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STUDY OF UNSTEADY FLOW DISTURBANCES OF LARGE AND SMALL AMPLITUDES MOVING THROUGH SUPERSONIC OR SUBSONIC STEADY FLOWS

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SUMMARY

A study of unsteady flow disturbances of large and small amplitudes moving through supersonic or subsonic steady flows is presented in three parts.

In part I a point-by-point method is developed for the calculation of unsteady flows through tubes with variable cross section under the assumption of constant flow velocity at a given cross section. The paper extends the work done previously by giving a detailed treatment of the interaction of strong shocks and large temperature contact discontinuities and by presenting the shock calculations and the calculations of flows with initial entropy gradients in a form convenient for computation by use of computing machines. Under certain assumptions the formulas established may also be used for the calculation of flows with continuous heat addition over a large space.

In part II calculations are made of the flow pattern created by the bursting into a vacuum of a diaphragm at the minimum section of a supersonic nozzle without a second throat. The transition time from the starting of the flow to the attainment of approximately steady flow conditions is sufficiently short to permit the use of very-short-duration tests. The transition time for the specific nozzle is presented in such a form that a "similarity rule" can be established concerning the transition time for nozzles of different size but of the same or affine shape.

In part III integral relationships invariant with respect to time are developed which describe as a whole the behavior of unsteady flow disturbances of large and small amplitudes. The invariant integrals are the conservation laws for the mass, energy, potential, and pulse area. For the special case of disturbances of small amplitude and small length traveling through tubes with small cross-sectional gradient (short disturbances), growth and reflection of mass, energy, and pulse area of a disturbance traveling through a steady flow in variable cross section can be separated and presented as a function of the steady-flow Mach number. The calculations show the interesting result that the conditions for zero reflection of mass,

energy, and pulse area exist at a steady-flow Mach number $M_0 = \sqrt{\frac{2}{\gamma - 1}}$ (which equals $\sqrt{5}$ for air), where γ is the ratio of specific heats, rather than at $M_0 = 1$; they also show that for practical purposes for the range of Mach number from 1 to 5 (for air) the reflections are small enough so that the mass, energy, and pulse area of the original disturbance may be considered constant. At low subsonic and at high supersonic Mach numbers, however, the reflections may not be neglected.

INTRODUCTION

The study of unsteady-flow problems has recently achieved new significance in aeronautics, partly because of the possibilities of using unsteady-flow phenomena for the improvement of the performance of high-speed internal-flow systems and wind tunnels and partly because of the greater understanding of the basic nature of flow phenomena such a study has to offer.

A few of the unsteady-flow problems that are useful to the aeronautical engineer are indicated. The study of the stability of shocks in diffusers involves a whole series of problems. One such problem is concerned with the stability of a normal shock with respect to disturbances moving upstream; such disturbances may be produced, for example, by the fluctuations in a burner of a jet airplane or by other types of fluctuations occurring in the operation of a jet airplane or of a wind tunnel. Still another type of stability problem deals with the two possible equilibrium positions of a shock in front or inside of a diffuser; the possible "jumping" of the shock from one equilibrium position to the other in steady-flow terminology is actually an unsteady-flow phenomenon.

Aside from these stability problems a series of problems involving flow discontinuities other than shock waves are of interest. One type of discontinuity is the temperature contact discontinuity created by the sudden pressure increase in a fluid such as occurs, for example, by bursting a diaphragm, detonation, or the crossing of two unequally strong shocks. Other types of discontinuities are the flame front moving through a combustible gas or the condensation front produced in a wind tunnel. Aside from the problems of discontinuous heat addition, there also exist problems of continuous heat addition over a large space (for example, those due to combustion). Naturally, both cases, the discontinuous flame front and the continuous heat addition over a large space, are only idealized models of the combustion process (though often very useful). Finally, for stability problems and related problems the study of small or infinitesimal disturbances moving through the subsonic or supersonic portion of a wind tunnel or through the internal-flow system of a jet airplane is of importance.

The present paper is presented in three parts. Part I may be considered a manual in the procedure for the construction of unsteady-flow patterns involving strong shocks and large temperature contact discontinuities under the assumption of constant flow velocity at a given cross section (one-dimensional flow). The flow variables are given in simple expressions in a form convenient for computation by use of computing machines. The emphasis in part I is on the method of performing the calculations, rather than on the elucidation of all their physical and mathematical meanings. The method is an extension of the methods developed in references 1 to 4.

In part II an example is calculated for the motion of a disturbance of large amplitude through a tube with variable cross section. The large-amplitude disturbance is produced by the bursting into a vacuum of a diaphragm at the minimum section of a supersonic nozzle without a second throat. The unsteady-flow pattern thus created is calculated up to the time that conditions very close to steady flow are approached in the nozzle (the actual approach is asymptotic). The transition time from the starting of the flow to the attainment of approximately steady-flow conditions for the specific nozzle is presented in such a form that a "similarity rule" can be established concerning the transition time for nozzles of different size but of the same or affine shape.

In part III integral relationships are developed to describe as a whole the behavior of unsteady flow disturbances of large and small amplitudes traveling in steady subsonic or supersonic flows through tubes with variable cross section. The development of such relationships is possible, in spite of the fact that, because of the inclined walls, a very complicated pattern of deformations, reflections, and re-reflections may occur within the disturbance, since integrals over the whole disturbance exist which are independent of time. First, one such integral relation is derived that represents the conservation of the velocity potential of an isentropic disturbance. Since the expression for the potential contains the first power of the velocity, it is obviously related to the momentum, which, in contrast to the tube with constant cross section, is not conserved because of unknown pressures at the inclined walls. The other two conservation laws for mass and energy can be used directly. The value of such invariant integrals is doubtful as long as the original disturbance, its reflections, and re-reflections still belong to the integrand. Therefore, as a first step for the separating of the original disturbances and their reflections, the concept of pulse area is introduced. The reason for the usefulness of the pulse areas is that they have as amplitudes certain linear combinations of the flow variables, that is, the parameters of the families of characteristics, which are associated with disturbances traveling with the speed of sound to the right and to the left relative to the fluid. The final step in separating the original disturbance and its reflection is achieved by restricting the length and the amplitude of the original disturbance and the inclination of the tube walls to small values (short disturbance). The investigations in part III are an extension and an elucidation of the studies made in references 5 and 6. In reference 5, the concept of pulse area for short disturbances is introduced without relating it to the potential and the momentum of disturbances of large and small amplitudes. The calculations are restricted to a short disturbance moving upstream in a subsonic diffuser in the proximity of a steady-flow Mach number of 1.0, and the result is obtained that, although the shape of the original pulse degenerates, its area is constant within the accuracy of the approximation used in reference 5. When the disappearing inclination of the walls is considered, this result can be misinterpreted to signify that the area of the short original pulse has a constant value near the minimum cross section of the nozzle. In part III, however, the area of the original pulse is shown to have finite and unique values as a function of the steady-flow Mach numbers with a finite derivative of the Mach number function at the Mach number of 1.0. Reference 6 presents the laws for the conservation of mass and energy (but not of the potential and the pulse area) for disturbances of large and small amplitudes in terms of the parameters of the characteristic families but is not extended to the case of short disturbances. The present analysis has the advantage of offering a clearer idea of the principal effects occurring in the motion of small disturbances.

SYMBOLS

a	velocity of sound
t	time
T	temperature
y	distance along tube axis; in example of part II y is referred to $\sqrt{F_{min}}$
F	cross-sectional area of tube; in example of part II F is referred to F_{min}
M	Mach number
v	flow velocity
Δu	speed of shock wave relative to flow
u	absolute speed of shock wave
Δv	velocity increment through shock
s	entropy
p	pressure

Π	total pressure recovery ratio
ρ	density
ψ	stream function
ϕ	potential
γ	ratio of specific heats
$\left.\begin{matrix}\lambda,\ \mu \\ \text{or} \\ P,\ Q\end{matrix}\right\}$	parameters of characteristic families; used as quantities for measuring amplitudes of disturbances of large and small amplitudes moving in steady flow or in gas at rest. Also used as mere labels for distinction between two groups of characteristic disturbances. The expressions "disturbance" and "pulse" are used interchangeably in present paper.
dP, dQ	amplitudes of growth and reflection of small-amplitude disturbances
m, n	slopes
D	constant
d	diameter

Subscripts:

0	reference conditions
1	condition ahead of shock
2	condition behind shock
o	steady-flow values
A,B,C	at points A,B,C

I.— A METHOD OF CALCULATING UNSTEADY FLOWS CONTAINING STRONG SHOCKS AND LARGE TEMPERATURE DISCONTINUITIES IN TUBES OF VARIABLE CROSS SECTION

GENERAL CONSIDERATIONS

A point-by-point method is developed for the calculation of unsteady flow disturbances of large amplitudes moving through tubes with variable cross section under the assumption of constant flow velocity at a given cross section. The method used in this part is based on the method of characteristics for steady rotational supersonic three-dimensional flows with axial symmetry developed in reference 1 and modified in references 2 and 3. The steady-flow method was applied to unsteady flow through tubes with variable cross section in reference 4.

