NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1852

REINFORCED CIRCULAR CUTOUTS IN PLANE SHEETS

By H. Reissner and M. Morduchow

Polytechnic Institute of Brooklyn

Washington

April 1949

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 1852

REINFORCED CIRCULAR CUTOUTS IN PLANE SHEETS

By H. Reissner and M. Morduchow

SUMMARY

The problem treated here is to design the reinforcement of a cutout in a plane sheet in such a way that it is as nearly as possible equivalent to the part of the structure which has been cut out. A perfect equivalence would mean that the stresses and displacements of the structure remain the same as those which would have appeared without the cutout.

General formulas are developed for the circumferential distribution of the cross-sectional moment of inertia I_r and of the area A_r of a circular reinforcement required for perfect equivalence. These formulas are then applied to some cases of external edge tractions: Hydrostatic stress, pure shear, uniaxial tension, and pure bending. It is found that in the first two cases, the required cross sections are physically possible (i.e., I_r and A_r come out positive), although the required moment of inertia is in some cases found to be quite high in comparison with the required area.

In the cases of uniaxial tension and pure bending, it is shown that constraint stresses, that is, additional stresses in the sheet due to the reinforced cutout, are practically unavoidable. Simple formulas are developed for calculating these "constraint" stresses for any given (constant) cross-sectional characteristics of the reinforcement ring. These formulas are derived on the basis of the assumption that the constraints diminish sufficiently rapidly with radial distance from the cutout so as to have little effect at the external edges of the sheet.

To check the influence which actual boundary conditions might have on the practical validity of these formulas, a test was made on a plane sheet with a reinforced circular cutout subjected to a tensile load causing constant displacements at the loaded edges. It was found that the values of the strains calculated from the exact formulas developed in this report for an infinite sheet were fairly similar to the values of the measured strains in the specimen, except along the loaded edges, where the actual strain decreased more rapidly toward the center of the sheet than the calculated strains. This discrepancy must be due, at least in part, to the actual condition of constant displacements at the loaded edges of finite length instead of, as in the analytical formulas, constant stress at the remote loaded edges of infinite length.

INTRODUCTION

This investigation is concerned with the problem of the reinforcement of cutouts in plane sheets. Such cutouts may serve either to make a structure lighter or to provide space for personnel or accessories. The problem of designing a reinforcement ring around the cutout so that it will be elastically equivalent to the portion of the sheet removed is treated here by an inverse method. The treatment is given first in general terms and then applied in detail for several types of edge loading.

A test was made on a plane sheet with a reinforced cutout to check the effect of some of the simplifying assumptions on the formulas developed.

This investigation was carried out at the Polytechnic Institute of Brooklyn under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics. Grateful acknowledgement is hereby also made to Professor N. J. Hoff and to Dr. B. Boley for giving the authors the benefit of the experience of the Laboratory for Aircraft Structures of the Polytechnic Institute of Brooklyn.

SYMBOLS

A_e	effective area of reinforcement ring with rivet holes
A_r	cross-sectional area of cutout reinforcement ring
a	radius of center line of ring
a_1	radius of outer circumference of ring
b	half width of sheet
c_{mn}	arbitrary constants in stress function
d	radial width of ring
e	distance from edge to neutral fiber of reinforcement ring
E_r	modulus of elasticity of ring material
E_s	modulus of elasticity of sheet material
G	bending moment in cross section of ring
h	height of web

I	moment of inertia; with subscripts e, F, and W, the effective moment of inertia, moment of inertia of the flanges, and moment of inertia of the web, respectively
I_r	cross-sectional moment of inertia of ring
i	dimensionless parameter of moment of inertia $\left(\frac{E_r}{E_s} \frac{I_r}{a^3 t} 24(1 + \nu)\right)$
l	half length of sheet
k_{mn}	abbreviations for products of c_{mn} and powers of a_1
M	bending moment about z-axis
N	radial shear force on cross section of reinforcement ring
n	order of terms of stress function
P	load
$P_n(r)$, $Q_n(r)$	coefficients in stress function (cf. equation (3a))
r	radial distance from center of cutout
r_o	radial distance from center of cutout to neutral fiber
t	thickness of sheet
T	normal stress resultant
u, v	displacements in x- and y-directions, respectively
u_r, u_t	same displacement system in radial and transverse directions, respectively
x, y, z	Cartesian coordinates, measured from center of cutout
Y, Z	tangential and radial loads on ring per unit of circumference
α	dimensionless parameter of area of ring $\left(\frac{E_r}{E_s} \frac{A_r}{at} 2(1 + \nu)\right)$
$\gamma_{r\varphi}$	shear strain
δ	dimensionless parameter of width of ring $(d/2a)$

ϵ_x, ϵ_y	strain in x- and y-directions, respectively
ϵ_r, ϵ_φ	radial and transverse strain, respectively
κ_r	change of curvature of ring
ν	Poisson's ratio
ρ	radius of curvature
σ_x, σ_y, τ_{xy}	normal and shear stresses in sheet (Cartesian coordinates)
σ_r, σ_φ, $\tau_{r\varphi}$	radial, transverse, and shear stresses in sheet (polar coordinates)
τ_o	original shear stress
φ	angular polar coordinate (fig. 1)
ψ	stress (Airy) function

Sub-subscripts:

o	original
1	constraint
T	total

GENERAL EQUATIONS AND GENERAL SOLUTIONS

In order to analyze the effect of a circular cutout in a plane sheet, the stresses, displacements, and strains in such a sheet may be represented in a polar-coordinate system (fig. 1). The original stresses and displacements, before being disturbed by the cutout, will then appear in the form

$$\left.\begin{array}{l} \sigma_{r_o} \\ \\ \sigma_{\varphi_o} \\ \\ \tau_{\varphi r_o} \\ \\ u_{r_o} \\ \\ u_{t_o} \end{array}\right\} \qquad (1)$$

The cutout edges are assumed to carry reinforcement strips or angles. It is the purpose of the analysis of this paper to choose the elastic properties of such an edge reinforcement so that the increments σ_{r_1}, σ_{φ_1}, and $\tau_{r\varphi_1}$ to the original stresses (the original stresses being given by the sheet edge loading) become zero or as small as possible.

Stresses

The stress in a plane sheet can be expressed by a stress function $\psi(r, \varphi)$ such that

$$\left.\begin{aligned} \sigma_r &= r^{-2}\ddot{\psi} + r^{-1}\psi' \\ \sigma_\varphi &= \psi'' \\ \tau_{r\varphi} &= -\left(r^{-1}\dot{\psi}\right)' \end{aligned}\right\} \qquad (2)$$

where the primes and dots are defined by

$$\frac{\partial}{\partial r} \equiv {}'$$

$$\frac{\partial}{\partial \varphi} \equiv \dot{}$$

The function ψ, moreover, must be such that it satisfies the equation

$$\Delta\Delta\psi = 0 \qquad (3)$$

where

$$\Delta\psi \equiv \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}$$

$$\equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2}$$

The complete solution of equation (3) for the case considered here, in which the edge stresses taken around the circumference of the circular cutout have zero resultant, may be written in the form:

$$\psi = \sum_{n=0} P_n \cos n\varphi + \sum_{n=1} Q_n \sin n\varphi \tag{3a}$$

where

$$\left.\begin{aligned}
P_0 &= c_{10} + c_{20}r^2 + c_{30} \log_e r + c_{40}r^2 \log_e r \\
P_1 &= c_{11}r + c_{21}r^3 + c_{31}r^{-1} + c_{41}r \log_e r \\
P_{n\geq 2} &= c_{1n}r^n + c_{2n}r^{-n} + c_{3n}r^{n+2} + c_{4n}r^{-(n-2)}
\end{aligned}\right\} \tag{3b}$$

The expressions for the functions Q are the same, except with possibly different numerical values of the constants c.

From equations (2) and (3a) it follows that

$$\left.\begin{aligned}
\sigma_r &= \sum_{n=0} \left(-n^2r^{-2}P_n + r^{-1}P_n'\right) \cos n\varphi \\
\sigma_\varphi &= \sum_{n=0} P_n'' \cos n\varphi \\
\tau_{r\varphi} &= \sum_{n=0} \left(r^{-1}P_n\right)' n \sin n\varphi
\end{aligned}\right\} \tag{4}$$

The terms in Q may be obtained from equations (4) by exchanging cosine for sine and vice versa and by making the right-hand side minus in the last equation for $\tau_{r\varphi}$.

Displacements

The radial and transverse strains ϵ_r and ϵ_φ in the sheet follow from the stresses by Hooke's law:

$$\left.\begin{aligned}E_s\epsilon_r &= E_s u_r' = \sigma_r - \nu\sigma_\varphi \\ E_s\epsilon_\varphi &= E_s r^{-1}\left(\dot{u}_t + u_r\right) = \sigma_\varphi - \nu\sigma_r\end{aligned}\right\} \qquad (5)$$

where E_s is the modulus of elasticity of the sheet material.

The stress-strain relations of the reinforcement ring can be expressed in the form (see appendix A):

$$\epsilon_r a = \frac{Ta}{E_r A_r} = \dot{u}_t + u_r + \ddot{u}_r \frac{d}{2a} \qquad (6a)$$

$$\kappa_r a^2 = -\frac{Ga^2}{E_r I_r} = \ddot{u}_r + u_r \qquad (6b)$$

where, for the ring, ϵ_r and κ_r are, respectively, the extension strain and curvature, T is the normal stress resultant, G the bending moment (at any φ), d the width of the ring, A_r and I_r the cross-sectional area and moment of inertia of the ring (which may be functions of φ), and u_t and u_r the circumferential and radial displacements of the ring at its outer circumference ($r = a_1$).

Since the displacements of the ring must be the same as those of the sheet where the ring is joined to the sheet, it follows that the displacements in equations (6a) and (6b) can be obtained by putting $r = a_1$ in equations (5) for the displacements in the sheet.

The stress and moment resultants T and G in the ring can be derived from the equations of equilibrium of the ring with the use of the fact that the unit loads acting on the reinforcement ring are due to the radial and shear stresses σ_r and $\tau_{r\varphi}$ in the sheet along the circumference of the cutout ring.

Cross-Sectional Properties in Terms of the Airy Function

Substituting the expressions, as obtained in the preceding discussion (see appendix B), for the displacements and the stress resultants into equations (6a) and (6b) and solving these equations for A_r and I_r yield the following expressions for the cross-sectional area and moment of inertia of the cutout reinforcement ring:

$$A_r = at\frac{E_s}{E_r}\,\frac{\sum\limits_{n=0}\left(P_n' \cos n\varphi\right)_{r=a_1,\,n\neq 1} + \frac{c_2}{t}\sin\varphi + \frac{c_3}{t}\cos\varphi}{\sum\limits_{n=0}\cos n\varphi\left\{n^2\nu\frac{P_n}{a_1} - \nu P_n' + a_1 P_n'' + n^2\frac{d}{2a}\left[\nu P_n' - \frac{P_n}{a_1} + \left(n^2 - 1\right)\int\frac{P_n}{r^2}\,dr\right]\right\}_{r=a_1}} \tag{7a}$$

$$I_r = a^3 t\frac{E_s}{E_r}\,\frac{\sum\limits_{n=0}\cos n\varphi\left[P_n' a_1 - \left(P_n' a\right)_{n\neq 1} - P_n\right]_{r=a_1} a^{-1} - \frac{c_2}{t}\sin\varphi - \frac{c_3}{t}\cos\varphi + \frac{c_1}{at}}{\sum\limits_{n=0}\cos n\varphi\left(n^2 - 1\right)\left[P_n a_1^{-1} - \nu P_n' - \left(n^2 - 1\right)\int P_n r^{-2}\,dr\right]_{r=a_1}} \tag{7b}$$

where c_1, c_2, and c_3 are integration constants. Sine terms are included in equations (7a) and (7b) by merely adding terms exactly similar to the corresponding cosine terms, with P_n replaced by Q_n.

General Possibility of Exact Equivalence or Zero Constraints

Equations (7a) and (7b) have the following significance. Suppose any type of original stress distribution as represented by an Airy function in the general form, equation (3a), is given (i.e., given by the loading at the external edges of the sheet). This means that, for an equivalent cutout reinforcement, P_n and Q_n are given. Then substitution for P_n and Q_n into equations (7a) and (7b) gives the distribution of cross-sectional cutout-ring area and moment of inertia required to give zero constraints, that is, required to replace with elastic equivalence the portion of the sheet cut out. If the values of A_r and I_r thus obtained are positive for any φ, then these are the correct and only values for zero constraints. If, on the other hand, the resulting values of A_r and I_r are negative or infinite for certain values of φ, then an elastically equivalent reinforcement cannot be obtained. This procedure is considered in more detail in the following discussion.

In equations (7a) and (7b), the P's (or Q's) in the general case represent the total stresses, that is, original plus "constraint" stresses, in the sheet. In case the reinforcement ring around the cutout is equivalent to the cutout portion of the sheet, then the constraint stresses will be zero and only the original stress distribution will exist in the sheet. Hence in that case, and in that case only, the P's will represent the original stresses. The original stresses in a sheet due to any loading of the external edges must be finite at all finite values of r including the center $(r = 0)$; hence they can be represented in general by an Airy function of the form given by equation (3a) in which the coefficients P_n (or Q_n) have the form (cf. equations (3b)):

$$\left.\begin{aligned} P_0 &= c_{20}r^2 \\ P_1 &= c_{21}r^3 \\ P_{n\geq 2} &= c_{1n}r^n + c_{3n}r^{n+2} \end{aligned}\right\} \tag{3c}$$

The expressions (3c) can, if desired, be put into equations (7a) and (7b) to give slightly more explicit expressions for A_r and I_r (as functions of φ) in terms of the constants c which depend on the given edge-loading conditions.

