convergent, two equations for the equilibrium of B can be established and solved for the required amounts of the combined operations.

Thus the foregoing procedures for both the symmetrical and antisymmetrical loading can be applied to any ring singly symmetrical with only one joint between the center line of symmetry joints. It may, therefore, be advantageous in some ring problems to combine several bars, as in the method of the growing unit, such that only one joint between the boundary joints has independent degrees of freedom. This will permit use of the foregoing procedure.

Sufficient accuracy for most engineering purposes can be obtained in the computations of this procedure by the use of a slide rule throughout. Although the combined operations shown herein were obtained by the use of a computing machine carrying five significant figures, the procedure was first demonstrated with the use of a slide rule for all calculations. The results of the two sets of calculations are in good agreement, thus indicating the sufficiency of slide-rule accuracy.

Egg-Shaped Ring

Figure 7 shows the dimensions of, and loading on, a ring which is analyzed in reference 5. The operations table for this ring is given as table 9. In this ring there are two points B and C between the center line of symmetry points A and D. By making the degrees of freedom of either point B or C dependent on the other and on the adjacent center line of the symmetry point, one point with independent degrees of freedom is established between A and D and the method discussed previously can be used.

However, in order to demonstrate the simplicity and effectiveness of group operations, another approach is used. The center lines of symmetry points A and D are balanced by simple radial displacements of A and D. The midpoint of bar BC is assumed restrained tangentially so that only equal and opposite tangential displacements of B and C are undertaken. Because of the large extensional stiffness of bar BC as compared with the bending rigidity of the circular segments and because the ring is almost symmetrical about a horizontal axis, such displacements of B and C liquidate approximately equal and opposite tangential residual forces at B and C, such as those which will be obtained at these points when the residuals associated with the other degrees of freedom are small.

If the balance at A and D is preserved by appropriate combinations of the radial displacements of A and D with the required displacements of B and C and if the tangential residuals at B and C are not considered until the foregoing operation will liquidate them both, main attention is focused on the radial force and moment residuals at B and C. In order to balance these, no specific method of convergence is used but the state of the residuals after each step is considered before the next operation is selected. In this problem of egg-shaped rings and many other rings and in the complete cylinder problems this approach, utilizing physical properties of the system and eliminating or reducing extraneous forces and moments at each step in the relaxation process, may be the most satisfactory method of solution.

Table 10 is the relaxation table for the ring in question. The first two operations involve only radial displacements which balance the 500-pound forces at A and D. The largest residual then is the radial force of 451 pounds at C. If point C is displaced radially so as to balance this residual, a large moment and a large radial force are introduced at B. In order to reduce these extraneous forces and moments and to keep joints A and D balanced, radial displacements of A, B, and D and a rotation of B are combined as shown by the following operations:

$$\left.\begin{aligned} -3.34833v_A + 8.92216w_B - 2.69614v_B &= 0 \\ 8.92216v_A - 327.866w_B + 11.4697v_B + 8.10267 \times 10^{-4} &= 0 \\ -2.69614v_A + 11.4697w_B - 4.00991v_B + 0.66158 \times 10^{-4} &= 0 \\ -12.2400v_D - 1.11900 \times 10^{-4} &= 0 \end{aligned}\right\} \quad (20)$$

The solution of this system of equations is: $v_A = -0.26384 \times 10^{-4}$ inch, $w_B = 0.03279 \times 10^{-4}$ radian, $v_B = 0.43618 \times 10^{-4}$ inch, and $v_D = -0.9024 \times 10^{-4}$ inch.

The forces and moments introduced by each of the individual operations and by the combination are given in the following table:

Operation \ Forces and moments	R_A	N_B	R_B	T_B
$v_C = 10^{-4}$ in.	0	8.10267	0.66158	0
$v_A = -0.26384 \times 10^{-4}$ in.	0.88343	-2.3540	0.71136	-1.0468
$w_B = 0.03279 \times 10^{-4}$ radian	0.29258	-10.751	0.37612	-0.42963
$v_B = 0.43618 \times 10^{-4}$ in.	-1.1760	5.0028	-1.7491	1.4984
$v_D = -0.90242 \times 10^{-4}$ in.	0	0	0	0
$\sum \rightarrow$ Operation F = 1	0	0	0	0.0219
Operation \ Forces and moments	N_C	R_C	T_C	R_D
$v_C = 10^{-4}$ in.	-2.95622	-1.90205	-0.88929	-1.1190
$v_A = -0.26384 \times 10^{-4}$ in.	0	0	0	0
$w_B = 0.03279 \times 10^{-4}$ radian	-2.0082	0.26570	0	0
$v_B = 0.43618 \times 10^{-4}$ in.	-3.5342	0.28857	0	0
$v_D = -0.90242 \times 10^{-4}$ in.	6.6350	1.00981	0.93626	1.1190
$\sum \rightarrow$ Operation F = 1	-1.8636	-0.33798	0.04697	0

The use of combined operation F is desirable in balancing the radial residual force at C, since it also reduces the moment residual at C and adjusts the tangential residuals at B and C in the desired manner.

The residual considered after use of operation F is $R_B = 402$ pounds. In order to balance it by a displacement v_B while the balance at A is preserved, a v_A displacement must be undertaken as well. If $v_B = 10^{-4}$ inch, then $v_A = \frac{2.69614}{-3.34833 \times 10^{4}} = -0.80522 \times 10^{-4}$ inch. The forces and moments introduced by these individual operations as well as by the combinations are given in the following table:

Operation \ Forces and moments	R_A	N_B	R_B	T_B	N_C	R_C	T_C	R_D
$v_B = 10^{-4}$ in.	-2.69614	11.4697	-4.00991	3.4352	-8.10267	0.66158	0	0
$v_A = -0.80522 \times 10^{-4}$ in.	2.6961	-7.1843	2.1710	-3.1949	0	0	0	0
$\sum \rightarrow$ Operation G = 1	0	4.2854	-1.83891	0.2403	-8.10267	0.66158	0	0

Consider the effect of eliminating the R_B residual by use of operation G. The moment residual N_B would also be reduced by roughly 1000 inch-pounds, the T_B residual would be brought in closer agreement with the T_C residual, an R_C residual of about 30 percent of the previous R_C residual of 451 pounds would be introduced, and a large N_C residual would be introduced. The last two effects are undesirable. However, by use of operation F again, the R_C residual can be balanced without introducing a new R_B residual. The large N_C residual is not so easily balanced unless a new combination involving joints A, B, and D, is evolved.

Suppose, therefore, that a rotation of C and a radial displacement of D are combined so that a moment at C can be eliminated and so that the balance of D is preserved by use of the combination. The individual operations and the combinations are given in the following table:

Operation \ Forces and moments	R_A	N_B	R_B	T_B	N_C	R_C	T_C	R_D
$w_C = 10^{-4}$ radian	0	-61.242	-8.10267	0	-288.367	-2.95622	-5.24667	-7.3524
$u_D = -5.9294 \times 10^{-4}$ in.	0	0	0	0	43.595	6.63500	6.1517	7.3524
$\sum \rightarrow$ Operation H = 1	0	-61.242	-8.1027	0	-244.772	3.6788	0.90508	0

If operations G and H are combined so that the moment at C introduced by the combination is zero, the resulting forces and moments are given in the following table:

Operation \ Forces and moments	R_A	N_B	R_B	T_B	N_C	R_C	T_C	R_D
1 × (G)	0	4.2854	-1.83891	0.2403	-8.10267	0.66158	0	0
-0.033103 × (H)	0	2.0273	0.26822	0	8.10267	-0.12178	-0.02996	0
$\sum \rightarrow$ Operation I = 1	0	6.3127	-1.5707	0.2403	0	0.53980	-0.02996	0

Use of operation I results in liquidation of the R_B residual, in reduction in the N_B residual, in adjustment of the T_B and T_C residuals toward the desired equality, and in introduction of an R_C residual of 138 pounds. The latter can be balanced by the use of operation F, which will preserve the balance of A and D and will not affect the N_B and R_B residuals.

After this fifth operation the T_B and T_C residuals are approximately equal and opposite as desired. Therefore, a group operation, involving equal and opposite tangential displacements of B and C and sufficient radial displacements of A and D so that the latter remain balanced, is developed in the following table:

Operation \ Forces and moments	R_A	N_B	R_B	T_B	N_C
$u_B = 10^{-4}$ in.	3.96771	-13.1014	3.4352	-30.9566	0
$u_C = -10^{-4}$ in.	0	0	0	-26.2058	5.24667
$v_A = 1.1850 \times 10^{-4}$ in.	-3.9677	10.573	-3.1949	4.7017	0
$v_D = 0.83669 \times 10^{-4}$ in.	0	0	0	0	-6.1517
$\sum \rightarrow$ Operation J = 1	0	-2.5284	0.2403	-52.461	-0.9050
Operation \ Forces and moments	R_C	T_C	R_D		
$u_B = 10^{-4}$ in.	0	26.2058	0		
$u_C = -10^{-4}$ in.	0.88929	27.0833	1.0375		
$v_A = 1.1850 \times 10^{-4}$ in.	0	0	0		
$v_D = 0.83669 \times 10^{-4}$ in.	-0.93626	-0.86807	-1.0375		
$\sum \rightarrow$ Operation J = 1	0.04697	52.421	0		

Use of operation J liquidates the T_B and T_C residuals and affects little the balance in the other degrees of freedom. The remaining residuals are considered negligibly small, the moment of 309 inch-pounds being approximately 3 percent of the maximum moment in the ring. As in the previous problem the individual displacements can be determined from the magnitudes of the group operations and thus the unknown moments and tangential forces at A and D calculated.

Although the calculations of the group operations shown herein have been carried out on a computing machine with five significant figures maintained wherever possible, sufficient accuracy for engineering purposes can be obtained by the use of a slide rule. In developing this procedure a slide rule was used for all computations and the results agreed satisfactorily with those shown herein.

Oval-Shaped Ring with Internal Bracing

The ring shown in figure 8 is used as a third example of the new relaxation procedures. As a check on the results of this procedure the system of equations given by the operations table and external forces is also solved by the exact mathematical methods of matrix calculus and of the growing-unit method. In order that the charts and tables of reference 6 could be used in determining the influence coefficients, the following physical characteristics of the elements of the ring are assumed:

Segments AB and EF:

$$\gamma = \frac{AL^2}{I} = 500$$

$$\xi = \frac{A^*}{A} = 0.10$$

$$\beta = 45^\circ$$

$$EI = 10^6 \text{ lb-in.}^2$$

$$L = 18.85 \text{ in.}$$

Segments BC, CD, and DE:

$$\gamma = \frac{AL^2}{I} = 500$$

$$\xi = \frac{A^*}{A} = 0.10$$

$$\beta = 30^\circ$$

$$EI = 10^6 \text{ lb-in.}^2$$

$$L = 18.85 \text{ in.}$$

Segment EG:

$$\gamma = \frac{AL^2}{I} = 400$$

$$\xi = \frac{A^*}{A} = 0.10$$

$$\beta = 0$$

$$EI = 10^5 \text{ lb-in.}^2$$

$$L = 16.97 \text{ in.}$$

Because of the symmetry about a line through AGF only one-half of the ring need be considered. Joints A, G, and F are then restrained from rotating or displacing tangentially and cannot be subjected to radial forces. The assumed positive directions of the displacements and of the forces and moments at each joint are shown in figure 8. From the foregoing assumptions, the influence coefficients and the operations table given in table 11 are determined.

The horizontal external forces of 1000 pounds at C and D are resolved into their tangential and radial components. Thus the external forces are:

$$\left.\begin{aligned} R_C &= 965.93 \text{ lb} \\ T_C &= -258.82 \text{ lb} \\ R_D &= -965.93 \text{ lb} \\ T_D &= -258.82 \text{ lb} \end{aligned}\right\} \quad (21)$$

The matrix calculus solution of the system of equations given by these external forces and by the operations table is first obtained so that the equilibrium of the ring as given by this solution will provide a check on the whole setup. Joint G is considered fixed so that a unique solution to this system of equations is obtained; thus there are 14 degrees of freedom to be considered. The 14 unknowns are found by the method of reference 9 to be:

$$\left.\begin{aligned}
v_A &= -605.73 \times 10^{-3} \text{ in.}\\
w_B &= 40.825 \times 10^{-3} \text{ radian}\\
v_B &= 35.144 \times 10^{-3} \text{ in.}\\
u_B &= -300.06 \times 10^{-3} \text{ in.}\\
w_C &= -11.445 \times 10^{-3} \text{ radian}\\
v_C &= 664.55 \times 10^{-3} \text{ in.}\\
u_C &= -72.282 \times 10^{-3} \text{ in.}\\
w_D &= -22.975 \times 10^{-3} \text{ radian}\\
v_D &= -94.734 \times 10^{-3} \text{ in.}\\
u_D &= 90.130 \times 10^{-3} \text{ in.}\\
w_E &= 6.2337 \times 10^{-3} \text{ radian}\\
v_E &= -42.621 \times 10^{-3} \text{ in.}\\
u_E &= 32.513 \times 10^{-3} \text{ in.}\\
v_F &= -44.648 \times 10^{-3} \text{ in.}
\end{aligned}\right\} \qquad (22)$$

These displacements give the following values of the unknown moments and tangential reactions at A, F, and G on the bars rather than on the joints:

$$\left.\begin{aligned} N_A &= -\ 3118.8 \text{ in.-lb} \\ T_A &= 402.51 \text{ lb} \\ N_F &= 105.43 \text{ in.-lb} \\ T_F &= -\ 182.57 \text{ lb} \\ N_G &= -\ 371.06 \text{ in.-lb} \\ R_G &= 0.43 \text{ lb} \\ T_G &= 584.98 \text{ lb} \end{aligned}\right\} \qquad (23)$$

Figure 9 is the bending-moment diagram for the ring with these reactions applied.

