NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1849

USE OF CHARACTERISTIC SURFACES FOR UNSYMMETRICAL SUPERSONIC FLOW PROBLEMS

By W. E. Moeckel

Lewis Flight Propulsion Laboratory
Cleveland, Ohio

Washington
March 1949

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 1849

USE OF CHARACTERISTIC SURFACES FOR UNSYMMETRICAL SUPERSONIC FLOW PROBLEMS

By W. E. Moeckel

SUMMARY

The three-dimensional nonlinear partial differential equation for the velocity potential in a supersonic stream is transformed by the method of characteristics to obtain a system of three ordinary differential equations wherein all quantities are known or can be calculated except the three velocity differentials. When the ordinary differential equations are converted to difference equations, the velocity components at an unknown point in a network of characteristics can be calculated if the velocity components at three neighboring points of the network are known. Use of the difference equations for computing the supersonic potential flow past unsymmetrical boundaries is discussed.

Application of the method of characteristics to the linearized three-dimensional equation results in a relatively simple system of difference equations that can be used to compute the supersonic flow past boundaries for which no other linearized solution is available.

INTRODUCTION

The velocity potential in a steady supersonic stream is described by a nonlinear partial differential equation of second order with three independent variables. If the velocity throughout the disturbed portion of the flow field is assumed to differ very little from the velocity in the undisturbed stream, the complete equation can be reduced to a linear equation whose form is identical to the wave equation of physics. Many solutions of the linearized equation have been obtained for particular types of boundary condition. The pressure distributions on thin wings, for example, can now be determined for many plan forms and profiles (references 1 to 5). Solutions are also available for determining the pressure distribution on axially symmetric bodies (references 6 and 7) or on cones that may be unsymmetrical (reference 8). No linearized solutions are known, however, for general body forms or for general wing-body combinations.

When the Mach number and the velocity change appreciably as the flow passes a solid bounding surface, the complete differential equation must be solved if accurate results are required. For a few special cases, such as plane flow (reference 9) and axially symmetric conical flow (reference 10), the exact equation has been solved without use of the method of characteristics. These solutions were obtainable because the problems to which they apply are completely determined by a single independent variable. For plane flow, the local Mach number is determined by the angle of the streamline with respect to the base plane and, for axially symmetric conical flow, the local Mach number depends only on the angle of the conical ray with respect to the axis of symmetry. When reduction of the problem to a single independent variable is impossible, solutions for the exact equation are available by application of the method of characteristics.

In references 11 and 12, the method of characteristics is applied to the calculation of the supersonic flow in a field that is plane or has axial symmetry. For plane flow, the procedure is essentially a step-by-step development of the Prandtl-Meyer solution (reference 9), whereby the effect of interacting characteristics is determined. Each characteristic has a constant strength throughout the field and the inclination of the characteristic with respect to the local stream direction is the local Mach angle. For axially symmetric flows, the strength of a characteristic varies with the distance from the axis of symmetry; hence the problem cannot be reduced to a single variable. The computations are therefore somewhat more laborious than for plane flows, but the simplicity of geometry is maintained. The flow can still be completely represented in a single coordinate plane and the angle between the characteristic and the local stream direction is still the local Mach angle. For plane and axially symmetric flows, the method of characteristics has been extended to include rotational flow (reference 12).

When the flow variables vary simultaneously in all three coordinate directions, graphical representation is more difficult and numerical computations are more lengthy. The relation between the characteristic surfaces and the coordinate planes is no longer simple. A formulation of the procedure for unsymmetrical flows has been published by Ferrari (reference 13), who derived the ordinary differential equations that determine the variation of velocity components along intersections of characteristic surfaces with meridian planes and discussed the procedure for the case of an axially symmetric body at angle of attack.

A reformulation and generalization of the methods described in reference 13 was completed at the NACA Lewis laboratory in March 1948 and is presented herein. The physical concepts used in the method of characteristics for general flows are described in some detail and the required difference equations are derived for the three common orthogonal coordinate systems. With these equations and concepts, the potential flow past three-dimensional boundaries can be determined by point-to-point computation to any desired degree of accuracy. By application of this method to the linearized differential equation for supersonic potential flow, solutions can be obtained for boundaries for which no other solutions are yet available or for which other solutions involve more computation than the procedure described. No attempt is made to determine the effect of shock waves on the flow field. All shock waves are assumed to be replaced by isentropic compression waves in order to retain the assumption of potential flow. To the extent that this assumption is valid, the resulting solution can satisfy the boundary conditions of the real flow field.

SYMBOLS

The following symbols are used in this report:

$\bar{A}$	partial differential equation of characteristics
A_1,A_2,A_3; B_1,B_2,B_3; C_1,C_2,C_3; D_1,D_2,D_3; P_1,P_3	quantities depending on velocity components and coordinates at points of characteristic network
$A^o,A',A''\ldots A^m$	source of characteristic surface that extends downstream and away from solid boundary (type I)
$a_{i,k}$	coefficients of $D_{i,k}\Psi$ in potential equation
$B^o,B',B''\ldots B^m$	source of characteristic surface that extends downstream and toward solid boundary (type II)
$C^o,C',C''\ldots C^m$	source curve determined by intersection of two characteristic surfaces
$D_{i,k}\Psi$	$= \dfrac{\partial^2\Psi}{\partial x_i \partial x_k}$ $(i,k = 1,2,3,$ table I)
$D^o,D',D''\ldots D^m$	source curve defined by intersection of characteristic of type II with x_1 = constant surface

$E^{o}, E', E'' \ldots E^{m}$	source curve defined by intersection of characteristic of type I with x_1 = constant surface
$F^{o}, F', F'' \ldots F^{m}$	derived source curve on solid boundary and lying in x_1 = constant surface
$f(x_1, x_3)$	function defining integral surface $S = x_2 - f(x_1, x_3) = 0$
$G^{o}, G', G'' \ldots G^{m}$	intersection curve of characteristic of type II with solid boundary
$H(v_i, x_i)$	function in potential equation whose nature depends on coordinate system used
k	ratio of critical speed to speed of sound
M	Mach number
R	ratio of coordinate differences required to solve simultaneous difference equations for velocity components
r, θ, φ	spherical coordinates
S	integral surface of potential equation
T	function of x_i, v_i, $\dfrac{dv_i}{dx_1}$, and $\dfrac{dv_i}{dx_3}$
V	ratio of local flow velocity to critical speed
v_i	first partial derivatives of Ψ with respect to x_i $(i = 1,2,3)$
$\left.\begin{array}{l} v_x, v_y, v_z \\ v_x, v_r, v_\theta \\ v_r, v_\theta, v_\varphi \end{array}\right\}$	ratio of velocity components to critical speed in Cartesian, cylindrical, and spherical coordinates, respectively
x, r, φ	cylindrical coordinates
x, y, z	Cartesian coordinates

x_1, x_2, x_3	orthogonal coordinates used in derivation
γ	ratio of specific heats ($\gamma = 1.40$ for air)
δ	angle between tangent to solid boundary and x_3 = constant surface
ϵ	angle between tangent to solid boundary and x_1 = constant surface
η	factor required to eliminate coordinate factors from v_1 to establish boundary condition (table I)
λ	$\frac{\partial x_2}{\partial x_1}$ along characteristic
μ	$\frac{\partial x_2}{\partial x_3}$ along characteristic
Ψ	ratio of velocity potential to critical speed

Subscripts:

$A^o, A', A'' \ldots G^m$	values of quantity at point of characteristic network
1,2,3	coordinate directions, x_1, x_2, x_3

Superscripts:

m	last point of source curve
n	general intermediate point of source curve

THEORY

Physical Concepts

The physical basis of the method of characteristics is the nature of wave propagation in a supersonic stream. Consider a moving stream of compressible fluid that is uniform except for the effects of a single stationary point source of disturbance (fig. 1). If the stream is moving subsonically, the wave front of a disturbance (shown at successive intervals of time in fig. 1) has no stationary envelope; when steady-state conditions are obtained, the discontinuous wave front has disappeared from the field of interest and a continuous disturbed flow field remains. If the stream is moving supersonically (fig. 1(b)), the wave front possesses a stationary conical envelope. The vertex of this envelope is the point source, its axis is the stream direction, and its conical half-angle is the Mach angle. When steady-state conditions are obtained, the initial spherical surface of discontinuity has disappeared, but the conical envelope of the disturbed portion of the field (Mach cone) remains. The Mach cone is a surface of infinitesimal discontinuity upon which the stream variables have two values, that of the free stream and that of the disturbed stream. Such a surface, upon which two solutions simultaneously exist, is called a characteristic and its differential equation can be directly obtained from the differential equation of the entire flow field.

Consider now a space curve that acts as a source of disturbance in a supersonic stream (not necessarily uniform) (fig. 2). The shape of the space-curve source may be arbitrary except that each point of the curve is assumed to lie outside the Mach cones emanating from all other points of the curve. In figure 2 the curve is assumed, for convenience, to lie in a plane perpendicular to a coordinate axis and the flow is in the general direction of the x_1-axis. The envelope of all disturbances from this source forms two distinct surfaces. These surfaces are characteristics and the stream variables may have either of two solutions at any point on the surfaces. If the line source is regarded as a series of point disturbances, the shapes of the characteristics are clearly such that they form an envelope to the Mach cones emanating from all point sources on the curve. If only very small regions of the flow field are considered, the characteristics are closely approximated by their tangent planes. These tangent planes and their intersections with coordinate planes and with each other are used in the method of characteristics to construct continuously varying flow fields (reference 13). (For spherical coordinates, the surfaces x_1 = constant become spheres rather than planes, but because most problems are best adapted to the use of Cartesian or cylindrical coordinates, the term "coordinate plane" is sometimes used to describe a surface for which either x_1 or x_3 is constant.)

The initial sources of the characteristics are actual or assumed discontinuities at the boundaries of the flow. These boundaries consist of the body past which, or through which, the flow is to be computed and the surface that separates the disturbed and undisturbed regions of the flow field. Smoothly varying boundaries are replaced by a succession of tangent planes whose spatial dimensions are small so that the original boundary is closely approximated by the resulting polyhedral surface. For boundary sources, it is convenient in most cases to choose curves that lie in surfaces for which one of the coordinates is constant. If Cartesian or cylindrical coordinates are used, for example, all initial sources can be so chosen that they lie in x_1 = constant planes. If the flow variables are known on an initial source lying on the solid boundary, a characteristic surface of type I (fig. 2) is determined. Intersection of this surface with a characteristic of type II, whose source is a small distance from the initial source, determines another curve that may be regarded as a secondary source of two more characteristic surfaces (one of each type). The secondary source so determined does not, in general, lie in an x_1 = constant plane, but a procedure is subsequently described whereby all secondary sources can be transferred to planes parallel to the initial source.

When each characteristic is replaced by its tangent planes, the flow field is represented by a network of points, each of which is a junction of two or three lines of intersection of planes tangent to the characteristic with coordinate planes. The mathematical theory of characteristics provides equations that determine the velocity components at these junction points when conditions at neighboring junction points are known.

Mathematical Development

The theory of characteristics is presented in many texts dealing with differential equations (for example, references 14 to 16). The following development considers the theory for nonlinear partial differential equations of the second order in three independent variables. This type represents the equation for compressible potential flow in a three-dimensional orthogonal coordinate system. Although the following development differs from the derivation given for cylindrical coordinates in reference 13, the final total-differential equations obtained are identical.

The differential equation of the velocity potential for compressible, nonviscous, irrotational flow can be written as

$$a_{11}D_{11}\Psi + a_{12}D_{12}\Psi + a_{13}D_{13}\Psi + a_{22}D_{22}\Psi + a_{23}D_{23}\Psi + a_{33}D_{33}\Psi + H = 0 \tag{1}$$

where $a_{i,k}$ and H are functions of velocity components and coordinates and the symbols $D_{i,k}\Psi$ represent the second partial derivatives of Ψ with respect to the coordinates. The nature of the quantities in equation (1) for Cartesian, cylindrical, and spherical coordinates is indicated in table I. The irrotational-flow conditions required to derive equation (1) are

$$
\begin{aligned}
D_{12}\Psi &= D_{21}\Psi \\
D_{13}\Psi &= D_{31}\Psi \\
D_{23}\Psi &= D_{32}\Psi
\end{aligned}
\tag{2}
$$

The feasibility of integrating equation (1) depends on the existence of surfaces in the flow field along which the derivatives of Ψ are continuous. If discontinuities are possible in the field, these surfaces (often called integral surfaces) must be so defined that the flow variables may be discontinuous across the surfaces but not along them. The differential equation that defines such a family of surfaces is derived as follows: Assume that the surface $S(x_1,x_2,x_3) = 0$ is an integral surface of equation (1). If S is solved for one of its independent variables, such as x_2, then

$$S = x_2 - f(x_1,x_3) = 0 \tag{3}$$

and the relation between the increments dx_1, dx_2, and dx_3 on this surface is

$$dx_2 = \lambda dx_1 + \mu dx_3 \tag{4}$$

where

$$\lambda = \frac{\partial x_2}{\partial x_1}$$

$$\mu = \frac{\partial x_2}{\partial x_3}$$

The differential of v_i may be written

$$dv_i = D_{i,1}\Psi dx_1 + D_{i,2}\Psi dx_2 + D_{i,3}\Psi dx_3 \qquad (i = 1,2,3) \tag{5}$$

If one of the coordinate increments, for example dx_2, is eliminated from equation (5) by means of equation (4), then equation (5) becomes

$$dv_1 = (D_{1,1}\Psi + \lambda D_{1,2}\Psi)dx_1 + (D_{1,3}\Psi + \mu D_{1,2}\Psi)dx_3 \tag{6}$$

Because x_1 and x_3 are independent variables, the increments dx_1 and dx_3 may be alternately set equal to zero to obtain the relations

$$D_{1,1}\Psi = \left(\frac{dv_1}{dx_1}\right)_{x_3 = \text{constant}} - \lambda D_{1,2}\Psi \tag{7}$$

$$D_{1,3}\Psi = \left(\frac{dv_1}{dx_3}\right)_{x_1 = \text{constant}} - \mu D_{1,2}\Psi \tag{7a}$$

where $\frac{dv_1}{dx_1}$ and $\frac{dv_1}{dx_3}$ are taken along the intersections of S with x_3 = constant planes and x_1 = constant planes, respectively.