SPECIAL CASES

Cases of Zero Constraint Stresses

The problem of the choice of an exactly equivalent reinforcement ring around a circular cutout in a plane sheet is, in principle, completely solved in GENERAL EQUATIONS AND GENERAL SOLUTIONS. It remains to show some of the practical implications of this solution. For this purpose, the special case will first be considered in which the edge loading on the sheet is such that the original stress distribution can be represented by an Airy function with only a single trigonometric term, that is,

$$\psi = P_n \cos n\varphi$$

or

$$\psi = Q_n \sin n\varphi$$

where P_n and Q_n are given by expressions (3c). This class of cases will lead to cross sections constant along the circumference of the ring.

For $n = 0$, equations (7a) and (7b) $\left(c_2 = c_3 = 0 \text{ because of the radial symmetry}\right)$ lead to the result:

$$A_r = \frac{at}{1 - \nu} \frac{E_s}{E_r} \tag{8a}$$

$$I_r = \frac{a^3 t}{2(1 - \nu)} \left(1 - \frac{d}{2a} - \frac{a_1}{atc_{20}} c_1\right) \tag{8b}$$

The fact that the constant c_1 is arbitrary shows that in this case the moment of inertia I_r is arbitrary; but the cross-sectional area must, for zero constraints, have the value given by equation (8a).

For $n = 1$ there results (assuming symmetry with respect to the y-axis and setting, therefore, $c_2 = 0$):

$$A_r = at\frac{E_s}{E_r} \frac{k_3}{k_{21}\left[6 - 2\nu - \frac{d}{2a}(1 - 3\nu)\right]}$$

$$I_r = a^3 t\frac{E_s}{E_r} \frac{\left[2k_{21}\left(1 + \frac{d}{2a}\right) - k_3\right] \cos \varphi + \frac{c_1}{at}}{0}$$

where, for abbreviation,

$$k_3 \equiv \frac{c_3}{t}$$

$$k_{21} \equiv c_{21} a^2$$

In order that I_r be finite, it is necessary that

$$c_1 = 0$$

$$k_3 = 2k_{21}\left(1 + \frac{d}{2a}\right)$$

Whence,

$$A_r = at\frac{E_s}{E_r}\frac{2\left(1 + \frac{d}{2a}\right)}{6 - 2\nu - \frac{d}{2a}(1 - 3\nu)} \tag{9}$$

Again, as for the case $n = 0$, I_r may be chosen as desired, but the area A_r required for zero constraints is fixed by equation (9). For $n \geq 2$, by setting $c_1 = c_2 = c_3 = 0$ to obtain a constant cross section, equations (7a) and (7b) lead to:

$$A_r = at\frac{E_s}{E_r}\frac{Kn + n + 2}{K\left[n\left(n - 1 + \frac{d}{2a}n^2\right)(1 + \nu)\right] + (n + 1)\left[(n - 2)\nu + n + 2\right] + \frac{d}{2a}\left[(n + 2)\nu + n - 2\right]} \tag{10a}$$

$$I_r = a^3t\frac{E_s}{E_r}\frac{1}{n^2 - 1}\frac{K\left[(n - 1)\frac{d}{2a} - 1\right] + \left[(n + 1)\frac{d}{2a} - 1\right]}{-n(1 + \nu)K + \left[2 - n - \nu(2 + n)\right]} \tag{10b}$$

where

$$K \equiv \frac{c_{1n}}{c_{3n}a_1^{\ 2}}$$

and K is prescribed by the original stress distribution. Here both the moment of inertia and the area are fixed if zero constraints are to be achieved.

Two technically important cases of exactly equivalent reinforcements having constant cross sections are included in equations (8a), (8b), (10a), and (10b) and are those of hydrostatic stress and of pure shear. This can be readily seen in the following discussion.

Homogeneous hydrostatic stress $\sigma_r = \sigma_\varphi$, $\tau_{r\varphi} = 0$.- In this case the Airy function, as may be verified by equations (2), is given by equations (3b) as

$$\psi = P_0 = c_{20}r^2$$

$$\sigma_r = \sigma_\varphi = 2c_{20}$$

From equations (7a) and (7b) it follows, as in equations (8a) and (8b), that

$$A_r = \frac{at}{1 - \nu}\frac{E_s}{E_r} \tag{11a}$$

$$I_r = \text{Arbitrary value} \tag{11b}$$

These equations show that the equivalence of the reinforcement is assured for the definite value of the cross section A_r given by equation (11a) but for an arbitrary value of the moment of inertia I_r of the ring. The value of I_r will then, of course, be chosen as small as is compatible with A_r and with buckling considerations.

Homogeneous shear distribution.- This (original) stress distribution can be expressed by $\tau_{xy} = -\tau_o$ in Cartesian coordinates. In polar coordinates,

$$\left.\begin{aligned}\sigma_r &= -\tau_o \sin 2\varphi\\ \sigma_\varphi &= \tau_o \sin 2\varphi\\ \tau_{r\varphi} &= -\tau_o \cos 2\varphi\end{aligned}\right\} \tag{12}$$

The corresponding Airy function, from equations (2), is seen to be

$$\psi = \tau_o \frac{r^2}{2} \sin 2\varphi \tag{12a}$$

signifying that (cf. equations (3c))

$$Q_2 = c_{12}r^2 + c_{32}r^4$$

where

$$c_{12} = \tau_o/2$$

$$c_{32} = 0$$

Using equations (7a) and (7b), but with sine terms instead of cosine terms and with $c_1 = c_2 = c_3 = 0$ to obtain a constant cross section, gives

$$A_r = \frac{at}{(1 + \nu)\left(1 + \frac{4d}{2a}\right)} \frac{E_s}{E_r} \tag{13}$$

$$I_r = \frac{a^3t\left(1 - \frac{d}{2a}\right)}{6(1 + \nu)} \frac{E_s}{E_r} \tag{14}$$

States of Stress with Imperfectly Equivalent Reinforcement Rings

In the previous special cases of one trigonometric term for the Airy function corresponding to the original stress distribution in the sheet, it was seen that a reinforcement around a cutout could, theoretically, be designed so as to produce no additional stresses due to the cutout. The required cross sections were, in fact, constant along the circumference. For most other cases of external edge loading, however, it will be found that equations (7a) and (7b), intended to give full equivalence, will lead to physically impossible (i.e., negative or infinite) values of A_r and I_r for at least some values of φ and that zero constraints are therefore impossible. Moreover, even in cases where zero constraints can be theoretically achieved, it may be found (as in the case of pure shear, discussed further in the following section) that the required geometric properties of the cross sections are in practice not realizable. In such cases it is of interest to determine how small the constraint stresses can be made for (preferably) constant cross sections of a reinforcement ring.

The constraint stresses can be calculated without difficulty by substituting some given constant values of A_r and I_r into equations (7a) and (7b), using expressions (3b) for the P_n (or Q_n), including both the original (given) and constraint (unknown) terms, and then determining the unknown constants c_{2n} and c_{4n} (corresponding to the constraint stresses) so that equations (7a) and (7b) are identically satisfied for any angle φ. The constraint stresses will be calculated here under the assumption that they vanish at infinity. A number of constants then disappear in the expressions (3b). This condition is selected to give practically negligible constraint stresses near the edges of the sheet. Because of this condition the constraint stresses in the sheet as derived in this report must be considered as approximate (they would be exact for an infinite sheet), but the approximation will be good if, as is commonly the case, the constraint stresses diminish sufficiently rapidly with distance from the cutout so as to have a very small or negligible magnitude at the edges of the sheet.

Three special cases of edge loading are now considered in detail.

Pure shear.- It will be found, upon closer examination of expressions (13) and (14), that the moment of inertia of the ring required for an exactly equivalent reinforcement in the case of pure shear is very high if its cross-sectional area be not higher than that required by equation (13). In actual design, therefore, the cross-sectional properties given by equations (13) and (14) cannot ordinarily be realized, and constraint stresses must be allowed. In the following paragraphs the method of calculating these stresses is given in detail.

For pure shear it suffices, except for the addition of terms in r, to employ only the trigonometric term in the series for the Airy stress function ψ which was used in the case of no constraints, namely:

$$\psi = Q_2 \sin 2\varphi$$

where (cf. equations (3b))

$$Q_2 = c_{12}r^2 + c_{22}r^{-2} + c_{42} \qquad (15)$$

and, as before,

$$c_{12} = \frac{\tau_o}{2}$$

The constraints are given by the terms in c_{22} and c_{42}, which are the unknowns. In expressions (3b), the constant c_{32} must be made zero because otherwise it would give constraint stresses which do not decrease with distance from the cutout. Putting equation (15) into equations (7a) and (7b) and introducing for abbreviation

$$\left.\begin{aligned} \delta &\equiv \frac{d}{2a} \\ k_{22} &\equiv c_{22}a_1^{-4} \\ k_{42} &\equiv c_{42}a_1^{-2} \end{aligned}\right\} \tag{15a}$$

the expressions for A_r and I_r become:

$$A_r = \frac{E_s}{E_r}\frac{at}{1 + \nu}\frac{\tau_o - 2k_{22}}{\tau_o(1 + 4\delta) + (6 - 8\delta)k_{22} - \left(\frac{16\delta}{1 + \nu} - \frac{3\nu}{1 + \nu}\right)k_{42}} \tag{16a}$$

$$I_r = \frac{E_s}{E_r}\frac{a^3t}{6(1 + \nu)}\frac{\tau_o(1 - \delta) + (2 + 6\delta)k_{22} + (2 + 2\delta)k_{42}}{\tau_o - 2k_{22} - \frac{4}{1 + \nu}k_{42}} \tag{16b}$$

Let

$$\frac{E_r}{E_s}A_r \equiv \frac{at}{2(1 + \nu)}\alpha$$

$$\frac{E_r}{E_s}I_r \equiv \frac{a^3t}{24(1 + \nu)}i$$

Then solving equations (16a) and (16b) for k_{22} and k_{42} with $\nu = 0.3$ gives

$$\left.\begin{aligned}
k_{22} &= \tau_o \frac{(1 - \delta - 0.25i)(0.462 - 6.15\delta)\alpha - (2 + 2\delta + 0.77i)\left[\alpha(0.5 + 2\delta) - 1\right]}{(2 + 6\delta + 0.5i)(6.15\delta - 0.462)\alpha + (2 + 2\delta + 0.77i)\left[2 + \alpha(3 - 4\delta)\right]} \\
k_{42} &= \tau_o \frac{(2 + 6\delta + 0.5i)\left[\alpha(0.5 + 2\delta) - 1\right] - (1 - \delta - 0.25i)\left[2 + \alpha(3 - 4\delta)\right]}{(2 + 6\delta + 0.5i)(6.15\delta - 0.462)\alpha + (2 + 2\delta + 0.77i)\left[2 + \alpha(3 - 4\delta)\right]}
\end{aligned}\right\} \qquad (17)$$

Substituting the terms in c_{22} and c_{42} of equation (15) into the stress expressions and using the abbreviations of expressions (15a), the expressions for the constraint stresses become:

$$\left.\begin{aligned}
\sigma_{r_1} &= -\left[6k_{22}\left(\frac{r}{a_1}\right)^{-4} + 4k_{42}\left(\frac{r}{a_1}\right)^{-2}\right]\sin 2\varphi \\
\sigma_{\varphi_1} &= 6k_{22}\left(\frac{r}{a_1}\right)^{-4}\sin 2\varphi \\
\tau_{r\varphi_1} &= -\left[6k_{22}\left(\frac{r}{a_1}\right)^{-4} + 2k_{42}\left(\frac{r}{a_1}\right)^{-2}\right]\cos 2\varphi
\end{aligned}\right\} \qquad (18)$$

For a given set of values of A_r and I_r, the constants k_{22} and k_{42} can be evaluated by means of equations (17), and the total stresses in the sheet can then be calculated by determining the constraint stresses as given by equations (18) and adding these to the original stresses, equations (12).

The following table gives the values of k_{22} and k_{42} for several reasonably practical sets of values of thickness, area, and the stiffness ratios δ, α, and i. The entry values for $\delta = 0$ must be understood to be an approximation for a very small width of the ring. The approximation $i = 0$ for very small values of δ introduces only negligible errors.