By examining the equilibrium of one-half the ring under these reactions and the external forces, the accuracy of the operations table is established. Since R_A and R_F are zero, the summation of forces in the vertical direction is simply:

$$\sum F_V = R_G = 0.43 \text{ lb} \qquad (24)$$

The summation of forces in the horizontal direction is:

$$\sum F_H = T_A - T_F - T_G = 0.10 \text{ lb} \qquad (25)$$

The summation of moments about point G is:

$$\sum M_G = N_A + T_A \left[24 + \frac{12}{0.70711} + 24(0.70711)\right]$$

$$+ N_G + N_F + T_F(24)(1 - 0.70711) - 1000(2 \times 36 \times 0.25882) \quad (26)$$

$$= - 3118.91 + 402.49(57.941) - 371.06 + 105.43$$

$$- 182.56(7.0294) - 18635.04$$

$$= 17.45 \text{ in.-lb}$$

The equilibrium conditions for the half ring are approximately satisfied, the maximum percent error being a moment of less than 0.1 percent of the applied couple of 18,635 inch-pounds. It is considered that the accuracy of the operations table is established by this equilibrium check.

Approximately 20 man-hours by an unskilled computing-machine operator were required to solve this system of 14 equations. It is estimated that a skilled operator familiar with the Crout method would require about 10 man-hours.

In applying the Crout method to this problem the coefficients of the linear equations are assumed to be mathematically exact and, therefore, as many figures as could be carried on the 10-bank computing machine are used throughout the computation. In this way an accurate solution is obtained and the additional computing work is not great. Afterward the values of the unknowns can be rounded off to the physically correct number of significant figures.

Use of the growing-unit method of solution on this ring is demonstrated as follows. This method is described in detail on pages 39 to 46 of reference 5. It is demonstrated on this new ring as an application of the procedure to a ring with many intermediate joints between the center line of symmetry points. In applying the growing-unit method to this ring the units are combined into bars of increasing length until displacements of all points are known such that the only unbalanced forces remaining act in the radial direction at A and F when unit radial displacements are undertaken at A and F. Then these forces at A and F can be eliminated by appropriate radial displacements of A and F and the final distorted shape determined.

The first units to be combined are AB and BC. In order to effect this combination, the displacements of B required to maintain the balance of B during a unit radial displacement of A and unit radial, tangential, and rotational displacements of C must be determined. The displacements of B required to maintain the balance of B while point A is displaced radially 10^{-3} inch are given by the equations:

$$\left.\begin{aligned} N_B &= -454.34w_B + 6.7238v_B - 78.411u_B + 5.9020 \times 10^{-3} = 0 \\ R_B &= 6.7238w_B - 12.093v_B + 0.55690u_B - 4.5778 \times 10^{-3} = 0 \\ T_B &= -78.411w_B + 0.55690v_B - 84.510u_B + 14.662 \times 10^{-3} = 0 \end{aligned}\right\} \quad (27)$$

The solution to these equations is: $w_B = -0.026434 \times 10^{-3}$ radian, $v_B = -0.38424 \times 10^{-3}$ inch, and $u_B = 0.19549 \times 10^{-3}$ inch.

If the forces and moments at points A and C due to a displacement $v_A = 10^{-3}$ inch and due to the foregoing displacements w_B, v_B, and u_B are summed, the following equations are obtained:

$$\left.\begin{aligned} R_A &= -2.6618 \text{ lb} \\ N_C &= 10.699 \text{ in.-lb} \\ R_C &= -1.8871 \text{ lb} \\ T_C &= 3.2614 \text{ lb} \end{aligned}\right\} \quad (28)$$

N_B, R_B, and T_B are zero since that is the condition satisfied by equations (27).

The displacements of B required to maintain balance at B during unit rotational, radial, or tangential displacements of C are determined in a similar manner and are collected in table 12.

The forces and moments given in the last seven rows of this table constitute the influence coefficients for a new unit of the ring, *namely*, the segment $\overline{ABC}$. This unit is not a bar, the center line of which is an arc of a circle, but rather one composed of two arcs of circles. This combining of units, extended until the entire ring is one segment, is the main principle of the growing-unit method.

Each column of table 12 represents a group displacement made up of individual displacements of points A, B, and C. Let these group displacements be identified by the Roman numeral given at the head of each column. For example, group II is made up of the displacements $w_C = 10^{-3}$ radian, $w_B = -0.21631 \times 10^{-3}$ radian, $v_B = -0.23971 \times 10^{-3}$ inch, $u_B = 0.74197 \times 10^{-3}$ inch, and $v_A = v_C = u_C = 0$. The moment at C, for instance, caused by the application of x_{II} units of the group displacement II is then

$$N_C = -389.56x_{II} \tag{29}$$

With a similar notation for all other forces and group displacements, equations (30) may be set up representing the requirements for equilibrium of joint C under the external forces acting at that point, balance of B being maintained.

$$\left.\begin{aligned}
N_C &= -389.56x_{II} - 10.093x_{III} - 53.771x_{IV} = 0 \\
R_C &= -10.093x_{II} - 6.7529x_{III} - 10.615x_{IV} + 965.93 = 0 \\
T_C &= -53.771x_{II} - 10.615x_{III} - 54.199x_{IV} - 258.82 = 0
\end{aligned}\right\} \tag{30}$$

The solution to this system is $x_{II} = 1.0476$, $x_{III} = 217.61$, and $x_{IV} = -48.436$, and the following forces and moments are introduced at A and D:

$$\left.\begin{aligned}
R_A &= -557.43 \text{ lb} \\
N_D &= -2666.6 \text{ in.-lb} \\
R_D &= 184.66 \text{ lb} \\
T_D &= 626.46 \text{ lb}
\end{aligned}\right\} \tag{31}$$

The forces and moments at D are added to the external forces applied to the ring at D and are balanced after the unit problem for the segment at $\overline{ABCD}$ is established. The R_A force is not balanced until the complete ring is one segment and until the R_A and R_F residuals can be balanced together.

The next unit to be considered is the combination of the $\overline{ABC}$ segment with bar CD into the segment $\overline{ABCD}$. The problem is to find the forces and moments at A, D, and E due (1) to a unit radial displacement of A with joint D fixed and (2) to unit radial, tangential, and rotational displacements of D with A and E fixed. Joints B and C are free to displace so as to maintain the balance at B and C in each of these four cases. By determining the magnitudes of x_{II}, x_{III}, and x_{IV} required to balance C in each of these four cases, the required displacements of both B and C are implicitly determined and the unit problem for segment $\overline{ABCD}$ solved.

The magnitudes of the x_{II}, x_{III}, and x_{IV} operations required to balance joint C when A is displaced radially 10^{-3} inch and B permitted to displace so as to remain in balance are given by the following equations:

$$\left.\begin{aligned}
N_C &= -389.56x_{II} - 10.093x_{III} - 53.771x_{IV} + 10.699 = 0 \\
R_C &= -10.093x_{II} - 6.7529x_{III} - 10.615x_{IV} - 1.8871 = 0 \\
T_C &= -53.771x_{II} - 10.615x_{III} - 54.199x_{IV} + 3.2615 = 0
\end{aligned}\right\} \qquad (32)$$

The forces and moments at C to be balanced are given in group I in table 12. The solution to these equations is $x_{II} = 0.021497$, $x_{III} = -0.53843$, and $x_{IV} = 0.14430$. Use of these multiples of operations II, III, and IV and of a unit amount of group I results in the following forces and moments at A and D:

$$\left.\begin{aligned}
R_A &= -0.94505 \text{ lb} \\
N_D &= 6.7928 \text{ in.-lb} \\
R_D &= -0.74924 \text{ lb} \\
T_D &= 0.77749 \text{ lb}
\end{aligned}\right\} \qquad (33)$$

The forces and moments given by groups V, VI, VII, and VIII in table 13 are the influence coefficients for segment $\overline{ABCD}$. For example, the forces and moments introduced at A, D, and E due to a unit radial displacement of D with A and E fixed and with B and C in balance are given by VII. With these sets of coefficients it is possible to balance joint D while the balance of B and C is preserved. The forces and moments to be balanced at D are (1) the external forces on the ring at D and (2) the forces and moments which are introduced at D by the balancing of C and which are given by equations (31). The residuals to be balanced at D are thus:

$$\left.\begin{aligned} N_D &= -2666.6 \text{ in.-lb} \\ R_D &= -965.93 + 184.66 = -781.27 \text{ lb} \\ T_D &= -258.82 + 626.46 = 367.64 \text{ lb} \end{aligned}\right\} \qquad (34)$$

The equations which condition the balancing of joint D, from consideration of groups VI, VII, and VIII, are seen to be:

$$\left.\begin{aligned} N_D &= -346.88x_{VI} - 16.697x_{VII} - 43.745x_{VIII} - 2666.6 = 0 \\ R_D &= -16.697x_{VI} - 5.6500x_{VII} - 12.458x_{VIII} - 781.27 = 0 \\ T_D &= -43.745x_{VI} - 12.458x_{VII} - 50.817x_{VIII} + 367.64 = 0 \end{aligned}\right\} \qquad (35)$$

The solution to these equations is $x_{VI} = -3.3100$, $x_{VII} = -328.09$, and $x_{VIII} = 90.518$, which give the following forces and moments:

$$\left.\begin{aligned} R_A &= 293.71 \text{ lb} \\ N_E &= 4889.4 \text{ in.-lb} \\ R_E &= -522.69 \text{ lb} \\ T_E &= -149.69 \text{ lb} \end{aligned}\right\} \qquad (36)$$

As in the balancing of C, a tangential force and moment are introduced at A by this balancing of D, but because of symmetry the equilibrium of A is not disturbed by these. The R_A forces will be balanced later and the residuals at E will be balanced when the influence coefficients for segment $\overline{ABCDE}$ have been determined.

In order to find the influence coefficients for bar $\overline{ABCDE}$, the forces and moments at A and E due to a radial displacement of A with E fixed and at A, E, and F due to unit radial, tangential, and rotational displacements of E with A and F fixed must be determined. By determining the magnitudes of groups VI, VII, and VIII required to balance D in each of these four cases, the required displacements of B, C, and D and the required forces and moments are determined.

The magnitudes of the groups VI, VII, and VIII required to balance D when joint A is moved radially 10^{-3} inch are given by the following equations:

$$\left.\begin{aligned} N_D &= -346.88x_{VI} - 16.697x_{VII} - 43.745x_{VIII} + 6.7928 = 0 \\ R_D &= -16.697x_{VI} - 5.6500x_{VII} - 12.458x_{VIII} - 0.74924 = 0 \\ T_D &= -43.745x_{VI} - 12.458x_{VII} - 50.817x_{VIII} + 0.77749 = 0 \end{aligned}\right\} \quad (37)$$

The forces and moments at D to be balanced by groups VI, VII, and VIII are given by V in table 13 and are the constant terms in equation (37). The solution of these equations is $x_{VI} = 0.028042$, $x_{VII} = -0.42660$, and $x_{VIII} = 0.095744$. The summation of forces and moments due to a unit magnitude of group V and the foregoing multiples of groups VI, VII, and VIII are:

$$\left.\begin{aligned} R_A &= -0.36050 \text{ lb} \\ N_E &= 4.1055 \text{ in.-lb} \\ R_E &= -0.31703 \text{ lb} \\ T_E &= 0.19226 \text{ lb} \end{aligned}\right\} \quad (38)$$

In a similar manner the complete set of influence coefficients for segment $\overline{ABCDE}$ is determined and is given in table 14. For example, the forces and moments in group XI are the forces and moments introduced at A, E, F, and G by a unit radial displacement of E with A, F, and G fixed and with points B, C, and D free to displace so as to remain in equilibrium. With these influence coefficients joint E can be balanced while the balance of B, C, and D is preserved.

The forces and moments to be balanced at joint E are those introduced by the balancing of joint D with groups VI, VII, and VIII and are given by equation (36).