The present paper extends the work reported in reference 4 by giving a detailed treatment of the interaction of strong shocks and large temperature contact discontinuities and by presenting the shock calculations in a form convenient for computation by use of computing machines. In the course of the analysis, expressions are also given for the effect of the entropy gradient behind a shock created by the varying energy losses in the motion of the shock with varying speed on disturbances traveling through this entropy gradient.

In the literature, problems of discontinuous heat addition in a flame front and problems of continuous heat addition over a large space have also been treated. The problem of discontinuous heat addition in the flame front (or condensation front) which has a nature similar to other discontinuity problems as, for example, the motion of a temperature discontinuity discussed in case C of part I has been treated, for example, in references 7 and 8. The problem of continuous heat addition over a large space has been treated in reference 9. The calculations in the present paper may be easily modified to include the case of continuous heat addition over a large space by treating the effects of heat addition and the compressibility effects separately. Since the energy equation (see, for example, equation (3) of reference 9) indicates that the effects of heat addition are independent of the compressibility effects for the case of small amounts of heat added at constant volume, the continuous heat addition over a large space may be substituted by the distribution in space of small amounts of heat added suddenly (at constant volume). During the time intervals between these sudden heat additions, the entropy along the time histories of the fluid particles is assumed furthermore to be constant. The scheme of sudden heat additions also permits a simple inclusion of further additions like those of fuel mass.

A few remarks seem pertinent at this point concerning the applicability to practical cases of the two previously mentioned combustion models, that of discontinuous heat addition in a flame front and that of continuous heat addition over a large space. The first model applies to problems of combustion for which one or several separate flame fronts exist or where the length of the burner is small enough relative to the entire length of the internal flow system to permit the continuous heat addition being averaged by a single flame front without essentially changing the unsteady-flow pattern as a whole. The choice of proper combustion model or models for the analysis of the cycle of a pulse jet or of the instabilities of a steady-flow ram jet depends on the merits of the individual case.

The method of calculation given in part I is mainly based on the presentation in the form of difference equations of the variation in convenient linear combinations of the flow variables along the time history of disturbances traveling with the speed of sound relative to the fluid (compression and expansion waves). In mathematics, the time histories of the disturbances which do not exceed the order of magnitude of discontinuities in the velocity gradient across the time histories are called the families of characteristics. The characteristics have furthermore the significance that they also exist when no disturbances are moving along them. The characteristics represent thus, in effect, the "characteristics" of the flow structure. The families of characteristics are hereinafter called for brevity the "time histories of the disturbances" in contrast to "the time history of the shock" (or strong shock) for which the discontinuity occurs in the velocity itself. The linear combinations of the flow variables are called "the parameters of the characteristics." They are used as quantities for measuring amplitudes of disturbances of large and small amplitudes moving in steady flow or in a gas at rest. They represent also mere labels for distinction between the two groups of characteristic disturbances. The labels of the characteristic disturbances refer to two different reference systems, one for the subsonic case and one for the supersonic case. The reason for the difference in reference systems for the characteristic parameters in the subsonic and supersonic cases lies in the following: For the subsonic case $(v < a)$ the fluid velocity may change between down-tube or up-tube direction; whereas for the supersonic case $(v > a)$ the fluid velocity may have only one direction. (The down-tube and up-tube directions or positive and negative y-directions may be arbitrarily designated in the upstream or downstream directions.) Thus, for the subsonic case the proper reference system for the distinction between the two groups of characteristic disturbances is one at rest with respect to the tube; whereas for the supersonic case the proper reference system is one moving with the fluid velocity v. In most literature on the subject of unsteady flow these parameters are represented, with the use of different signs, by the pair of symbols λ and μ, P and Q, or r and s. In part I the symbols λ and μ are used in agreement with reference 4. As well as being identified with the motion of a small disturbance, the characteristic parameters have the significance that they remain constant along the time histories of the disturbances for isentropic flow through constant cross section. For the general case of unsteady flow through variable cross section with entropy gradient and with or without gradual heat addition, the time histories of the disturbances moving relative to the fluid (compression and expansion waves) are not the only characteristic families of the flow. Since the entropy is integrable or constant along the time histories of the fluid particles, these time histories form a third family of characteristics.

METHODS OF CALCULATIONS

General Remarks

Unsteady-flow problems through a tube deal with the variation of the flow variables (for example, the velocity of flow v and the velocity of sound a) along the length of the tube with time. In order that the unsteady-flow phenomena may be determined as a function of time, the flow conditions along the tube must be known for a given time (initial conditions). In addition, the changes in flow boundaries, including flow changes at ends of the tube (boundary conditions), must also be known as a function of time. In most problems considered, the cross section does not vary with time and, thus, the boundary conditions apply only to the ends of the tube.

The calculations are applied to the problem of one-dimensional unsteady flows through tubes of variable cross section. The assumption of one-dimensional flow is made on the basis that the rate of change of cross-sectional area with respect to distance is small and that the flow velocity at a given cross section can be assumed constant. The tube cross section is assumed not to vary with time. As in previously developed methods, the effects of friction and heat transfer have been neglected. (The physical nature of these effects does not permit treatment with the theories dealing with wave propagation.)

By application of the method of characteristics to the three basic flow equations, the equations of the characteristic families and of the variation of the most significant combination of the flow variables along them are established. For the case under consideration, the characteristic families consist of the two families of the possible time histories of the disturbances propagated with the speed of sound relative to the fluid and the single family of time histories of convective variations in entropy transported with the velocity of the fluid particles. Since the purpose of the calculations is partly to determine the equations of the time histories of the disturbances and the convective entropy variations, naturally these equations cannot be given in advance. The slopes of the unknown time histories, at any point of the flow field, are the velocities of the disturbances and of the convective variations in entropy (or of the fluid particles). The particle velocity at a given time t expressed in the convenient y,a_0t coordinate system is given by

$$\frac{1}{a_0}\frac{dy}{dt} = \frac{v}{a_0} \tag{1}$$

The velocity of a downstream-moving disturbance at a given time t is given by

$$\frac{1}{a_0}\frac{dy}{dt} = \frac{v + a}{a_0} \tag{2}$$

The velocity of an upstream-moving disturbance at a given time t is given by

$$\frac{1}{a_0}\frac{dy}{dt} = \frac{v - a}{a_0} \tag{3}$$

The variation of the most significant combination of flow variables along the three families of characteristics is given next. For the single family of time histories of convective variations in entropy, the entropy itself is naturally the significant combination of the flow variables. For the families of the time histories of the disturbances traveling relative to the fluid, the most significant combination of the flow variables is represented by the parameters λ and μ, where

$$\lambda = \frac{2}{\gamma - 1}\left(\frac{a}{a_0} - 1\right) + \frac{v}{a_0} \tag{4}$$

and

$$\mu = \frac{2}{\gamma - 1}\left(\frac{a}{a_0} - 1\right) - \frac{v}{a_0} \tag{5}$$

The method of calculation is developed and applied to the following three cases:

Case A – Unsteady flow with an entropy gradient in a tube of variable cross section

Case B – Unsteady flow containing a strong shock through a tube of variable cross section

Case C – Unsteady flow containing a strong shock and a large entropy or temperature contact discontinuity through a tube of variable cross section

General Formulas

Guderley (reference 4) resolves the problems of unsteady flow into one of integration of a system of three partial-differential equations by means of the method of characteristics. The three equations are: the equation of motion

$$\frac{1}{\rho}\frac{\partial p}{\partial y} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t} = 0$$

the continuity equation

$$v\frac{\partial \rho}{\partial y} + \rho\frac{\partial v}{\partial y} + \frac{\partial \rho}{\partial t} + \rho v\frac{d \log F}{dy} = 0$$

and the energy equation

$$v\frac{\partial s}{\partial y} + \frac{\partial s}{\partial t} = 0$$

Since Guderley only calculates the cases for which an entropy gradient exists behind a shock due to varying energy losses in the shock, the energy equation states that the entropy is constant during the time history of a fluid particle.