PURE SHEAR

α	0	1	2	4	8
$\delta = 0;\ i = 0$					
k_{22}/τ_o	0.5	0.161	0.0652	------	-------
k_{42}/τ_o	-1.0	-.661	-.565	------	-------
$\delta = 0.1;\ i = 0.1$					
k_{22}/τ_o	----	0.0505	-0.0682	------	-------
k_{42}/τ_o	----	-.440	-.304	------	-------
$\delta = 0.1;\ i = 0.2$					
k_{22}/τ_o	----	------	-0.0675	-0.154	-------
k_{42}/τ_o	----	------	-.284	-.183	-------
$\delta = 0.1;\ i = 0.4$					
k_{22}/τ_o	----	------	-------	-0.153	-0.226
k_{42}/τ_o	----	------	-------	-.149	-.0884

It will be seen from this table and equations (18) and (12) that in actual design it is difficult to avoid entirely some stress concentration at the cutout $\left(\frac{r}{a_1} = 1\right)$. Without any reinforcement (i.e., for $\delta = i = \alpha = 0$), the stress concentration at the cutout is extremely high, for

$$\sigma_{\varphi_T} = 4\sigma_{\varphi_o}$$

where σ_{φ_T} is the total tangential normal stress with cutout and σ_{φ_o} is the original stress (without cutout). It will be found that reinforcements with practical cross sections, such as those in the foregoing table, will relieve this normal-stress concentration, although they will introduce a smaller shear-stress concentration $\tau_{r\varphi}$. From the point of view of minimum ratios of total stresses to local original stresses, the most suitable cross section given in the foregoing table is $\delta = 0.1$, $i = 0.1$, $\alpha = 1$. The stresses for this section are given in the following table. (Also, see table I.)

PURE SHEAR

$$\left[\delta = 0.1;\ A_r = \frac{at}{2(1 + \nu)};\ I_r = \frac{a^3t}{240(1 + \nu)};\ \nu = 0.3\right]$$

r/a_1	σ_{r_1}/τ_o	$\sigma_{\varphi_1}/\tau_o$	$\tau_{r\varphi_1}/\tau_o$	$\sigma_{r_T}/\sigma_{\varphi_o}$	$\sigma_{\varphi_T}/\sigma_{\varphi_o}$	$\tau_{r\varphi_T}/\tau_{r\varphi_o}$
1.0	1.46	0.303	0.577	-0.46	1.303	1.577
1.25	1.00	.124	.438	.00	1.124	1.438

<u>Uniform single tension or compression</u>.- In Cartesian coordinates, this state of original stress is given by

$$\sigma_{x_o} = \sigma_o$$

$$\sigma_{y_o} = \tau_{xy_o} = 0$$

where σ_o is a constant.

In polar coordinates, the same state of stress is:

$$\left.\begin{aligned}\sigma_{r_o} &= \frac{\sigma_o}{4}(2 - 2\cos 2\varphi)\\ \sigma_{\varphi_o} &= \frac{\sigma_o}{4}(2 + 2\cos 2\varphi)\\ \tau_{r\varphi_o} &= \frac{\sigma_o}{4}\,2\sin 2\varphi\end{aligned}\right\} \tag{19}$$

The Airy function following from equations (2) and (19) has the form

$$\psi = c_{20}r^2 + c_{12}r^2\cos 2\varphi$$

Here, $c_{20} = c_{12} = \frac{\sigma_o}{4}$. Thus,

$$\psi_o = \frac{\sigma_o}{4}r^2(1 + \cos 2\varphi) \tag{20}$$

It is easy to show that in this case, a perfectly equivalent reinforcement ring is not realizable (see appendix C). The Airy function for the final state of stress in the sheet will therefore contain terms representing stresses due to the reinforced cutout, that is, constraint terms, in addition to those corresponding to the original state of stress. Referring to equation (20), it is seen that the Airy function for the original (superscript o) state of stress is of the form

$$\psi_o = P_0^{\,o} + P_2^{\,o}\cos 2\varphi$$

where

$$P_0^{\,o} = c_{20}r^2$$

$$P_2^{\,o} = c_{12}r^2$$

with

$$c_{20} = c_{12} = \frac{\sigma_o}{4}$$

The constraint stresses, which will occur in the sheet with the reinforced cutout, are included by merely adding to the expressions for $P_0^{\,0}$ and $P_2^{\,0}$ the additional terms appearing in expressions (3b). Thus,

$$\psi_T = P_0 + P_2 \cos 2\varphi$$

where

$$\left.\begin{aligned} P_0 &= c_{20}r^2 + c_{30}\log_e r \\ P_2 &= c_{12}r^2 + c_{22}r^{-2} + c_{42} \end{aligned}\right\} \tag{21}$$

in which

$$c_{20} = c_{12} = \frac{\sigma_o}{4}$$

The constants c_{40} and c_{32} in expressions (3b) can and must be set equal to zero, since otherwise they would give constraint stresses which do not diminish with increasing distance r from the cutout. The constant c_{10} has been omitted since it would not contribute anything to the stresses or displacements. In expressions (21), c_{20} and c_{12} are given $\left(\text{equal to } \frac{\sigma_o}{4}\right)$ by the original state of stress, while c_{30}, c_{22}, and c_{42} are the unknowns. These unknowns are determined for a given reinforcement ring by putting expressions (21) into equations (7a) and (7b), which express A_r and I_r, and by satisfying equations (7a) and (7b) in every trigonometric term. With the abbreviations:

$$\left.\begin{aligned}
c_{30}a_1^{-2} &\equiv k_{30} \\
c_{22}a_1^{-4} &\equiv k_{22} \\
c_{42}a_1^{-2} &\equiv k_{42} \\
\frac{d}{2a} &\equiv \delta \\
\frac{I_r}{a^3 t}\frac{E_r}{E_s}6(1 + \nu) &\equiv i_1 \\
\frac{A_r}{at}\frac{E_r}{E_s}(1 + \nu) &\equiv \alpha_1
\end{aligned}\right\} \qquad (22)$$

the following result is obtained (see appendix D):

$$\left.\begin{aligned}
k_{30} &= \frac{\sigma_o}{1 + \alpha_1}\left[\frac{\alpha_1(1 - \nu)}{2(1 + \nu)} - \frac{1}{2}\right] \\
k_{22} &= \sigma_o \frac{\frac{1}{4}\left(i_1 - 1 + \delta\right)\alpha_1\left(\frac{4\nu - 16\delta}{1 + \nu}\right) - \frac{1}{2}\left(1 + \delta + \frac{2i}{1 + \nu}\right)\left[1 - \alpha_1(1 + 4\delta)\right]}{\left(1 + 3\delta + i_1\right)\alpha_1\left(\frac{4\nu - 16}{1 + \nu}\right) - \left(1 + \delta + \frac{2i}{1 + \nu}\right)\left[\alpha_1(6 - 8\delta) + 2\right]} \\
k_{42} &= \sigma_o \frac{\frac{1}{2}\left(1 + 3\delta + i_1\right)\left[1 - \alpha_1(1 + 4\delta)\right] - \frac{1}{4}\left(i_1 - 1 + \delta\right)\left[\alpha_1(6 - 8\delta) + 2\right]}{\left(1 + 3\delta + i_1\right)\alpha_1\left(\frac{4\nu - 16\delta}{1 + \nu}\right) - \left(1 + \delta + \frac{2i}{1 + \nu}\right)\left[\alpha_1(6 - 8\delta) + 2\right]}
\end{aligned}\right\} \qquad (23)$$

Putting expressions (21) into equations (4) and omitting the terms in c_{20} and c_{12}, the expressions for the constraint stresses are seen to be:

$$\left.\begin{aligned}
\sigma_{r_1} &= k_{30}\left(\frac{r}{a_1}\right)^{-2} - \left[6k_{22}\left(\frac{r}{a_1}\right)^{-4} + 4k_{42}\left(\frac{r}{a_1}\right)^{-2}\right]\cos 2\varphi \\
\sigma_{\varphi_1} &= -k_{30}\left(\frac{r}{a_1}\right)^{-2} + 6k_{22}\left(\frac{r}{a_1}\right)^{-4}\cos 2\varphi \\
\tau_{r\varphi_1} &= -\left[6k_{22}\left(\frac{r}{a_1}\right)^{-4} + 2k_{42}\left(\frac{r}{a_1}\right)^{-2}\right]\sin 2\varphi
\end{aligned}\right\} \qquad (24)$$

In order to calculate the stresses in a plane sheet with a reinforced cutout under an external edge loading causing an original state of stress (i.e., state of stress if there were no cutout at all in the sheet) given by $\sigma_x = \sigma_o = \text{Constant}$, it is necessary to calculate δ, i_1, and α_1 in accordance with expressions (22) from the dimensions of the reinforcement ring, determine k_{30}, k_{22}, and k_{42} directly from equations (23), find the constraint-stress distribution from equations (24), and add this to the original stress distribution, equations (19).

The case of uniform tension corresponds approximately to the experiments carried out for this report and, with the analytical calculation of the stresses in the test specimen (appendix G), serves as an illustrative example.

Pure-bending stress.- If an I-beam consisting of a rectangular web and flanges, as shown in figure 1, is subjected to a pure-bending moment M about an axis z perpendicular to the plane of the plate, then the stress distribution in the plate will be:

$$\sigma_{x_o} = -\frac{M}{I_W}y$$

$$\sigma_{y_o} = \tau_{xy_o} = 0$$

where

$$I_W = \frac{th^3}{12} + I_F$$

In polar coordinates this state of stress is given by:

$$\left.\begin{aligned}
\sigma_{r_o} &= \frac{M}{4I_W} r(\cos 3\varphi - \cos \varphi) \\
\sigma_{\varphi_o} &= -\frac{M}{4I_W} r(3 \cos \varphi + \cos 3\varphi) \\
\tau_{r\varphi_o} &= -\frac{M}{4I_W} r(\sin \varphi + \sin 3\varphi)
\end{aligned}\right\} \tag{25}$$

It can be verified by equations (2) that the Airy function for this original state of stress is given by

$$\psi_o = c_{21} r^3 \cos \varphi + c_{13} r^3 \cos 3\varphi$$

where

$$c_{21} = 3c_{13} = -\frac{M}{8I_W}$$

By substitution into equations (7a) and (7b) it is found (see appendix E) that it is not possible to design a reinforcement ring producing exactly zero constraint stresses.

To determine the approximate additional stresses (in this case unavoidable) due to a cutout reinforced by a ring of given dimensions, there must, as in the preceding case, be added to the Airy function the additional terms in expressions (3b) corresponding to the same trigonometric orders as occur in ψ_o. Thus, for the stresses in the sheet under pure bending with a reinforced cutout

$$\left.\begin{aligned}
\psi &= P_1 \cos \varphi + P_3 \cos 3\varphi \\
P_1 &= c_{21} r^3 + c_{31} r^{-1} \\
P_3 &= c_{13} r^3 + c_{23} r^{-3} + c_{43} r^{-1}
\end{aligned}\right\} \tag{26}$$

The constants c_{41} and c_{33} in expressions (3b) have been omitted since they would otherwise violate the assumed condition of constraint stresses decreasing to small values at the edges of the sheet. The constant c_{11} has been omitted since it has no effect on the calculations.

Substituting equations (26) into equations (7a) and (7b), determining the constants c_{31}, c_{23}, and c_{43} (c_{21} and c_{13}, as stated, are fixed by the applied bending moment) so that equations (7a) and (7b) are identically satisfied (see appendix F), and using the abbreviations

$$\left.\begin{aligned}
\frac{c_3}{a_1 t} &\equiv k_3 \\
c_{23} a_1^{-5} &\equiv k_{23} \\
c_{43} a_1^{-3} &\equiv k_{43} \\
c_{31} a_1^{-3} &\equiv k_{31} \\
\frac{Ma_1}{24 I_W} &\equiv K \\
\frac{E_r}{E_s} \frac{A_r}{at} 2(1 + \nu) &\equiv \alpha \\
\frac{E_r}{E_s} \frac{I_r}{a^3 t} 24(1 + \nu) &\equiv i
\end{aligned}\right\} \qquad (27)$$

and the assumption $\nu = 0.3$, the following results are obtained:

$$
\left.
\begin{aligned}
k_3 &= -(1 + \delta)\left(6K + 2k_{31}\right) \\
k_{43} &= -K\frac{(i - 1 + 2\delta)\left[\alpha(2 - 4.5\delta) + 1\right] + (i + 1 + 4\delta)\left[\alpha(1 + 4.5\delta) - 1\right]}{(1.36i + 1 + 2\delta)\left[\alpha(2 - 4.5\delta) + 1\right] - \left[\alpha(0.641 - 6.12\delta) + 0.333\right](i + 1 + 4\delta)} \\
k_{23} &= -K\frac{(i - 1 + 2\delta)\left[\alpha(0.641 - 6.12\delta) + 0.333\right] + \left[\alpha(1 + 4.5\delta) - 1\right](1.36i + 1 + 2\delta)}{(i + 1 + 4\delta)\left[\alpha(0.641 - 6.21\delta) + 0.333\right] - \left[\alpha(2 - 4.5\delta) + 1\right](1.36i + 1 + 2\delta)} \\
k_{31} &= -K\frac{\left[\alpha(2.78 + 40.6\delta) - 3 + 6\delta\right] + k_{23}\left[\alpha(18 - 40.4\delta) + 9\right] + k_{43}\left[\alpha(5.76 - 55\delta) + 3\right]}{\alpha\left(1 - \frac{\delta}{2}\right) + 2 + 2\delta}
\end{aligned}
\right\} \quad (28)
$$

Putting equations (26) into equations (4) and omitting the terms in c_{21} and c_{13} (which give the original stresses), the constraint stresses in the sheet due to the reinforced cutout are found to be:

$$
\begin{aligned}
\sigma_{r_1} &= -2k_{31}\left(\frac{r}{a_1}\right)^{-3}\cos\varphi - \left[12k_{23}\left(\frac{r}{a_1}\right)^{-5} + 10k_{43}\left(\frac{r}{a_1}\right)^{-3}\right]\cos 3\varphi \\
\sigma_{\varphi_1} &= 2k_{31}\left(\frac{r}{a_1}\right)^{-3}\cos\varphi + \left[12k_{23}\left(\frac{r}{a_1}\right)^{-5} + 2k_{43}\left(\frac{r}{a_1}\right)^{-3}\right]\cos 3\varphi \\
\tau_{r\varphi_1} &= 2k_{31}\left(\frac{r}{a_1}\right)^{-3}\sin\varphi + \left[12k_{23}\left(\frac{r}{a_1}\right)^{-5} + 6k_{43}\left(\frac{r}{a_1}\right)^{-3}\right]\sin 3\varphi
\end{aligned}
\quad (29)
$$

With the constants k_{43}, k_{23}, and k_{31} determined from equations (27) and (28), the stresses in the sheet can be readily calculated by adding the constraint stresses, as given by equations (29), to the original stresses, as given by equations (25).