The equations in x_X, x_{XI}, and x_{XII} balancing joint E under these loads are:

$$\left.\begin{aligned} N_E &= -533.92x_X - 38.099x_{XI} - 29.579x_{XII} + 4889.4 = 0 \\ R_E &= -38.099x_X - 49.295x_{XI} - 53.432x_{XII} - 522.69 = 0 \\ T_E &= -29.579x_X - 53.432x_{XI} - 76.322x_{XII} - 149.69 = 0 \end{aligned}\right\} \quad (39)$$

The solution to these equations is $x_X = 11.184$, $x_{XI} = -51.518$, and $x_{XII} = 29.771$ and the forces at A, F, and G introduced by this balancing of E are:

$$\left.\begin{aligned} R_A &= 67.974 \text{ lb} \\ R_F &= -266.68 \text{ lb} \\ R_G &= 70.668 \text{ lb} \end{aligned}\right\} \quad (40)$$

The tangential forces and the moments introduced at A, F, and G are not considered in this balancing of the half ring, since these are equilibrated by the forces and moments from the other half of the ring.

The final combination of units will be the combination of bar EF with the unit $\overline{ABCDE}$. When this union is effected, the influence coefficients for the half ring as a unit will have been determined and the radial forces at A and F can be balanced simultaneously. The radial forces at joints A and F due to a unit radial displacement of A with F fixed and to a unit radial displacement of F with A fixed must be determined. In both cases joints B, C, D, and E are displaced so as to remain balanced.

The equations giving the magnitudes of groups X, XI, and XII required to balance joints B, C, D, and E when joint A is displaced radially as in group IX are:

$$\left.\begin{aligned} N_E &= -533.92x_X - 38.099x_{XI} - 29.579x_{XII} + 4.1055 = 0 \\ R_E &= -38.099x_X - 49.295x_{XI} - 53.432x_{XII} - 0.31703 = 0 \\ T_E &= -29.579x_X - 53.432x_{XI} - 76.322x_{XII} + 0.19226 = 0 \end{aligned}\right\} \quad (41)$$

The solution to these equations is $x_X = 0.0094398$, $x_{XI} = -0.051804$, and $x_{XII} = 0.035128$ and the forces introduced by a unit magnitude of IX and by these multiples of groups X, XI, and XII are:

$$\left.\begin{aligned} R_A &= -0.29857 \text{ lb} \\ R_F &= -0.33361 \text{ lb} \\ R_G &= 0.035176 \text{ lb} \end{aligned}\right\} \quad (42)$$

The equations giving the magnitudes of groups X, XI, and XII required to balance joints B, C, D, and E when joint F is displaced radially 10^{-3} inch are:

$$\left.\begin{aligned} N_E &= -533.92x_X - 38.099x_{XI} - 29.579x_{XII} - 5.9020 = 0 \\ R_E &= -38.099x_X - 49.295x_{XI} - 53.432x_{XII} - 4.5778 = 0 \\ T_E &= -29.579x_X - 53.432x_{XI} - 76.322x_{XII} - 14.662 = 0 \end{aligned}\right\} \quad (43)$$

The solution to these equations is $x_X = -0.017182$, $x_{XI} = 0.50354$, and $x_{XII} = -0.53797$ and the forces introduced at A, F, and G by a radial displacement of F of 10^{-3} inch and by the foregoing multiples of groups X, XI, and XII are:

$$\left.\begin{aligned} R_A &= -0.33361 \text{ lb} \\ R_F &= -1.4470 \text{ lb} \\ R_G &= 1.1156 \text{ lb} \end{aligned}\right\} \quad (44)$$

The forces given by equations (42) and (44) represent the influence coefficients for the entire half ring and are labeled groups XIII and XIV, respectively. These forces permit calculation of the multiples of groups XIII and XIV required to balance the radial forces at A and F. These forces are the total forces remaining from the balancing of C, D, and E; R_A is given by the sum of the R_A forces of equations (31), (36), and (40) and is:

$$R_A = -557.43 + 293.71 + 67.974 = -195.75 \text{ lb}$$

The R_F force is the force introduced by the balancing of E alone and is given by equation (40). It is:

$$R_F = -266.68 \text{ lb}$$

The equations giving the magnitudes of groups XIII and XIV required to balance joints A and F under those loads are:

$$\left.\begin{aligned} R_A &= -0.29857x_{XIII} - 0.33361x_{XIV} - 195.75 = 0 \\ R_F &= -0.33361x_{XIII} - 1.4470x_{XIV} - 266.68 = 0 \end{aligned}\right\} \quad (45)$$

The solution to these equations is $x_{XIII} = -605.73$ and $x_{XIV} = -44.646$. The radial force at G introduced by this balancing is - 71.114 pounds, but the R_G force given by equation (40) in the balancing of E is 70.668 pounds. The difference between the two, - 0.446 pound, is considered negligibly small compared to the applied loads of 1000 pounds.

With the balancing of joints A and F and the substantiation of the balance at G, the entire half ring is balanced. The total deflections in each degree of freedom can now be calculated and used to determine the unknown bending moments and tangential forces at A, F, and G. In order to calculate these deflections the balancing equations (30), (35), (39), and (45) give the magnitudes of the group operations involved while the equations determining the group influence coefficients give the individual operations involved in each group.

Table 15 gives the magnitude of all group displacements from I to XIV implied in a unit application of any one group. For example, row X in this table indicates that a unit magnitude of group X (that is, $x_X = 1$) is equivalent to the sum of the effects of $X_{VIII} = 2.1885$, $X_{VII} = -4.6760$, $X_{VI} = -0.16186$, and $w_E = 10^{-3}$ radian, or the sum of the effects of $X_{IV} = 2.9219$, $X_{III} = 1.7552$, $X_{II} = -0.19736$, $w_E = 10^{-3}$ radian, $w_D = -0.16186 \times 10^{-3}$ radian, $v_D = -4.6760 \times 10^{-3}$ inch, and $u_D = 2.1885 \times 10^{-3}$ inch. During the solution of the problem the magnitude of group X which was explicitly used was 11.184, as given in the last column of table 15.

From table 15 the total magnitudes of each group operation may be found. For example, the total magnitude of group VI is:

$$\begin{aligned} x_{VI} &= (1)(-3.3100) + (0.028042)(0) + (-0.16186)(11.184) \\ &\quad + (0.037474)(-51.518) + (-0.0022427)(29.771) \\ &\quad + (0.024494)(-605.73) + (0.022857)(-44.646) \qquad (46) \\ &= -22.975 \end{aligned}$$

The total displacement w_D is:

$$w_D = (x_{VI}) \times 10^{-3} = -22.975 \times 10^{-3} \text{ radian} \qquad (46a)$$

Similarly the displacements of all points except point B may be calculated from table 15 and are given in the last row of that table.

Point B was displaced during the application of groups I, II, III, and IV, and therefore the magnitude of its displacement must be calculated as indicated in the following example:

$$\begin{aligned} w_B &= (-0.026434 \times 10^{-3})(-605.73) + (-0.21631 \times 10^{-3})(-11.444) \\ &\quad + (0.035554 \times 10^{-3})(664.55) + (0.017847 \times 10^{-3})(-72.286) \\ &= 40.825 \times 10^{-3} \text{ inch} \end{aligned} \tag{47}$$

where the first number in each product is the magnitude of w_B involved in each unit application of groups I, II, III, and IV, respectively.

The total displacements used are assembled in equations (47a).

$$\left.\begin{aligned} v_A &= -605.73 \times 10^{-3} \text{ in.} \\ w_B &= 40.825 \times 10^{-3} \text{ radian} \\ v_B &= 35.144 \times 10^{-3} \text{ in.} \\ u_B &= -300.06 \times 10^{-3} \text{ in.} \\ w_C &= -11.444 \times 10^{-3} \text{ radian} \\ v_C &= 664.55 \times 10^{-3} \text{ in.} \\ u_C &= -72.286 \times 10^{-3} \text{ in.} \\ w_D &= -22.975 \times 10^{-3} \text{ radian} \\ v_D &= -94.731 \times 10^{-3} \text{ in.} \\ u_D &= 90.127 \times 10^{-3} \text{ in.} \\ w_E &= 6.2331 \times 10^{-3} \text{ radian} \\ v_E &= -42.620 \times 10^{-3} \text{ in.} \\ u_E &= 32.511 \times 10^{-3} \text{ in.} \\ v_F &= -44.646 \times 10^{-3} \text{ in.} \end{aligned}\right\} \tag{47a}$$

These total displacements constitute the unknowns of the system of equations given by the operations table and the external forces; comparison between this growing-unit and the matrix calculus solutions given by equations (47a) and (22), respectively, indicates good agreement for the displacements. In fact, the forces and moments given by the two methods differ by less than 1 percent and therefore are given only for the matrix method (equation (23)).

Several general remarks are made about the growing-unit method:

(a) In determining the influence coefficients and in balancing the external forces and moments, sets of equations with the same left-hand sides but with different constant terms are used several times. This simplifies solution of the equations and reduces the computational work considerably.

(b) In order to obtain sufficient accuracy of solution for rings with many joints, calculating machines must be used; five significant figures were carried throughout the calculations. However, on the simpler rings such as the circular ring and the egg-shaped ring discussed previously, slide-rule accuracy for determining the displacements in a combined operation is probably sufficient for engineering purposes.

(c) A check on the influence coefficients for composite bars is obtained by applying Maxwell's theorem of reciprocal deflections. This is a valuable device for assuring accuracy at each stage.

In applying the new relaxation procedures to this ring, it would have been possible to use the general method described for the egg-shaped ring, that is, to consider the residuals after each operation and develop a satisfactory combined operation to reduce as many residuals as possible. However, the number of degrees of freedom involved in this ring is large and, therefore, the number of residuals to be considered in testing the efficacy of a particular operation is large.

The loading on the ring provides a clue to overcoming this difficulty. No external loads are applied at A, B, E, F, and G; moreover, A, F, and G are points along the center line of symmetry. Therefore, if in balancing D and E the balance at the other joints is preserved by suitable displacements, attention is fixed on the two joints D and E and the procedure described for the egg-shaped ring can be used effectively. It will be recognized that this procedure is essentially a combination of growing-unit and relaxation methods of solution.

In executing the proposed method the bar $\overline{ABCD}$, free only to displace radially at A and fixed at D, is considered first. The equations giving the displacements of A and B required to maintain balance of these points while joint C is rotated through 10^{-3} radian are:

$$\left.\begin{aligned}
R_A &= -7.1310v_A + 5.9020w_B - 4.5778v_B + 14.662u_B = 0\\
N_B &= 5.9020v_A - 454.34w_B + 6.7238v_B - 78.411u_B - 38.489 \times 10^{-3} = 0\\
R_B &= -4.5778v_A + 6.7238w_B - 12.093v_B + 0.55690u_B - 1.8576 \times 10^{-3} = 0\\
T_B &= 14.662v_A - 78.411w_B + 0.55690v_B - 84.510u_B + 45.876 \times 10^{-3} = 0
\end{aligned}\right\} \quad (48)$$

The solution to these equations is $v_A = 4.0195 \times 10^{-3}$ inch, $w_B = -0.32255 \times 10^{-3}$ radian, $v_B = -1.7842 \times 10^{-3}$ inch, and $u_B = 1.5277 \times 10^{-3}$ inch. These displacements combined with the unit rotation of C yield:

$$\left.\begin{aligned}
N_C &= -346.56 \text{ in.-lb}\\
R_C &= -17.678 \text{ lb}\\
T_C &= -40.663 \text{ lb}\\
N_D &= -38.489 \text{ in.-lb}\\
R_D &= 1.8576 \text{ lb}\\
T_D &= 45.876 \text{ lb}
\end{aligned}\right\} \quad (49)$$

The moment and tangential force introduced at A are not considered until the balancing of the ring is complete.

In a similar manner the forces and moments for unit radial and tangential displacements of C are determined, as shown in table 16. The forces and moments given by groups XV, XVI, and XVII constitute the influence coefficients for the displacements of C with A and D fixed and with joints A and B balanced. Use of these coefficients permits focusing of attention on C and D, the joints at which the external forces are applied when C is being balanced.

The forces and moments introduced at C and D when D is displaced a unit amount in each degree of freedom and when E, F, and G are displaced so as to maintain the balance thereof are calculated and shown in table 17.

Table 18 is an operations table consisting of unit magnitudes of group operations XV to XX. Table 19 is the relaxation table for this ring which uses these group operations. The external forces applied at C and D are given in the first row of table 19.

A discussion of each step in the relaxation process is given as follows.

<u>Step 1</u>.- Because of the antisymmetry of the loading and of the quasisymmetry of the ring about a horizontal axis operations $x_{XVI} = 1$ and $x_{XIX} = -1$ are applied as a first approximation to the deflected shape. The forces and moments introduced are as given in the following table:

Operation \ Forces and moments	N_C	R_C	T_C	N_D	R_D	T_D
(XVI) = 1	-17.678	-5.4150	-12.928	-1.8576	-2.2277	13.781
(XIX) = -1	-1.8576	2.2277	13.781	-11.535	6.2441	-12.706
$\sum$ → Operation K = 1	-19.536	-3.1873	0.853	-13.393	4.0164	1.075

Operation K is used to balance the R_C residual; the same operation reduces the other force residuals but introduces large N_C and N_D residuals.