With equations (7) and (7a), all second derivatives of Ψ except one can be eliminated from equation (1); thus, if $D_{22}\Psi$ is the exception, equation (1) becomes

$$\bar{A}D_{22}\Psi + T = 0 \tag{8}$$

where T is a function of v_1, x_1, $\frac{dv_1}{dx_1}$, and $\frac{dv_1}{dx_3}$, and the coefficient of $D_{22}\Psi$ is

$$\bar{A} = a_{11}\lambda^2 - a_{12}\lambda + a_{13}\lambda\mu + a_{22} - a_{23}\mu + a_{33}\mu^2 \tag{9}$$

Equation (9) is called the characteristic form of equation (1). If $\bar{A} \neq 0$, then $D_{22}\Psi$ and consequently all other partial derivatives of Ψ are uniquely determined by equation (8) and are consequently single-valued. If $\bar{A} = 0$, however, the partial derivatives are not uniquely determined by equation (8) and may therefore be multivalued. Hence the possibility that discontinuities may exist in the flow field is established by the condition $\bar{A} = 0$. The integral surfaces defined by the relation $\bar{A} = 0$ are called the characteristics of equation (1). If the characteristics exist and are real, they constitute a family of surfaces along which equation (1) is necessarily integrable. Other surfaces may cross the characteristics and hence contain discontinuities in the derivatives of Ψ. In general, if the flow is supersonic, real characteristic surfaces exist. An interesting exception is the equation for conical flow, which is discussed in appendix A.

If the coefficient of $D_{22}\Psi$ (equation (9)) is set equal to zero and the resulting equation is solved for λ, the differential equation of the characteristics of equation (1) becomes

$$\lambda = \frac{-(a_{12} - \mu a_{13}) \pm \sqrt{(\mu a_{13} - a_{12})^2 - 4a_{11}(a_{22} - \mu a_{23} + \mu^2 a_{33})}}{-2a_{11}} \tag{10}$$

where the + and - are used for characteristics of types I and II, respectively (fig. 2). The quantities λ and μ may be called intersection parameters of the characteristic surfaces because they determine the rate of change of x_2 with respect to x_1 and x_3, respectively, along the characteristics. (For plane and axially symmetric flows, the velocity vector can always be represented in a single plane. The parameter μ is then zero and equation (10) can be reduced to

$$\lambda = \tan(\theta \pm \beta)$$

where θ is the angle between the local velocity vector and the x-axis and β is the local Mach angle.)

For three-dimensional flow the significance of λ and μ is shown in figure 3, where x_1, x_2, and x_3 are represented as Cartesian coordinates. If in figure 3(a) the intersection of the characteristic with an x_1 = constant surface is given (line $\overline{A^oA'}$), then two values of λ at point A^o can be calculated from equation (10), in which the approximate value of μ at A is given by the difference ratio

$$\mu \equiv \left(\frac{dx_2}{dx_3}\right)_{x_1 = \text{constant}} \cong \frac{x_{2,A'} - x_{2,A^o}}{x_{3,A'} - x_{3,A^o}}$$

These two values of λ determine the slope of the intersection at A^o of a characteristic of each type with the x_3 = constant plane containing A^o. If the intersection of a characteristic with an x_3 = constant plane is given (line $\overline{A^oC^o}$, fig. 3(b)), two values of μ can be calculated from the equation obtained by solving the characteristic equation (10) for μ. Because the x_1-axis is assumed to be in the general direction of the flow, the parameter μ will usually be given and the two values of λ will be computed from equation (10).

The expressions for the changes in the velocity components in passing from one point to a neighboring point along a characteristic intersection can be determined by setting the function T, obtained in equation (8), equal to zero. Because the nature of this function has not yet been specified, the derivation will be repeated in more detail. With the aid of the irrotational flow conditions (equations (2)), the following relations are obtained from equations (7) and (7a):

$$\left.\begin{aligned}
D_{11}\Psi &= \frac{dv_1}{dx_1} - \lambda\frac{dv_2}{dx_1} + \lambda^2 D_{22}\Psi \\
D_{12}\Psi &= \frac{dv_2}{dx_1} - \lambda D_{22}\Psi \\
D_{13}\Psi &= \frac{dv_3}{dx_1} - \lambda D_{23}\Psi \\
D_{23}\Psi &= \frac{dv_2}{dx_3} - \mu D_{22}\Psi \\
D_{33}\Psi &= \frac{dv_3}{dx_3} - \mu\frac{dv_2}{dx_3} + \mu^2 D_{22}\Psi
\end{aligned}\right\} \tag{11}$$

and equation (8) becomes

$$\begin{aligned}
T = a_{11}dv_1 + (a_{12} - \lambda a_{11})dv_2 + a_{13}dv_3 \\
+ \left[(a_{23} - \lambda a_{13} - \mu a_{33})\frac{dv_2}{dx_3} + a_{33}\frac{dv_3}{dx_3} + H\right]dx_1 = 0
\end{aligned} \tag{12}$$

where the coefficient of $D_{22}\Psi$ has been set equal to zero in accordance with the characteristic condition. Equation (12) provides a relation between the velocity increments in moving a small distance dx_1 along intersections of characteristics with x_3 = constant surfaces. If, for example, the velocity components and coordinates are known at A^o and A' in figure 3(a), all quantities in equation (12) are known except dv_1, dv_2, and dv_3. With the aid of equations (7) and (7a), the relation

$$\lambda \frac{dv_2}{dx_3} = \mu \frac{dv_2}{dx_1} + \frac{dv_3}{dx_1} - \frac{dv_1}{dx_3}$$

is obtained and equation (12) can be converted to

$$a_{13}dv_1 + (a_{23} - \mu a_{33})dv_2 + a_{33}dv_3 + \left[(a_{12} - \lambda a_{11} - \mu a_{13}) \frac{dv_2}{dx_1} + a_{11} \frac{dv_1}{dx_1} + H\right] dx_3 = 0 \qquad (13)$$

which relates the velocity components in moving a distance dx_3 along intersections of characteristics with x_1 = constant planes. For cylindrical coordinates, equations (12) and (13) are identical to those derived by Ferrari (reference 13). When equations (12) and (13) are converted to difference equations, the supersonic flow past arbitrary boundaries can be constructed by a point-to-point process of numerical integration.

Formulation of difference equations along characteristic intersections. - The problem of constructing the supersonic flow past arbitrary boundary surfaces may be divided into two parts: (1) Given two neighboring source curves, both of which lie in surfaces for which x_1 = constant, determine a third source curve lying in another x_1 = constant surface slightly downstream; (2) given one source curve on the solid boundary and another curve just off the solid boundary, find another source curve (slightly downstream) on the solid boundary. (All source curves are again to lie in surfaces for which x_1 = constant.) These problems and the geometry of the solution are illustrated in figure 4, where x_1, x_2, and x_3 are represented as Cartesian coordinates.

Suppose that the points A^o, A', A'', F^o, F', F'', G^o, G', and G'' are located on a body that represents a boundary of the flow. The rectangles $A^oA'F'F^o$ and $A'A''F''F'$ are assumed to be sufficiently small to closely approximate the actual contour of the body in the region $A^oA''F''F^o$. The rectangles $A^oA'E'E^o$ and $A'A''E''E'$ are tangent planes that closely approximate a characteristic surface of type I whose source is the curve $A^oA'A''$ on the body; the rectangles $B^oB'D'D^o$ and $B'B''D''D'$ are tangent planes that closely approximate a characteristic of type II whose source is $B^oB'B''$. It is desired first to determine the coordinates and velocity components at points E^o, E', and E'' (or D^o, D', and D'') when the coordinates and velocity components are known at points A^o, A', A'', B^o, B', and B''. When the new field source $E^oE'E''$ has been determined, the second problem requires the determination of the velocity components at the new surface source $F^oF'F''$.

The fact that lines such as $\overline{A'E'}$ and $\overline{B'D'}$ do not generally intersect at the same value of x_1 as the lines $\overline{A^oE^o}$ and $\overline{B^oD^o}$ must be noted in solving the first problem. If the body is very unsymmetrical, the x_1 value of the intersection point changes appreciably as the successive points $E', E'' \ldots E^m$ are computed. Such lines as $\overline{A'E'}$ and $\overline{E^oE'}$, however, which are intersections of the same characteristic tangent plane with x_3 = constant surfaces and x_1 = constant surfaces, respectively, naturally intersect at the chosen values of x_1 and x_3.

The velocity components along the derived source curve $E^oE'E''$ can be determined with equations (12) and (13), which may be written as follows: Along an intersection of a characteristic with an x_3 = constant plane,

$$a_{11}dv_1 + (a_{12} - \lambda a_{11})dv_2 + a_{13}dv_3 + P_1dx_1 = 0 \tag{14}$$

where

$$P_1 = (a_{23} - \lambda a_{13} - \mu a_{33}) \frac{dv_2}{dx_3} + a_{33} \frac{dv_3}{dx_3} + H$$

Along an intersection of a characteristic with an x_1 = constant plane,

$$a_{13}dv_1 + (a_{23} - \mu a_{33})dv_2 + a_{33}dv_3 + P_3dx_3 = 0 \tag{15}$$

where

$$P_3 = (a_{12} - \mu a_{13} - \lambda a_{11}) \frac{dv_2}{dx_1} + a_{11} \frac{dv_1}{dx_1} + H$$

If the velocity gradients and the coordinates change only slightly in moving from one point to a neighboring point on the network of characteristic intersections, the differentials in equations (14) and (15) may be replaced by differences. The coordinate differences dx_1 and dx_3 are easily obtained. For any point such as C^o (or C' or C''), which is a junction of two intersections of characteristics with an x_3 = constant plane, the two relations

$$\lambda_{B^o} = \frac{x_{2,C^o} - x_{2,B^o}}{x_{1,C^o} - x_{1,B^o}}$$

and

$$\lambda_{A^o} = \frac{x_{2,C^o} - x_{2,A^o}}{x_{1,C^o} - x_{1,A^o}}$$

can be used to determine x_{1,C^o} and x_{2,C^o} (x_{3,C^o} is given for such points). Explicitly the required coordinates are

$$\left.\begin{aligned} x_{1,C^o} &= \frac{x_{2,A^o} - x_{2,B^o} + \lambda_{B^o} x_{1,B^o} - \lambda_{A^o} x_{1,A^o}}{\lambda_{B^o} - \lambda_{A^o}} \\ x_{2,C^o} &= \frac{\lambda_{B^o} x_{2,A^o} - \lambda_{A^o} x_{2,B^o} + \lambda_{B^o}\lambda_{A^o}\left(x_{1,B^o} - x_{1,A^o}\right)}{\lambda_{B^o} - \lambda_{A^o}} \end{aligned}\right\} \tag{16}$$

For such points as E', D', E'', and D'', which are junctions of intersections of the same characteristic tangent plane with an x_1 = constant plane and with an x_3 = constant plane, the coordinates x_1 and x_3 are known from preceding points and x_2 can be determined from relations such as, for point E',

$$\lambda_{A'} = \frac{x_{2,E'} - x_{2,A'}}{x_{1,E'} - x_{1,A'}} \tag{17}$$

In order to determine the velocity components at points such as E', three equations are required. From equations (14) and (15) the following difference equations may be obtained:

Along A'E'

$$(a_{11})_{A'} (v_{1,E'} - v_{1,A'}) + (a_{12} - \lambda a_{11})_{A'} (v_{2,E'} - v_{2,A'})$$
$$+ (a_{13})_{A'} (v_{3,E'} - v_{3,A'}) + P_{1,A'} (x_{1,E'} - x_{1,A'}) = 0 \tag{18}$$

Along B'D'

$$(a_{11})_{B'} (v_{1,C'} - v_{1,B'}) + (a_{12} - \lambda a_{11})_{B'} (v_{2,C'} - v_{2,B'})$$
$$+ (a_{13})_{B'} (v_{3,C'} - v_{3,B'}) + P_{1,B'} (x_{1,C'} - x_{1,B'}) = 0 \tag{19}$$

Along E^oE'

$$(a_{13})_{E^o} (v_{1,E'} - v_{1,E^o}) + (a_{23} - \mu a_{33})_{E^o} (v_{2,E'} - v_{2,E^o})$$
$$+ (a_{33})_{E^o} (v_{3,E'} - v_{3,E^o}) + P_{3,E^o} (x_{3,E'} - x_{3,E^o}) = 0 \tag{20}$$

Because only two characteristic intersections join at points such as E', three independent relations between the three velocity components at E' cannot be obtained unless the assumption that the quantities v_i vary linearly for small distances along characteristic intersections is used to eliminate $v_{i,C'}$ from equation (19). This assumption is expressed by the relation

$$\frac{v_{i,C'} - v_{i,A'}}{v_{i,E'} - v_{i,A'}} = \frac{x_{1,C'} - x_{1,A'}}{x_{1,E'} - x_{1,A'}} = R \tag{21}$$