By application of the method of characteristics to the three basic flow equations, the variation of the parameters λ and μ along the time histories of disturbances moving in an entropy gradient produced behind a shock because of varying energy losses in the shock is obtained as (see references 4 and 10)

$$\frac{1}{a_0}\frac{d\lambda}{dt} = -\frac{a}{a_0}\frac{v}{a_0}\frac{d\log F}{dy} + \frac{a}{\gamma R}\frac{ds}{dt} \tag{6a}$$

and

$$\frac{1}{a_0}\frac{d\mu}{dt} = -\frac{a}{a_0}\frac{v}{a_0}\frac{d\log F}{dy} + \frac{a}{\gamma R}\frac{ds}{dt} \tag{6b}$$

or in terms of the stream function ψ and the total pressure recovery Π

$$\frac{1}{a_0}\frac{d\lambda}{dt} = -\frac{a}{a_0}\frac{v}{a_0}\frac{d\log F}{dy} - \frac{1}{\gamma}\left(\frac{a}{a_0}\right)^{\frac{2\gamma}{\gamma-1}} F\frac{d\Pi}{d\psi} \tag{7a}$$

and

$$\frac{1}{a_0}\frac{d\mu}{dt} = -\frac{a}{a_0}\frac{v}{a_0}\frac{d\log F}{dy} + \frac{1}{\gamma}\left(\frac{a}{a_0}\right)^{\frac{2\gamma}{\gamma-1}} F\frac{d\Pi}{d\psi} \tag{7b}$$

From equations (6) and (7), the parameters λ and μ can be seen to have the added significance that they remain constant along the time histories of disturbances for isentropic flow through tubes with constant cross section. (See also reference 8.) The physical significance of this constancy is that if one chooses the characteristic parameters as the amplitudes of the disturbances, the amplitudes of disturbances moving downstream and upstream through tubes with constant cross section do not interfere with each other.

The stream function ψ is defined at each point C of the diagram of y against a_0t by the line integral

$$\psi = \int_0^C \left(\frac{\partial\psi}{\partial y}\,dy + \frac{\partial\psi}{\partial t}\,dt\right)$$

$$= \int_0^C \left(F\,\frac{\rho}{\rho_0}\,dy - F\,\frac{\rho}{\rho_0}\,v\,dt\right) \tag{8}$$

Equations (6) and (7) differ by the factor F_0 from the corresponding equations of reference 4. This difference is due to the fact that Guderley uses for the stream function the dimension of a length as is frequently done for the calculation of steady supersonic flows. For the present calculations, using a volume as the dimension of the stream function seemed more convenient; the factor F_0 was thus eliminated. The parameters Π and ψ are both constant along the time history of a particle between the shocks. The parameter Π for the total pressure recovery of the shock is given by

$$\Pi = \left(\frac{a_2}{a_1}\right)^{-\frac{2}{\gamma-1}} \frac{\rho_2}{\rho_1} \tag{9}$$

The quantity ρ/ρ_0 appearing in the expression for ψ (equation (8)) is consequently related to Π by

$$\frac{\rho}{\rho_0} = \Pi\left(\frac{a}{a_0}\right)^{\frac{2}{\gamma-1}} \tag{10}$$

For the purpose of setting up the computational equations, the quantities $\frac{v}{a_0}$, $\frac{a}{a_0}$, $\frac{v+a}{a_0}$, $\frac{v-a}{a_0}$, $\frac{a}{a_0}\,\frac{v}{a_0}\,\frac{d\log F}{dy}$, and $\frac{1}{\gamma}\left(\frac{a}{a_0}\right)^{\frac{2}{\gamma-1}} F\,\frac{d\Pi}{d\psi}$ may be expressed in terms of the parameters λ and μ as follows:

$$\frac{v}{a_0} = \frac{\lambda - \mu}{2} \tag{11}$$

$$\frac{a}{a_0} = 1 + \frac{\gamma - 1}{4}(\lambda + \mu) \tag{12}$$

$$\frac{v + a}{a_0} = 1 + \frac{\gamma + 1}{4}\lambda - \frac{3 - \gamma}{4}\mu \tag{13}$$

$$\frac{v - a}{a_0} = -1 - \frac{\gamma + 1}{4}\mu + \frac{3 - \gamma}{4}\lambda \tag{14}$$

$$\frac{a}{a_0}\frac{v}{a_0}\frac{d \log F}{dy} = \left[1 + \frac{\gamma - 1}{4}(\lambda + \mu)\right]\left(\frac{\lambda - \mu}{2}\frac{1}{F}\frac{dF}{dy}\right) \tag{15}$$

and

$$\frac{1}{\gamma}\left(\frac{a}{a_0}\right)^{\frac{2\gamma}{\gamma-1}} F\frac{d\Pi}{d\psi} = \frac{1}{\gamma}\left[1 + \frac{\gamma - 1}{4}(\lambda + \mu)\right]^{\frac{2\gamma}{\gamma-1}} F\frac{d\Pi}{d\psi} \tag{16}$$

In order to calculate the flow field, equations (7a) and (7b) are evaluated in difference form; this evaluation permits the construction of the unsteady flow field by means of a point-by-point method in the following manner:

The flow variables v/a_0 and a/a_0 are known for two locations in the tube y_A and y_B at two known times a_0t_A and a_0t_B (see fig. 1). From this information, the flow variables are calculated at a third location in the tube y_C at a corresponding time a_0t_C. The location y_C, the corresponding time a_0t_C, and the values of the flow variables at C are determined by the intersection of the time-history curves of the waves through A and B and the knowledge of the variation of the flow variables along these time-history curves. Since the curvature of the time histories, however, depends on the variation of the flow variables along them, an iteration process was used in references 4 and 10 for the simultaneous determination of y_C, a_0t_C, and the flow variables v_C/a_0 and a_C/a_0.

In reference 4 the known time-history curves of disturbances traveling through constant cross section were used as a first approximation. (The curvature is due to interference of the two families of time histories of disturbances.) Since, however, the curvature of time histories of the waves traveling through a tube of variable cross section is still unknown, the use of the curvature for constant cross section as a first approximation introduces unnecessary complications.

The method of the present study is based on the substitution of the very slightly curved time histories through the points A and B by the tangents at these points as a first approximation. In this first approximation, the point C obtained by the intersection of the tangents is used for the calculation of the flow variables. Within the order of accuracy of the calculations, the point C obtained by intersection of the tangents is identical with the point obtained by the intersection of the time-history curves in the first approximation of reference 4. The second approximation consists in using the arithmetic mean between a_A/a_0 and a_C/a_0, v_A/a_0 and v_C/a_0, a_B/a_0 and a_C/a_0, and v_B/a_0 and v_C/a_0 for determining new time histories between A and C' and B and C', respectively. The values a_C'/a_0 and v_C'/a_0 for the second approximation are found by integrating along the newly found time-history curves.

It should be emphasized at this point that the use of a second approximation does not imply the use of very large steps in the construction of the net of time histories, since, for the success of the second iteration, terms of the second order must be small. If small steps are used initially, however, the use of a second approximation may be unimportant by engineering standards. Also, for the use of more elaborate computing machines such as the Bell Telephone Laboratories X-66744 relay computer in use at the Langley Laboratory, the first approximation used with many very small steps may prove more advantageous.

In developing the present method, the possibility of directly obtaining more exact values at C was also considered. These values could be obtained by intersecting lines AC and BC (fig. 1) given by the average directions between A and C and B and C, respectively. Furthermore, the average conditions between the flow variables at A and C and B and C, respectively, could also be taken into account. The use of average conditions, however, results in cubic or quartic equations for the unknowns instead of the linear equations. The solution of these higher-order equations would itself require an iteration process.

Case A – Unsteady Flow with Entropy Gradient in a Tube with Variable Cross Section

The purpose of the following calculations is to determine the effect of unsteady flow disturbances produced isentropically by compression or expansion on the change of the flow variables with time as the disturbance travels in a tube with variable cross section through the entropy gradient behind a shock. With the aid of assumptions stated in the section entitled "General Considerations," the expressions lend themselves also to the calculation of problems of continuous heat addition over a large space. The calculations are presented conveniently in the following manner (see fig. 1):

Two points in a tube with given variation of cross section $F = f(y)$

$$A\left(a_0t_A,\ y_A,\ F_A,\ \frac{dF_A}{dy_A},\ \frac{v_A}{a_0},\ \frac{a_A}{a_0},\ \Pi_A,\ \psi_A\right)$$

and

$$B\left(a_0t_B,\ y_B,\ F_B,\ \frac{dF_B}{dy_B},\ \frac{v_B}{a_0},\ \frac{a_B}{a_0},\ \Pi_B,\ \psi_B\right)$$

are given. The point

$$C\left(a_0t_C,\ y_C,\ F_C,\ \frac{dF_C}{dy_C},\ \frac{v_C}{a_0},\ \frac{a_C}{a_0},\ \Pi_C,\ \psi_C\right)$$

is to be determined.