<u>Step 2</u>.- In order to reduce these moment residuals an antisymmetrical combination of w_C and w_D is made, as shown as group operation L:

Operation \ Forces and moments	N_C	R_C	T_C	N_D	R_D	T_D
(XV) = 1	-346.56	-17.678	-40.662	-38.489	1.8576	45.876
(XVIII) = 1	-38.489	-1.8576	45.876	-392.46	11.535	-42.305
$\sum$ → Operation L = 1	-385.04	-19.536	5.214	-430.95	13.393	3.571

However, use of operation L by itself would reintroduce large R_C and R_D residuals, and therefore operations K and L are combined so that the R_C residual will be smaller and the R_D residual eliminated, as shown in the following table:

Operation \ Forces and moments	N_C	R_C	T_C	N_D	R_D	T_D
Operation L = 1	-385.04	-19.536	5.214	-430.95	13.393	3.571
-3.3346 × Operation K	65.145	10.628	-2.844	44.660	-13.393	-3.5846
$\sum$ → Operation M = 1	-319.90	-8.908	2.370	-386.29	0	-0.0137

The new force residuals introduced by operation M are less than 30 percent of the original residuals and, therefore, the rate of convergence is felt to be adequate.

Step 3.- The radial residuals at C and D have the same sign and, therefore, symmetrical displacements v_C and v_D are undertaken. It is seen that such a combination would introduce large tangential residuals at C and D. Therefore, a tangential displacement of C (D could have been chosen instead) such as to eliminate the T_C and T_D forces is undertaken. The forces and moments introduced by the individual displacements and by the combination are denoted as operation N.

Operation \ Forces and moments	N_C	R_C	T_C	N_D	R_D	T_D
(XVI) = 1	-17.678	-5.4150	-12.928	-1.8576	-2.2277	13.781
(XIX) = 1	1.8576	-2.2277	-13.781	11.535	-6.2441	12.706
-0.53201 × (XVII)	21.632	6.8773	26.709	-24.406	7.3316	-26.587
$\sum$ → Operation N = 1	5.811	-0.7653	0	-14.730	-1.1400	-0.100

The use of operation N reduces substantially all the residuals except N_C.

Step 4.- In order to reduce N_C and at the same time keep the T_C and T_D residuals small, a combination of groups XV and XX is made. Group XX is included since a force increasing the residual T_D would be introduced by the use of XV alone.

Operation \ Forces and moments	N_C	R_C	T_C	N_D	R_D	T_D
(XV) = 1	-346.55	-17.678	-40.662	-38.489	1.8576	45.876
0.91279 × (XX)	41.875	12.579	45.616	-38.616	11.597	-45.876
$\sum$ → Operation O = 1	-304.67	-5.099	4.954	-77.105	13.455	0

Steps 5 and 6.- After operation O is used, the largest force residual is approximately 6 percent of the applied forces and the moment residuals are small. It was considered desirable to reduce further the force residuals. Therefore, operation I was used again so as to reduce R_D the largest force residual, and then XVII was used so as to reduce the resulting T_C residual. After this sixth step the largest residual of 4 percent of the external force is considered small enough.

A check table using the total displacements is used as a check on the accuracy of the combined operations and on the relaxation table. The total individual displacements are calculated as discussed in the previous two examples and are as follows:

$$
\left.
\begin{aligned}
v_A &= -596.18 \times 10^{-3} \text{ in.} \\
w_B &= 36.779 \times 10^{-3} \text{ radian} \\
v_B &= 0.22394 \times 10^{-3} \text{ in.} \\
u_B &= -304.69 \times 10^{-3} \text{ in.} \\
w_C &= -14.32 \times 10^{-3} \text{ radian} \\
v_C &= 561.66 \times 10^{-3} \text{ in.} \\
u_C &= -114.61 \times 10^{-3} \text{ in.} \\
w_D &= -18.4 \times 10^{-3} \text{ radian} \\
v_D &= -133.66 \times 10^{-3} \text{ in.} \\
u_D &= 3.7242 \times 10^{-3} \text{ in.} \\
w_E &= 7.4813 \times 10^{-3} \text{ radian} \\
v_E &= 30.507 \times 10^{-3} \text{ in.} \\
u_E &= -41.708 \times 10^{-3} \text{ in.} \\
v_F &= 59.979 \times 10^{-3} \text{ in.} \\
v_G &= 114.52 \times 10^{-3} \text{ in.}
\end{aligned}
\right\} \qquad (50)
$$

It is pointed out that certain of these displacements differ considerably from those given by the exact solutions of the matrix calculus and growing-unit methods, mainly because the relaxation solution is approximate and in it joint G is permitted to displace radially.

The unknown reactions given by the foregoing relaxation procedure are:

$$\left.\begin{aligned} N_A &= -2851.1 \text{ in.-lb} \\ T_A &= 380.21 \text{ lb} \\ N_F &= 140.16 \text{ in.-lb} \\ T_F &= -224.09 \text{ lb} \\ N_G &= -440.45 \text{ in.-lb} \\ T_G &= 648.18 \text{ lb} \end{aligned}\right\} \qquad (51)$$

Consideration of the equilibrium of the half ring gives:

$$\left.\begin{aligned} \sum F_H &= 380.21 + 224.09 - 648.18 = -43.88 \text{ lb} \\ \sum F_V &= 0 \\ \sum M_G &= -2851.1 + 380.21(57.941) - 440.45 + 140.16 \\ &\quad - 224.09(7.0294) - 18{,}635 \\ &= -1331.8 \text{ in.-lb} \end{aligned}\right\} \qquad (52)$$

The moment equilibrium unbalance is approximately 7 percent of the applied moment and is considered satisfactory for engineering purposes. If a more accurate representation of the final deflected shape and consequently of the bending-moment diagram is desired, several more operations in the relaxation table could be undertaken and the residuals at C and D further reduced.

The bending-moment diagram given by the reactions of equation (51) is shown in figure 9 along with that of the exact solutions. The external unbalanced moment of 1331.8 inch-pounds is applied linearly along the ring as a distributed moment. If this unbalance is not distributed in this manner, it would be concentrated at either joint A or joint F, depending on the direction in which the bending moments are calculated, and would lead to large errors in the bending moment in the neighborhood of that joint. It is seen from figure 9 that the agreement between the exact and relaxation solutions is good.

It is pointed out that, by slightly modifying the determination of the influence coefficient for joint D when E is fixed and F and G free to displace radially, a table similar to table 17 could be established and solved by matrix calculus methods. The slight modification is to make $V_G = 0$ in the equations corresponding to table 17. Such a solution is essentially the growing-unit method, except that the ring is combined from joints C and D to A and F, respectively, rather than from A to F. The total displacements in each degree of freedom will be the same in each approach.

CONCLUSIONS

This report contains recommendations as to the choice of the most expeditious method of solution of the simultaneous linear equations represented by the operations table and the external loads. The operations table is first established in accordance with Southwell's suggestions and, together with the external loads, defines completely the problem of stress distribution in a reinforced panel or of the moment distribution in a fuselage ring. However, the following generalized suggestions can be made:

1. In most reinforced panel problems the use of the relaxation procedure is advantageous.

2. Solution of the equations defining a reinforced panel problem by means of the electric analogue is advisable when many closely related problems have to be investigated.

3. Ring problems are best solved by matrix methods.

4. In very complicated ring problems a combination of matrix methods with the growing-unit and relaxation methods may become advisable.

Polytechnic Institute of Brooklyn
Brooklyn, N. Y., June 25, 1947

REFERENCES

1. Cross, Hardy, and Morgan, Newlin Dolbey: Continuous Frames of Reinforced Concrete. John Wiley & Sons, Inc., 1932.

2. Southwell, R. V.: Relaxation Methods in Engineering Science. A Treatise on Approximate Computations. The Clarendon Press (Oxford), 1940.

3. Hoff, N. J., Levy, Robert S., and Kempner, Joseph: Numerical Procedures for the Calculation of the Stresses in Monocoques. I - Diffusion of Tensile Stringer Loads in Reinforced Panels. NACA TN No. 934, 1944.

4. Hoff, N. J., and Kempner, Joseph: Numerical Procedures for the Calculation of the Stresses in Monocoques. II - Diffusion of Tensile Stringer Loads in Reinforced Flat Panels with Cut-Outs. NACA TN No. 950, 1944.

5. Hoff, N. J., Libby, Paul A., and Klein, Bertram: Numerical Procedures for the Calculation of the Stresses in Monocoques. III - Calculation of the Bending Moments in Fuselage Frames. NACA TN No. 998, 1946.

6. Hoff, N. J., Klein, Bertram, and Libby, Paul A.: Numerical Procedures for the Calculation of the Stresses in Monocoques. IV - Influence Coefficients of Curved Bars for Distortions in Their Own Plane. NACA TN No. 999, 1946.

7. Hoff, N. J., Boley, Bruno A., and Klein, Bertram: Stresses and General Instability of Monocoque Cylinders with Cutouts. II - Calculation of the Stresses in a Cylinder with a Symmetric Cutout. NACA TN No. 1014, 1946.

8. Hoff, N. J., and Klein, Bertram: Stresses in and General Instability of Monocoque Cylinders with Cutouts. V - Calculation of the Stresses in Cylinders with Side Cutout. NACA TN No. 1435, 1948.

9. Crout, Prescott D.: A Short Method for Evaluating Determinants and Solving Systems of Linear Equations with Real or Complex Coefficients. Trans. A.I.E.E., vol. 60, 1941, pp. 1235-1240.

10. Niles, Alfred S.: Overbalancing in Residual-Liquidation Computations. NACA TN (to be published).

11. Newton, R. E.: Electrical Analogy for Shear Lag. Rep. No. R-140, Curtiss-Wright Corp., Airplane Div. (St. Louis), April 21, 1944.

12. Engle, M. E., and Newton, R. E.: Experimental Solutions of Shear Lag Problems by an Electrical Analogy. Rep. No. R-149, Curtiss-Wright Corp., Airplane Div. (St. Louis), Oct. 27, 1944.

TABLE 1.- OPERATIONS TABLE FOR REINFORCED PANEL

[Forces are in lb; displacements, in in. × 10^{-4}]

Operation \ Force	Y_A	Y_B	Y_E	Y_F	Y_J	Y_K	Y_N	Y_O
$v_A = 1$	-50.8	2.00	46.8	2.00				
$v_B = 1$	2.00	-52.2	2.00	51.2				
$v_E = 1$	46.8	2.00	-101.6	4.00	46.8	2.00		
$v_F = 1$	2.00	51.2	4.00	-110.4	2.00	51.2		
$v_J = 1$			46.8	2.00	-101.6	4.00	46.8	2.00
$v_K = 1$			2.00	51.2	4.00	-110.4	2.00	51.2
$v_N = 1$					46.8	2.00	-50.8	2.00
$v_O = 1$					2.00	51.2	2.00	-53.2
$v_{Block\ 1} = 1$; $v_A = v_E = v_J = v_N = 1$	-4.00	4.00	-8.00	8.00	-8.00	8.00	-4.00	4.00
$v_{Block\ 2} = 1$; $v_B = v_F = v_K = v_O = 1$	4.00	-4.00	8.00	-8.00	8.00	-8.00	4.00	-4.00
(1) $v_A = v_E = 1$	-4.00	4.00	-54.8	6.00	46.8	2.00		
(2) $v_A = v_E = v_J = 1$	-4.00	4.00	-8.00	8.00	-54.8	6.00	46.8	2.00
(3) $v_B = v_F = 1$	4.00	-4.00	6.00	-59.2	2.00	51.2		
(4) $v_B = v_F = v_K = 1$	4.00	-4.00	8.00	-8.00	6.00	-59.2	2.00	51.2

NACA

TABLE 2.- RELAXATION TABLE FOR REINFORCED PANEL - PROCEDURE 1

[Cycles of operations shown should be repeated until residuals are considered negligibly small. Forces are in lb; displacements, in in. × 10^{-4}]