With equation (21), equation (19) becomes

$$
\begin{aligned}
(a_{11})_{B'} & \left[Rv_{1,E'} - v_{1,B'} + (1-R)v_{1,A'}\right] \\
& + (a_{12} - \lambda a_{11})_{B'} \left[Rv_{2,E'} - v_{2,B'} + (1-R)v_{2,A'}\right] \\
& + (a_{13})_{B'} \left[Rv_{3,E'} - v_{3,B'} + (1-R)v_{3,A'}\right] \\
& + P_{1,B'} \left(x_{1,C'} - x_{1,B'}\right) = 0 \qquad (22)
\end{aligned}
$$

The solutions of equations (18), (22), and (20) are

$$
v_{1,E'} = \frac{\begin{vmatrix} B_1 & C_1 & D_1 \\ B_2 & C_2 & D_2 \\ B_3 & C_3 & D_3 \end{vmatrix}}{-\Delta} \qquad
v_{2,E'} = \frac{\begin{vmatrix} A_1 & C_1 & D_1 \\ A_2 & C_2 & D_2 \\ A_3 & C_3 & D_3 \end{vmatrix}}{\Delta} \qquad
v_{3,E'} = \frac{\begin{vmatrix} A_1 & B_1 & D_1 \\ A_2 & B_2 & D_2 \\ A_3 & B_3 & D_3 \end{vmatrix}}{-\Delta} \qquad (23)
$$

where

$$\Delta = A_1 (B_2C_3 - B_3C_2) - A_2 (B_1C_3 - B_3C_1) + A_3 (B_1C_2 - B_2C_1)$$

$$
\begin{aligned}
A_1 &= (a_{11})_{A'} & B_1 &= (a_{12} - \lambda a_{11})_{A'} & C_1 &= (a_{13})_{A'} \\
A_2 &= (a_{11})_{B'} & B_2 &= (a_{12} - \lambda a_{11})_{B'} & C_2 &= (a_{13})_{B'} \\
A_3 &= (a_{13})_{E^o} & B_3 &= (a_{23} - \mu a_{33})_{E^o} & C_3 &= (a_{33})_{E^o}
\end{aligned}
$$

$$D_1 = - A_1v_{1,A'} - B_1v_{2,A'} - C_1v_{3,A'} + P_{1,A'} (x_{1,E'} - x_{1,A'})$$

$$D_2 = \frac{1-R}{R}\left(A_2 v_{1,A'} + B_2 v_{2,A'} + C_2 v_{3,A'}\right)$$

$$- \frac{1}{R}\left(A_2 v_{1,B'} + B_2 v_{2,B'} + C_2 v_{3,B'}\right) + \frac{P_{1,B'}}{R}\left(x_{1,C'} - x_{1,B'}\right)$$

$$D_3 = - A_3 v_{1,E^o} - B_3 v_{2,E^o} - C_3 v_{3,E^o} + P_{3,E^o}\left(x_{3,E'} - x_{3,E^o}\right)$$

$$P_{1,A'} = \left[a_{33} \frac{dv_3}{dx_3} + \left(a_{23} - \lambda a_{13} - \mu a_{33}\right) \frac{dv_2}{dx_3} + H\right]_{A'}$$

$$P_{1,B'} = \left[a_{33} \frac{dv_3}{dx_3} + \left(a_{23} - \lambda a_{13} - \mu a_{33}\right) \frac{dv_2}{dx_3} + H\right]_{B'}$$

$$P_{3,E^o} = \left[a_{11} \frac{dv_1}{dx} + \left(a_{12} - \mu a_{13} - \quad a_{11}\right) \frac{dv_2}{dx} + H\right]_{E^o}$$

The difference forms of the derivatives in the expressions for $P_{1,A'}$, $P_{1,B'}$, and P_{1,E^o} are given by equations (B16) and (B17) in appendix B.

By means of equation (23), the velocity components at any point E^n of a derived source curve can be determined provided that the velocity components at points A^n, B^n and E^{n-1} are known. Inasmuch as the source curves $A^oA' \ldots A^m$ and $B^oB' \ldots B^m$ are given, the velocity components at all A and B points can be considered known. The determination of the E^o point, however, is also required before the computation can proceed. Because only two equations are available to determine the velocity components at E^o, (one along $\overline{A^oE^o}$ and one along $\overline{B^oE^o}$), one of the velocity components at E^o must be known or assumed. If the problem has a plane of symmetry, that plane can be used as a reference plane on which the cross component of the velocity v_3 is zero. For most practical problems, such a reference plane can usually be found. If none is available, however, the computations must proceed by trial and error, that is, an initial value must be assumed for one of the velocity components at E^o and the new source $E^oE' \ldots E^m$ must then be computed entirely around the body. If the final computed value of the velocity component is considerably different from the assumed value, this value must be correspondingly modified and the computations repeated. The resulting process may be much too lengthy in practice. The value of v_3 in the reference plane $x_3 = 0$ is therefore assumed to be known and equal to zero. From equations (14) and (15), the following difference equations for the velocity components at E^o are then obtained:

$$(a_{11})_{A^o} (v_{1,E^o} - v_{1,A^o}) + (a_{12} - \lambda a_{11})_{A^o} (v_{2,E^o} - v_{2,A^o}) + P_{1,A^o} (x_{1,E^o} - x_{1,A^o}) = 0 \tag{24}$$

$$(a_{11})_{B^o} (v_{1,E^o} - v_{1,B^o}) + (a_{12} - \lambda a_{11})_{B^o} (v_{2,E^o} - v_{2,A^o}) + P_{1,B^o} (x_{1,E^o} - x_{1,A^o}) = 0 \tag{25}$$

whose solutions are

$$v_{1,E^o} = \frac{\begin{vmatrix} B_1 & D_1 \\ B_2 & D_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}} ; \quad v_{2,E^o} = - \frac{\begin{vmatrix} A_1 & D_1 \\ A_2 & D_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}} \tag{26}$$

where

$$A_1 = (a_{11})_{A^o} \qquad B_1 = (a_{12} - \lambda a_{11})_{A^o}$$

$$A_2 = (a_{11})_{B^o} \qquad B_2 = (a_{12} - \lambda a_{11})_{B^o}$$

$$D_1 = - A_1 v_{1,A^o} - B_1 v_{2,A^o} + P_{1,A^o} (x_{1,E^o} - x_{1,A^o})$$

$$D_2 = - A_2 v_{1,B^o} - B_2 v_{2,B^o} + P_{1,B^o} (x_{1,E^o} - x_{1,A^o})$$

$$P_{1,A^o} = \left(-\mu a_{33}\frac{dv_2}{dx_3} + a_{33}\frac{dv_3}{dx_3} + H\right)_{A^o}$$

$$P_{1,B^o} = \left(-\mu a_{33}\frac{dv_2}{dx_3} + a_{33}\frac{dv_3}{dx_3} + H\right)_{B^o}$$

This solution can also be obtained from equation (23) by setting $A_3 = B_3 = D_3 = 0$ and $C_3 = R = 1$.

With equations (26) and (23), the new field source $E^oE' \ldots E^m$ can be computed. This new source curve determines two new characteristics, one of which extends toward the body whereas the other extends away from the body. The characteristic that extends away from the body may intersect another characteristic of type II whose source is a curve above $B^oB' \ldots B^m$ in figure 4. This intersection can be calculated by the procedure used to calculate $E^oE' \ldots E^m$; thus the entire characteristic whose initial source is $A^oA' \ldots A^m$ can be computed if conditions on a surface a small distance upstream of it are known. The determination of a new initial source on the body, such as $F^oF' \ldots F^m$, must now be considered.

The x_1-coordinate of the surface in which the new initial source $F^oF' \ldots F^m$ is to be located may be chosen as the junction of the type-II-characteristic intersection from E^o with the body. This value of x_{1,F^o} is determined from the relations

$$\tan \delta_{F^o} = \frac{\frac{1}{\eta_{2,A^o}}\left(x_{2,F^o} - x_{2,A^o}\right)}{x_{1,F^o} - x_{1,A^o}}$$

$$\lambda_{E^o} = \frac{x_{2,F^o} - x_{2,E^o}}{x_{1,F^o} - x_{1,E^o}}$$

where $\tan \delta_{F^o}$ is the slope of $\overline{A^oF^o}$ and λ_{E^o} is calculated from equation (10) with

$$\mu_{E^o} = \frac{x_{2,E'} - x_{2,E^o}}{x_{3,E^o} - x_{3,E^o}}$$

The factor $1/\eta_{2,A^o}$ is given in table I for each of the coordinate systems.

The coordinates of points that are junctions of the characteristic intersection from E' with the body can be determined from the relations

$$\tan \delta_{G'} = \frac{\frac{1}{\eta_{2,A'}}(x_{2,G'} - x_{2,A'})}{x_{1,G'} - x_{1,A'}}$$

$$\lambda_{E'} = \frac{x_{2,G'} - x_{2,E'}}{x_{1,G'} - x_{1,E'}}$$

For determining the velocity components at F', two characteristic equations are available, one along $\overline{E'G'}$ and one along $\overline{F^oF'}$. The third independent relation is obtained from the condition that the velocity vector at any point on the body must be in a plane tangent to the body at that point. This condition is evidently given by the relation (fig. 5)

$$\eta_2 v_2 = \eta_3 v_3 \cot \epsilon + v_1 \tan \delta \tag{27}$$

where ϵ is the angle between the x_2-direction and the trace of the body tangent plane in an $x_1 = \text{constant}$ surface and δ is the angle between the x_1-direction and the trace of the tangent plane in an $x_3 = \text{constant}$ surface. The quantities η_2 and η_3 are required to eliminate the coordinate factor from v_i when angular coordinates are used (table I). Along the characteristic source $\overline{F^oF'}$, the following relation is obtained from equation (15):

$$\begin{aligned}(a_{13})_{F^o}(v_{1,F'} - v_{1,F^o}) + (a_{23} - \mu a_{33})_{F^o}(v_{2,F'} - v_{2,F^o}) \\ + (a_{33})_{F^o}(v_{3,F'} - v_{3,F^o}) + P_{3,F^o}(x_{3,F'} - x_{3,F^o}) = 0\end{aligned} \tag{28}$$

Equations (27) and (28) are two of the relations required to determine the velocity components at F'. In order to obtain the third relation, an assumption similar to that made in determining E' is required to eliminate $v_{i,G'}$ from the characteristic relation along E'G'. The required assumption is

$$\frac{v_{1,G'} - v_{1,A'}}{v_{1,F'} - v_{1,A'}} = \frac{x_{1,G'} - x_{1,A'}}{x_{1,F'} - x_{1,A'}} = R \tag{29}$$

When equation (29) is used to replace $v_{1,G'}$ in the relation

$$(a_{11})_{E'} (v_{1,G'} - v_{1,E'}) + (a_{12} - \lambda a_{11})_{E'} (v_{2,G'} - v_{2,E'}) + (a_{13})_{E'} (v_{3,G'} - v_{3,E'}) + P_{1,E'} (x_{1,G'} - x_{1,E'}) = 0 \tag{30}$$

the third required equation becomes

$$\begin{aligned}(a_{11})_{E'} &\left[Rv_{1,F'} - v_{1,E'} + (1-R)v_{1,A'}\right] \\ &+ (a_{12} - \lambda a_{11})_{E'} \left[Rv_{2,F'} - v_{2,E'} + (1-R)v_{2,A'}\right] \\ &+ (a_{13})_{E'} \left[Rv_{3,F'} - v_{3,E'} + (1-R)v_{3,A'}\right] \\ &+ P_{1,E'} (x_{1,G'} - x_{1,E'}) = 0\end{aligned} \tag{31}$$

The solution of equations (27), (28), and (31) is

$$v_{1,F'} = \frac{\begin{vmatrix} B_1 & C_1 & 0 \\ B_2 & C_2 & D_2 \\ B_3 & C_3 & D_3 \end{vmatrix}}{-\Delta}; \quad v_{2,F'} = \frac{\begin{vmatrix} A_1 & C_1 & 0 \\ A_2 & C_2 & D_2 \\ A_3 & C_3 & D_3 \end{vmatrix}}{\Delta}; \quad v_{3,F'} = \frac{\begin{vmatrix} A_1 & B_1 & 0 \\ A_2 & B_2 & D_2 \\ A_3 & B_3 & D_3 \end{vmatrix}}{-\Delta} \tag{32}$$

where

$$\Delta = A_1 (B_2C_3 - B_3C_2) - A_2 (B_1C_3 - B_3C_1) + A_3 (B_1C_2 - B_2C_1)$$

$A_1 = \tan \delta_{F'}$ $\qquad B_1 = -\eta_{2,F'}$ $\qquad C_1 = (\eta_3 \cot \epsilon)_{F'}$

$A_2 = (a_{13})_{F^o}$ $\qquad B_2 = (a_{23} - \mu a_{33})_{F^o}$ $\qquad C_2 = (a_{33})_{F^o}$

$A_3 = (a_{11})_{E'}$ $\qquad B_3 = (a_{12} - \lambda a_{11})_{E'}$ $\qquad C_3 = (a_{13})_{E'}$

$$D_2 = -A_2v_{1,F^o} - B_2v_{2,F^o} - C_2v_{3,F^o} + P_{3,F^o}\left(x_{3,F'} - x_{3,F^o}\right)$$

$$D_3 = \frac{1-R}{R}\left(A_3v_{1,A'} + B_3v_{2,A'} + C_3v_{3,A'}\right)$$

$$- \frac{1}{R}\left(A_3v_{1,E'} + B_3v_{2,E'} + C_3v_{3,E'}\right) + \frac{P_{1,E'}}{R}\left(x_{1,G'} - x_{1,E'}\right)$$

$$P_{3,F^o} = \left[(a_{12} - \mu a_{13} - \lambda a_{11})\frac{dv_2}{dx_1} + a_{11}\frac{dv_1}{dx_1} + H\right]_{F^o}$$

$$P_{1,E'} = \left[(a_{23} - \lambda a_{13} - \mu a_{33})\frac{dv_2}{dx_3} + a_{33}\frac{dv_3}{dx_3} + H\right]_{E'}$$

For point F^o, which lies in the reference plane, v_3 is assumed to be zero and equation (27) becomes

$$(\eta_2 v_2)_{F^o} = v_{1,F^o} \tan \delta_{F^o} \tag{33}$$

The second relation required to determine v_{2,F^o} and v_{1,F^o} is obtained from equation (14), which, for $v_3 = 0$, becomes

$$(a_{11})_{E^o}\left(v_{1,F^o} - v_{1,E^o}\right) + (a_{12} - \lambda a_{11})_{E^o}\left(v_{2,F^o} - v_{2,E^o}\right)$$

$$+ P_{1,E^o}\left(x_{1,F^o} - x_{1,E^o}\right) = 0 \tag{34}$$

where

$$P_{1,E^o} = \left[a_{33}\left(\frac{dv_3}{dx_3} - \mu\frac{dv_2}{dx_3}\right) + H\right]_{E^o}$$

The solution of equations (33) and (34) for v_{1,F^o} is

$$v_{1,F^o} = \frac{(a_{11})_{E^o} v_{1,E^o} + (a_{12} - \lambda a_{11})_{E^o} v_{2,E^o} - P_{1,E^o}\left(x_{1,F^o} - x_{1,E^o}\right)}{(a_{11})_{E^o} + (a_{12} - \lambda a_{11})_{E^o}\left(\frac{\tan \delta}{\eta_2}\right)_{F^o}} \tag{35}$$

With equations (32), (33), and (35), new initial sources on the surface can be computed when conditions on a field source and on another initial source (both slightly upstream) are known.