The intersection of the tangents to the time-history curves through points A and B results in

$$a_0t_C = \frac{e - d}{l - k}$$

and

$$y_C = la_0t_C + d$$

where the constants in terms of λ and μ (equations (4) and (5)) are

$$l = \left(\frac{v + a}{a_0}\right)_A$$

$$= 1 + \frac{\gamma + 1}{4}\lambda_A - \frac{3 - \gamma}{4}\mu_A$$

$$k = \left(\frac{v - a}{a_0}\right)_B$$

$$= -1 - \frac{\gamma + 1}{4} \mu_B + \frac{3 - \gamma}{4} \lambda_B$$

$$d = -l a_0 t_A + y_A$$

$$e = -k a_0 t_B + y_B$$

The quantity y_C determines F_C and dF_C/dy_C since the shape of the tube is given. The next step is to express equations (7a) and (7b) in difference form. Based on the first approximation of tangency, λ_C and μ_C are given in the following form:

$$\lambda_C = \lambda_A - \frac{1}{2} \frac{v_A}{a_0} \frac{a_A}{a_0}\left(\frac{1}{F_A} \frac{dF_A}{dy_A} + \frac{1}{F_C} \frac{dF_C}{dy_C}\right) a_0 \Delta t_{CA}$$
$$- \frac{1}{\gamma}\left(\frac{a_A}{a_0}\right)^{\frac{2\gamma}{\gamma-1}} \frac{F_C + F_A}{2} \frac{\Pi_C - \Pi_A}{\psi_C - \psi_A} a_0 \Delta t_{CA} \tag{17}$$

and

$$\mu_C = \mu_B - \frac{1}{2} \frac{v_B}{a_0} \frac{a_B}{a_0}\left(\frac{1}{F_B} \frac{dF_B}{dy_B} + \frac{1}{F_C} \frac{dF_C}{dy_C}\right) a_0 \Delta t_{CB}$$
$$+ \frac{1}{\gamma}\left(\frac{a_B}{a_0}\right)^{\frac{2\gamma}{\gamma-1}} \frac{F_C + F_B}{2} \frac{\Pi_C - \Pi_B}{\psi_C - \psi_B} a_0 \Delta t_{CB} \tag{18}$$

The flow variables v_C/a_0 and a_C/a_0 are determined from λ_C and μ_C by means of equations (4) and (5).

All quantities on the right-hand side of equations (17) and (18) are now known, except Π_C and ψ_C which may be obtained by assuming a linear variation of the values of Π and ψ from A to B. Since Π and ψ are constant along the time-history curve of a particle, and because the time-history curve of the particle is here substituted for the tangent, the condition of linear variation of Π and ψ along AB yields

$$\frac{\Pi_B - \Pi_C}{\Pi_C - \Pi_A} = \frac{\psi_B - \psi_C}{\psi_C - \psi_A}$$

$$= \frac{c_1}{c_2}$$

The fact that Π_C and ψ_C are subtracted in the numerator is necessitated by the physical condition that the velocities of particles flowing through BC have to be of the same sign as the velocities of the particles flowing through CA. Since the tangent to the time-history curve of the particle bisects the angle formed by the tangents to the time-history curves of the disturbances at C, the ratio c_1/c_2 is equivalent to the ratio b/a. (See fig. 1.) The quantities Π_C and ψ_C are then given by the following formulas:

$$\Pi_C = \frac{\Pi_B a + \Pi_A b}{a + b}$$

$$\psi_C = \frac{\psi_B a + \psi_A b}{a + b}$$

where

$$a = \sqrt{(a_0 t_C - a_0 t_A)^2 + (y_C - y_A)^2}$$

and

$$b = \sqrt{(a_0 t_C - a_0 t_B)^2 + (y_C - y_B)^2}$$

Finally, a brief note is made concerning the scheme of sudden heat additions (at constant volume) distributed over a large space. The sudden additions of heat are made at the intersections of the net of time histories of the disturbances. During the time intervals between these sudden heat additions, the entropy of the fluid along the time histories of the fluid particles is assumed furthermore to be constant. As is shown in figure 1, the time interval between the sudden additions of heat is the time it takes a particle to move along its time history from the connecting line of the two former net points to the new point. The variations of λ and μ along the time histories of the downstream and upstream moving disturbances are calculated by the use of equations (6a) and (6b) in difference form.

Case B – Unsteady Flow Containing a Strong Shock through a Tube with Variable Cross Section

The purpose of the calculations in this section is to determine the effect of a strong shock on the change of the flow variables with time as the shock travels in a tube with variable cross section through a steady isentropic flow or through a gas at rest. The more complicated case of a shock crossing an unsteady isentropic flow is not developed herein. A method very similar to that used for the "interweaving" into the flow of a temperature contact discontinuity (see case C) can, however, be used in that case.

The expressions for the flow variables are developed for the case of a shock moving upstream through a steady isentropic flow in a tube with variable cross section. The presentation of the flow problem is given in figure 2. A steady flow is assumed to exist in the direction of the negative y-axis and a shock is assumed to be produced traveling in the direction of the postive y-axis (through decreasing cross section). The production of a shock of given strength is identical with the condition that the following values are known: $\Delta u_A/a_{A_1}$, $\Delta v_A/a_{A_1}$, and a_{A_2}/a_{A_1}. These values in turn are used to determine the absolute speed of the shock u_A/a_0 and the conditions behind the shock v_{A_2}/a_0 and a_{A_2}/a_0 by the following fundamental relationships:

$$\frac{u_A}{a_0} = \frac{v_{A_1}}{a_0} + \frac{\Delta u_A}{a_0}$$

where v_{A_1}/a_0 is negative and $\Delta u_A/a_0$ is positive

$$\frac{v_{A_2}}{a_0} = \frac{v_{A_1}}{a_0} + \frac{\Delta v_A}{a_0}$$

where $\Delta v_A/a_0$ is positive, and, finally,

$$\frac{a_{A_2}}{a_0} = \frac{a_{A_2}}{a_{A_1}} \frac{a_{A_1}}{a_0}$$

The quantities $\Delta u_A/a_0$ and $\Delta v_A/a_0$ are obtained by multiplying $\Delta u_A/a_{A_1}$ and $\Delta v_A/a_{A_1}$, respectively, by a_{A_1}/a_0.

In order to find the changes in the behavior of the shock as it travels in the positive y-direction, a tangent to the time-history curve at B is drawn and brought to an intersection with u_A/a_0 at point C. The position of the point B in the diagram of y against a_0t is known, as well as the flow variables v_B/a_0 and a_B/a_0 at B. The point B may, generally, be assumed to lie on the time-history curve of a disturbance of previous construction. In this particular case, which presents the initial calculations for the shock movement in a diffuser, the point B is assumed to lie on the time-history curve of the particle v_{A_2}/a_0.

The calculations of the flow variables at the point C are based on the knowledge of their variation during the time history AC of the shock and the time history BC of the disturbance. The variation of the flow variables during the time history BC of the disturbance is given by equation (17) since the small disturbance is moving down tube, the flow is subsonic, and the parameter λ is chosen to be associated with the down-tube direction. The variation of the flow variables during the time history of the shock is developed as follows:

Two points

$$A\left(a_0t_A,\ y_A,\ F_A,\ \frac{dF_A}{dy_A},\ \frac{v_{A_1}}{a_0},\ \frac{v_{A_2}}{a_0},\ \frac{a_{A_1}}{a_0},\ \frac{a_{A_2}}{a_0},\ \frac{u_A}{a_0},\ \Pi_A,\ \psi_A\right)$$

and

$$B\left(a_0t_B,\ y_B,\ F_B,\ \frac{dF_B}{dy_B},\ \frac{v_B}{a_0},\ \frac{a_B}{a_0},\ \Pi_B,\ \psi_B\right)$$

are given. The point

$$C\left(a_0t_C,\ y_C,\ F_C,\ \frac{dF_C}{dy_C},\ \frac{v_{C_1}}{a_0},\ \frac{v_{C_2}}{a_0},\ \frac{a_{C_1}}{a_0},\ \frac{a_{C_2}}{a_0},\ \frac{u_C}{a_0},\ \Pi_C,\ \psi_C\right)$$

is to be determined. In the present method, the location of point C in the a_0t,y plane is given by

$$a_0t_C = \frac{f - e}{b - d}$$

and

$$y_C = ba_0t_C + d$$

where

$$b = \frac{u_A}{a_0}$$

$$\begin{aligned} d &= \left(\frac{v + a}{a_0}\right)_B \\ &= 1 + \frac{\gamma + 1}{4} \lambda_B - \frac{3 - \gamma}{4} \mu_B \end{aligned}$$

$$e = -ba_0t_A + y_A$$

and

$$f = -da_0t_B + y_B$$

The location of point C, given by y_C, determines F_C and dF_C/dy_C from the shape of the tube; furthermore, if y_C is known, the quantities v_{C_1}/a_0 and a_{C_1}/a_0 are determined since a given steady flow exists in the tube. Only the determination of v_{C_2}/a_0, a_{C_2}/a_0, and u_C/a_0 thus remains.