Operation \ Force	Y_A	Y_B	Y_E	Y_F	Y_J	Y_K	Y_N	Y_O
External forces	-120						60	60
$v_{Block\ 1} = -2.5$	10	-10	20	-20	20	-20	10	-10
	-110	-10	20	-20	20	-20	70	50
$v_E = 2.35$	110	5	-238	9	110	5	0	0
	0	-5	-218	-11	130	-15	70	50
$v_J = 4.65$	0	0	218	9	-473	19	218	9
	0	-5	0	-2	-343	4	288	59
$v_N = 7.33$	0	0	0	0	343	14	-372	14
	0	-5	0	-2	0	18	-84	73
$v_{Block\ 2} = 3.5$	14	-14	28	-28	28	-28	14	-14
	14	-19	28	-30	28	-10	-70	59
$v_F = 0.371$	1	19	1	-41	1	19	0	0
	15	0	29	-71	29	9	-70	59
$v_K = 1.385$	0	0	3	71	5	-153	3	71
	15	0	32	0	34	-144	-67	130
$v_O = 2.81$	0	0	0	0	5	144	5	-156
	15	0	32	0	39	0	-62	-26
$v_{Block\ 1} = 1.0$	-4	4	-8	8	-8	8	-4	4
	11	4	24	8	31	8	-66	-22
$v_E = -0.235$	-11	0	24	-1	-11	0	0	0
	0	4	48	7	20	8	-66	-22
$v_J = -1.025$	0	0	-48	-2	104	-4	-48	-2
	0	4	0	5	124	4	-114	-24
$v_N = -2.65$	0	0	0	0	-124	-5	135	-5
	0	4	0	5	0	-1	21	-29

NACA

TABLE 3.- RELAXATION TABLE FOR REINFORCED PANEL - PROCEDURE 2

[Forces are in lb; displacements, in in. × 10^{-4}]

Operation \ Force	Y_A	Y_B	Y_E	Y_F	Y_J	Y_K	Y_N	Y_O
External forces $v_{Block\ 1}$ = -2.5	-120 10	 -10	 20	 -20	 20	 -20	60 10	60 -10
v_A = -2.165	-110 110	-10 -4.3	20 -101.3	-20 -4.3	20 0	-20 0	70 0	50 0
(1) = -1.482	0 5.9	-14.3 -5.9	-81.3 81.3	-24.3 -8.9	20 -69.4	-20 -3.0	70 0	50 0
(2) = -0.901	5.9 3.6	-20.2 -3.6	0 7.2	-33.2 -7.2	-49.4 49.4	-23.0 -5.4	70 -42.2	50 -1.8
$v_{Block\ 2}$ = -1.850	9.5 -7.4	-23.8 7.4	7.2 -14.8	-40.4 14.8	0 -14.8	-28.4 14.8	27.8 -7.4	48.2 7.4
v_B = -0.297	2.1 -0.6	-16.4 16.4	-7.6 -0.6	-25.6 -15.2	-14.8 0	-13.6 0	20.4 0	55.6 0
(3) = -0.689	1.5 -2.8	0 2.8	-8.2 -4.1	-40.8 40.8	-14.8 -1.4	-13.6 -35.3	20.4 0	55.6 0
(4) = -0.825	-1.3 -3.3	2.8 3.3	-12.3 -6.6	0 6.6	-16.2 -5.0	-48.9 48.9	20.4 -1.6	55.6 -42.3
$v_{Block\ 1}$ = -1.08	-4.6 4.3	6.1 -4.3	-18.9 8.6	6.6 -8.6	-21.2 8.6	0 -8.6	18.8 4.3	13.3 -4.3
(1) = -0.188	-0.3 0.8	1.8 -0.8	-10.3 10.3	-2.0 -1.1	-12.6 -8.8	-8.6 -0.4	23.1 0	9.0 0
(2) = -0.390	0.5 1.6	1.0 -1.6	0 3.1	-3.1 -3.1	-21.4 21.4	-9.0 -2.3	23.1 -18.3	9.0 -0.8
$v_{Block\ 2}$ = -0.412	2.1 -1.6	-0.6 1.6	3.1 -3.3	-6.2 3.3	0 -3.3	-11.3 3.3	4.8 -1.6	8.2 1.6
(4) = -0.135	0.5 -0.5	1.0 0.5	-0.2 -1.1	-2.9 1.1	-3.3 -0.8	-8.0 8.0	3.2 -0.3	6.6 -6.9
	0	1.5	-1.3	-1.8	-4.1	0	2.9	-0.3

NACA

TABLE 4.- RELAXATION TABLE FOR REINFORCED PANEL - FIXED ENDS

[Forces are in lb; displacements, in in. × 10^{-4}]

Force / Operation	Y_A	Y_B	Y_E	Y_F	Y_J	Y_K	Y_N	Y_O
External forces v_A = -2.36	-120 120	-4.7	-110.6	-4.7				
(1) = -2.02	0 8.1	-4.7 -8.1	-110.6 110.6	-4.7 -12.1	-94.4	-4.0		
(2) = -1.723	8.1 6.9	-12.8 -6.9	0 13.8	-16.8 -13.8	-94.4 94.4	-4.0 -10.3	-80.6	-3.4
v_B = -0.357	15.0 -0.8	-19.7 19.7	13.8 -0.8	-30.6 -18.3	0	-14.3	-80.6	-3.4
(3) = -0.83	14.2 -3.4	0 3.4	13.0 -5.1	-48.9 48.9	0 -1.7	-14.3 42.3	-80.6	-3.4
(4) = -0.959	10.8 -3.8	3.4 3.8	7.9 -7.6	0 7.6	-1.7 -5.7	-56.6 56.6	-80.6 -1.9	-3.4 -49.1
v_A = 0.1375	7.0 -7.0	7.2 0.3	0.3 6.4	7.6 0.3	-7.4	0	-82.5	-52.5
(1) = 0.124	0 -0.5	7.5 0.5	6.7 -6.7	7.9 0.7	-7.4 5.8	0 0.2	-82.5	-52.5
(2) = -0.0292	-0.5 0.1	8.0 -0.1	0 0.2	8.6 -0.2	-1.6 1.6	0.2 -0.3	-82.5 -1.4	-52.5 -0.1
v_B = 0.143	-0.4 0.3	7.9 -7.9	0.2 0.3	8.4 7.3	0	-0.1	-83.9	-52.6
(3) = 0.265	-0.1 1.1	0 -1.1	0.5 1.6	15.7 -15.7	0 0.5	-0.1 13.6	-83.9	-52.6
(4) = 0.228	1.0 0.9	-1.1 -0.9	2.1 1.8	0 -1.8	0.5 1.4	13.5 -13.5	-83.9 0.5	-52.6 11.7
	1.9	-2.0	3.9	-1.8	1.9	0	-83.4	-40.9

TABLE 5.- OPERATIONS TABLE FOR REINFORCED PANEL - GROWING-UNIT METHOD

[Forces are in lb; displacements, in in. × 10^{-4}]

Operation \ Forces and moments	Y_A	Y_B	Y_E	Y_F	Y_J	Y_K	Y_N	Y_O	Y_R	Y_S
(1) $v_A = 1$	-50.8	2	46.8	2						
(2) $v_B = 1$	2	-55.2	2	51.2						
(3) $v_E = 1$	46.8	2	-101.6	4	46.8	2				
(4) $v_F = 1$	2	51.2	4	-110.4	2	51.2				
(5) $v_J = 1$			46.8	2	-101.6	4	46.8	2		
(6) $v_K = 1$			2	51.2	4	-110.4	2	51.2		
(7) $v_N = 1$					46.8	2	-101.6	4	46.8	2
(8) $v_O = 1$					2	51.2	4	-110.4	2	51.2
(9) 0.0362 × (2)	0.07	-2	0.07	1.86						
(10) (1) + (9)	-50.7	0	46.9	3.8						
(11) 0.921 × (1)	-46.9	1.85	43.2	1.85						
(12) 0.0695 × (2)	0.1	-3.8	0.1	3.5						
(13) (3) + (11) + (12)	0	0	-58.3	9.4	46.8	2				
(14) 0.0758 × (1)	-3.9	0.2	3.5	0.2						
(15) 0.923 × (2)	1.85	-51.3	1.85	47.6						
(16) (4) + (14) + (15)	0	0	9.4	-62.6	2	51.2				
(17) 0.828 × (13)			-48.3	7.9	38.8	1.7				
(18) 0.158 × (16)			1.5	-9.9	0.3	8.1				
(19) (5) + (17) + (18)			0	0	-62.5	13.8	46.8	2		
(20) 0.170 × (13)			-9.9	1.6	8.0	0.3				
(21) 0.845 × (16)			7.9	-52.8	1.7	43.2				
(22) (6) + (20) + (21)			0	0	13.7	-66.9	2	51.2		
(23) 0.79 × (19)					-49.4	10.9	37.0	1.59		
(24) 0.19 × (22)					2.6	-12.7	0.38	9.83		
(25) (7) + (23) + (24)					0	0	-64.2	15.4	46.8	2
(26) 0.21 × (19)					-13.1	2.9	9.8	0.42		
(27) 0.81 × (22)					11.1	-54.1	1.61	41.2		
(28) (8) + (26) + (27)					0	0	15.4	-68.8	2	51.2

TABLE 6.- RELAXATION TABLE FOR REINFORCED PANEL - GROWING-UNIT METHOD

[Forces are in lb; displacements, in in. × 10^{-4}]

Operation \ Force	Y_A	Y_B	Y_E	Y_F	Y_J	Y_K	Y_N	Y_O	Y_R	Y_S
External loads -2.37 × (10)	-120 120		-111	-9						
-1.975 × (13) -0.444 × (16)	0		-111 115.2 -4.2	-9 -18.8	-92.5 -0.9	-4.0 -22.7				
-1.657 × (19) -0.742 × (28)			0	0	-93.4 103.7 -10.2	-26.7 -22.9 49.6	-77.6 -1.5	-3.3 -37.9		
-1.45 × (25) -0.925 × (28)					0	0	-79.1 93.2 -14.2	-41.2 -22.4 63.6	-67.9 -1.8	-2.9 -47.4
							0	0	-69.7	-50.3

TABLE 7.- OPERATIONS TABLE FOR CIRCULAR RING

Operation \ Forces and moments	N_A (in.-lb)	T_A (lb)	N_B (in.-lb)	R_B (lb)	T_B (lb)	N_C (in.-lb)	T_C (lb)
(1) $w_A = 10^{-3}$ radian	-281.95	-49.079	-29.966	-4.733	64.675		
(2) $u_A = 10^{-3}$ in.	-49.079	-52.296	64.675	-22.441	51.516		
(3) $w_B = 10^{-3}$ radian	-29.966	64.675	-439.849	31.443	-50.642	56.5117	6.632
(4) $v_B = 10^{-3}$ in.	-4.733	-22.441	31.443	-12.338	20.14	8.842	0.524
(5) $u_B = 10^{-3}$ in.	64.675	51.516	-50.642	20.14	-52.618	6.632	0.0685
(6) $w_C = 10^{-3}$ radian			56.5117	8.842	6.632	-157.899	-1.563
(7) $u_C = 10^{-3}$ in.			6.632	0.524	0.0685	-1.563	-0.322

TABLE 8.- RELAXATION TABLE FOR CIRCULAR RING

Forces and moments / Operation	N_A (in.-lb)	T_A (lb)	N_B (in.-lb)	R_B (lb)	T_B (lb)	N_C (in.-lb)	T_C (lb)
External Forces -0.00778 × (A)	-1.84 1.84	-8.75 0	-55.0 0.7	59.5 -0.1	38.1 -0.1	-53.1 0	-23.9 0
-0.2 × (B)	0 0	-8.75 8.75	-54.3 -14.0	59.4 4.3	38.0 -8.0	-53.1 0	-23.9 0
-0.353 × (C)	0 0	0 0	-68.3 -8.6	63.7 -2.2	30.0 -2.2	-53.1 53.1	-23.9 0
-77.8 × (D)	0 0	0 0	-76.9 -472	61.5 -34.0	27.8 -0.2	0 0	-23.9 23.9
-2.98 × (E)	0 0	0 0	-549 549	27.5 -27.5	27.6 -27.5	0 0	0 0
	0	0	0	0	0.1	0	0
Check-table results	0.0171	0.0030	0.5095	-0.0043	-0.0053	-0.0726	-0.0121

NACA

TABLE 9.- OPERATIONS TABLE FOR EGG-SHAPED RING

Forces and moments / Operation	R_A (lb)	M_B (in.-lb)	R_B (lb)	T_B (lb)	M_C (in.-lb)	R_C (lb)	T_C (lb)	R_D (lb)
(1) $v_A = 10^{-4}$ in.	-3.34833	8.92216	-2.69614	3.96771				
(2) $w_B = 10^{-4}$ radian	8.92216	-327.866	11.4697	-13.1014	-61.242	8.10267	0	
(3) $v_B = 10^{-4}$ in.	-2.69614	11.4697	-4.00991	3.4352	-8.10267	0.66158	0	
(4) $u_B = 10^{-4}$ in.	3.96771	-13.1014	3.4352	-30.9566	0	0	26.2058	
(5) $w_C = 10^{-4}$ radian		-61.242	-8.10267	0	-288.367	-2.95622	-5.24667	-7.3524
(6) $v_C = 10^{-4}$ in.		8.10267	0.66158	0	-2.95622	-1.90205	-0.88929	-1.11900
(7) $u_C = 10^{-4}$ in.		0	0	26.2058	-5.24667	-0.88929	-27.0833	-1.0375
(8) $v_D = 10^{-4}$ in.					-7.3524	-1.11900	-1.0375	-1.2400