Equations (23), (26), (32), (33), and (35) constitute the solution to the two basic problems mentioned at the beginning of the section. With these equations, the entire disturbed flow field about a body placed in a supersonic stream can be calculated provided that (1) the flow remains everywhere supersonic, (2) the rotation and the viscosity of the flow are zero or negligible, (3) conditions are known along the surface that separates the uniform flow field from the disturbed flow field, and (4) a reference surface exists for which $v_3 = 0$. Condition (4) is a practical limitation that will eliminate trial-and-error procedure. It may be replaced by some other known condition satisfied by one of the velocity components at points of types E^o and F^o if the equations for these points are appropriately altered.

The initial surface required by condition (3) may be, for example, the shock surface attached to the nose of a pointed body or to the lip of an open-nosed body. For unsymmetrical bodies, the determination of the form and intensity of the initial shock surface is difficult and the replacement of the shock by a characteristic surface may be essential for solution of some problems. This procedure neglects not only the rotationality of the real flow field, but also the abrupt compression at the foremost boundary of the disturbed portion of the field. The resulting solution therefore fails to satisfy exactly the real boundary conditions. For such problems, the linearized characteristic equations (to be derived subsequently) may yield a solution as accurate as that obtainable with the nonlinearized equations.

If the flow over a pointed body is to be computed, some assumption must initially be made concerning the velocity components near the point of the body because the vertex itself is singular. Perhaps the simplest assumption is that the flow is conical for a small distance beyond the vertex. Any reasonable assumption is valid, however, because the effect of that initial assumption can be made negligible by scaling up the body so that the portion near the vertex is a very small part of the entire body. The initiation of the characteristic solution for pointed bodies is discussed in more detail in appendix A. For open-nosed bodies the starting process is somewhat more straightforward because the velocity components a small distance from the lip are accurately obtainable from two-dimensional flow theory.

The equations required to carry out computations are summarized in appendix B, whereas in appendix C the first steps involved in the computation of the flow past a cone that has an elliptic cross section are described.

Solution of the linearized equation by means of characteristic surfaces. - When the changes in the velocity components throughout the disturbed flow field are small relative to the critical velocity, equation (1) can be reduced to a linear equation, which, for Cartesian and cylindrical coordinates, has the form

$$a_{11}D_{11}\Psi + a_{22}D_{22}\Psi + a_{33}D_{33}\Psi + H = 0 \tag{36}$$

where

$$a_{11} = (1-M^2)$$

and where the quantities $k^2{v_1}^2$ are neglected in the expressions for a_{22}, a_{33}, and H (table I). The value of M is now assumed to remain constant throughout the flow field.

The linearized theory for supersonic flow consists in finding solutions to equation (36) for various boundary conditions. For some types of solid boundary, such as thin wings and conical bodies, a variety of solutions are available. These solutions permit the determination of flow parameters at the surface of the boundary without determining the flow in the entire disturbed field. Solutions that have been obtained for bodies whose contour varies in the stream direction or for body-wing combinations postulate some restriction as to the shape of the boundary and in themselves involve considerable computation. For bodies that are not axially symmetric, solutions have as yet been obtained only for the condition that the flow is conical (reference 8). The following discussion develops a method of characteristics for solution of the linearized differential equation for general boundaries. The resulting equations are quite simple relative to those obtained for nonlinear flow, although the method involves, as with nonlinear flow, computation of the entire flow field. The method is consequently not intended to be used when boundary-surface solutions are available unless these solutions involve more computation than the one presented herein.

The differential equation for the characteristic surfaces of equation (36) is

$$a_{11}\lambda^2 + a_{22} + a_{33}\mu^2 = 0 \tag{37}$$

which, for Cartesian coordinates, becomes

$$\lambda = \pm\sqrt{\frac{1 + \mu^2}{M^2 - 1}} \tag{38}$$

In equation (37), λ and μ have the same significance as in equation (4); hence equations (7) and (7a) may be used to eliminate the partial derivatives from equation (36) in the manner used for nonlinear flow. The resulting ordinary differential equation that relates the velocity components in moving from one point to a neighboring point along an intersection of a characteristic with an x_3 = constant surface is

$$a_{11}\left(dv_1 - \lambda dv_2\right) + a_{33}\left(\frac{dv_3}{dx_3} - \mu\frac{dv_2}{dx_3}\right)dx_1 + H dx_1 = 0 \tag{39}$$

The corresponding relation along intersections of a characteristic with an x_1 = constant surface becomes

$$a_{33}\left(dv_3 - \mu dv_2\right) + a_{11}\left(\frac{dv_1}{dx_1} - \lambda\frac{dv_2}{dx_1}\right)dx_3 + H dx_3 = 0 \tag{40}$$

Equations (39) and (40) can also be obtained by setting a_{12}, a_{13}, and a_{23} equal to zero in equations (14) and (15). The difference equations for each type of source point may consequently be derived simply by setting these coefficients equal to zero in the nonlinearized equations. The resulting equations are given in appendix B for a general field-source point E^n (equation (B6)) and for a general solid-boundary source point F^m (equation (B7)). If v_3 is again assumed to be zero in the $x_3 = 0$ surface, the velocity components for points E^o can be obtained from equation (B6) by setting $D_3 = \mu_{E^{n-1}} = 0$ and $R = 1$. The velocity components for point F^o are given by equations (B8) and (B9). The expressions for the coordinates of the various types of source point are the same as for nonlinear theory and are given by equations (B12) to (B15).

With the expressions given in appendix B, the linearized solution for the supersonic flow past general boundaries can be obtained provided that, to the conditions mentioned for the nonlinear solutions, the additional condition is added that the velocity is nowhere greatly different from the velocity of the undisturbed stream. The initial surface that is required to start the computations is, for linearized theory, the foremost characteristic surface of the disturbed field, upon which it may be assumed that the velocity components have their free-stream values. An initial source curve on the boundary surface must be assumed, but the effect of this initial assumption can again be made negligible by scaling up the initial section of the body.

SUMMARY OF ANALYSIS

The method of characteristics has been applied to the nonlinear and linearized partial differential equations for the velocity potential in a supersonic stream. By use of the resulting difference equations, the velocity components can be computed throughout an irrotational supersonic flow field for arbitrary boundary conditions. The solution for the linearized equation involves considerably less computation than the solution for the nonlinearized equation, although both require the determination of the entire flow field that influences the flow variables at the boundaries. The use of the linearized solution is suggested in those problems for which the disturbances resulting from the presence of boundary surfaces are everywhere small and for which no simpler linearized solution is available. The nonlinearized solution is required, for accurate results, when the presence of solid boundaries results in large changes in the velocity components. No procedure has been given for treating the effects of an initial shock surface; however, the replacement of the initial shock surface with a characteristic surface should not lead to serious error unless the shock is intense.

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, December 29, 1948.

APPENDIX A

TREATMENT OF UNSYMMETRICAL CONICAL FLOW FIELDS

If the supersonic flow over pointed bodies is to be determined, the initial portion of the body near the point can usually be regarded as a small cone. If the nose is axially symmetric, the velocity components at the surface of the initial cone and throughout the conical portion of the field can be calculated by the method of Taylor and Maccoll (reference 10). If the initial cone is not axially symmetric, however (if it has, for example, an elliptic cross section), no exact solution other than the method of characteristics is known for determining the flow throughout the conical field. If a linearized solution is desired, the procedure presented in reference 8 may be used for determining the velocity components at the surface of an unsymmetrical cone. This method may be used to obtain the initial source curve required to start the solution by means of characteristics if the conical nose is only a small part of the entire body. If the conical nose is a considerable part of the body, however, flow conditions must be accurately known over an entire surface ahead of the characteristic whose source is the first variation of the body from the conical shape of the nose. A method for determining the velocity components throughout an unsymmetrical conical flow field must therefore be developed.

The method of characteristics cannot be simplified to solve this problem because the assumption that the flow is conical leads to an equation that has no real characteristic solutions. If the conical-flow conditions are imposed on equation (1), then for spherical coordinates, equation (1) becomes

$$a_{22}D_{22}\Psi + a_{23}D_{23}\Psi + a_{33}D_{33}\Psi + H' = 0 \tag{A1}$$

where

$$H' = \frac{v_r}{r}\left(2 - k^2v_\theta^{\,2} - k^2v_\varphi^{\,2}\right) + \frac{v_\theta}{r}\cot\theta\left(1 + k^2v_\varphi^{\,2}\right)$$

The characteristic form of equation (A1) is

$$\bar{A} = a_{22} - \mu a_{23} + \mu^2 a_{33} \tag{A2}$$

If equation (A2) is set equal to zero, the characteristics can be shown to be imaginary unless

$$k^2 v_\theta^{\,2} + k^2 v_\varphi^{\,2} \geq 1$$

Because v_θ and v_φ are generally quite small near the cone surface, the inequality is invalid throughout a conical field.

The following procedure may be used, however, to determine the flow past unsymmetrical cones by means of characteristics. The conditions at an initial source curve near the point can be fixed at some reasonable values, possibly based on a linear variation of velocity components with the conical half-angle θ. The normal procedure described in the text can then be used to develop the flow field until there is no further variation of the velocity components along a given radius vector from the vertex of the cone. The number of computations required to establish the conical solution will depend on the accuracy of the initial assumption. The use of this procedure to compute the flow over an elliptic cone is described in appendix C. Other procedures, such as the direct numerical integration of equation (A1) in hodograph planes or along prescribed conical surfaces, appears to be feasible and their development should constitute a profitable field for future research.

APPENDIX B

SUMMARY OF EQUATIONS

Nonlinear Theory

The following equations are used for field-source points E^n:

$$v_{1,E^n} = \frac{B_1(C_2D_3-C_3D_2)-B_2(C_1D_3-C_3D_1)+B_3(C_1D_2-C_2D_1)}{-\Delta} \qquad \text{(B1)}$$

$$v_{2,E^n} = \frac{A_1(C_2D_3-C_3D_2)-A_2(C_1D_3-C_3D_1)+A_3(C_1D_2-C_2D_1)}{\Delta} \qquad \text{(B2)}$$

$$v_{3,E^n} = \frac{A_1(B_2D_3-B_3D_2)-A_2(B_1D_3-B_3D_1)+A_3(B_1D_2-B_2D_1)}{-\Delta}$$

$$= -\frac{1}{C_1}\left(A_1v_{1,E^n}+B_1v_{2,E^n} + D_1\right) \qquad \text{(B3)}$$

where

$$\Delta = A_1(B_2C_3-B_3C_2)-A_2(B_1C_3-B_3C_1)+A_3(B_1C_2-B_2C_1)$$

$$A_1 = (a_{11})_{A^n} \qquad B_1 = (a_{12}-\lambda\, a_{11})_{A^n} \qquad C_1 = (a_{13})_{A^n}$$

$$A_2 = (a_{11})_{B^n} \qquad B_2 = (a_{12}-\lambda\, a_{11})_{B^n} \qquad C_2 = (a_{13})_{B^n}$$

$$A_3 = (a_{13})_{E^{n-1}} \qquad B_3 = (a_{23}-\mu\, a_{33})_{E^{n-1}} \qquad C_3 = (a_{33})_{E^{n-1}}$$

$$D_1 = -A_1v_{1,A^n} - B_1v_{2,A^n} - C_1v_{3,A^n} + P_{1,A^n}(x_{1,E^n} - x_{1,A^n})$$

$$D_2 = \frac{1-R}{R}\left(A_2v_{1,A^n} + B_2v_{2,A^n} + C_2v_{3,A^n}\right)$$

$$- \frac{1}{R}\left(A_2v_1 + B_2v_2 + C_2v_3\right)_{B^n} + \frac{P_{1,B^n}}{R}(x_{1,C^n} - x_{1,B^n})$$

$$D_3 = -\left(A_3v_1 + B_3v_2 + C_3v_3\right)_{E^{n-1}} + P_{3,E^{n-1}}\left(x_{3,E^n}-x_{3,E^{n-1}}\right)$$

$$P_{1,A^n} = \left[a_{33}\frac{dv_3}{dx_3} + (a_{23}-\lambda\ a_{13}-\mu a_{33})\frac{dv_2}{dx_3} + H\right]_{A^n}$$

$$P_{1,B^n} = \left[a_{33}\frac{dv_3}{dx_3} + (a_{23}-\lambda\ a_{13}-\mu a_{33})\frac{dv_2}{dx_3} + H\right]_{B^n}$$

$$P_{3,E^{n-1}} = \left[a_{11}\frac{dv_1}{dx_1} + (a_{12}-\mu a_{13}-\lambda\ a_{11})\frac{dv_2}{dx_1} + H\right]_{E^{n-1}}$$

$$R = \frac{x_{1,C^n}-x_{1,A^n}}{x_{1,E^n}-x_{1,A^n}}$$

For point E^o $(v_3 = 0)$: $A_3 = B_3 = D_3 = 0$, $C_3 = R = 1$.