In order to relate the values of the flow variables at A and C, a relation is set up between the shock parameters $\Delta v/a_1$ and a_2/a_1; the relation is obtained with the aid of the shock equations (reference 11). Thus,

$$\frac{\Delta v}{a_1} = \frac{2}{\gamma + 1}\left(\frac{\Delta u}{a_1} - \frac{a_1}{\Delta u}\right) \tag{19}$$

and

$$\frac{a_2}{a_1} = \sqrt{1 + \frac{2(\gamma - 1)}{(\gamma + 1)^2}\left[\left(\frac{\Delta u}{a_1}\right)^2 - 1\right]\left[\gamma + \left(\frac{a_1}{\Delta u}\right)^2\right]} \tag{20}$$

A plot of a_2/a_1 against $\Delta v/a_1$ is given for air ($\gamma = 1.4$) in figure 3 and more extensive values are presented in table I.

For the determination of the change of the flow variables along the time-history curve of the shock, the variation of the shock parameters $\Delta v/a_1$ and $\Delta a/a_1$ (where $\Delta a = a_2 - a_1$) is assumed to be linear. Thus,

$$\frac{\Delta a_C}{a_{C_1}} - \frac{\Delta a_A}{a_{A_1}} = m_A\left(\frac{\Delta v_C}{a_{C_1}} - \frac{\Delta v_A}{a_{A_1}}\right) \tag{21}$$

Since

$$\frac{\Delta a_A}{a_{A_1}} = \frac{a_{A_2}}{a_{A_1}} - 1$$

and

$$\frac{\Delta a_C}{a_{C_1}} = \frac{a_{C_2}}{a_{C_1}} - 1$$

equation (21) may be written as

$$\frac{a_{C_2}}{a_{C_1}} - \frac{a_{A_2}}{a_{A_1}} = m_A\left(\frac{\Delta v_C}{a_{C_1}} - \frac{\Delta v_A}{a_{A_1}}\right) \tag{22}$$

The change of the flow variables during the time history of the shock may be conveniently expressed in the form

$$\frac{a_{C_2}}{a_0} = m_A \frac{v_{C_2}}{a_0} + \text{Constant}$$

This expression is obtained by multiplying equation (22) by a_{C_1}/a_0. The following equation results:

$$\frac{a_{C_2}}{a_0} = m_A \frac{v_{C_2}}{a_0} - R \tag{23}$$

where

$$R = m_A \frac{v_{C_1}}{a_0} + \frac{a_{C_1}}{a_0}\left(m_A \frac{\Delta v_A}{a_{A_1}} - \frac{a_{A_2}}{a_{A_1}}\right)$$

For the special case of a shock moving through variable cross section with air at rest in front of it $\left(a_{A_1} = a_{C_1} = a_0 \text{ and } \frac{v_{A_1}}{a_0} = \frac{v_{C_1}}{a_0} = 0\right)$, equation (23) becomes

$$\frac{a_{C_2}}{a_0} = m_A \frac{v_{C_2}}{a_0} - R_1 \tag{24}$$

where

$$R_1 = m_A \frac{v_{A_2}}{a_0} - \frac{a_{A_2}}{a_0}$$

The linearization in the shock equations is equivalent to substituting for the curve in figure 3 a tangent with the direction m for a point with the coordinates $\Delta v/a_1$ and a_2/a_1. For the purpose of determining accurate values for m, an analytic expression for m which was developed from equations (19) and (20) is given in appendix A.

A plot of m against $\Delta v/a_1$ for air ($\gamma = 1.4$) is presented in figure 4 with more extensive values given in table II. The error involved in the linearization can be seen from table II to be small for small steps. The lower limit of m is identical with the value of m obtained from the laws for isentropic disturbances

$$\frac{a_2}{a_1} = \frac{\gamma - 1}{2}\frac{\Delta v}{a_1} + 1$$

$$\frac{d\left(\frac{a_2}{a_1}\right)}{d\left(\frac{\Delta v}{a_1}\right)} = \frac{\gamma - 1}{2}$$

for air ($\gamma = 1.4$)

$$\frac{d\left(\frac{a_2}{a_1}\right)}{d\left(\frac{\Delta v}{a_1}\right)} = 0.2$$

whereas the upper limit is reached asymtotically as $\Delta v/a_1$ approaches infinity. The upper limit given in reference 11 is $\sqrt{\frac{\gamma(\gamma - 1)}{2}}$ (0.52915 for air). This upper limit, however, is reached asymtotically only for a few practical cases, since γ may vary greatly across the shock.

The variation of the flow variables during the time history BC of the disturbance can be expressed in terms of the same two unknowns v_{C_2}/a_0 and a_{C_2}/a_0. From equation (17) and equation (4), the following expression is obtained:

$$\frac{2}{\gamma - 1}\frac{a_{C_2}}{a_0} + \frac{v_{C_2}}{a_0} - \frac{2}{\gamma - 1}\frac{a_B}{a_0} - \frac{v_B}{a_0} = -\frac{1}{2}\frac{a_B}{a_0}\frac{v_B}{a_0}\left(\frac{1}{F_B}\frac{dF_B}{dy_B} + \frac{1}{F_C}\frac{dF_C}{dy_C}\right)a_0\Delta t_{CB}$$

$$- \frac{1}{\gamma}\frac{F_B + F_C}{2}\left(\frac{a_B}{a_0}\right)^{\frac{2\gamma}{\gamma-1}}\frac{\Pi_C - \Pi_B}{\psi_C - \psi_B}a_0\Delta t_{CB} \quad (25)$$

Finally, ψ_C and Π_C have to be expressed in terms of known quantities. For the purpose of expressing ψ_C, $\psi_C - \psi_B$ is written in the form $\psi_C - \psi_A + (\psi_A - \psi_B)$. In accordance with equation (8)

$$\psi_C - \psi_A = \frac{F_A + F_C}{2} \frac{\rho_{A_1} + \rho_{C_1}}{2\rho_0}\left(\Delta y_{AC} - \frac{v_{A_1} + v_{C_1}}{2a_0} a_0 \Delta t_{AC}\right) \tag{26}$$

and ψ_A and ψ_B are known from previous calculations. For the special case of a shock moving into a gas at rest

$$\psi_C - \psi_A = \frac{F_A + F_C}{2} \Delta y_{AC}$$

The only value which still must be expressed in terms of known quantities is Π_C. If a linear relation is assumed, therefore, between Π_A and a_{A_2}/a_{A_1}, and Π_C and a_{C_2}/a_{C_1}, then

$$\Pi_C = n\left(\frac{a_{C_2}}{a_{C_1}} - \frac{a_{A_2}}{a_{A_1}}\right) + \Pi_A \tag{27}$$

where n is the slope of the curve $\Pi = f(a_2/a_1)$ for a_{A_2}/a_{A_1} and Π_A. The derivation of the formula for the slope n is given in appendix B. A plot of $\Pi = f(a_2/a_1)$ is given in figure 5 and the values are given in table I. A plot of the slope $\frac{d\Pi}{d(a_2/a_1)} = n$ is given in figure 6, and the variation of n with a_2/a_1 for a shock is given more completely in table III.

Substituting equations (27) and (23) into equation (25) gives finally

$$\frac{a_{C_2}}{a_0} = \frac{-\frac{a_0\Delta t_{CB}}{2}\frac{a_B}{a_0}\frac{y_B}{a_0}\left(\frac{1}{F_B}\frac{dF_B}{dy_B} + \frac{1}{F_C}\frac{dF_C}{dy_C}\right) + \frac{2}{\gamma - 1}\frac{a_B}{a_0} + \frac{v_B}{a_0} - \frac{R}{m_A}}{\frac{2}{\gamma - 1} + \frac{1}{m_A} + \frac{1}{\gamma}\frac{\frac{F_B + F_C}{2} a_0\Delta t_{CB}\left(\frac{a_B}{a_0}\right)^{\frac{2\gamma}{\gamma-1}} n_A \frac{a_0}{a_{C_1}}}{\frac{F_A + F_C}{2}\frac{\rho_{A_1} + \rho_{C_1}}{2\rho_0}\left(\Delta y_{AC} - \frac{v_{A_1} + v_{C_1}}{2a_0} a_0\Delta t_{AC}\right) + \psi_A - \psi_B}}$$

$$- \frac{\frac{1}{\gamma}\frac{F_B + F_C}{2} a_0\Delta t_{CB}\left(\frac{a_B}{a_0}\right)^{\frac{2\gamma}{\gamma-1}} \frac{\Pi_A - n_A\frac{a_{A_2}}{a_{A_1}} - \Pi_B}{\frac{F_A + F_C}{2}\frac{\rho_{A_1} + \rho_{C_1}}{2\rho_0}\left(\Delta y_{AC} - \frac{v_{A_1} + v_{C_1}}{2a_0} a_0\Delta t_{AC}\right)}}{\frac{2}{\gamma - 1} + \frac{1}{m_A} + \frac{1}{\gamma}\frac{\frac{F_B + F_C}{2} a_0\Delta t_{CB}\left(\frac{a_B}{a_0}\right)^{\frac{2\gamma}{\gamma-1}} n_A \frac{a_0}{a_{C_1}}}{\frac{F_A + F_C}{2}\frac{\rho_{A_1} + \rho_{C_1}}{2\rho_0}\left(\Delta y_{AC} - \frac{v_{A_1} + v_{C_1}}{2a_0} a_0\Delta t_{AC}\right) + \psi_A - \psi_B}} \tag{28}$$

and

$$\frac{v_{C_2}}{a_0} = \frac{1}{m_A} \frac{a_{C_2}}{a_0} + \frac{R}{m_A}$$

Case C – Unsteady Flow Containing a Strong Shock and a Large Entropy or Temperature Contact Discontinuity through a Tube with Variable Cross Section