NACA

TABLE 10.- RELAXATION TABLE FOR EGG-SHAPED RING

Forces and moments / Operation	R_A (lb)	N_B (in.-lb)	R_B (lb)	T_B (lb)	N_C (in.-lb)	R_C (lb)	T_C (lb)	R_D (lb)
External forces -149.2 × (1)	-500 500	0 -1330	0 402	0 -592	0 0	0 0	0 0	-500 0
-403 × (8)	0 0	-1330 0	402 0	-592 0	0 2960	0 451	0 418	-500 500
1336 × (F)	0 0	-1330 0	402 0	-592 29	2960 -2485	451 -451	418 63	0 0
256 × (I)	0 0	-1330 1615	402 -402	-563 62	475 0	0 138	481 -8	0 0
409 × (F)	0 0	285 0	0 0	-501 9	475 -760	138 -138	473 19	0 0
-9.4 × (J)	0 0	285 24	0 -2	-492 493	-285 9	0 0	492 -492	0 0
	0	309	-2	1	-276	0	0	0
Check table	-0.465	309.971	-2.231	0.938	-280.405	-0.202	-0.351	-0.279

NACA

TABLE 11.- OPERATIONS TABLE FOR OVAL-SHAPED RING

Operation \ Forces and moments	R_A (lb)	M_B (in.-lb)	R_B (lb)	T_B (lb)	M_C (in.-lb)	R_C (lb)	T_C (lb)	M_D (in.-lb)
$v_A = 10^{-3}$ in.	-7.1310	5.9020	-4.5778	14.662				
$w_B = 10^{-3}$ radian	5.9020	-454.34	6.7238	-78.411	-38.489	1.8576	45.876	
$v_B = 10^{-3}$ in.	-4.5778	6.7238	-12.093	0.55690	-1.8576	-2.2277	13.781	
$u_B = 10^{-3}$ in.	14.662	-78.411	0.55690	-84.510	45.876	-13.781	49.974	
$w_C = 10^{-3}$ radian		-38.489	-1.8576	45.876	-432.37	0	-77.623	-38.489
$v_C = 10^{-3}$ in.		1.8576	-2.2277	-13.781	0	-9.9231	0	-1.8576
$u_C = 10^{-3}$ in.		45.876	13.781	49.974	-77.623	0	-100.34	45.876
$w_D = 10^{-3}$ radian					-38.489	-1.8576	45.876	-432.37
$v_D = 10^{-3}$ in.					1.8576	-2.2277	-13.781	0
$u_D = 10^{-3}$ in.					45.876	13.781	49.974	-77.623
$w_E = 10^{-3}$ radian								-38.489
$v_E = 10^{-3}$ in.								1.8576
$u_E = 10^{-3}$ in.								45.876
$v_F = 10^{-3}$ in.								
$v_G = 10^{-3}$ in.								

Operation \ Forces and moments	R_D (lb)	T_D (lb)	M_E (in.-lb)	R_E (lb)	T_E (lb)	R_F (lb)	R_G (lb)
$v_A = 10^{-3}$ in.							
$w_B = 10^{-3}$ radian							
$v_B = 10^{-3}$ in.							
$u_B = 10^{-3}$ in.							
$w_C = 10^{-3}$ radian	1.8576	45.876					
$v_C = 10^{-3}$ in.	-2.2277	13.781					
$u_C = 10^{-3}$ in.	-13.781	49.974					
$w_D = 10^{-3}$ radian	0	-77.623	-38.489	1.8576	45.876		
$v_D = 10^{-3}$ in.	-9.9231	0	-1.8576	-2.2277	13.781		
$u_D = 10^{-3}$ in.	0	-100.34	45.876	-13.781	49.974		
$w_E = 10^{-3}$ radian	-1.8576	45.876	-649.24	-18.056	-67.080	-5.9020	16.025
$v_E = 10^{-3}$ in.	-2.2277	-13.781	-18.056	-53.957	-40.533	-4.5778	1.3355
$u_E = 10^{-3}$ in.	13.781	49.974	-67.080	-40.533	-126.37	-14.662	-1.3355
$v_F = 10^{-3}$ in.			-5.9020	-4.5778	-14.662	-7.1310	0
$v_G = 10^{-3}$ in.			16.025	1.3355	-1.3355	0	-1.8886

TABLE 12.- GROUP OPERATIONS IN GROWING-UNIT METHOD FOR SEGMENT $\overline{ABC}$

$$\begin{bmatrix} N_B = -454.34w_B + 6.7238v_B - 78.411u_B - (\text{R.H.S. in } N_B \text{ equation}) = 0 \\ R_B = 6.7238w_B - 12.093v_B + 0.55690u_B - (\text{R.H.S. in } R_B \text{ equation}) = 0 \\ T_B = -78.411w_B + 0.55690v_B - 84.510u_B - (\text{R.H.S. in } T_B \text{ equation}) = 0 \end{bmatrix}$$

Group	I	II	III	IV
Displacement / Operation	$v_A = 10^{-3}$ in. $w_C = v_C = u_C = 0$	$w_C = 10^{-3}$ radian $v_A = v_C = u_C = 0$	$v_C = 10^{-3}$ in. $v_A = w_C = u_C = 0$	$u_C = 10^{-3}$ in. $v_A = w_C = v_C = 0$
(-10^3) × right-hand side in equation				
for: N_B, in.-lb	5.9020	-38.489	1.8576	45.876
R_B, lb	-4.5778	-1.8576	-2.2277	13.781
T_B, lb	14.662	45.876	-13.781	49.974
(10^3) × displacements of joint B:				
w_B, radian	-0.026434	-0.21631	0.035554	0.017847
v_B, in.	-0.38424	-0.23971	-0.17353	1.1763
u_B, in.	0.19549	0.74197	-0.19720	0.58253
Resultant forces and moments				
R_A, lb	-2.6618	10.699	-1.8871	3.2615
N_C, in.-lb	10.699	-389.56	-10.093	-53.771
R_C, lb	-1.8871	-10.093	-6.7529	-10.615
T_C, lb	3.2615	-53.771	-10.615	-54.199
N_D, in.-lb	0	-38.489	-1.8576	45.876
R_D, lb	0	1.8576	-2.2277	-13.781
T_D, lb	0	45.876	13.781	49.974

NACA

TABLE 13.- GROUP OPERATIONS IN GROWING-UNIT METHOD FOR SEGMENT $\overline{ABCD}$

$$\begin{bmatrix} N_C = -389.56x_{II} - 10.093x_{III} - 53.771x_{IV} - (\text{R.H.S. in } N_C) = 0 \\ R_C = -10.093x_{II} - 6.7529x_{III} - 10.615x_{IV} - (\text{R.H.S. in } R_C) = 0 \\ T_C = -53.771x_{II} - 10.615x_{III} - 54.199x_{IV} - (\text{R.H.S. in } T_C) = 0 \end{bmatrix}$$

Group	V	VI	VII	VIII
Displacement / Operation	(I) = 1 $w_D = v_D = u_D = 0$	$w_D = 10^{-3}$ radian $(I) = v_D = u_D = 0$	$v_D = 10^{-3}$ in. $(I) = w_D = u_D = 0$	$u_D = 10^{-3}$ in. $(I) = w_D = v_D = 0$
(-1) × right-hand side in equation for: N_C, in.-lb	10.699	-38.489	1.8576	45.876
R_C, lb	-1.8871	-1.8576	-2.2277	13.781
T_C, lb	3.2615	45.876	-13.781	49.974
Magnitudes of (II), (III), and (IV): x_{II}	0.021497	-0.25291	0.046327	-0.0099019
x_{III}	-0.53843	-2.3436	0.10521	0.85346
x_{IV}	0.14430	1.5564	-0.32084	0.76472
Forces and moments: R_A, lb	-0.94505	6.7928	-0.74924	0.77749
N_D, in.-lb	6.7928	-346.88	-16.697	-43.745
R_D, lb	-0.74924	-16.697	-5.6500	-12.458
T_D, lb	0.77749	-43.745	-12.458	-50.817
N_E, in.-lb	0	-38.489	-18.576	45.876
R_E, lb	0	1.8576	-2.2277	-13.781
T_E, lb	0	45.876	13.781	49.974

NACA

TABLE 14.- GROUP OPERATIONS IN GROWING-UNIT METHOD FOR SEGMENT $\overline{ABCDE}$

$$\left[\begin{aligned} N_E &= -346.88x_{VI} - 16.69\,x_{VII} - 43.745x_{VIII} - (\text{R.H.S. in } N_E) = 0 \\ R_E &= -16.697x_{VI} - 5.6500x_{VII} - 12.458x_{VIII} - (\text{R.H.S. in } R_E) = 0 \\ T_E &= -43.745x_{VI} - 12.458x_{VII} - 50.817x_{VIII} - (\text{R.H.S. in } T_E) = 0 \end{aligned}\right]$$

Group	IX	X	XI	XII
Displacement / Operation	$(V) = 1$ $w_E = v_E = u_E = 0$	$w_E = 10^{-3}$ radian $(V) = v_E = u_E = 0$	$v_E = 10^{-3}$ in. $(V) = w_E = u_E = 0$	$u_E = 10^{-3}$ in. $(V) = w_E = v_E = 0$
(-1) × right-hand side in equation for: N_E, in.-lb	6.7928	-38.489	1.8576	45.876
R_E, lb	-0.74924	-1.8576	-2.2277	13.781
T_E, lb	0.77749	45.876	-13.781	49.974
Magnitudes of VI, VII, and VIII: x_{VI}	0.028042	-0.16186	0.037474	-0.0022427
x_{VII}	-0.42660	-4.6760	0.35713	0.59436
x_{VIII}	0.095744	2.1885	-0.39100	0.83963
Forces and moments: R_A, lb	-0.36050	4.1055	-0.31703	0.19226
N_E, in.-lb	4.1055	-533.92	-38.099	-29.579
R_E, lb	-0.31703	-38.099	-49.295	-53.432
T_E, lb	0.19226	-29.579	-53.432	-76.322
R_F, lb	0	-5.9020	-4.5778	-14.662
R_G, lb	0	16.025	1.3355	-1.3355

NACA

TABLE 13.- DETERMINATION OF TOTAL DISPLACEMENTS IN GROWING-UNIT METHOD

Related tables	w_B, v_B, and u_B from table 12				(II), (III), and (IV) from tables 13 or 15				
Prime displacement / Group operations	$v_A = 10^{-3}$ in. I	$w_C = 10^{-3}$ radian II	$v_C = 10^{-3}$ in. III	$u_C = 10^{-3}$ in. IV	$x_I = 1$ V	$w_D = 10^{-3}$ radian VI	$v_D = 10^{-3}$ in. VII	$u_D = 10^{-3}$ in. VIII	Magnitudes of group displacements explicitly used in balancing
I	1								0
II		1							1.0476
III			1						217.61
IV				1					-48.436
V	1	0.021497	-0.53843	0.14430	1				0
VI		-0.25291	-2.3436	1.5564		1			-3.3100
VII		0.046327	0.10521	-0.32084			1		-328.09
VIII		-0.0099019	0.85346	0.76472				1	90.518
IX	1	-0.0063062	-0.56732	0.39803	1	0.028042	-0.42660	0.095744	0
X		-0.19736	1.7552	2.9219		-0.16186	-4.6760	2.1885	11.184
XI		0.010939	-0.38395	-0.35526		0.037474	0.35713	-0.39100	-51.518
XII		0.019788	0.78438	0.44790		-0.0022427	0.59436	0.83963	29.771
XIII	1	-0.0080407	-0.50331	0.45975	1	0.024494	-0.46836	0.16615	-605.73
XIV		-0.0017462	-0.64546	-0.47005		0.022857	-0.059576	-0.68618	-44.646
Total displacements	-605.73	-11.444	664.55	-72.286	-605.73	-22.975	-94.731	90.127	

Related tables	(VI), (VII), and (VIII) from tables 14 or 15				(X), (XI), and (XII) from table 15		
Prime displacement / Group operations	$x_V = 1$ IX	$w_E = 10^{-3}$ radian X	$v_E = 10^{-3}$ in. XI	$u_E = 10^{-3}$ in. XII	$x_{IX} = 1$ XIII	$v_F = 10^{-3}$ in. XIV	Magnitudes of group displacements explicitly used in balancing
I							0
II							1.0476
III							217.61
IV							-48.436
V							0
VI							-3.3100
VII							-328.09
VIII							90.518
IX	1						0
X		1					11.184
XI			1				-51.518
XII				1			29.771
XIII	1	0.0094398	-0.051804	0.035128	1		-605.73
XIV		-0.017182	0.50354	-0.53797		1	-44.646
Total displacements	-605.73	6.2331	-42.620	32.511	-605.73	-44.646	