Equations for the velocity components v_{1,F^n}, v_{2,F^n}, and v_{3,F^n} at the solid-boundary source point F^n have the same forms as equations (B1), (B2), and (B3), respectively, where the coefficients are given by the following expressions:

$A_1 = \tan\delta_{F^n}$ $\quad B_1 = -\eta_{2,F^n}$ $\quad C_1 = (\eta_3 \cot \epsilon)_{F^n}$

$A_2 = (a_{13})_{F^{n-1}}$ $\quad B_2 = (a_{23}-\mu a_{33})_{F^{n-1}}$ $\quad C_2 = (a_{33})_{F^{n-1}}$

$A_3 = (a_{11})_{E^n}$ $\quad B_3 = (a_{12}-\lambda\ a_{11})_{E^n}$ $\quad C_3 = (a_{13})_{E^n}$

$$D_1 = 0$$

$$D_2 = -\left(A_2v_1 + B_2v_2 + C_2v_3\right)_{F^{n-1}} + P_{3,F^{n-1}}\left(x_{3,F^n} - x_{3,F^{n-1}}\right)$$

$$D_3 = \frac{1-R}{R}\left(A_3v_{1,A^n} + B_3v_{2,A^n} + C_3v_{3,A^n}\right)$$

$$-\frac{1}{R}\left(A_3v_1 + B_3v_2 + C_3v_3\right)_{E^n} + \frac{P_{1,E^n}}{R}\left(x_{1,G^n}-x_{1,E^n}\right)$$

$$P_{3,F^{n-1}} = \left[(a_{12}-\mu a_{13}-\lambda a_{11})\ \frac{dv_2}{dx_1} + a_{11}\ \frac{dv_1}{dx_1} + H\right]_{F^{n-1}}$$

$$P_{1,E^n} = \left[(a_{23}-\lambda a_{13}-\mu a_{33})\ \frac{dv_2}{dx_3} + a_{33}\ \frac{dv_3}{dx_3} + H\right]_{E^n}$$

$$R = \frac{x_{1,G^n}-x_{1,A^n}}{x_{1,F^n}-x_{1,A^n}}$$

For point F^o $(v_3=0)$, the following equations are used:

$$v_{1,F^o} = \frac{(a_{11})_{E^o}v_{1,E^o}+(a_{12}-\lambda a_{11})_{E^o}v_{2,E^o}-P_{1,E^o}(x_{1,F^o}-x_{1,E^o})}{(a_{11})_{E^o}+(a_{12}-\lambda a_{11})_{E^o}\left(\frac{\tan\delta}{\eta_2}\right)_{F^o}} \tag{B4}$$

$$P_{1,E^o} = \left[a_{33}\left(\frac{dv_3}{dx_3} - \mu\ \frac{dv_2}{dx_3}\right) + H\right]_{E^o}$$

$$(\eta_2 v_2)_{F^o} = (v_1\ \tan\delta)_{F^o} \tag{B5}$$

Linearized Theory

The following equations are used for field-source points E^n:

$$\left.\begin{aligned}
v_{1,E^n} &= \frac{\lambda_{A^n}D_2-\lambda_{B^n}D_1}{A_1(\lambda_{B^n}-\lambda_{A^n})} \\
v_{2,E^n} &= \frac{D_2-D_1}{A_1(\lambda_{B^n}-\lambda_{A^n})} \\
v_{3,E^n} &= D_3 - \frac{\mu_{E^{n-1}}(D_1-D_2)}{A_1(\lambda_{B^n}-\lambda_{A^n})} = D_3 + \mu_{E^{n-1}}\ v_{2,E^n}
\end{aligned}\right\} \tag{B6}$$

where

$$A_1 = M^2-1$$

$$D_1 = -A_1(v_1-\lambda v_2)_{A^n} + P_{1,A^n}(x_{1,E^n}-x_{1,A^n})$$

$$D_2 = \frac{1-R}{R} A_1(v_{1,A^n}-\lambda_{B^n} v_{2,A^n}) - \frac{A_1}{R} (v_1-\lambda v_2)_{B^n} + \frac{P_{1,B^n}}{R} (x_{1,C^n}-x_{1,B^n})$$

$$D_3 = (v_3-\mu v_2)_{E^{n-1}} + P_{3,E^{n-1}} (x_{3,E^n}-x_{3,E^{n-1}})$$

$$P_{1,A^n} = -\left[a_{33}\left(\frac{dv_3}{dx_3} - \mu\frac{dv_2}{dx_3}\right) + H\right]_{A^n}$$

$$P_{1,B^n} = -\left[a_{33}\left(\frac{dv_3}{dx_3} - \mu\frac{dv_2}{dx_3}\right) - H\right]_{B^n}$$

$$P_{3,E^{n-1}} = \left[\frac{A_1}{a_{33}}\left(\frac{dv_1}{dx_1} - \lambda\frac{dv_2}{dx_1}\right) - \frac{H}{a_{33}}\right]_{E^{n-1}}$$

The following equations are used for boundary-surface source points F^n:

$$\left.\begin{aligned}
v_{1,F^n} &= \frac{A_1\lambda_{E^n}D_2(\eta_3 \cot \epsilon)_{F^n}-D_3\left[\eta_{2,F^n}-\mu_{F^{n-1}}(\eta_3 \cot \epsilon)_{F^n}\right]}{\Delta} \\
v_{2,F^n} &= \frac{A_1D_2 (\eta_3 \cot \epsilon)_{F^n}-D_3 \tan \delta_{F^n}}{\Delta} \\
v_{3,F^n} &= \frac{A_1D_2(\eta_{2,F^n}-\lambda_{E^n} \tan \delta_{F^n})-\mu_{F^{n-1}} D_3 \tan \delta_{F^n}}{\Delta} \\
&= \frac{(\eta_2 v_2)_{F^n}-v_{1,F^n} \tan \delta_{F^n}}{(\eta_3 \cot \epsilon)_{F^n}}
\end{aligned}\right\} \quad \text{(B7)}$$

where

$$\Delta = A_1 \left[\eta_{2,F^n} - \mu_{F^{n-1}} (\eta_3 \cot \epsilon)_{F^n} - \lambda_{E^n} \tan \delta_{F^n}\right]$$

$$A_1 = M^2 - 1$$

$$D_2 = (v_3 - \mu v_2)_{F^{n-1}} + \left[\frac{A_1}{a_{33}}\left(\frac{dv_1}{dx_1} - \lambda\frac{dv_2}{dx_1}\right) - \frac{H}{a_{33}}\right]_{F^{n-1}} (x_{3,F^n} - x_{3,F^{n-1}})$$

$$D_3 = \frac{1-R}{R} A_1 (v_{1,A^n} - \lambda_{E^n} v_{2,A^n}) - \frac{A_1}{R} (v_1 - \lambda v_2)_{E^n} + \frac{P_{1,E^n}}{R} (x_{1,G^n} - x_{1,E^n})$$

$$P_1 = -\left[a_{33}\left(\frac{dv_3}{dx_3} - \mu\frac{dv_2}{dx_3}\right) + H\right]_{E^n}$$

For point F^o $(v_3=0)$, the following equations apply:

$$v_{1,F^o} = \frac{(v_1 - \lambda v_2)_{E^o} - \dfrac{P_{1,E^o}}{A_1}(x_{1,F^o} - x_{1,E^o})}{1 - \lambda_{E^o}\left(\dfrac{\tan \delta}{\eta_2}\right)_{F^o}} \tag{B8}$$

$$P_{1,E^o} = -\left[a_{33}\left(\frac{dv_3}{dx_3} - \mu\frac{dv_2}{dx_3}\right) + H\right]_{E^o}$$

$$(\eta_2 v_2)_{F^o} = (v_1 \tan \delta)_{F^o} \tag{B9}$$

Expressions Involving Coordinates and Derivations of

Velocity Components (Nonlinear or Linear Theory)

The difference equations for μ and λ are:

$$\left.\begin{aligned}
\mu_{A^n} &= \frac{x_{2,A^{n+1}} - x_{2,A^n}}{x_{3,A^{n+1}} - x_{3,A^n}} \\
\mu_{B^n} &= \frac{x_{2,B^{n+1}} - x_{2,B^n}}{x_{3,B^{n+1}} - x_{3,B^n}} \\
\mu_{E^{n-1}} &= \frac{x_{2,E^n} - x_{2,E^{n-1}}}{x_{3,E^n} - x_{3,E^{n-1}}}
\end{aligned}\right\} \quad \text{(B10)}$$

Nonlinear,

$$\lambda = \frac{-(a_{12}-\mu a_{13}) \pm \sqrt{(\mu a_{13}-a_{12})^2 - 4a_{11}(a_{22}-\mu a_{23}+\mu^2 a_{33})}}{-2a_{11}}$$

Linear,

$$\lambda = \pm\sqrt{\frac{a_{22}+\mu^2 a_{33}}{-a_{11}}} \quad \text{(B11)}$$

where v_y^2, v_z^2, v_r^2, v_θ^2 and v_φ^2 are neglected in $a_{i,k}$.

For computing E^n, + is used for λ_{A^n} and $\lambda_{E^{n-1}}$ and - for λ_{B^n}.

For computing F^n, + is used for $\lambda_{F^{n-1}}$ and - for λ_{E^n}.

The expressions for the coordinates of the points in the network are as follows:

Coordinates of C^n are

$$\left.\begin{aligned}
x_{1,C^n} &= \frac{x_{2,B^n}-x_{2,A^n}+\lambda_{A^n}x_{1,A^n}-\lambda_{B^n}x_{1,B^n}}{\lambda_{A^n}-\lambda_{B^n}} \\
x_{2,C^n} &= x_{2,A^n}+\lambda_{A^n}\left(x_{1,C^n}-x_{1,A^n}\right) \\
x_{3,C^n} &= x_{3,A^n} = x_{3,B^n}
\end{aligned}\right\} \tag{B12}$$

Coordinates of E^n are

$$\left.\begin{aligned}
x_{1,E^n} &= x_{1,E^o} = x_{1,C^o} \\
x_{2,E^n} &= x_{2,A^n}+\lambda_{A^n}\left(x_{1,E^n}-x_{1,A^n}\right) \\
x_{3,E^n} &= x_{3,A^n} = x_{3,B^n}
\end{aligned}\right\} \tag{B13}$$

Coordinates of G^n are

$$\left.\begin{aligned}
x_{1,G^n} &= \frac{x_{2,E^n}-x_{2,A^n}+x_{1,A^n}\tan\delta_{G^n}-\lambda_{E^n}x_{1,E^n}}{\tan\delta_{G^n}-\lambda_{E^n}} \\
x_{2,G^n} &= x_{2,E^n}+\lambda_{E^n}\left(x_{1,G^n}-x_{1,E^n}\right) \\
x_{3,G^n} &= x_{3,E^n}
\end{aligned}\right\} \tag{B14}$$

Coordinates of F^n are

$$\left.\begin{aligned}
x_{1,F^n} &= x_{1,F^o} = x_{1,G^o} \\
x_{2,F^n} &= x_{2,A^n} + (\eta_2)_{A^n} \tan \delta_{F^n}\left(x_{1,F^n} - x_{1,A^n}\right) \\
x_{3,F^n} &= x_{3,E^n}
\end{aligned}\right\} \qquad \text{(B15)}$$

The velocity derivatives in difference form are as follows:

For P_1,

$$\left.\begin{aligned}
\left(\frac{dv_3}{dx_3}\right)_{A^n} &= \frac{v_{3,A^{n+1}} - v_{3,A^n}}{x_{3,A^{n+1}} - x_{3,A^n}} \\
\left(\frac{dv_2}{dx_3}\right)_{A^n} &= \frac{v_{2,A^{n+1}} - v_{2,A^n}}{x_{3,A^{n+1}} - x_{3,A^n}}
\end{aligned}\right\} \qquad \text{(B16)}$$

Similar expressions apply for points B^n and E^n.

For P_3,

$$\left.\begin{aligned}
\left(\frac{dv_1}{dx_1}\right)_{E^{n-1}} &= \frac{v_{1,E^{n-1}} - v_{1,A^{n-1}}}{x_{1,E^{n-1}} - x_{1,A^{n-1}}} \\
\left(\frac{dv_2}{dx_1}\right)_{E^{n-1}} &= \frac{v_{2,E^{n-1}} - v_{2,A^{n-1}}}{x_{1,E^{n-1}} - x_{1,A^{n-1}}}
\end{aligned}\right\} \qquad \text{(B17)}$$

Similar expressions apply for points F^{n-1}

APPENDIX C

COMPUTATION OF FLOW OVER CONE

WITH ELLIPTIC CROSS SECTION

As an example of the use of the nonlinearized method of characteristics for unsymmetrical bodies, the first steps in determining the flow over an elliptic cone are described in detail. This example illustrates the starting procedure for pointed bodies in general, because the shape of the body can be modified progressively downstream once the initial conical field at the point has been determined.