In many actual cases, the reinforcement rings are quite narrow, so that if the approximations $\delta = 0$ and $i = 0$ are made only negligible errors will be introduced. The stresses in the sheet are then functions of only the cross-sectional area of the ring. This is true, of course, for any type of external edge loading on the sheet as well as for a pure-bending load. The following table gives the values of the constants k for pure bending for different values of the area concentrated in such a line reinforcement.

PURE BENDING

$\left[\delta = 0; \ i = 0\right]$

$\alpha \equiv \frac{A_r}{at} 2(1 + \nu)$	0	1/2	1	2
k_{43}/K	3.00	1.857	1.48	1.182
k_{23}/K	-2.00	-.857	-.486	-.182
$2k_{31}/K$	-6.00	-2.31	.426	3.22

Figure 2 shows the stress distribution in the sheet for $\alpha = 1$ and, for comparison, for $\alpha = 0$ (i.e., no reinforcement around the cutout).

Summary of Special Cases

Table I gives a brief summary of the numerical results of the theoretical investigation of the special cases treated here. In this table, W.R. denotes the ratio of the weight of material added (in order to form the reinforcement ring) to the weight of material cut out. Assuming that material of the same density is used for both ring and sheet, this weight ratio is given by

$$\text{W.R.} = \frac{2\left(\frac{A_r}{at} - \frac{d}{a}\right)}{\left(1 - \frac{d}{2a}\right)^2}$$

Table I shows also the ratios, at the cutout $\left(\frac{r}{a_1} = 1\right)$, of maximum total (i.e., original plus constraint) stresses to maximum original stresses. In the column headed "Remarks," the term "satisfactory" means that the cross section concerned has practical dimensions and that at the same time it prevents high stresses in the sheet.

An examination of table I shows that it is theoretically possible to obtain, for the cases of hydrostatic stress, pure shear, single tension (or compression), and pure bending, zero or minimum constraints and minimum total stresses by reinforcement rings. However, except for hydrostatic stress, the required cross sections of such rings would have very high moments of inertia but very small widths and areas. Such sections are extremely difficult to design in practice. The table also shows, on the other hand, that in all cases considered here, not the minimum but at least fairly low total stresses in the sheet can be obtained by using appropriate reinforcement rings of practical dimensions. It appears, moreover, that in several instances, such rings may weigh less than the material removed from the sheet to form the cutout.

TEST ON PLANE SHEET WITH REINFORCED CUTOUT

For the experimental part of this research a test was made on a plane sheet with a reinforced circular cutout under a tensile load, realized by a heavy I-beam transmitting four concentrated loads to the sheet (see figs. 3 and 4). The loads were produced by adjustable jack-screws and measured by calibrated strain gages on the eight connecting links.

Before carrying out the main test, a preliminary test was made with a sheet of the same dimensions but without cutout. The purpose of this preliminary test was to ascertain whether the means of application of the load would produce a uniform tension stress $\left(\sigma_x = \sigma_o\right)$. Two different total-load stages were used, with the loads distributed as evenly as possible over the length of the I-beam. The total loads for the preliminary test were 1475 and 3016 pounds. For the main test (i.e., with cutout), the loads were 3000 and 4020 pounds.

The locations of the strain gages (including rosettes) together with the results of experimental measurements in the sheet without cutout are shown in figures 5 and 6.

From the measured values of the test on the sheet without cutout, it is seen that the axial strain ϵ_x was distributed fairly uniformly throughout the sheet. At the higher load, the maximum deviations from the average measured value were -10 and 6.4 percent while, from the expected value of $\epsilon_x = \frac{4P}{2btE_s} = \frac{3016}{0.052 \times 18.5 \times 10.5 \times 10^6} = 2.98 \times 10^{-4}$, the maximum deviation was 10.4 percent.

Near the loaded edges rosette strain gages were attached to show the influence of the lateral constraint of the riveted joint of sheet and I-beam. The small average magnitude of the transverse strain ϵ_y in the sheet near the loaded edges indicated that in fact the load was transmitted by the heavy I-bars in such a way that the transverse strain ϵ_y (and not the stress σ_y) was practically zero at these edges. If the stress σ_y had been zero at the edges, then the magnitude of ϵ_y would have been much higher, namely $\epsilon_y = -\nu\epsilon_x = -0.3\epsilon_x$.

Having checked this type of original stress (or strain) distribution in the sheet obtained with the particular loading used here (see fig. 3), a circular hole was made in the center of the same sheet specimen and was reinforced by a ring of dimensions given in figure 3. Strain rosettes and simple strain gages were then placed on the sheet in the symmetric positions indicated in figure 7. Two different loads were applied, 3000 and 4020 pounds. Since both loads produced proportional results, only those corresponding to the higher load are given here. The results of the measurements are shown in figure 7.

It may be of interest to compare the strains measured in the actual test specimen with the strains calculated, in accordance with the theory, on the basis of an infinite sheet. Since the boundary conditions of the test piece are obviously different from those of an infinite sheet, the strains may be expected to be different in these two cases. A comparison of the strains for these cases may nevertheless be instructive in indicating the influence which the finite-edge conditions have on the strain (or stress) distribution in the sheet. This comparison is given by the strain diagram in figure 7. Because of the symmetry about both x- and y-axes, the theoretical values for the infinite sheet are given only at the points indicated in the diagram.

From figure 7, it will be observed that, qualitatively, the finite sheet behaves quite similarly to the infinite sheet. For example, in both cases the transverse normal strain $\left(\epsilon_\varphi\right)_{\frac{r}{a_1}=1.38}$ decreases steadily with φ from a maximum at $\varphi = 0^{\circ}$ to a minimum at $\varphi = 90^{\circ}$. Also $\left(\epsilon_x\right)_{x=9.2 \text{ in.}}$ near the rigid bars varies in both cases from a maximum at the edge of the sheet to a minimum in the center. (See graph below strain diagram, fig. 7.) The radial strains were all of a relatively low order of magnitude, but it can be stated that here, also, both the order of magnitude and the variation of ϵ_r corresponding to the calculations for an infinite sheet were similar to those of the test specimen.

The effect of the particular boundary conditions of the test piece appears to be pronounced at two places. The maximum transverse normal strain ϵ_φ (perpendicular to the radius and, at $\varphi = 0$, in the direction of the general tension stress) near the cutout ring $\left(\frac{r}{a_1} = 1.38\right)$ is higher for the test piece than for the infinite sheet, the percentage difference being $\frac{5.06 - 4.07}{5.06} \times 100 = 19.6$ percent. The axial strains near the loaded bars (see graph below strain diagram, fig. 7) decrease more sharply in the actual specimen than in the specimen calculated as an infinite sheet, the local percentage difference at the center being $\frac{3.17 - 2.32}{2.32} \times 100 = 36.6$ percent, although the analytical values were practically equal to the experimental values at the quarter points of the sheet. The difference at the center must be due to the fact that in the actual test piece the rigid bars caused constant axial displacements and zero transverse displacements at the finite loaded edges, whereas in the theoretical work an infinite sheet was treated with constant axial stress at the remote loaded edges.

The maximum strain measured in the sheet was the shearing strain at $\varphi = 45^{\circ}$, and it will be observed that this value $\left(7.66 \times 10^{-4}\right)$ was practically unaffected by the difference in the boundary conditions between the actual and the theoretical specimen, since the experimental and the theoretical values are practically equal.

The calculations for the infinite-sheet specimen, based on the formulas developed in the analysis preceding the experiments, are shown in Appendix G.

It may be remarked that in these calculations account was taken of the rivet holes in the cutout ring. These holes had the effect of reducing the actual cross-sectional area of the ring. To determine the amount of this reduction, a test with the same size and spacing of the rivet holes as in the sheet specimen was carried out.

The ratio of the effective area to the full geometric area was thus found to be 0.80. Therefore in the calculation of the theoretical stresses in the test specimen used in the experiment the cross-sectional area of the cutout ring was taken as 0.80 times the full geometric area.

CONCLUDING DISCUSSION

The problem first considered was what must be the cross section of a ring reinforcing a circular cutout in a plane sheet in order that the stresses in the sheet remain unchanged by the cutout. The general solution to this problem is given by equations (7a) and (7b). In these equations the required distributions of moment of inertia I_r and of cross-section area A_r along the circumference of the reinforcement ring are expressed in terms of the stress function (with coefficients P_n and Q_n as defined by equations (3a) and (3b)) for the original stress distribution, that is, for the stresses in the sheet without cutout or reinforcement rings.

It was found that these expressions in terms of the circumferential angle φ for I_r and A_r lead to physically possible (i.e., positive) values only in a limited number of cases (for example, when the original stress function has no or only one trigonometric term, as in centric symmetry and in pure shear, respectively). In the other cases, which include uniaxial tension and pure bending, constraint stresses, that is, additional stresses due to the reinforced cutout, are unavoidable. A method of calculating such stresses in the sheet for a given (constant) cross section of the reinforcement ring was developed, based on the requirement that the constraint stresses and displacements vanish sufficiently rapidly with increasing distance from the cutout so as to have a negligible influence at the edges of the sheet. The formulas, which are straightforward and convenient to apply, were derived in detail for the cases of pure shear, uniform axial tension (or compression), and pure bending. For example, in the case of uniform axial tension, it is necessary merely to calculate the values of the dimensionless constants i_1 and α_1 (proportional, respectively, to the cross-sectional moment of inertia and the area of the reinforcement ring) from the given data in accordance with the definitions, equations (22), of i_1 and α_1. The values of the constants k_{30}, k_{22}, and k_{42} follow from the elastic properties of the ring and are determined by equations (23). The constraint stresses (which must be added to the original stress distribution $\sigma_x = \sigma_o$ = Constant) then follow readily from equations (24).

In the formulas derived for the cases deemed technically important, it can be seen that for the narrow reinforcement rings commonly used only small errors will be introduced in the numerical calculations if the moment of inertia is taken as zero.[1] The results of the analysis therefore show that the stresses in a plane sheet with a cutout are a function only of the cross-sectional area of the ring reinforcing the cutout, while the moment of inertia is of practically no influence. This means that the ring experiences primarily tensile or compressive stress resultants and not bending moments.

In general the approximation given by the formulas of the report will be closer the smaller the constraint stresses are at the edges of the finite sheet.

In the experimental part, a test was made of a plane sheet with a reinforced cutout subjected to a uniform tensile displacement. The purpose of this experiment was to see how great an error is produced by boundary conditions which differ from those for which the theoretical formulas are exact. In particular, the sheet specimen was (of course) not infinite, while at the loaded edges, the axial displacements, and not the stresses, were constant (with zero transverse displacements there). On the two other opposite edges the normal and the tangential edge tractions were zero.

It was found that qualitatively the strain (and therefore stress) distribution in the sheet specimen was quite similar to that predicted by the theoretical formulas for an infinite specimen with the same reinforced cutout. Quantitatively, the chief effect of the actual boundary conditions seemed to be at the loaded edges where the axial strains decreased more sharply toward the center of the sheet than predicted by the formulas for the infinite sheet.

The transverse strain ϵ_φ at the transverse center line of the sheet near the cutout $\left(\varphi = 0, \frac{r}{a_1} = 1.38\right)$ was actually about 20 percent

---

[1]This, of course, is true only if there is no concentration of the original stress along any radius of the cutout circle, a case which appears for a concentrated load at an external edge but which was not considered in this report.

higher than according to the theory for the infinite sheet. In other respects, however, the quantitative results appeared not to be greatly affected by the particular boundary conditions of the test specimen, since the experimental values of the strains were fairly similar to the analytical values based on an infinite sheet.