TABLE 16.- GROUP OPERATIONS IN RELAXATION METHOD FOR SEGMENT $\overline{ABCD}$

$$\left[\begin{array}{l} R_A = -7.1310v_A + 5.9020w_B - 4.5778v_B + 14.662u_B - (\text{R.H.S. in } R_A) = 0 \\ N_B = 5.9020v_A - 454.34w_B + 6.7238v_B - 78.411u_B - (\text{R.H.S. in } N_B) = 0 \\ R_B = -4.5778v_A + 6.7238w_B - 12.093v_B + 0.55690u_B - (\text{R.H.S. in } R_B) = 0 \\ T_B = 14.662v_A - 78.411w_B + 0.55690v_B - 84.510u_B - (\text{R.H.S. in } T_B) = 0 \end{array}\right]$$

Group	XV	XVI	XVII
Displacement / Operation	$w_C = 10^{-3}$ radian $v_C = u_C = w_D = v_D = u_D = 0$	$v_C = 10^{-3}$ in. $w_C = u_C = w_D = v_D = u_D = 0$	$u_C = 10^{-3}$ in. $w_C = v_C = w_D = v_D = u_D = 0$
(-10^3) × right-hand side in equation for:			
R_A, lb	0	0	0
N_B, in.-lb	-38.489	1.8576	45.876
R_B, lb	-1.8576	-2.2277	13.781
T_B, lb	45.876	-13.781	49.974
(10^3) × displacements of joints A and B:			
v_A, in.	4.0195	-0.70895	1.2253
w_B, radian	-0.32255	0.054293	-0.014540
v_B, in.	-1.7842	0.098882	0.70554
u_B, in.	1.5277	-0.33579	0.82205
Forces and moments:			
N_C, in.-lb	-346.56	-17.678	-40.662
R_C, lb	-17.678	-5.4150	-12.928
T_C, lb	-40.662	-12.928	-50.203
N_D, in.-lb	-38.489	-1.8576	45.876
R_D, lb	1.8576	-2.2277	-13.781
T_D, lb	45.876	13.781	49.974

TABLE 17.- GROUP OPERATIONS IN RELAXATION METHOD FOR SEGMENT $\overline{CDEF\text{-}G}$

$$\left[\begin{array}{l} M_E = -649.24w_E - 18.056v_E - 67.080u_E - 5.9020v_F + 16.025v_G - (\text{R.H.S. in } M_E) = 0 \\ R_E = -18.056w_E - 53.957v_E - 40.533u_E - 4.5778v_F + 1.3355v_G - (\text{R.H.S. in } R_E) = 0 \\ T_E = -67.080w_E - 40.533v_E - 126.37u_E - 14.662v_F - 1.3355v_G - (\text{R.H.S. in } T_E) = 0 \\ R_F = -5.9020w_E - 4.5778v_E - 14.662u_E - 7.1310v_F - (\text{R.H.S. in } R_F) = 0 \\ R_G = 16.025w_E + 1.3355v_E - 1.3355u_E - 1.8886v_G - (\text{R.H.S. in } R_G) = 0 \end{array}\right]$$

Group	XVIII	XIX	XX
Displacement / Operation	$w_D = 10^{-3}$ radian $w_C = v_C = u_C = v_D = u_D = 0$	$v_D = 10^{-3}$ in. $w_C = v_C = u_C = w_D = u_D = 0$	$u_D = 10^{-3}$ in. $w_C = v_C = u_C = w_D = u_D = 0$
(-10^3) × right-hand side in equation:			
M_E, in.-lb	-38.489	-1.8576	45.876
R_E, lb	1.8576	-2.2277	-13.781
T_E, lb	45.876	13.781	49.974
R_F, lb	0	0	0
R_G, lb	0	0	0
(10^3) × displacements of joints E, F, and G:			
w_E, radian	-0.1703	-0.032775	-0.008775
v_E, in.	-0.42970	-0.19092	-0.78327
u_E, in.	0.74456	0.23168	0.79421
v_F, in.	-1.1141	-0.32667	-1.1229
v_G, in.	-2.2753	-0.57676	-1.1900
Forces and moments:			
M_C, in.-lb	-38.489	1.8576	45.876
R_C, lb	-1.8576	-2.2277	13.781
T_C, lb	45.876	-13.871	49.974
M_D, in.-lb	-392.46	11.535	-42.305
R_D, lb	11.535	-6.2441	12.706
T_D, lb	-42.305	12.706	-50.259

NACA

TABLE 18.- GROUP-OPERATIONS TABLE FOR RELAXATION METHOD

[Forces and moments at joints A, B, E, F, and G are zero for all operations]

Operation \ Forces and moments	N_C (in.-lb)	R_C (lb)	T_C (lb)	N_D (in.-lb)	R_D (lb)	T_D (lb)
(XV) = 1	-346.45	-17.678	-40.662	-38.489	1.8576	45.876
(XVI) = 1	-17.678	-5.4150	-12.928	-1.8576	-2.2277	13.781
(XVII) = 1	-40.662	-12.928	-50.203	45.876	-13.781	49.974
(XVIII) = 1	-38.489	-1.8576	45.876	-392.46	11.535	-42.305
(XIX) = 1	1.8576	-2.2277	-13.781	11.535	-6.2441	12.706
(XX) = 1	45.876	13.781	49.974	-42.305	12.706	-50.259

NACA

TABLE 19.- RELAXATION TABLE FOR GROUP OPERATIONS

Step	Operation \ Forces and moments	R_A (lb)	M_B (in.-lb)	R_B (lb)	T_B (lb)	M_C (in.-lb)	R_C (lb)	T_C (lb)	M_D (in.-lb)
1	External loads 302 × (K)					0 -5900	966 -966	-259 258	0 -4050
2	-18.4 × (M)					-5900 5900	0 164	-1 -44	-4050 7100
3	214 × (N)					0 1246	164 -164	-45 0	3050 -3150
4	4.08 × (O)					1246 -1246	0 -21	-45 20	-100 -314
5	-15.7 × (K)					0 306	-21 50	-25 -13	-414 210
6	-0.76 × (XVII)					306 31	29 10	-38 38	-204 -35
Check table		0.001	0.310	0.025	0.156	337 324.76	39 42.600	0 -0.450	-239 -228.26

Step	Operation \ Forces and moments	R_D (lb)	T_D (lb)	M_E (in.-lb)	R_E (lb)	T_E (lb)	R_F (lb)	R_G (lb)
1	External loads 302 × (K)	-966 1218	-259 326					
2	-18.4 × (M)	252 0	67 0					
3	214 × (N)	252 -244	67 -21					
4	4.08 × (O)	8 55	46 0					
5	-15.7 × (K)	63 -63	46 -17					
6	-0.76 × (XVII)	0 10	29 -38					
Check table		10 5.390	-9 -10.650	-1.734	-0.001	-0.750	-0.003	0.047

NACA

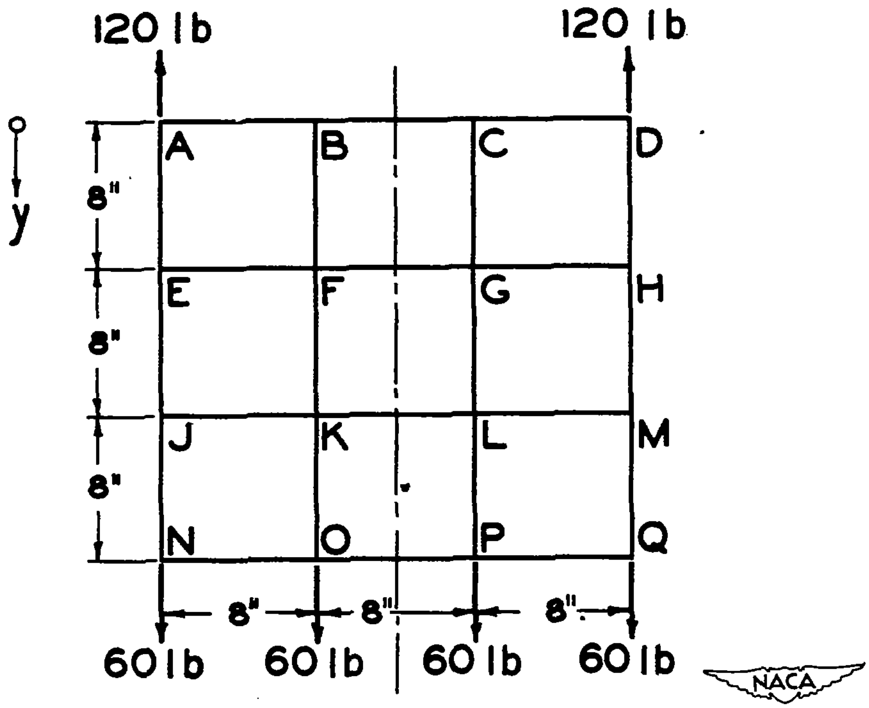

Figure 1.- Reinforced panel with conditions at both ends specified in terms of force.

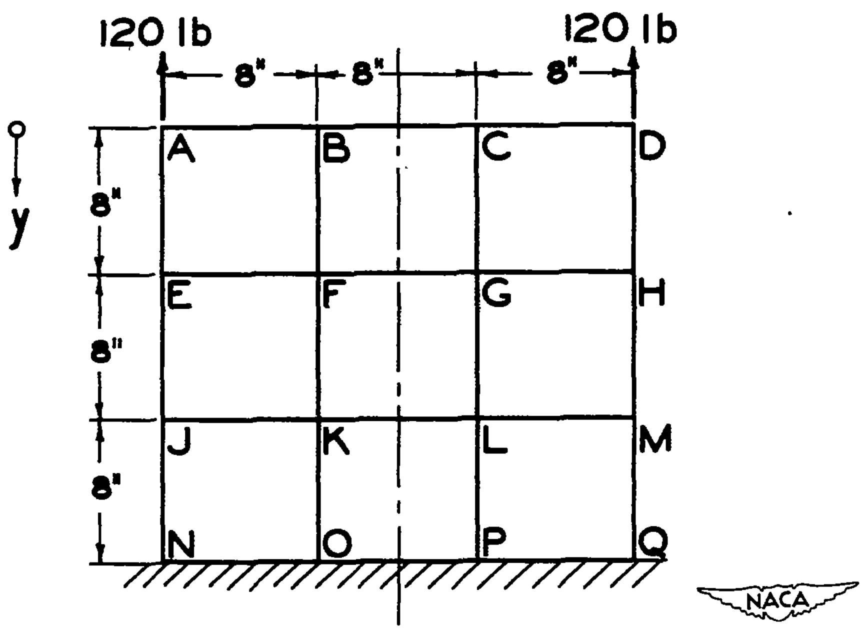

Figure 2.- Reinforced panel with conditions at one end specified in terms of force and at the other in terms of displacements.

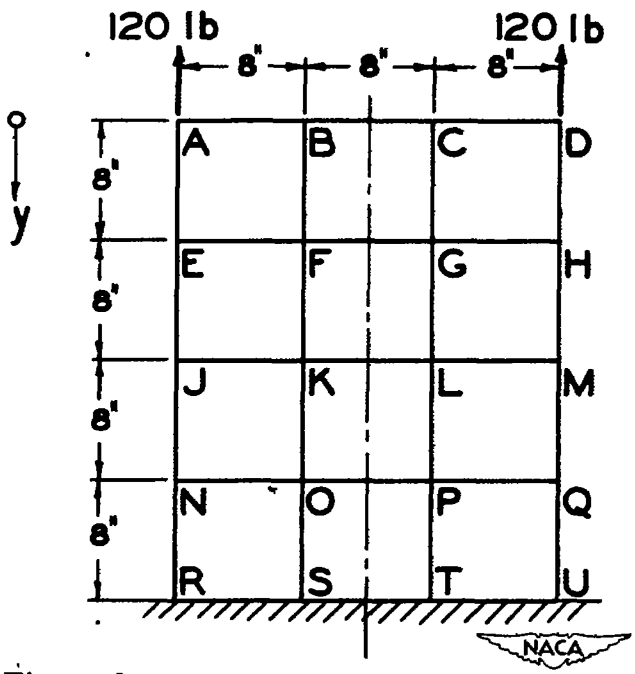

Figure 3.- Reinforced panel with 12 bays.

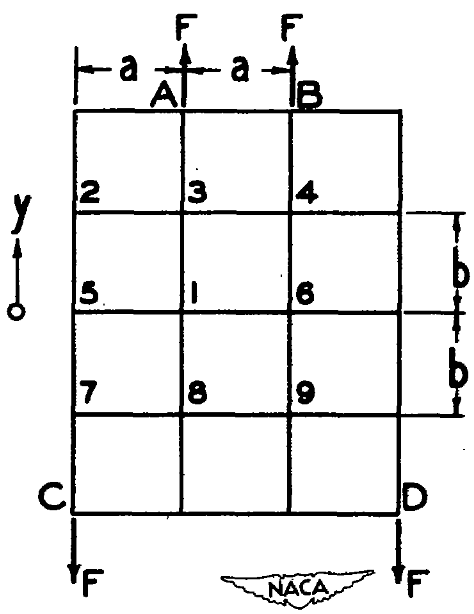

Figure 4.- Forces transmitted through structural elements of reinforced panel.