The elliptic cone used is similar to that described in reference 8 and is shown in figure 6. The ratio of the major to the minor axis of the cross section is 3 and the slope of the cone in the plane $\varphi = 90^{\circ}$ is 0.315; thus the cone is represented by the equation

$$\frac{r}{x} = \tan \delta = \frac{0.315}{\sqrt{9 \cos^2\varphi + \sin^2\varphi}} \tag{C1}$$

and the expression for $\cot \epsilon$ is

$$\cot \epsilon = \frac{dr}{rd\varphi} = \frac{8 \sin\varphi \cos\varphi}{9 \cos^2\varphi + \sin^2\varphi} \tag{C2}$$

The free-stream Mach number was assumed to be 1.90, as in reference 8.

A preliminary computation plan is illustrated in figure 7, where each point in the plane $\varphi = 0$ (fig. 7(a)) represents a source curve composed of 17 points circumferentially distributed at values of φ of 0, 10, 20, 30, 40, 50, 60, 65, 70, 73, 76, 79, 82, 85, 87, 89, and 90° (fig. 7(b)). As yet no basis other than experience is known for determining the number and distribution of points required for a prescribed degree of accuracy. It is evident, however, that a denser distribution of points is required where the shape of the surface varies most rapidly.

In order to start the computations, values of v_x, v_r, and v_φ were assumed at the points of source curves 1 to 6 (figs. 6 and 7). At group 1, on the cone surface, the velocity components were assigned according to the equations

$$v_{r,n} = v_{x,n} \tan \delta_n + v_{\varphi,n} \cot \epsilon_n$$

$$V_n^{\ 2} = v_{x,n}^{\ 2} + v_{r,n}^{\ 2} + v_{\varphi,n}^{\ 2} \tag{C3}$$

wherein V and v_φ were assumed at each of the 17 points and v_r and v_x were then calculated. For $n=0$ ($\varphi=0^\circ$), V was assumed to have the value resulting from two-dimensional compression of $\delta_0 = \tan^{-1} 0.105$; for $n = 16$ ($\varphi = 90^\circ$), V was assumed to have the value obtained on a symmetrical cone of half-angle $\delta_{16} = \tan^{-1} 0.315$. For $n = 0$ and $n = 16$, v_φ must be zero because the flow is symmetrical with respect to these planes. For other values of n, the values of V and v_φ were selected by drawing reasonable-looking curves between their values at $\varphi = 0^\circ$ and $\varphi = 90^\circ$.

At the Mach cone (group 6), $v_\varphi = v_r = 0$ and $v_x = 1.585$ (for the assumed free-stream Mach number 1.90). For groups 2 to 5 the velocity components and the values of r/x were assumed to vary linearly (for each value of n) between their values at the cone surface and at the Mach cone.

With the velocity components and coordinates thus assigned at groups 1 to 6, the velocity components and coordinates at groups 7, 9, 13, 19, and 27 can be simultaneously computed by equations (B1) to (B3) and (B10) to (B13) in appendix B. In order to compute group 7, for example, the points in group 1 are used as the A^n points and the points in group 2 are the B^n points. When group 7 (E^o, E^1, . . . E^{16}) has been computed, these points together with group 1 can be used to compute the new solid-boundary source (group 8).

An alternate procedure to the one outlined would consist in assigning values only at groups 1 and 6 and computing groups 31 and 36. This procedure has the advantage that values need be assumed only at one surface source curve and at the Mach cone, conditions at the Mach cone being known if no shock is assumed to occur. The increments of r between the groups, however, are very large relative to the value of r at the solid boundary; thus the assumption that the coordinate increments are small is violated. It is therefore doubtful whether the computed values of the velocity components would converge as rapidly to their conical-field values as with the procedure previously described. The alternate procedure has the additional disadvantage that only one source curve can be computed at a time.

Because v_φ is zero both for $\varphi = 0$ and for $\varphi = 90^\circ$, either of these planes can be used as the reference plane (fig. 7(b)). When two or more reference planes are available, any cumulative error involved in the computation of a source curve can be reduced by using each reference plane as a locus of E^o and F^o points and computing values of E^n and F^n from two directions. This procedure was used in the present preliminary computations. The difference between computed values of the velocity components resulting from the use of this procedure rather than computation in one direction is shown for group 9 in figure 8. Part of the difference between the two results is due to the difference in the value of x for the two groups of points, inasmuch as the $n = 0$ points determine the x-coordinate of the entire group. Most of the difference, however, probably results from the fact that the values of some of the total derivatives, when approximated by their difference form, depend somewhat on the direction of computation (equations (B10) and (B16)).

In order to determine whether the number of points used in a group greatly affects the computation results, group 7 was computed twice. For the first computation 17 points were used and for the second 31 points. In addition, the assumed values of v_φ in groups 1 and 2 were slightly altered to determine the sensitivity of the computed results to changes in initial values. The results of these computations are shown in figure 9. The assumed values for groups 1 and 2 are indicated with solid lines. The alteration in the assumed values of v_φ for the 17-point computation is indicated for group 1 by a broken line. The oscillation of the computed value of v_x about the assumed initial values occurred for both sets of computations, and no difference in results is apparent that could be attributed to the use of a different number of points per group. The abrupt changes in v_x at 0° and 20° for the 31-point group were attributable to computation errors. These errors were retained to determine their effect on succeeding points in the group. The results indicate that the influence of such errors is limited to the immediate vicinity of the point for which the error is made.

The difference in the initial values of v_φ for the two groups resulted in relatively large changes in the computed values of v_φ and smaller changes in the computed values of v_x and v_r. In the regions for which the assumed values of v_φ were the same (the leftward-running computations), the computed results were almost the same except near the peaks of the oscillations in v_x and v_φ.

The cause of the oscillations in the computed values of v_x and v_φ has not been definitely determined. The fact that an increase in the number of points did not reduce their amplitude indicates that the magnitude of the coordinate differences was not responsible. This conclusion is further substantiated by the fact that the oscillations occurred regardless of the direction of computation, although the peaks were somewhat displaced. The hypothesis is therefore advanced that the initial values chosen for the velocity components were unsatisfactory and tended to initiate strong compression and expansion waves in the flow field. Insufficient time was available to check this hypothesis by altering the initial values. If the hypothesis is correct, a more accurate knowledge of initial values for each of the velocity components would appear to be very desirable for the computation of unsymmetrical flow fields by the method presented. Convergence of the computations may be very slow if the assumed values can be corrected only by means of strong compression or expansion of the flow.

With ordinary calculating machines, a group of 17 points comprising a source curve could be computed and checked in about 80 man-hours. The computation setup consisted of 186 columns for the E^n groups and 130 columns for the F^n groups. For a linearized solution these figures may be reduced by about one half. The use of electronic calculators will, of course, reduce both computation time and the possibility of error.

REFERENCES

1. Puckett, Allen E.: Supersonic Wave Drag of Thin Airfoils. Jour. Aero. Sci., vol. 13, no. 9, Sept. 1946, pp. 475-484.

2. Evvard, John C.: Theoretical Distribution of Lift on Thin Wings at Supersonic Speeds (An Extension). NACA TN No. 1585, 1948.

3. Jones, Robert T.: Thin Oblique Airfoils at Supersonic Speed. NACA Rep. No. 851, 1946.

4. Stewart, H. J.: The Lift of a Delta Wing at Supersonic Speeds. Quarterly Appl. Math., vol. IV, no. 3, Oct. 1946, pp. 246-254.

5. Heaslet, Max A., Lomax, Harvard, and Jones, Arthur L.: Volterra's Solution of the Wave Equation as Applied to Three-Dimensional Supersonic Airfoil Problems. NACA TN No. 1412, 1947.

6. Brown, Clinton E., and Parker, Herman M.: A Method for the Calculation of External Lift, Moment, and Pressure Drag of Slender Open-Nosed Bodies of Revolution at Supersonic Speeds. NACA Rep. No. 808, 1945.

7. Tsien, Hsue-Shen: Supersonic Flow over an Inclined Body of Revolution. Jour. Aero. Sci., vol. 5, no. 12, Oct. 1938, pp. 480-483.

8. Maslen, Stephen: Method for Calculation of Pressure Distributions on Thin Conical Bodies of Arbitrary Cross Section in Supersonic Stream. NACA TN No. 1659, 1948.

9. Taylor, G. I., and Maccoll, J. W.: The Two-Dimensional Flow around a Corner; Two-Dimensional Flow past a Curved Surface. Vol. III of Aerodynamic Theory, div. H, ch. IV, secs. 5-6, W. F. Durand, ed., Julius Springer (Berlin), 1935, pp. 243-249. (Reprinted, C.I.T., Jan. 1943.)

10. Taylor, G. I., and Maccoll, J. W.: The Air Pressure on a Cone Moving at High Speeds. - I and II. Proc. Roy. Soc. (London), ser. A, vol. 139, no. 838, Feb. 1, 1933, pp. 278-311.

11. Sauer, R.: Method of Characteristics for Three-Dimensional Axially Symmetrical Supersonic Flows. NACA TM No. 1133, 1947.

12. Ferri, Antonio: Application of the Method of Characteristics to Supersonic Rotational Flow. NACA Rep. No. 841, 1946.

13. Ferrari, C.: Determination of the Pressure Exerted on Solid Bodies of Revolution with Pointed Noses Placed Obliquely in a Stream of Compressible Fluid at Supersonic Velocity. R.T.P. Trans. No. 1105, M.A.P.

14. Courant, R., und Hilbert, D.: Methoden der Mathematischen Physik. II. Julius Springer (Berlin), 1937, S. 290-307, 346-372.

15. Hadamard, Jacques: Lectures on Cauchy's Problem in Linear Partial Differential Equations. Book I. General Properties of Cauchy's Problem. Oxford Univ. Press (London), 1923, pp. 3-44.

16. Bateman, H.: Partial Differential Equations of Mathematical Physics. Dover Pub. (New York), 1944, pp. 132-135.

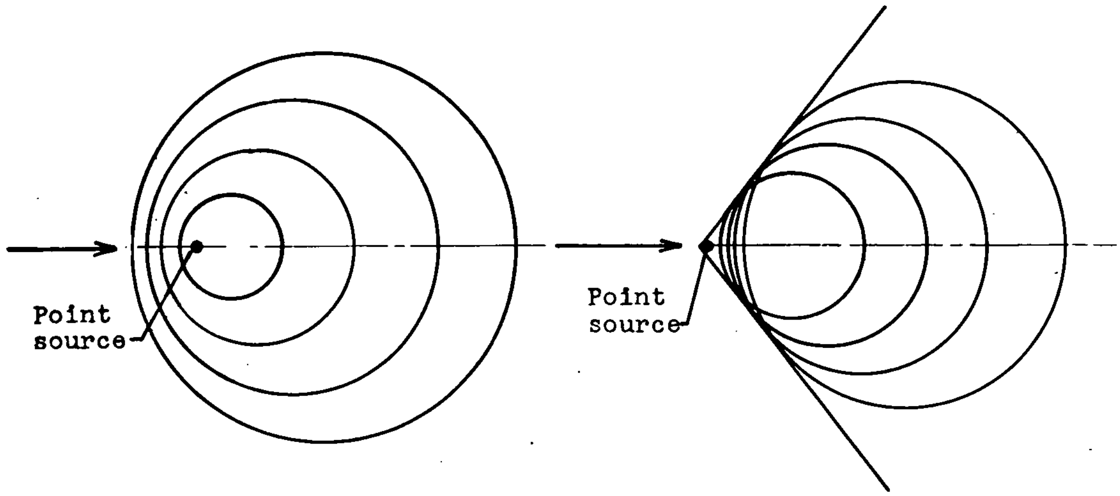
TABLE I - DEFINITION OF QUANTITIES IN EQUATION (1) FOR SEVERAL COORDINATE SYSTEMS

[All velocity components are made dimensionless by division with critical speed.