Polytechnic Institute of Brooklyn
Brooklyn, N. Y., June 27, 1947

APPENDIX A

STRESS-STRAIN RELATIONS OF REINFORCEMENT RING

The theory of curved beams states that the change of curvature κ_r of a ring due to a bending moment G is

$$\kappa_r = - \frac{G}{E_r I_e}$$

where the effective moment of inertia I_e is given by

$$I_e = r_o \int_{a_1-d}^{a_1} br\, dr - Ar_o^2$$

and where $r = r_o$ denotes the radius of the neutral fiber, while b is the width perpendicular to the plane of bending of a cross section, and r_o, moreover, is given by (cf., for example, reference 1)

$$r_o \equiv a_1 - e$$

$$= \frac{A}{\int_{a_1-d}^{a_1} \frac{b}{r}\, dr}$$

The following table, however, shows that for a cross section of I-shape (for example) with values of d/a up to 0.4, a ring may be treated as a straight beam, so that it is permissible to put

$$r_o = a_1 - \frac{d}{2} \equiv a$$

$$e = \frac{d}{2}$$

$$I_e = I_r$$

COMPARISON BETWEEN CURVED BEAM AND STRAIGHT BEAM

d/a	$a_1 - \frac{d}{2}$ (in.)	r_o (in.)	I_e (in.^{4})	I_r (in.^{4})
0.2	6	5.973049	0.041855	0.04187
.3	6	5.93044	.119638	.124966
.4	6	5.865331	.252761	.260266

It follows then that

$$\kappa_r = -\frac{G}{E_r I_r}$$

as in equation (6b). The extension strain ϵ_r and the curvature κ_r of a ring are given in terms of the radial and tangential displacements u_{r_o} and u_{t_o} at the neutral fiber $r = r_o$ by the following well-known relations:

$$\epsilon_r a = \dot{u}_{t_o} + u_{r_o}$$

$$\kappa_r a^2 = -\left(u_{r_o} + \ddot{u}_{r_o}\right)_{r=a_1}$$

Since the radial displacement u_r of the ring for any given φ will be constant along the width of the ring, it follows that

$$\kappa_r a^2 = -\left(u_r + \ddot{u}_r\right)_{r=r_o} = -\left(u_r + \ddot{u}_r\right)_{r=a_1}$$

The tangential strain, however, will vary along the width as, in fact, can be seen from figure 8,

$$\dot{u}_{t_o} = \left(\dot{u}_t\right)_{r=a_1} + \ddot{u}_r \frac{d}{2a}$$

Hence

$$\epsilon_r a = \left(\dot{u}_t + u_r + \ddot{u}_r \frac{d}{2a}\right)_{r=a_1}$$

Thus, equations (6a) and (6b) follow.

APPENDIX B

EXPLICIT EXPRESSIONS FOR STRESS RESULTANTS AND STRAINS OF REINFORCEMENT RING

From figure 1 the following equilibrium conditions between the stress resultants T, G, and N in a ring and the unit load forces Y and Z can be derived:

$$\dot{T} + N = -Ya_1 = -\left(\tau_{r\varphi}\right)_{r=a_1} a_1 t \tag{B1}$$

$$\dot{N} - T = -Za_1 = -\left(\sigma_r\right)_{r=a_1} a_1 t \tag{B2}$$

$$\dot{G} - Nr_o = -Ya_1 e = -\left(\tau_{r\varphi}\right)_{r=a_1} a_1 te$$

Noting that $r_o \approx a$, $e \approx \frac{d}{2}$ (see appendix A), the last equation can be written as

$$\dot{G} - Na = -\tau_{r\varphi} a_1 t \frac{d}{2} \tag{B3}$$

Equations (B1) to (B3) can be solved for T and G as follows:
From equations (B1) and (B2),

$$\ddot{T} + T = a_1 t\left(\sigma_r - \dot{\tau}_{r\varphi}\right)_{r=a_1} \tag{B4}$$

Moreover, from (B1) and (B3),

$$G = -Ta - a_1^2 t \int \left(\tau_{r\varphi}\right)_{r=a_1} d\varphi + c_1 \tag{B5}$$

By putting σ_r and $\tau_{r\varphi}$ in terms of the stress function ψ (relations (2)) equation (B4) can be directly integrated with the results:

for $n \neq 1$,

$$T = a_1 t \left(r^{-1}\psi'\right)_{r=a_1} + c_2 \sin \varphi + c_3 \cos \varphi$$

$$G = a_1 t \frac{d}{2}\left(r^{-1}\psi'\right)_{r=a_1} - a_1^2 t \left(r^{-2}\psi\right)_{r=a_1} + c_1 - c_2 a \sin \varphi - c_3 a \cos \varphi$$

for $n = 1$,

$$T_1 = c_2 \sin \varphi + c_3 \cos \varphi$$

$$G_1 = -c_2 a \sin \varphi - c_3 a \cos \varphi + c_1 + a_1^2 t \left(r^{-1}\psi' - r^{-2}\psi\right)_{r=a_1}$$

Substitution for ψ by means of relations (3) will give expressions for T and G in terms of P_n and φ.

Expressions in terms of P_n for the extension strain and curvature can be obtained by putting equations (2) into equations (5) and integrating the latter to find u_r and u_t.[2] Thus,

---

[2]Two arbitrary functions $f_1(r)$ and $f_2(\varphi)$ will appear as a result of this integration. However, it can be shown that, for compatibility between deformation values (involving also the shear strain $\gamma_{r\varphi}$) and the equilibrium conditions, the functions must have forms which are already included in the general expressions (3a) and (3b) for the Airy function.

$$E_s\left(u_r + \ddot{u}_r\right) = \int \frac{\ddddot{\psi} + 2\ddot{\psi} + \psi}{r^2}\, dr + \frac{1}{r}\left(\ddot{\psi} + \psi\right) - \nu\left(\ddot{\psi} + \psi\right)'$$

$$E_s\left(u_r + \dot{u}_t + \ddot{u}_r \frac{d}{2a}\right) = \frac{d}{2a}\int \frac{\ddddot{\psi} + \ddot{\psi}}{r^2}\, dr + r\psi'' - \frac{\ddot{\psi}}{r}\left(\nu - \frac{d}{2a}\right)$$

$$- \nu\left(\frac{\psi}{r} + \frac{d}{2a}\,\psi'\right)^{\cdot\cdot} - \nu\psi'$$

By using the expression (3a) for ψ, the strains in terms of P_n and φ are obtained.

APPENDIX C

IMPOSSIBILITY OF ZERO CONSTRAINTS IN CASE OF UNIFORM TENSION

$$\left(\sigma_x = \text{Constant}\right)$$

From the expression (20) for ψ_o,

$$P_0 = \frac{\sigma_o}{4} r^2$$

$$P_2 = \frac{\sigma_o}{4} r^2$$

Substituting these values into (7a) and (7b), putting $c_2 = c_3 = 0$ (for symmetry about both x- and y-axes), and simplifying give the following expressions for the distribution of cross-sectional area and moment of inertia:

$$A_r = at \frac{E_s}{E_r} \frac{1 + \cos 2\varphi}{(1 - \nu) + \left[1 + \nu + 4\delta(1 + \nu)\right] \cos 2\varphi}$$

$$I_r = a^3 t \frac{E_s}{E_r} \frac{(1 - \delta)(1 + \cos 2\varphi) - \dfrac{c_1}{a_1 at}}{2(1 - \nu) + 6(1 + \nu) \cos 2\varphi}$$

where $\delta \equiv d/2a$.

It is not difficult to see that I_r can be made positive for all values of φ (by choosing a proper value for c_1) but that A_r necessarily becomes negative and infinite for some values of φ. Hence a perfectly equivalent reinforcement is not realizable in this case.

APPENDIX D

DERIVATION OF EQUATIONS (23) PERTAINING TO AN ORIGINAL STATE OF UNIFORM TENSION

Putting equations (21) into equations (7a) and (7b) with the abbreviations (22), the following equations result:

$$\alpha_1 = \frac{\left(\frac{\sigma_o}{2} + k_{30}\right) + \left(\frac{\sigma_o}{2} - 2k_{22}\right)\cos 2\varphi}{\left[\frac{\sigma_o(1 - \nu)}{2(1 + \nu)} - k_{30}\right] + \left[\frac{\sigma_o}{2} + 6k_{22} + \frac{4\nu}{1 + \nu}k_{42} + \delta\left(2\sigma_o - 8k_{22} - \frac{16}{1 + \nu}k_{42}\right)\right]\cos 2\varphi}$$

$$i_1 = \frac{K_1 + \left[\frac{\sigma_o}{4}(1 - \delta) + k_{22} + k_{42} + \delta\left(3k_{22} + k_{42}\right)\right]\cos 2\varphi}{\left[\frac{1 - \nu}{12(1 + \nu)}\sigma_o - \frac{k_{30}}{6}\right] + \left[\frac{\sigma_o}{4} - k_{22} - \frac{2}{1 + \nu}k_{42}\right]\cos 2\varphi}$$

where

$$K_1 \equiv -k_1 a_1^{-1} + \frac{\sigma_o}{4}(1 - \delta) + k_{30}\left[(1 + \delta)\log a_1 - \delta\right]$$

Clearing each of the preceding two equations of fractions and equating the coefficients of equal trigonometric orders (viz, constant terms and terms with cos 2φ), the following four relations are obtained:

$$\frac{\sigma_o}{2} + k_{30} = \alpha_1 \left[\frac{\sigma_o(1 - \nu)}{2(1 + \nu)} - k_{30}\right]$$

$$\frac{\sigma_o}{2} - 2k_{22} = \alpha_1 \left[\sigma_o\left(\frac{1}{2} + 2\delta\right) + k_{22}(6 - 8\delta) + k_{42}\left(\frac{4\nu - 16\delta}{1 + \nu}\right)\right]$$

$$K_1 = i_1 \left[\frac{1 - \nu}{12(1 + \nu)}\sigma_o - \frac{k_{30}}{6}\right]$$

$$\frac{\sigma_o}{4}(1 - \delta) + k_{22}(1 + 3\delta) + k_{42}(1 + \delta) = i_1\left(\frac{\sigma_o}{4} - k_{22} - \frac{2}{1 + \nu}k_{42}\right)$$

The first of these four equations can be solved immediately for k_{30}. The third equation determines the value of K_1, which, however, has no influence on the stresses in the sheet. Hence this equation need not be considered any further here. The second and fourth equations can be solved simultaneously for k_{22} and k_{42}. The results are given by equations (23).

APPENDIX E

PRACTICAL IMPOSSIBILITY OF PERFECT EQUIVALENCE IN CASE OF PURE BENDING

From the expression for ψ_o, it is seen that, corresponding to the (original) state of stress due to pure bending,

$$P_1 = c_{21}r^3$$

$$P_3 = c_{13}r^3$$

$$c_{21} = 3c_{13} = -\frac{M}{8I_W}$$

Substitution into equations (7a) and (7b) leads to the following expressions for A_r and I_r $\left(c_1 = c_2 = 0\right.$, because of the antisymmetry about the x-axis and the symmetry about the y-axis$\left.\right)$:

$$A_r = \frac{at}{(2 + 9\delta)(1 + \nu)} \frac{\dfrac{c_3}{3a_1^{\,2}tc_{13}} \cos \varphi + \cos 3\varphi}{\left[\dfrac{6 - 2\nu + \delta(3\nu - 1)}{(2 + 9\delta)(1 + \nu)}\right] \cos \varphi + \cos 3\varphi}$$

$$I_r = \frac{a^3t}{24(1 + \nu)}(1 - 2\delta) \frac{\left\{\left[\dfrac{c_3}{a_1^{\,2}tc_{13}} - 6(1 + \delta)\right] \Big/ (1 - 2\delta)\right\} \cos \varphi + \cos 3\varphi}{\cos 3\varphi}$$

It will be shown now that A_r and I_r for zero constraints can both be made positive for any φ only if the value of the ratio δ (identical with $d/2a$) is extremely low, too low in fact for practical purposes.

From the second of the two equations, in order that I_r be everywhere positive it is necessary that

$$\frac{c_3}{a_1^2 t c_{13}} = 6(1 + \delta)$$

Substituting, then, for c_3 into the expression for the area A_r gives

$$A_r = \frac{at}{(2 + 9\delta)(1 + \nu)} \frac{2(1 + \delta) \cos \varphi + \cos 3\varphi}{\left[\frac{6 - 2\nu + \delta(3\nu - 1)}{(1 + \nu)(2 + 9\delta)}\right] \cos \varphi + \cos 3\varphi}$$

The value of this expression for A_r will be positive regardless of φ if and only if

$$2(1 + \delta) = \frac{6 - 2\nu + \delta(3\nu - 1)}{(1 + \nu)(2 + 9\delta)}$$

With $\nu = 0.3$, the positive root of this quadratic equation in δ is

$$\delta = 0.00684$$

In practical design, this low value for δ (identical with $d/2a$, fig. 1) will be incompatible with the large required moment of inertia

$$I_r = \frac{a^3 t}{24(1 + \nu)}(1 - 2\delta).$$

APPENDIX F

DERIVATION OF EQUATIONS (28) PERTAINING TO CASE OF PURE BENDING

Substitution of the expressions (26) for P_1 and P_3 into equations (7a) and (7b), with the abbreviations (27), leads to the following results:

$$i = \frac{\left[-k_3 - (1 + \delta)\left(6K + 2k_{31}\right)\right] \cos \varphi + \left[K(1 - 2\delta) - k_{23}(1 + 4\delta) - k_{43}(1 + 2\delta)\right] \cos 3\varphi}{\left[K + k_{23} + \frac{5 + \nu}{3(1 + \nu)} k_{43}\right] \cos 3\varphi}$$

$$\alpha = \frac{k_3 \cos \varphi - \left[3K + 3k_{23} + k_{43}\right] \cos 3\varphi}{\left\{K\left[-\frac{3(3 - \nu)}{1 + \nu} + \delta\frac{3 - 9\nu}{2(1 + \nu)}\right] + k_{31}\left(1 - \frac{\delta}{2}\right)\right\} \cos \varphi + \left[-K\left(3 + \frac{27}{2}\delta\right) + k_{23}\left(6 - \frac{27}{2}\delta\right) + k_{43}\frac{2 + 10\nu - 9\delta(5 + \nu)}{2(1 + \nu)}\right] \cos 3\varphi}$$

Clearing each of these two equations of fractions and equating coefficients of like trigonometric terms ($\cos \varphi$ and $\cos 3\varphi$) give four simultaneous equations:

$$-k_3 - (1 + \delta)\left(6K + 2k_{31}\right) = 0$$

$$K(1 - 2\delta) - k_{23}(1 + 4\delta) - k_{43}(1 + 2\delta) = i\left[K + k_{23} + \frac{5 + \nu}{3(1 + \nu)} k_{43}\right]$$

$$k_3 = \alpha\left\{K\left[-\frac{3(3 - \nu)}{1 + \nu} + \delta\frac{3 - 9\nu}{2(1 + \nu)}\right] + k_{31}\left(1 - \frac{\delta}{2}\right)\right\}$$

$$-\left(3K + 3k_{23} + k_{43}\right) = \alpha\left[-K\left(3 + \frac{27}{2}\delta\right) + k_{23}\left(6 - \frac{27}{2}\delta\right) + k_{43}\frac{2 + 10\nu - 9\delta(5 + \nu)}{2(1 + \nu)}\right]$$

The second and fourth equations are linear equations in k_{23} and k_{43}, which can therefore be readily solved for these two unknowns. By eliminating k_3 from the first and third equations, k_{31} can be solved for in terms of k_{23} and k_{43}. The results are given by equations (28).