The purpose of the calculations is to determine the interaction of a strong shock and a large entropy contact discontinuity or temperature contact discontinuity as they travel through a tube with variable cross section. Such temperature contact discontinuities are, for example, produced by bursting of a diaphragm in a tube. The bursting causes a shock and "centered" expansion disturbances, due to instantaneous expansion, to travel in the tube. The strength of the shock and the extent of the expansion are determined by the condition that equal pressures and velocities establish themselves in the flow between the shock and the expansion. A temperature contact discontinuity occurs because the expansion lowers the temperature; whereas the shock raises the temperature and, thus, causes two flow fields of different temperatures to be in contact. In figure 7 the results of bursting a diaphragm are presented in a diagram of y against a_0t.

In the calculations presented for case A, relations had to be established for the variation of the flow variables along the time-history curves of the disturbances moving upstream and downstream. In the calculation of the temperature discontinuity, the variation of the flow variables along the time history of the discontinuity must also be known. For this purpose the condition that the entropy is constant during the time history of a fluid particle must be considered.

As a result of this condition, the ratio of the sound velocities on both sides (1 and 1'; see fig. 7) of the temperature contact discontinuity is constant along the time-history curve. The variation of the flow variables along the time-history curve of the temperature contact discontinuity is, thus, given by:

$$\frac{v_1'}{a_0} = \frac{v_1}{a_0} \tag{29}$$

and

$$\frac{a_1'/a_0}{a_1/a_0} = D \tag{30}$$

where D is a constant. The ratio $\frac{a_1'/a_0}{a_1/a_0}$ is determined by the shock strength and the extent of the centered expansion disturbances. The relations may also be conveniently expressed in terms of λ and μ (equations (4) and (5)):

$$\lambda_1' - \mu_1' = \lambda_1 - \mu_1 \tag{31}$$

$$D = \frac{1 + \frac{\gamma - 1}{4}(\lambda_1' + \mu_1')}{1 + \frac{\gamma - 1}{4}(\lambda_1 + \mu_1)} \tag{32}$$

The flow variables along the time-history curve of the temperature contact discontinuity can now be calculated. (See fig. 8 in which the shock and the centered expansion waves at $t = 0$ are shown by dashed lines.) The following points are known from previous construction: Point 1 $\left(a_0t_1, y_1, F_1, \frac{dF_1}{dy_1}, \frac{v_1}{a_0}, \frac{a_1}{a_0}, \Pi_1, \psi_1\right)$ is situated somewhere along the time history of the temperature discontinuity (also indicated in fig. 7). Point 2 $\left(a_0t_2, y_2, F_2, \frac{dF_2}{dy_2}, \frac{v_2}{a_0}, \frac{a_2}{a_0}, \Pi_2, \psi_2\right)$ is situated either on the shock (see fig. 8) or along a disturbance traveling in the direction of the positive y-axis. Points 7 $\left(a_0t_7, y_7, F_7, \frac{dF_7}{dy_7}, \frac{v_7}{a_0}, \frac{a_7}{a_0}, \Pi_7, \psi_7\right)$ and 9 $\left(a_0t_9, y_9, F_9, \frac{dF_9}{dy_9}, \frac{v_9}{a_0}, \frac{a_9}{a_0}, \Pi_9, \psi_9\right)$ are situated on the time-history curve of a disturbance traveling in the direction of the negative y-axis. Points 4 $\left(a_0t_4, y_4, F_4, \frac{dF_4}{dy_4}, \frac{v_4}{a_0}, \frac{a_4}{a_0}, \Pi_4, \psi_4\right)$ and 5 $\left(a_0t_5, y_5, F_5, \frac{dF_5}{dy_5}, \frac{v_5}{a_0}, \frac{a_5}{a_0}, \Pi_5, \psi_5\right)$, lying on either side of the time-history curve of the temperature discontinuity, are to be calculated. The values of a_0t, y, and F for point 4 are identical to those of point 5.

The first operation in the calculation is concerned with the intersection of the tangent of the time-history curve of the disturbance at point 2 (traveling in the negative y-direction) with the tangent of the time-history curve of the temperature discontinuity at point 1. The intersection results in

$$a_0t_4 = \frac{s - r}{p - q}$$

and

$$y_4 = pa_0t_4 + r$$

where

$$p = \frac{\lambda_1 - \mu_1}{2}$$

$$q = \left(\frac{v - a}{a_0}\right)_2 - 1 - \frac{\gamma + 1}{4}\mu_2 + \frac{3 - \gamma}{4}\lambda_2$$

$$r = pa_0t + y_1$$

and

$$s = -qa_0t_2 + y_2$$

The values of a_0t_4 and y_4 are identical with the values of a_0t_5 and y_5, respectively.

Use is then made of the fact that the tangent to the time-history curve of a disturbance moving in the direction of the positive y-axis must satisfy the following conditions:

The quantities a_0t_3, y_3, F_3, $\frac{dF_3}{dy_3}$, λ_3, and μ_3 must assume values such that the tangent to the time-history curve of the disturbance in the positive y-direction at point 3 will pass through the previously determined point 5; thus,

$$\left(\frac{v + a}{a_0}\right)_3 = 1 + \frac{\gamma + 1}{4} \lambda_3 - \frac{3 - \gamma}{4} \mu_3$$

$$= \frac{y_5 - y_3}{a_0t_5 - a_0t_3} \tag{33}$$

Equation (33) contains four unknowns. Three additional equations for the quantities a_0t_3, y_3, λ_3, and μ_3 are obtained by assuming linear relationships between the quantities y, λ, μ, and a_0t at the points 7, 9, and 3. Thus,

$$\left.\begin{aligned} y_3 &= ma_0t_3 + b \\ \lambda_3 &= ka_0t_3 + d \\ \mu_3 &= la_0t_3 + e \end{aligned}\right\} \tag{34}$$

where

$$m = \frac{y_9 - y_7}{a_0t_9 - a_0t_7}$$

$$k = \frac{\lambda_9 - \lambda_7}{a_0t_9 - a_0t_7}$$

$$l = \frac{\mu_9 - \mu_7}{a_0t_9 - a_0t_7}$$

$$b = -ma_0t_7 + y_7$$

$$d = -ka_0t_7 + \lambda_7$$

and

$$e = -la_0t_7 + \mu_7$$

Substituting the values for y_3, λ_3, and μ_3 from equations (34) into equation (33) yields the following expression for a_0t_3

$$a_0t_3 = \frac{-H \pm \sqrt{H^2 - 4GK}}{2G}$$

where

$$G = \frac{3 - \gamma}{4} l - \frac{\gamma + 1}{4} k$$

$$H = -1 + \frac{\gamma + 1}{4} ka_0t_5 - \frac{\gamma + 1}{4} d - \frac{3 - \gamma}{4} la_0t_5 + \frac{3 - \gamma}{4} e + m$$

and

$$K = a_0t_5 + \frac{\gamma + 1}{4} a_0t_5d - y_5 + b - \frac{3 - \gamma}{4} a_0t_5e$$

The quantities λ_4, μ_4, λ_5, and μ_5 can now be obtained with the aid of the relations for the changes of the flow variables along the time history of the waves $\overline{3,5}$, $\overline{2,4}$, and the temperature contact discontinuity $\overline{1,4}$.