The factor k is defined by $k^2 = \frac{2}{\gamma+1}\left(1 - \frac{\gamma-1}{\gamma+1} V^2\right)^{-1}$]

Coordinate system			
General	Cartesian	Cylindrical	Spherical
x_1	x	x	r
x_2	y	r	θ
x_3	z	φ	φ
v_1	v_x	v_x	v_r
v_2	v_y	v_r	rv_θ
v_3	v_z	rv_φ	$r \sin \theta \, v_\varphi$
D_{11}^Ψ	$\frac{\partial v_x}{\partial x}$	$\frac{\partial v_x}{\partial x}$	$\frac{\partial v_r}{\partial r}$
D_{12}^Ψ	$\frac{\partial v_x}{\partial y}$	$\frac{\partial v_x}{\partial r}$	$\frac{\partial v_r}{\partial \theta}$
D_{13}^Ψ	$\frac{\partial v_x}{\partial z}$	$\frac{\partial v_x}{\partial \varphi}$	$\frac{\partial v_r}{\partial \varphi}$
D_{22}^Ψ	$\frac{\partial v_y}{\partial y}$	$\frac{\partial v_r}{\partial r}$	$\frac{\partial (rv_\theta)}{\partial \theta}$
D_{23}^Ψ	$\frac{\partial v_y}{\partial z}$	$\frac{\partial v_r}{\partial \varphi}$	$\frac{\partial (rv_\theta)}{\partial \varphi}$
D_{33}^Ψ	$\frac{\partial v_z}{\partial z}$	$\frac{\partial (rv_\varphi)}{\partial \varphi}$	$\frac{\partial (r \sin \theta \, v_\varphi)}{\partial \varphi}$
a_{11}	$1-k^2{v_x}^2$	$1-k^2{v_x}^2$	$1-k^2{v_r}^2$

Coordinate system			
General	Cartesian	Cylindrical	Spherical
a_{12}	$-2k^2 v_x v_y$	$-2k^2 v_x v_r$	$\frac{-2k^2 v_r v_\theta}{r}$
a_{13}	$-2k^2 v_x v_z$	$\frac{-2k^2 v_x v_\varphi}{r}$	$\frac{-2k^2 v_r v_\varphi}{r \sin \theta}$
a_{22}	$1-k^2{v_y}^2$	$1-k^2{v_r}^2$	$\frac{1-k^2{v_\theta}^2}{r^2}$
a_{23}	$-2k^2 v_y v_z$	$\frac{-2k^2 v_r v_\varphi}{r}$	$\frac{-2k^2 v_\theta v_\varphi}{r^2 \sin \theta}$
a_{33}	$1-k^2{v_z}^2$	$\frac{1-k^2{v_\varphi}^2}{r^2}$	$\frac{1-k^2{v_\varphi}^2}{r^2 \sin^2 \theta}$
H	0	$\frac{v_r}{r}(1 + k^2{v_\varphi}^2)$	$\frac{v_r}{r}(2 + k^2{v_\theta}^2 + k^2{v_\varphi}^2)$ $+ \frac{v_\theta}{r} \cot \theta \, (1 + k^2{v_\varphi}^2)$
η_2	1	1	$\frac{1}{r}$
η_3	1	$\frac{1}{r}$	$\frac{1}{r \sin \theta}$



(a) Subsonic stream. (b) Supersonic stream.

Figure 1. - Illustration of propagation of initial disturbance in stream moving at subsonic and at supersonic velocities.

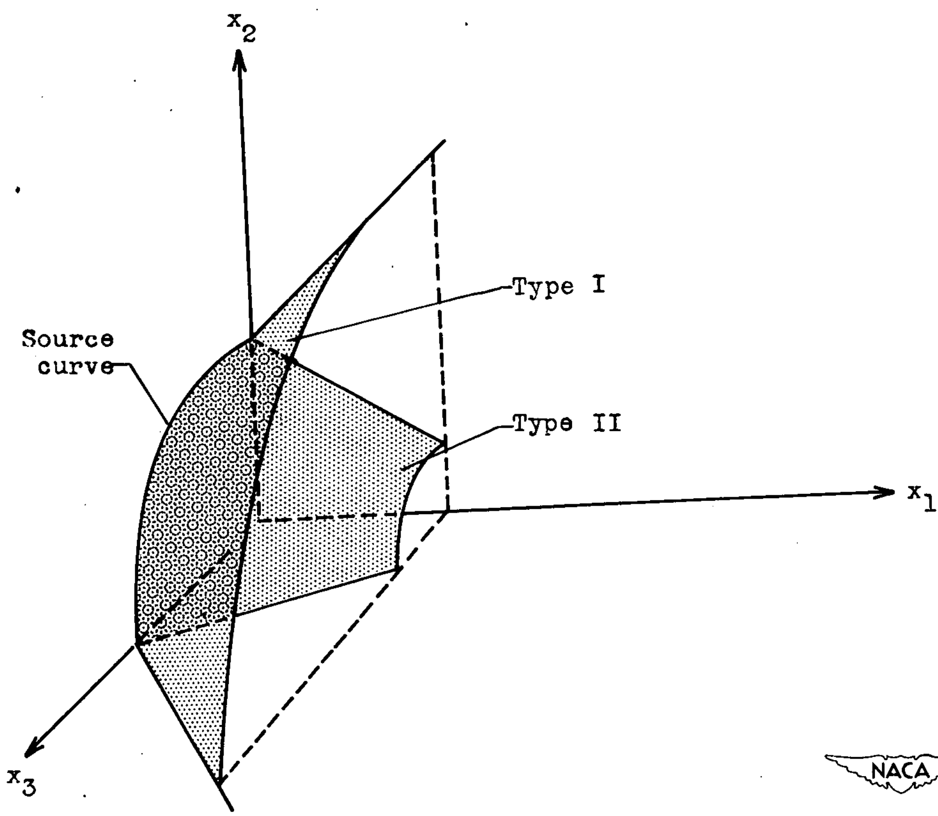
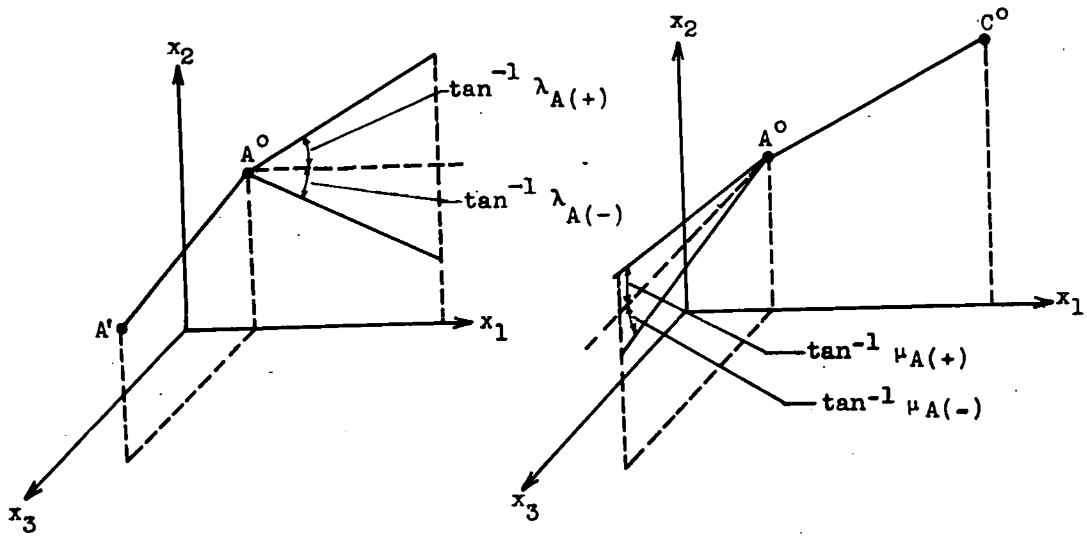


Figure 2. - Illustration of two possible characteristic surfaces originating at source curve.



(a) Velocity components given at points A^o and A'.

(b) Velocity components given at points A^o and C^o.

Figure 3. - Illustration of method of determining characteristic intersections when velocity components at two points are known.

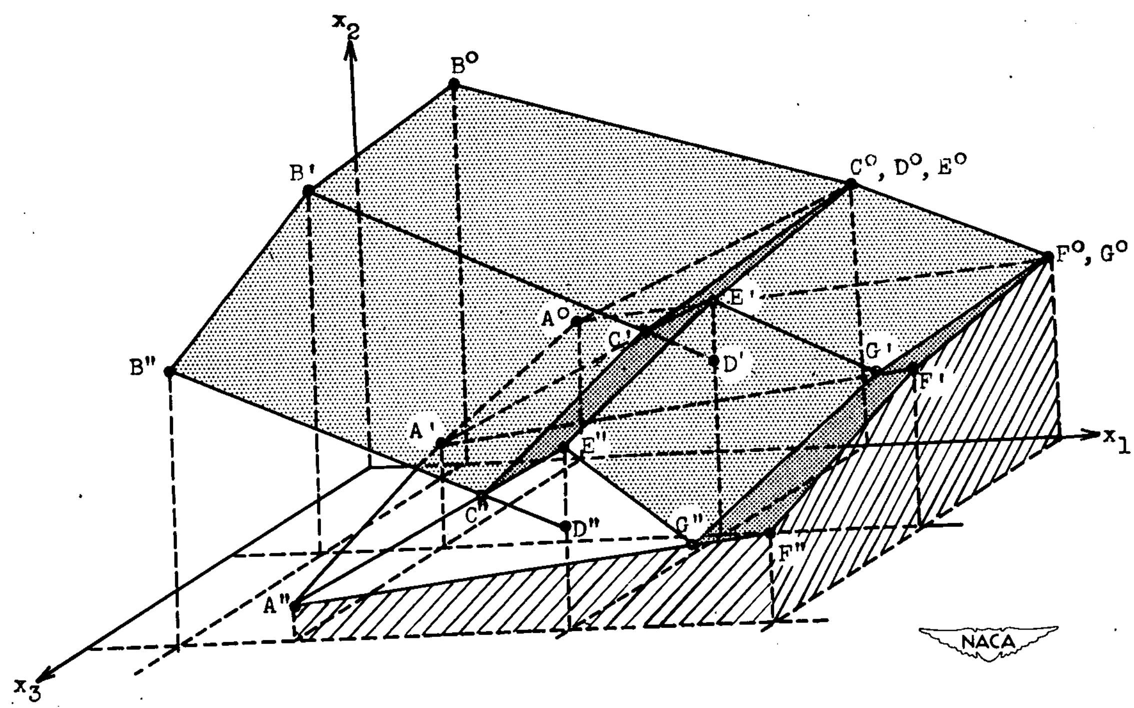


Figure 4. - Network of characteristic intersections near a solid boundary. All surfaces replaced by successive tangent planes.

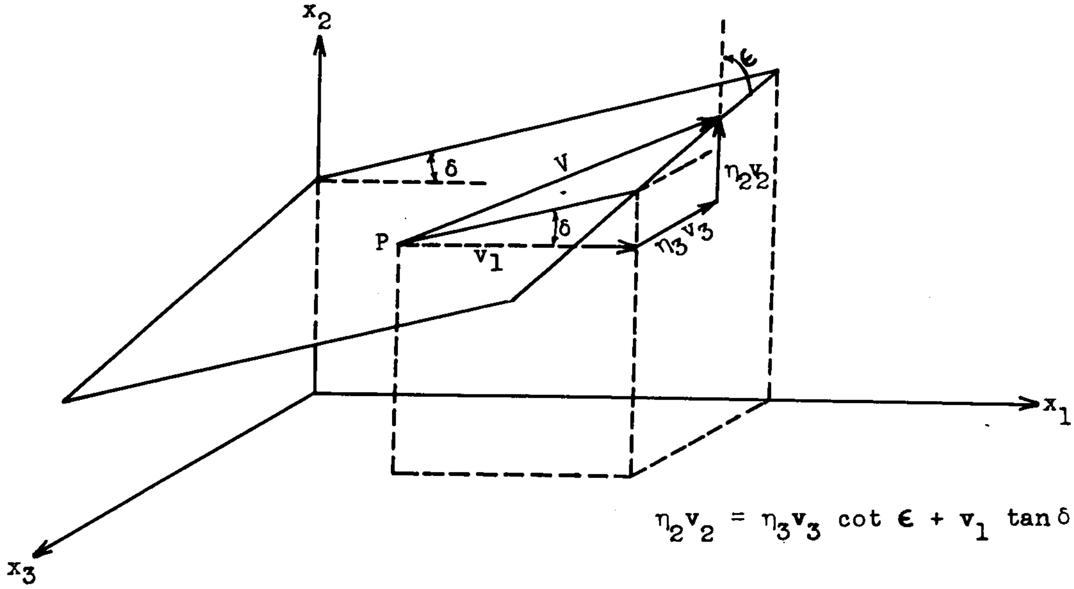


Figure 5. - Illustration of boundary condition on velocity components. Plane tangent to solid boundary at point P.