APPENDIX G

CALCULATION OF STRAINS IN INFINITE-SHEET SPECIMEN WITH SAME REINFORCED CUTOUT AS ACTUAL SPECIMEN

The dimensions of the specimen are given in figure 3. From the dimensions,

$$a = 3.66 \text{ inches}$$

$$a_1 = 3.5 + \frac{5}{16} = 3.81 \text{ inches}$$

$$t = 0.052 \text{ inch}$$

$$\delta \equiv \frac{d}{2a} = \frac{\frac{5}{16}}{2 \times 3.66} = 0.0426$$

$$A_e = 0.80\left(\frac{5}{16} \times 0.434\right) = 0.1085 \text{ square inch}$$

$$\frac{A_e(1 + \nu)}{at} = \frac{0.1085 \times 1.3}{3.66 \times 0.052} = 0.74$$

$$I_r = \frac{\left(\frac{5}{16}\right)^3 \times 0.434}{12} = 1.1 \times 10^{-3} \text{ inch}^4$$

$$i_1 \equiv 6(1 + \nu)\frac{I_r}{a^3 t} = \frac{7.8 \times 1.1 \times 10^{-3}}{(3.66)^3 \times 0.052} = 3.37 \times 10^{-3}$$

From equations (23) substitution of the preceding values leads to the following numerical results for the constants k_{30}, k_{22}, and k_{42}:

$$k_{30} = -0.173\sigma_o$$

$$k_{22} = 0.0228\sigma_o$$

$$k_{42} = -0.252\sigma_o$$

From equations (24) it then follows that the calculated constraint stresses in the sheet will be:

$$\frac{\sigma_{r_1}}{\sigma_o} = -0.173\left(\frac{r}{a_1}\right)^{-2} - \left[0.1368\left(\frac{r}{a_1}\right)^{-4} - 1.008\left(\frac{r}{a_1}\right)^{-2}\right] \cos 2\varphi$$

$$\frac{\sigma_{\varphi_1}}{\sigma_o} = 0.173\left(\frac{r}{a_1}\right)^{-2} + 0.1368\left(\frac{r}{a_1}\right)^{-4} \cos 2\varphi$$

$$\frac{\tau_{r\varphi_1}}{\sigma_o} = -\left[0.1368\left(\frac{r}{a_1}\right)^{-4} - 0.504\left(\frac{r}{a_1}\right)^{-2}\right] \sin 2\varphi$$

The calculated stresses without cutout (see equations (19)) are:

$$\sigma_{r_o} = \frac{\sigma_o}{2}(1 - \cos 2\varphi)$$

$$\sigma_{\varphi_o} = \frac{\sigma_o}{2}(1 + \cos 2\varphi)$$

$$\tau_{r\varphi_o} = \frac{\sigma_o}{2} \sin 2\varphi$$

Hence, adding the constraint to the original stresses, the total stresses in the sheet at any point (r, φ) are found to be:

$$\sigma_r = \sigma_o \left\{0.500 - 0.173\left(\frac{r}{a_1}\right)^{-2} - \left[0.500 + 0.1368\left(\frac{r}{a_1}\right)^{-4} - 1.008\left(\frac{r}{a_1}\right)^{-2}\right] \cos 2\varphi\right\}$$

$$\sigma_\varphi = \sigma_o \left\{0.500 + 0.173\left(\frac{r}{a_1}\right)^{-2} + \left[0.500 + 0.1368\left(\frac{r}{a_1}\right)^{-4}\right] \cos 2\varphi\right\}$$

$$\tau_{r\varphi} = \sigma_o \left[0.500 + 0.504\left(\frac{r}{a_1}\right)^{-2} - 0.1368\left(\frac{r}{a_1}\right)^{-4}\right] \sin 2\varphi$$

From these stresses, the strains expressed in polar coordinates at any point are:

$$\epsilon_r = \frac{1}{E_s}\left(\sigma_r - \nu\sigma_\varphi\right)$$

$$\epsilon_\varphi = \frac{1}{E_s}\left(\sigma_\varphi - \nu\sigma_r\right)$$

$$\gamma_{r\varphi} = \frac{2(1 + \nu)}{E_s}\tau_{r\varphi}$$

The strains ϵ_x in the direction of the load were obtained by first determining the Cartesian stresses σ_x and σ_y according to the relations:

$$\sigma_x = \frac{1}{2}\left[\left(\sigma_r + \sigma_\varphi\right) - \left(\sigma_r - \sigma_\varphi\right)\cos 2\varphi + 2\tau_{r\varphi}\sin 2\varphi\right]$$

$$\sigma_y = \frac{1}{2}\left[\left(\sigma_r + \sigma_\varphi\right) + \left(\sigma_r - \sigma_\varphi\right)\cos 2\varphi - 2\tau_{r\varphi}\sin 2\varphi\right]$$

The strain ϵ_x is then readily obtained by

$$\epsilon_x = \frac{1}{E_s}\left(\sigma_x - \nu\sigma_y\right)$$

The value of the original stress σ_o without cutout is

$$\sigma_o = \frac{4P}{2bt}$$

where $2b$ is the width of the sheet. For the test specimen and for a load of 4020 pounds,

$$\sigma_o = \frac{4020}{18.5 \times 0.052} = 4160 \text{ pounds per square inch}$$

For the strain this can be expressed as:

$$\epsilon_{x_o} = \frac{4160}{10.5 \times 10^6 \times 4.02} = 0.99 \times 10^{-4}$$

The actual measurement gave 1.04×10^{-4}, that is, an average inaccuracy of 5 percent in the strain gages.

The numerical results of the preceding calculations are given in the data diagram (fig. 7).

REFERENCE

1. Timoshenko, S.: Strength of Materials. Part II - Advanced Theory and Problems. Second ed., D. Van Nostrand Co., Inc., 1941, ch. II.

TABLE I

SUMMARY OF THEORETICAL INVESTIGATION OF SPECIAL CASES

Loading	$\frac{d}{2a}$	$\frac{A_r}{at}$	$\frac{I_r}{a^3t}$	W.R. (1)	Ratios of maximum total stresses to maximum original stresses at $r/a_1 = 1$				Remarks (3)
					$\frac{\sigma_r}{\sigma_{r_o}}$	$\frac{\sigma_\varphi}{\sigma_{\varphi_o}}$	$\frac{\tau_{r\varphi}}{\tau_{r\varphi_o}}$	$\frac{\sigma_\varphi}{\sigma_{\varphi_o}}$ (2)	
Hydrostatic σ_r = Constant	0	1.43	Any	2.86	1.00	1.00	1.00	2	Zero constraints; satisfactory
	.1	1.43	Any	3.04	1.00	1.00	1.00		
	.2	1.43	Any	3.21	1.00	1.00	1.00		
Pure shear τ_{xy} = Constant	.2	.428	0.1025	.0875	1.0	1.0	1.0	4	Zero constraints, but I_r too high to conform with A_r and d/a
	.1	.385	.0032	.457	-.46	1.30	1.58		Satisfactory
	.1	.550	.1155	.865	1.0	1.0	1.0		Zero constraints, but I_r too high to conform with A_r and d/a
Single uniform tensile (or compressive) stress σ_x = Constant	.1	.882	.115	1.69	.822	.872	1.31	3	Low constraints, but I_r too high to conform with A_r and d/a
	.2	.795	.1025	1.232	.783	.790	1.43		Low constraints, but I_r too high to conform with A_r and d/a
	0	.615	0	1.23	.750	1.44	1.63		Satisfactory
	.1	.461	0	.682	.512	1.34	1.59		Satisfactory
Pure bending	0	.385	.0320	.770	1.015	.995	1.015	2	Almost zero constraints, but I_r too high to conform with A_r and d/a
	0	.385	0	.770	1.01	1.10	1.34		Satisfactory

[1]W.R. ratio of the weight of material added (to form reinforcement ring) to the weight of material cut out.

[2]No reinforcement.

[3]"Satisfactory" means that the cross section has practical dimensions and that at the same time it prevents high stresses in the sheet. "Zero constraints" means that the reinforcement ring is exactly equivalent to the portion of the sheet which has been cut out.

NACA

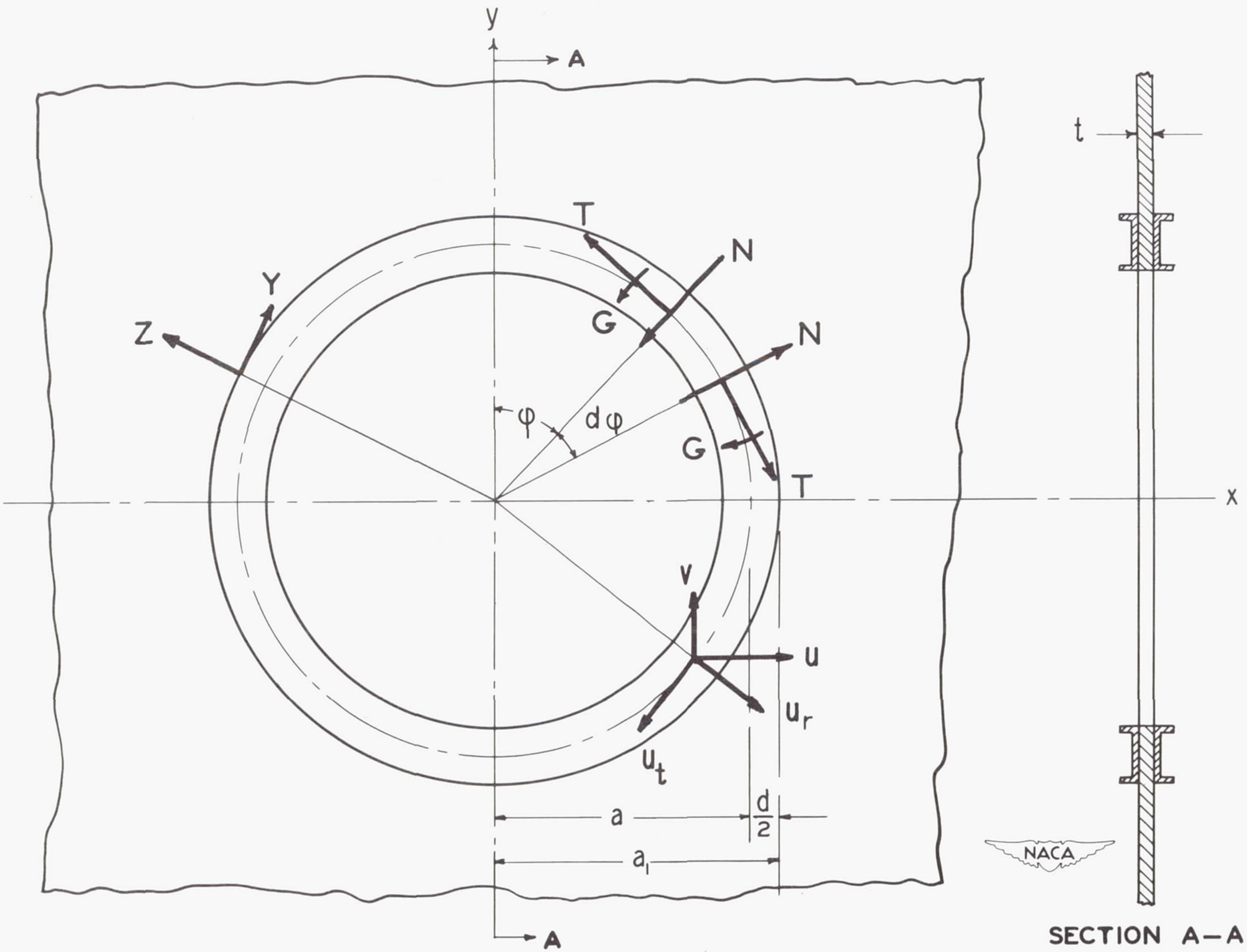

Figure 1.- Stress resultants and displacements in a reinforcement ring represented in a polar-coordinate system.