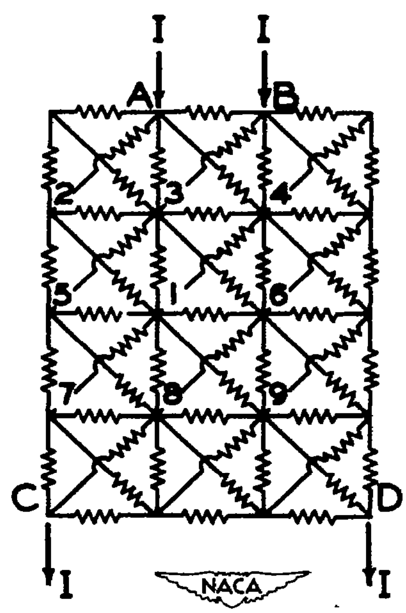

Figure 5.- Currents flowing through branches of direct-current network analagous to reinforced panel of figure 4.

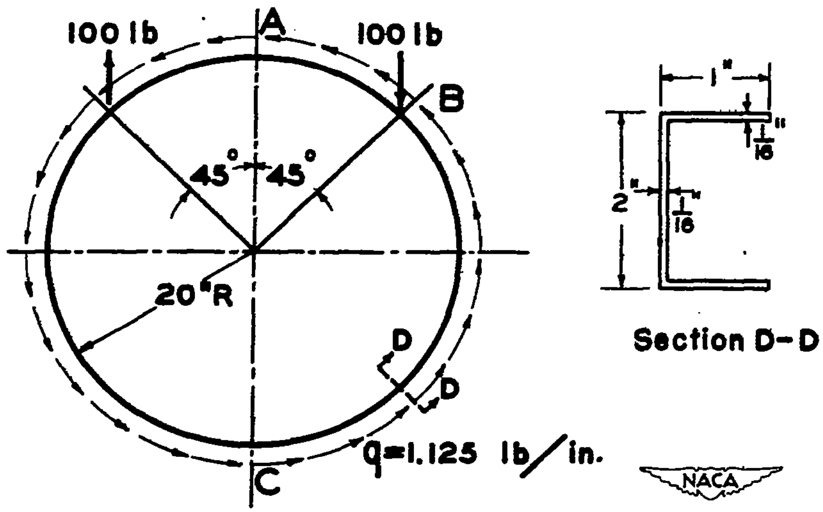

Figure 6.- Circular ring with antisymmetric loads.

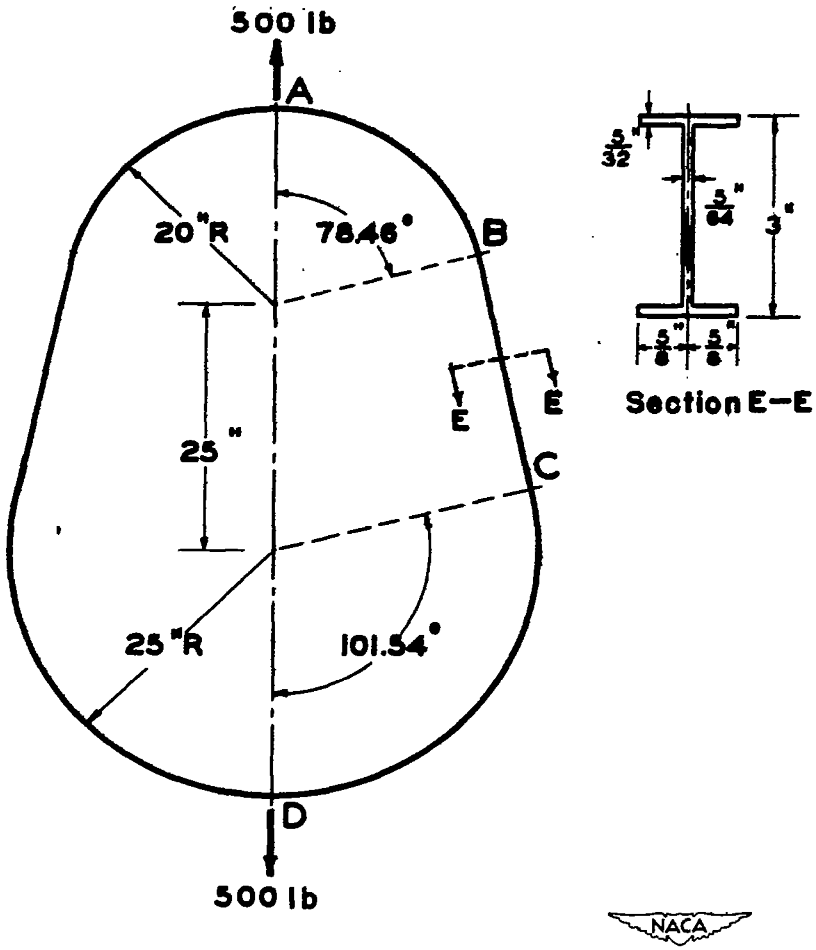

Figure 7.- Egg-shaped ring with symmetric loads.

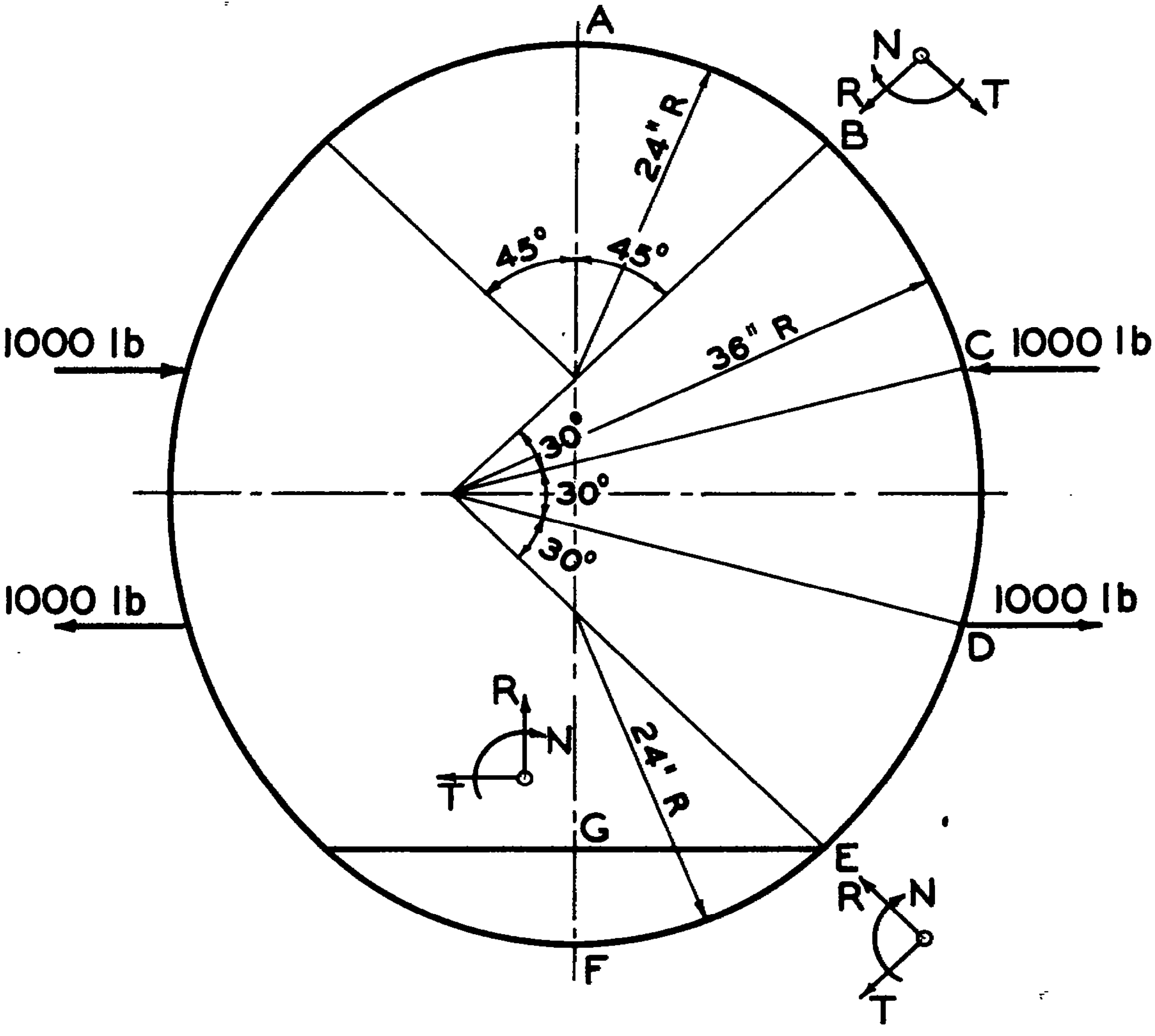

Figure 8.- Oval-shaped ring with positive directions of forces and moments shown.

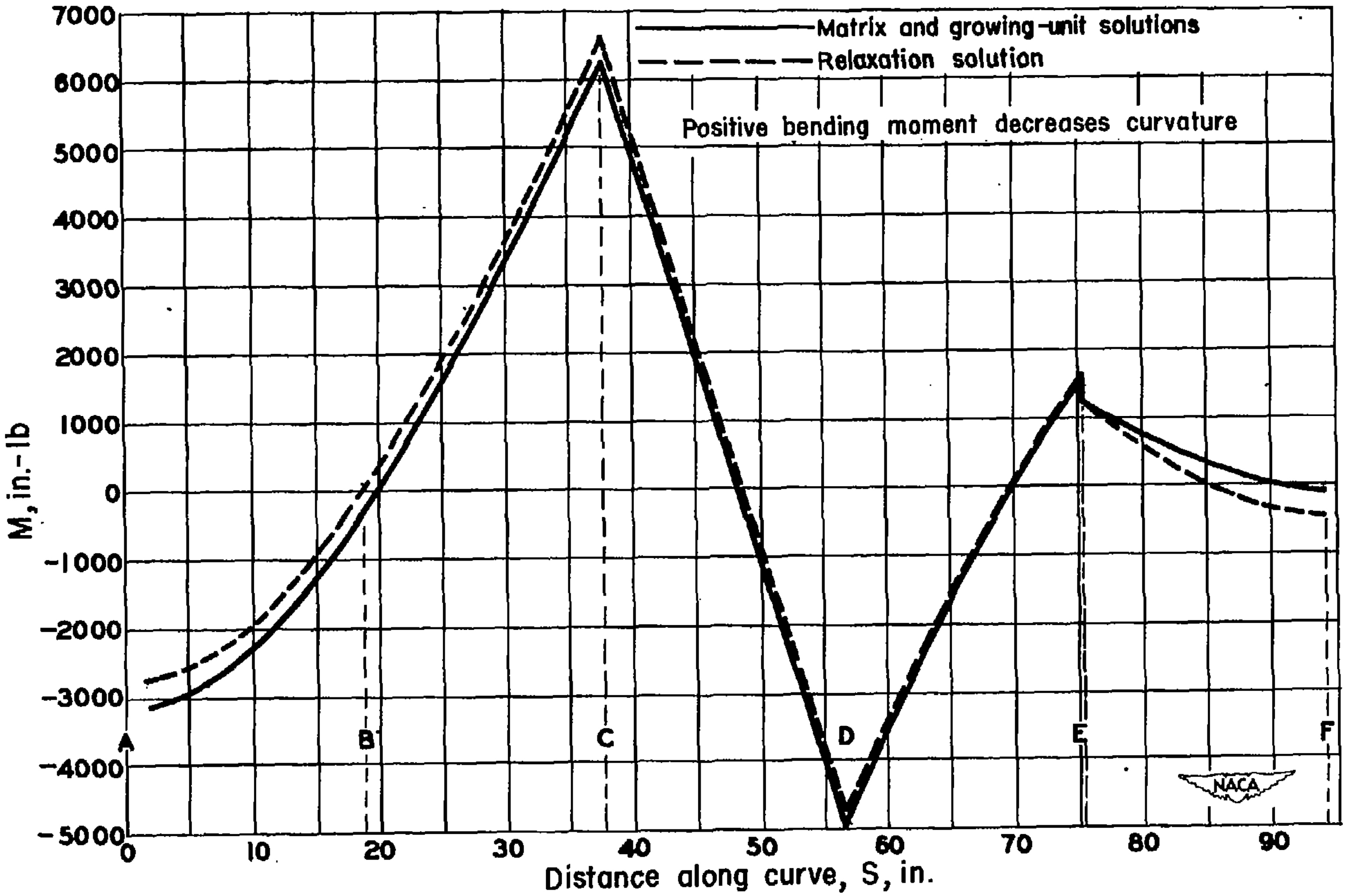

Figure 9.- Bending-moment diagram for oval-shaped ring with internal bracing.