The variations of the flow variables along the time-history curves of the downstream and upstream waves are given by equations (17) and (18). Equations (31) and (32), that is, the variation along the time-history curve of the temperature contact discontinuity, are written more conveniently as

$$\lambda_5 - \mu_5 = \lambda_4 - \mu_4$$

$$1 + \frac{\gamma - 1}{4}\left(\lambda_5 + \mu_5\right) = D\left[1 + \frac{\gamma - 1}{4}\left(\lambda_4 + \mu_4\right)\right]$$

The following equations are thus obtained for the two remaining unknowns:

$$\lambda_4 = \frac{2}{D + 1} \lambda_5 - \frac{D - 1}{D + 1}\left(\frac{4}{\gamma - 1} + \mu_4\right) \qquad (35a)$$

$$\mu_5 = \frac{D - 1}{2}\left(\frac{4}{\gamma - 1} + \lambda_4\right) + \frac{D + 1}{2} \mu_4 \qquad (35b)$$

The quantity Π_5 is on the expansion side of the temperature contact discontinuity and, hence, is equal to one; in this case $\psi_5 = 0$. The quantity Π_4 is constant during the time history of the temperature contact discontinuity (particle) and, hence, has the value corresponding to the shock which was initially produced by the bursting of the diaphragm; in this case $\psi_4 = 0$.

II.– STUDY OF THE FLOW PRODUCED BY THE BURSTING INTO A VACUUM OF A DIAPHRAGM AT THE MINIMUM CROSS SECTION OF A SUPERSONIC NOZZLE

GENERAL CONSIDERATIONS

As an introduction to the case of the bursting of a diaphragm in a nozzle, that is, a tube with variable cross section, the nature of the simpler problem of bursting a diaphragm in a tube with constant cross section is briefly discussed. A thorough account of this problem is given in reference 12; a statement is also made therein that seems a good introduction to the problem of bursting:

On first thought, one might be led to believe that at the instance of bursting the diaphragm the total pressure jump across the diaphragm would be propagated toward the low-pressure side as shock wave. However, this is not possible, for then the entire air mass on the high-pressure side of the diaphragm would have to be suddenly accelerated to the speed of the mass of air behind the shock.

The gradual acceleration of the mass of air on the high-pressure side is accomplished by the fact that the total-pressure jump across the diaphragm produces not only a shock traveling into the low-pressure side but also a sudden expansion spreading gradually into the high-pressure side.

In setting up the calculations for the present investigation of a nozzle for high supersonic Mach numbers, an infinite pressure ratio across the diaphragm was chosen. This is equivalent to having a vacuum on the low-pressure side of the diaphragm. Furthermore, the assumption was made that the high-pressure tank is sufficiently large to keep the pressure constant in it. The choice of a vacuum appears to be of considerable advantage since conditions for the existence of a shock are not fulfilled for that case, and thus, only the effect of the spreading of an isentropic sudden expansion of large amplitude into a tube with variable cross section must be calculated.

Figure 9 gives a presentation in the plot of y (distance along the tube) against a_0t of the sudden expansion into a vacuum for a tube with constant cross section. The plot of y against a_0t is a modification of the length-against-time diagram which is convenient because it presents the velocities of the disturbances and the fluid elements in nondimensional form. The lines radiating from the coordinate origin $\left(\frac{v-a}{a_0}\text{ lines}\right)$ in figure 9 and the curved line, substituting for a group of parallel lines, $\left(\frac{v+a}{a_0}\text{ lines}\right)$ represent the two groups of characteristics. Since initially the sudden expansion has the nature of a flow through constant cross section, only the radiating characteristics are carriers of disturbances. The group of curved parallel characteristics are not carriers of disturbances; however, they are necessary for the analysis of the flow structure. Since this problem is one of unsteady flow through constant cross section, one of the parameters λ and μ of the characteristic families has to be constant. In this particular case λ will be constant since it is the down-tube characteristic which cuts across the infinity of radiating characteristics carrying the small disturbances which build up the finite-amplitude disturbance. Thus, λ indicates the variation of the flow variables in the finite-amplitude disturbance. If the expansion starts from air at rest and the increase of fluid velocities in the plot of y against a_0t is assumed to be positive, the slope of the time histories of fluid particles increases as indicated by the v/a_0 line in figure 9; furthermore, for the case of the expansion starting from air at rest λ is equal to zero throughout the expansion. The flow variables are then related as follows (see equation (4)):

$$\lambda = \frac{2}{\gamma - 1}\left(\frac{a}{a_0} - 1\right) + \frac{v}{a_0} = 0$$

or

$$\frac{a}{a_0} = 1 - \frac{\gamma - 1}{2}\frac{v}{a_0} \tag{36}$$

The speed of the disturbances traveling along the radiating characteristics, and therewith the slope of these time histories, is given by

$$\frac{1}{a_0}\frac{dy}{dt} = \frac{v - a}{a_0} \tag{37}$$

It may be seen from equations (36) and (37) that the slope of the radiating time histories of disturbances varies from −1 to $\frac{2}{\gamma - 1}$ (which equals 5 for air); for $v = a$ the slope of the time history is zero. (Fig. 9, representing the initial flow conditions, is drawn in the scale of 2:5 in agreement with the final steady-flow pattern, fig. 13; the final flow pattern had to be drawn in that scale to avoid crowding the characteristic lines.) Aside from giving the limits indicated, equations (36) and (37) show that, in the subsonic domain of the expansion, the disturbances travel "up-tube" and in the supersonic domain they travel "down-tube"; whereas relative to the fluid they always travel upstream. The difference in behavior between the subsonic and the supersonic domain of the unsteady expansion can also be brought out well by observing its variation with time for two cross sections, one located at a negative y-value and the other at a positive y-value. At the negative y-value (subsonic domain) the expansion will cause a/a_0 (and thus the pressure) to decrease; whereas at the positive y-values (supersonic domain) the expansion will cause the pressure to increase. The group of $\frac{v + a}{a_0}$ characteristics carries zero disturbances for the case of a single large-amplitude expansion through constant cross section. (The same is true for the analogous problem of a single expansion – Prandtl-Meyer – for two-dimensional steady supersonic flow.) For the case of flow through variable cross section, however, the $\frac{v + a}{a_0}$ lines represent time histories of disturbances participating in the flow development due to the inclination of tube walls; these disturbances are often called reflected disturbances.

Since it is desirable to anticipate at least some of the results, a discussion of the physical nature of the problem precedes the detailed investigation of the calculations for the unsteady flow through variable cross section (nozzle). So far in this paper the behavior of the disturbance of large amplitude has been discussed from a consideration of the microscopic elements; that is, the behavior of small (characteristic) disturbances (the elements of a large-amplitude disturbance) moving along the characteristic families has been under scrutiny. While the combination of these microscopic elements is simple for the case of a large-amplitude disturbance traveling through constant cross section and, thus, permits the gaining of a thorough understanding of the physical aspects for this case, for the motion through variable cross section not much insight can be gained without making the actual calculations. The reason lies in the fact that for the case of variable cross section within the large-amplitude disturbance as it moves past the inclined tube walls, many local reflections and re-reflections are created which interfere with each other as they travel along different families of characteristics. The determination of the deformation of the large-amplitude disturbance consequently requires a point-by-point integration process and an iteration procedure. For the case of constant cross section, however, the correlation between the microscopic and macroscopic behavior will be simple since no reflections are created within the disturbance. It is thus necessary to look to viewpoints other than the microscopic in order to anticipate the results of the calculation.

A lead on some other physical aspect of the motion of an unsteady large disturbance may be easily obtained from consideration of the fact that, for the case of bursting a diaphragm in a supersonic nozzle, the central problem is concerned with the process of reaching steady flow. The problem of balance between unsteady and steady flow energy is the basic problem for the motion of large unsteady disturbances in general and not just for the special case discussed herein. The anticipated flow pattern in this special case is thus the steady-flow pattern given by Bernoulli's equation for one-dimensional flow.

In order to find the proper basis for comparison with the pattern of characteristics (time histories) of figure 9, representing the initial conditions, the final steady flow has also to be interpreted from consideration of the time histories of small (characteristic) disturbances moving through it in the downstream or in the upstream direction with the speeds $v + a$ and $v - a$, respectively. The nondimensional form of these speeds is easily obtained from Bernoulli's equation. For ideal gases Bernoulli's equation can be written in the form:

$$\frac{a_o}{a_0} = \sqrt{1 - \frac{\gamma - 1}{2}\left(\frac{v_o}{a_0}\right)^2} \tag{38}$$

or

$$\frac{a_o}{a_0} = \frac{1}{\sqrt{1 + \frac{\gamma - 1}{2} M_o^2}} \tag{39}$$

and

$$\frac{v_o}{a_0} = \frac{M_o}{\sqrt{1 + \frac{\gamma - 1}{2} M_o^2}} \tag{40}$$

then

$$\frac{v_o - a_o}{a_0} = \frac{M_o - 1}{\sqrt{1 + \frac{\gamma - 1}{2} M_o^2}} \tag{41}$$

and

$$\frac{v_o + a_o}{a_0} = \frac{M_o + 1}{\sqrt{1 + \frac{\gamma - 1}{2} M_o^2}} \tag{42}$$

For the nozzle shape used in this paper and for air as a medium, the time histories of small disturbances, used to correlate the steady flow with the unsteady flow conditions, are given in figure 10. In figure 10 only three time histories are drawn, two of these are the time histories of small disturbances moving upstream with the speed $v_o - a_o$ in the subsonic and supersonic steady flow. The third is representative of the time histories of small disturbances moving downstream with the speed $v_o + a_o$.