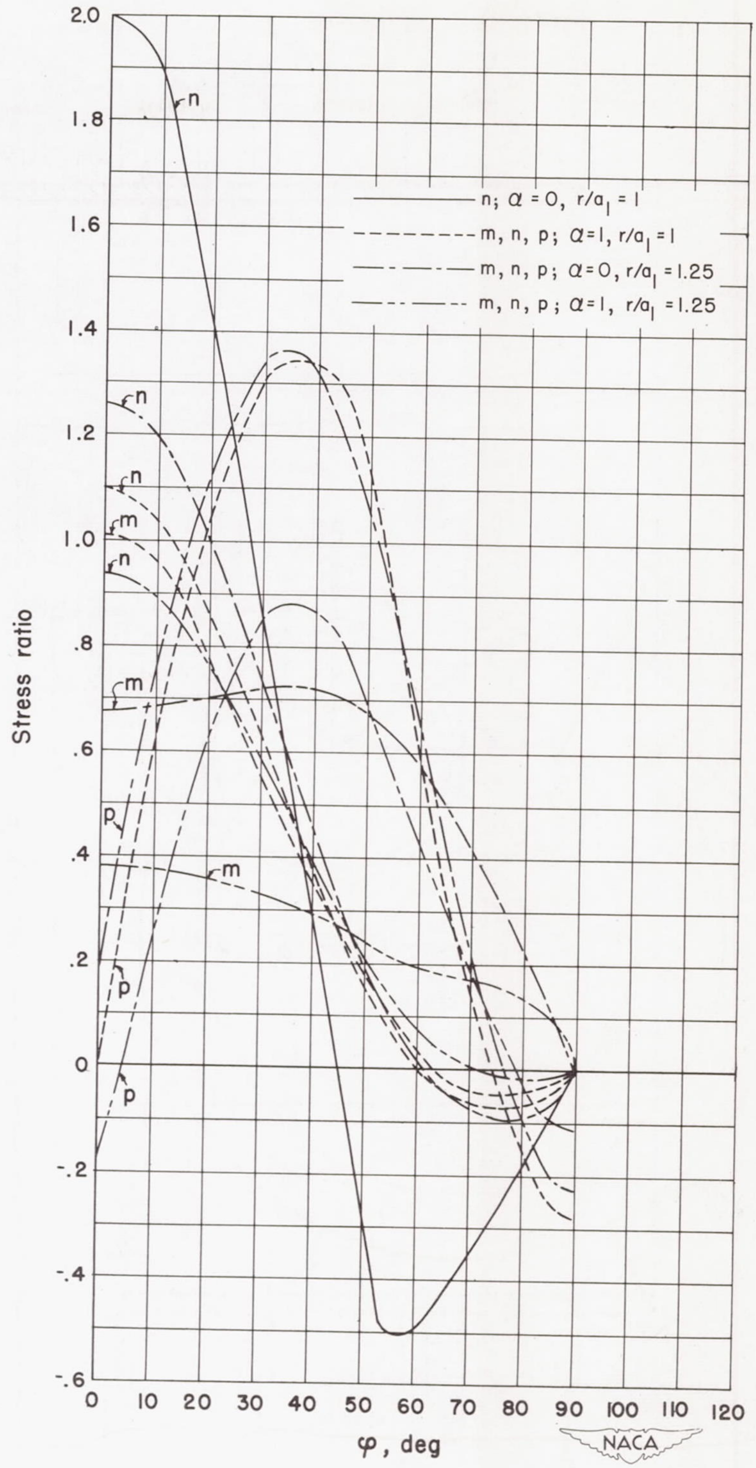

Figure 2.- Total-stress distribution in sheet produced by pure bending.

$I_r = 0.$ $m = \dfrac{\sigma_{r_1} + \sigma_{r_o}}{\sigma_{r_{max}}}$, $n = \dfrac{\sigma_{\varphi_1} + \sigma_{\varphi_o}}{\sigma_{\varphi_{max}}}$, $p = \dfrac{\tau_{r\varphi_1} + \tau_{r\varphi_o}}{\tau_{r\varphi_{max}}}$; $\delta \equiv \dfrac{d}{2a} = 0.$

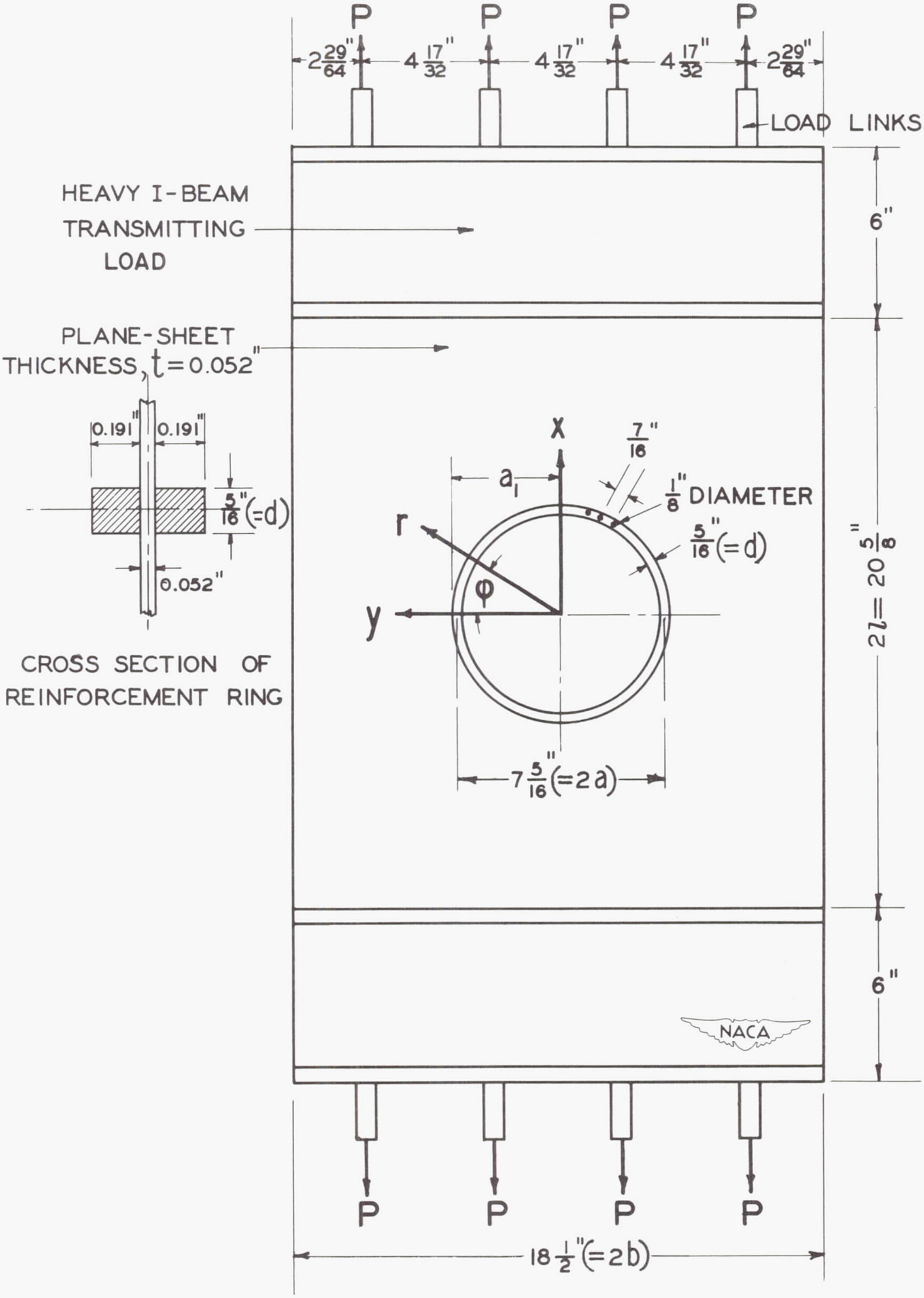

Figure 3.- Experimental setup.

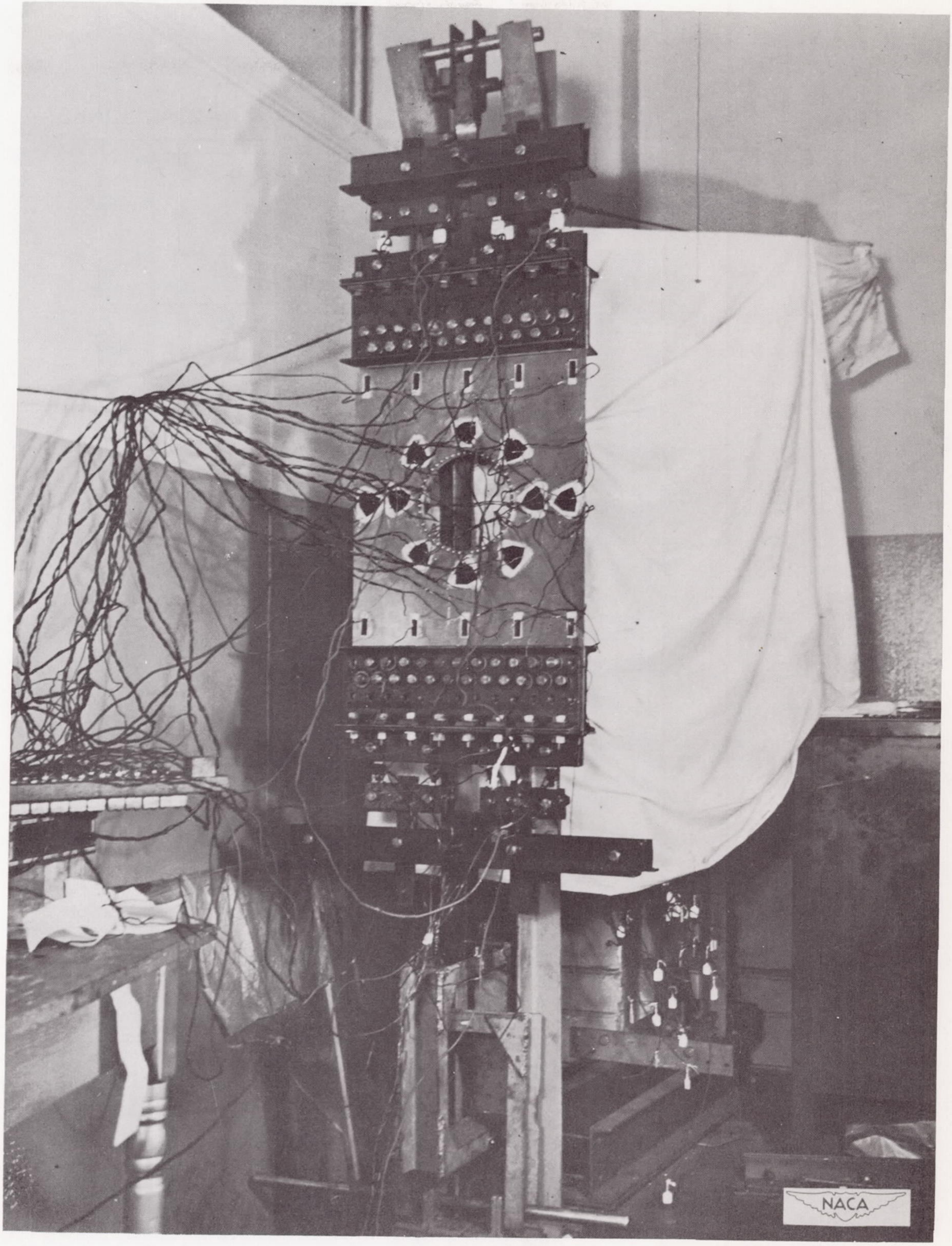

Figure 4.- Test specimen.