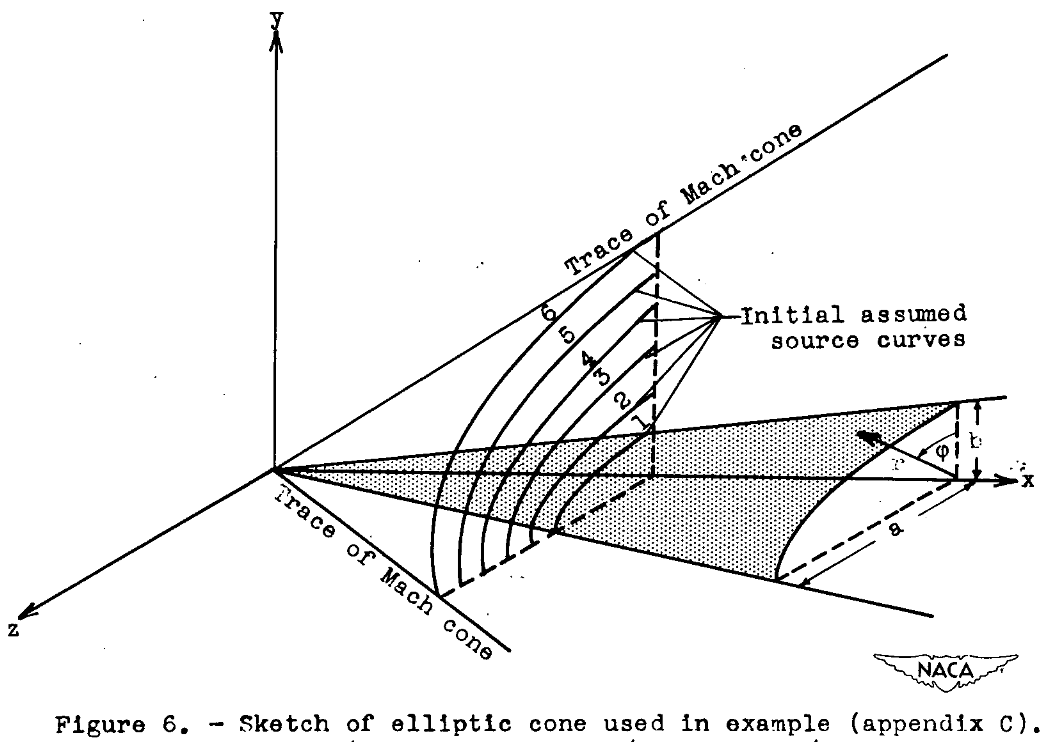
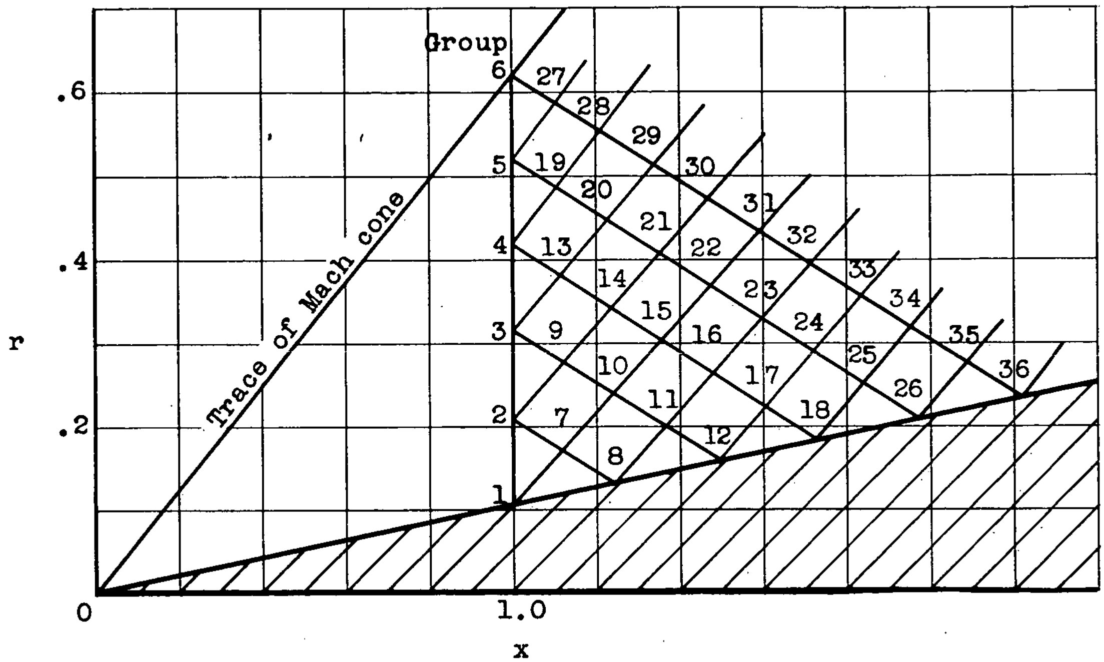
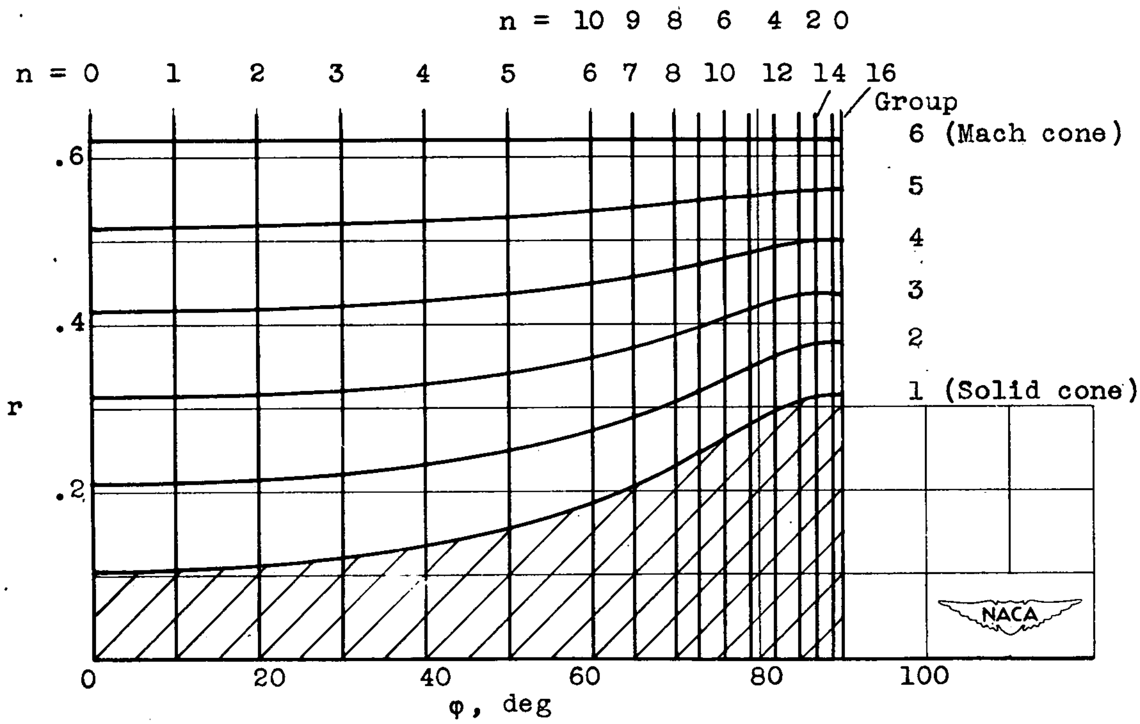


Figure 6. - Sketch of elliptic cone used in example (appendix C). Free-stream Mach number, 1.90; a/b = 3.0; a/x = 0.315.



(a) Plane $\varphi = 0$, showing source curves assumed and to be computed.



(b) Plane $x = 1.0$, showing distribution of points in assumed source curves.

Figure 7. - Illustration of preliminary computation plan.

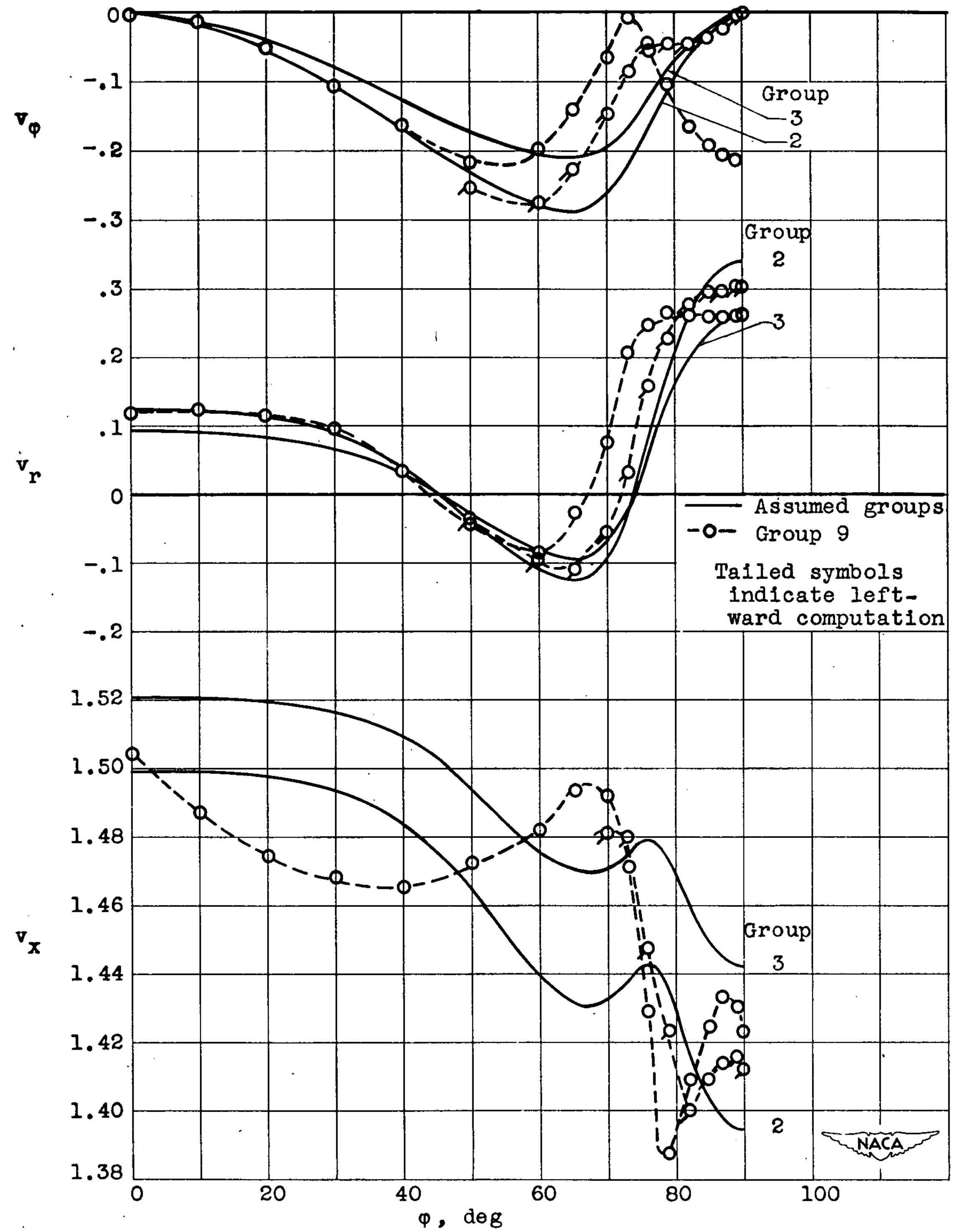


Figure 8. - Comparison of two methods of computing group 9 from assumed values in groups 2 and 3.

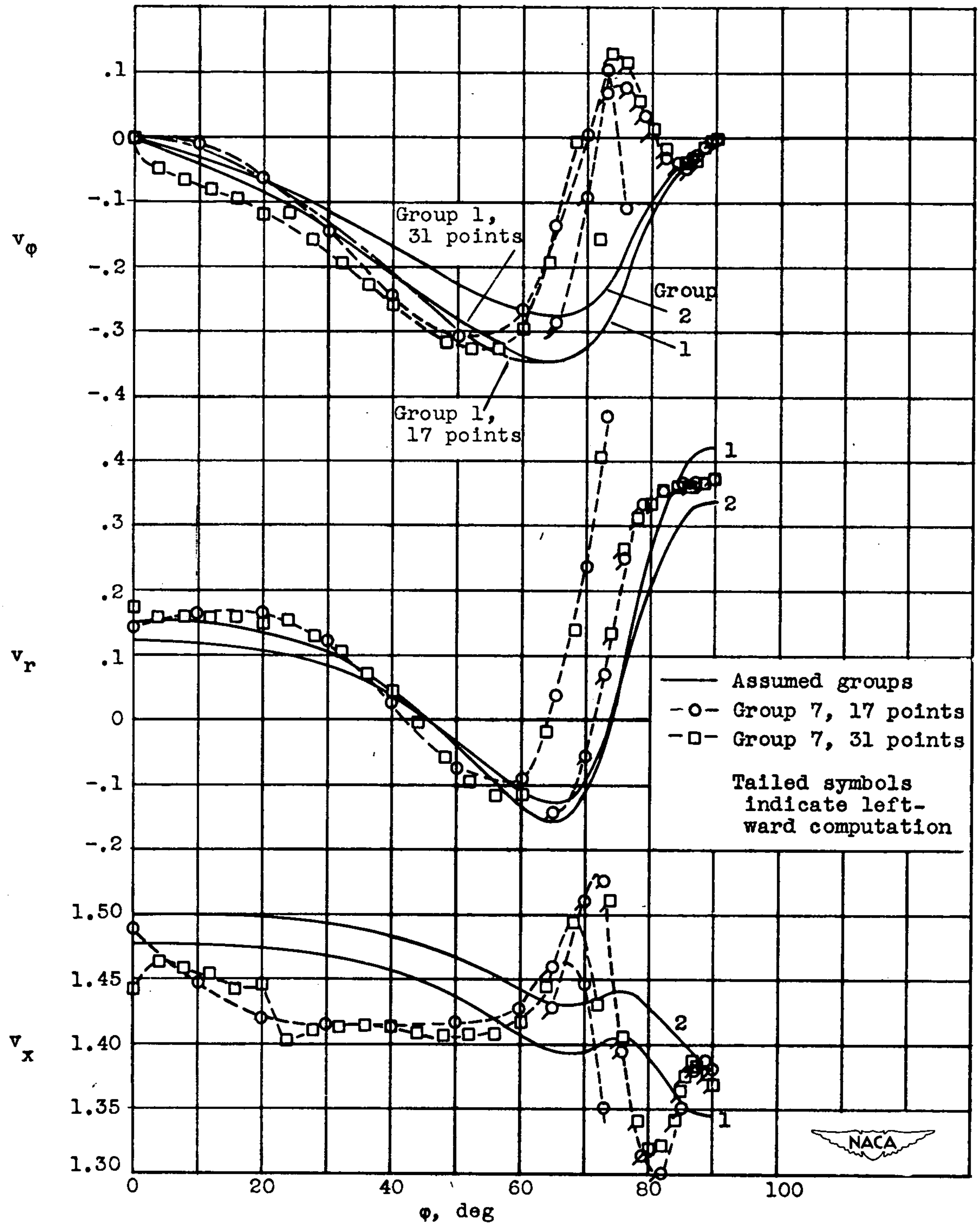


Figure 9. - Computed velocity components for group 7 from assumed values in groups 1 and 2.