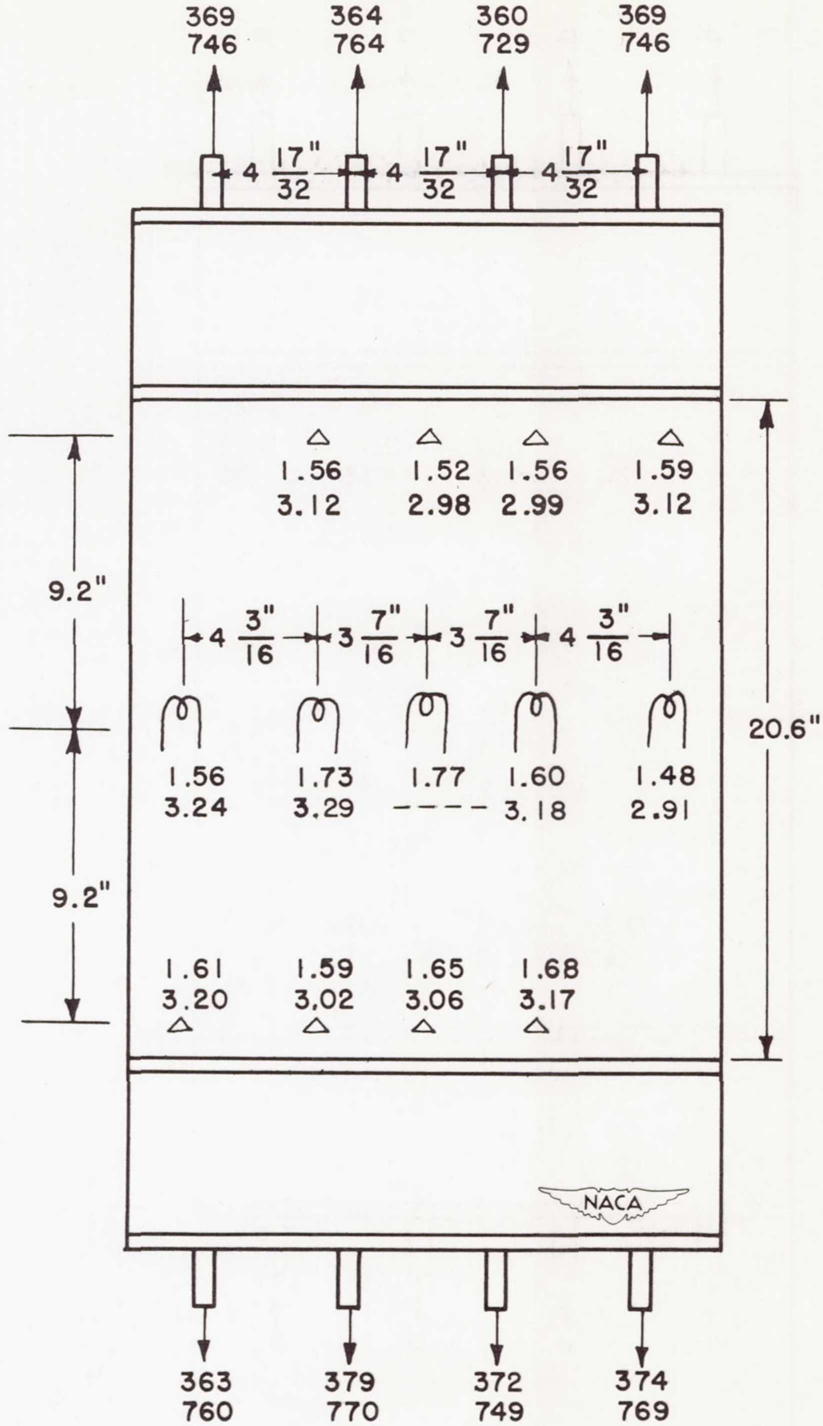

Figure 5.- Measured strains $\epsilon_x \times 10^4$ in direction of loading. Top values are for a total load 4P of 1475 pounds; bottom values are for a load of 3016 pounds. Rosette, Δ; simple strain gage, ⌈⌉. Drawing not exactly to scale.

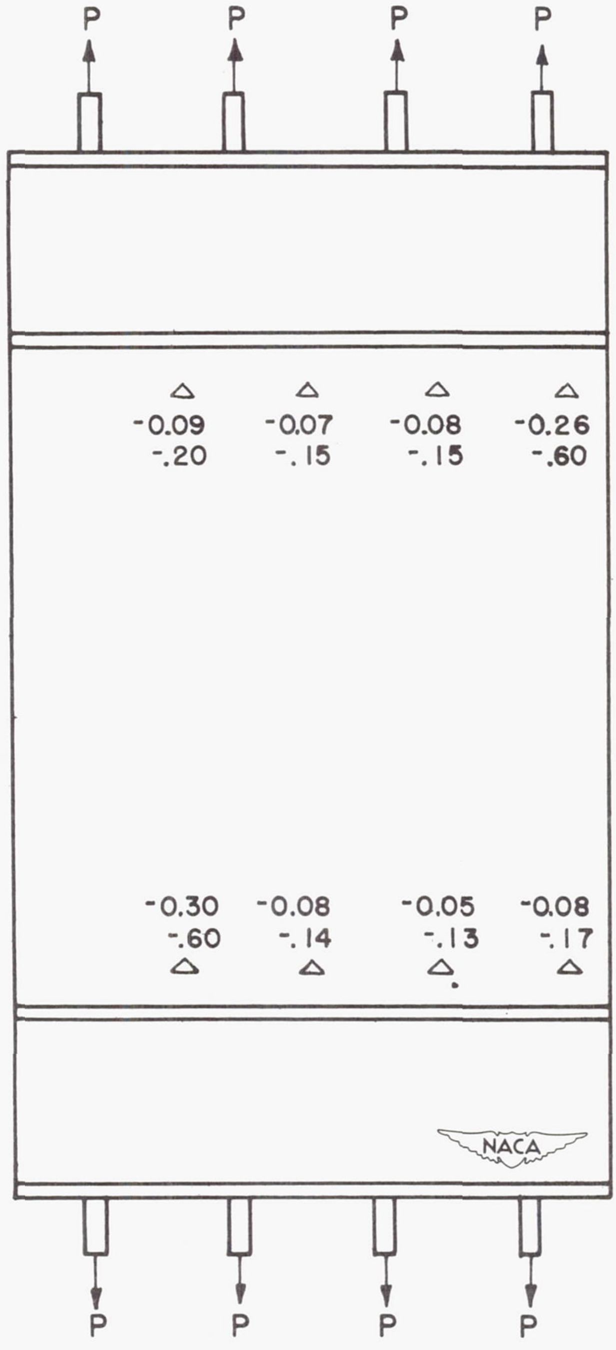

Figure 6.- Measured strains $\epsilon_y \times 10^4$ transverse to direction of loading.

Top values are for a total load 4P of 1475 pounds; bottom values are for a load of 3016 pounds. Rosette, Δ. For dimensions see figure 5. Drawing not exactly to scale.

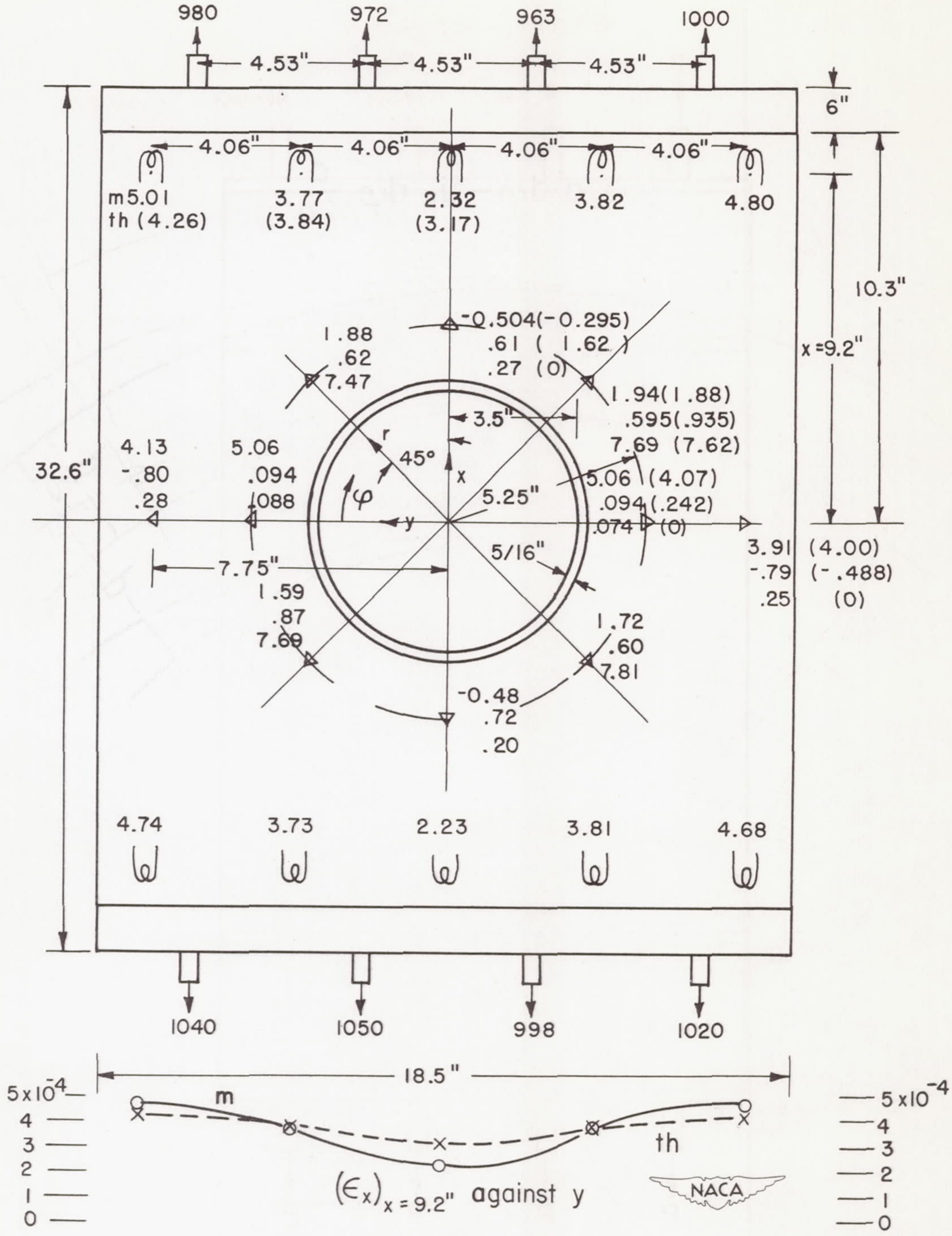

Figure 7.- Strain measurements on plane sheet with reinforced cutout. Total load, 4020 pounds; rosette, Δ ; simple strain gage, ⋒ ; values in parentheses are corresponding theoretical values for reinforced cutout but for an infinite sheet (see appendix G). For each rosette, the top, middle, and bottom values are, respectively, normal transverse, normal radial, and shear strain times 10^4. For each simple gage, values given are those of axial strains $\epsilon_x \times 10^4$.

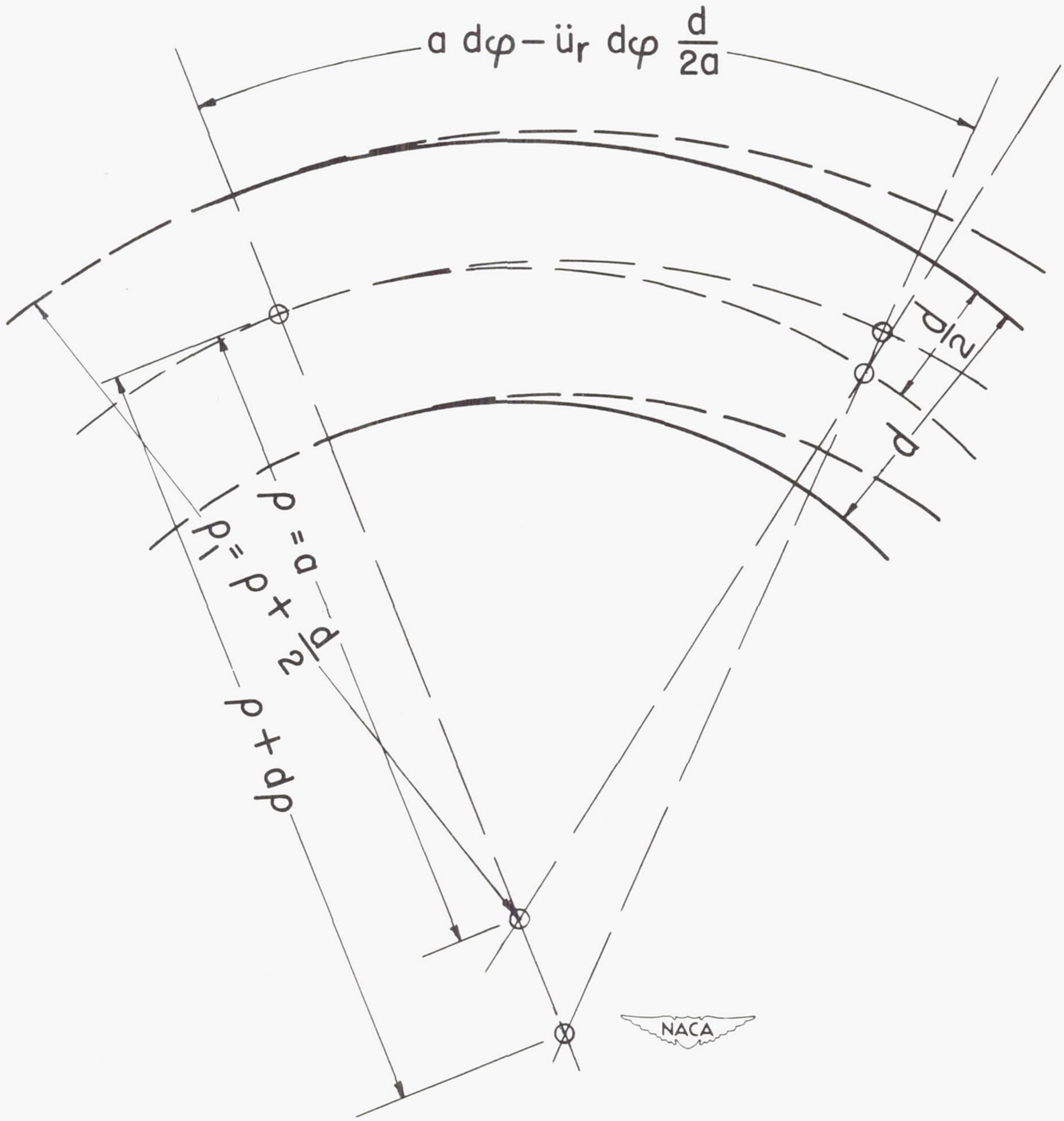

Figure 8.- Contribution of curvature of cutout ring to extension at outer boundary of ring. $\dot{u}_{t_1} = \dot{u}_t - \ddot{u}_r \frac{d}{2a}$.