For the anticipation of the results of the calculations it is further useful to present the transition from unsteady to steady flow in a plot which simply demonstrates the deviations of the flow variables from the steady and the unsteady state. Such a plot is one with the coordinates a/a_0 and v/a_0 (fig. 11). In this plot the sudden unsteady expansion into the vacuum for a tube with constant cross section (initial condition) is represented by a straight line given by equation (36)

$$\frac{a}{a_0} = 1 - \frac{\gamma - 1}{2} \frac{v}{a_0}$$

Bernoulli's equation for steady flow is represented by an ellipse with the equation

$$\left(\frac{a_o}{a_0}\right)^2 + \frac{\gamma - 1}{2}\left(\frac{v_o}{a_0}\right)^2 = 1 \tag{43}$$

It is important for the further discussion of transition from unsteady to steady flow to determine the limiting conditions for $\frac{v}{a_0}$, $\frac{v - a}{a_0}$, and $\frac{v + a}{a_0}$. From equation (36) the following values are obtained: maximum velocity for unsteady flow through constant cross section,

$$\frac{v}{a_0} = \frac{2}{\gamma - 1}$$

which equals 5 for air, and maximum velocity for steady flow

$$\frac{v_o}{a_0} = \sqrt{\frac{2}{\gamma - 1}}$$

which equals $\sqrt{5}$ for air. Since for the maximum velocity the velocity of sound a is zero, the maximum of the velocity is the same as that of $\frac{v - a}{a_0}$ and $\frac{v + a}{a_0}$. Further values of importance are those of the critical velocities. For unsteady flow through constant cross section, the critical velocity is

$$\frac{v}{a_0} = \frac{2}{\gamma + 1}$$

which equals 0.833 for air and for steady flow

$$\frac{v_o}{a_0} = \sqrt{\frac{2}{\gamma + 1}}$$

which equals 0.9129 for air.

The value of $\frac{v - a}{a_0}$ corresponding to the critical velocity is zero in both cases. Thus in the plot of y against a_0t the "critical" expansion wave coincides with the a_0t axis.

Two useful properties of the presentation of the transition problem in the coordinate system of a/a_0 against v/a_0 or the related system $\sqrt{2i}$ against v are now discussed. One of these useful properties is that the Mach numbers are represented by straight lines radiating from the center of the coordinate system. The other useful property concerns the geometric presentations of the speed of the waves for arbitrary gases as intersects of the $\frac{v}{a_0}$ axis. In one presentation (fig. 11) the intersects are obtained by subtending from the abscissa v/a_0 of the point A $\left(\frac{v}{a_0}, \frac{a}{a_0}\right)$ the length of the ordinate a/a_0 in the positive direction or in the negative direction depending on whether one desires to obtain the values of $\frac{v + a}{a_0}$ or that of $\frac{v - a}{a_0}$ at the point A. In the other presentation (fig. 12) the speeds of the waves for arbitrary gases are obtained by presenting the steady-flow ellipse with the equation

$$2i + v^2 = 2i_0 \tag{44}$$

where i is the enthalpy, as a circle with the radius $\sqrt{2i_0}$ in the coordinate system of $\sqrt{2i}$ against v. The intersects along the v-axis are made for this presentation by the normals to the characteristics $\lambda = f\left(\frac{a}{a_0}\right) + \frac{v}{a_0}$ and $\mu = f\left(\frac{a}{a_0}\right) - \frac{v}{a_0}$. The latter presentation was first given by Busemann (reference 13); it can also be extended to the presentation of the speed of shock waves in this coordinate system. The exact expressions for λ and μ for arbitrary gases are given in reference 4.

The reason for the simple geometrical picture of the speed of the waves lies in the following facts: The characteristic theory for arbitrary as well as for ideal gases is based on the fact that in the vicinity of each point of the flow field, the flow may be linearized (reference 14). Furthermore, use is made of the fact that the motion of the wave may be expressed in terms of a coordinate system moving with the wave. In such a coordinate system, the wave will stand still, but the fluid will be moving with a velocity $v - a$ or $v + a$. Since the steady flow is expressed by a circle in this diagram, the variation of the velocity of the locally superposed steady flow is given by the motion of the center of the circle along the v-axis.

METHOD OF CALCULATIONS

Design of Nozzle

The shape of the nozzle is given by a simple analytic expression that approximates the shape of a conventional wind tunnel designed to give a steady-flow Mach number of 5 in the test section. With the area ratio thus given as 25, the length and the shape of the nozzle still had to be determined. The length was determined such as to give the shortest nozzle for the given Mach number and was calculated to be $16.5887\sqrt{F_{min}}$ or $16.5887 \frac{\sqrt{\pi}}{2} d_{min}$. The shape of the nozzle was given by two analytic expressions. For lack of a more exact criterion, the subsonic part was chosen on the basis of the reasonable criterion that the maximum gradient of cross section of the nozzle presented as a one-dimensional flow should not be larger than the gradient of surface area of half a spherical wave traveling into a gas at rest. Thus, if F and y are made nondimensional by referring them to F_{min} and $\sqrt{F_{min}}$, respectively, the variation in cross-sectional area is given by the parabola

$$F = 2\pi y^2 + 1$$

The supersonic part of the nozzle was chosen to have the variation of cross section of a parabola tangent to the cross-sectional area of the test section and to the parabola $F = 2\pi y^2 + 1$ continued to positive y-values. The equation thus obtained was

$$F = -0.08843918(y^2 - 33.17786342y + 275.192655279) + 25$$

The point of tangency of the two parabolas turned out to be

$$F = 1.33312388$$

and

$$y = 0.230257055$$

The high number of decimal places has its basis in the fact that a smooth junction between the two analytic expressions requires considerable accuracy. This increased accuracy offers no difficulty to the Bell computer since it uses a high number of decimal places in all computations.

Flow Calculations

As previously mentioned in the section "General Considerations," the calculations are concerned with the problem of transition from the initial purely unsteady flow condition (fig. 9) to the final steady flow condition. The initial conditions for the construction of the net of characteristics in terms of the quantities v/a_0, a/a_0, λ, μ, $\frac{v + a}{a_0}$, $\frac{v - a}{a_0}$, and $\beta = \tan^{-1} \frac{v - a}{a_0}$ are given in table IV; their magnitudes are obtained from equation (36). In the last columns of the table the values for $\frac{v - a}{a_0}$ or $\tan \beta$ multiplied by the scale factor of 2/5 and the corresponding angle $\tan^{-1}\left(\frac{2}{5} \tan \beta\right)$ are given. As previously stated, these values had to be used in the construction of figure 9 in order to show a basis of comparison with the net of characteristics in figure 13. The calculations were started at the $\frac{v - a}{a_0}$ time history of the characteristic disturbance which moves into the gas at rest $\left(\frac{v - a}{a_0} = -1\right)$ by subtending from it $\frac{v + a}{a_0}$ characteristics at an interval of 0.1y. Starting with these initial conditions the flow field was calculated according to the step-by-step process given in part I with the aid of the Bell computer.

DISCUSSION OF RESULTS OF CALCULATIONS

The results of the calculations are presented in the plot of y against a_0t (fig. 13), as well as in the plot of a/a_0 against v/a_0 (fig. 14). In the plot of y against a_0t (fig. 13) the criterion for the attainment of steady flow is given by the condition that the $\frac{v - a}{a_0}$ and $\frac{v + a}{a_0}$ lines have to become parallel to those shown in figure 10 for steady flow. It can be seen that this will be true for a range of higher values of a_0t. The relative position of the two $\frac{v - a}{a_0}$ lines and the $\frac{v + a}{a_0}$ line of figure 10 has no significance. The proper comparison between figures 10 and 13 is obtained by comparing the $\frac{v-a}{a_0}$ lines and the $\frac{v+a}{a_0}$ line of figure 10 individually with the corresponding lines in figure 13 by shifting them individually along the a_0t axis. In the diagram of a/a_0 against v/a_0 (fig. 14) the transition from unsteady to steady flow conditions is conveniently expressed by the variation of the flow variables v/a_0 and a/a_0 at various locations y at the nozzle. Steady flow conditions are obtained when the y = Constant lines in the diagram of a/a_0 against v/a_0 reach the steady-flow ellipse or, in terms of Mach number, when the Mach numbers along the y = Constant lines reach the steady-flow Mach number corresponding to the nozzle cross section at y. Since the plot in figure 14 does not permit the reading of values with sufficient accuracy, the values of v/a_0, a/a_0, and M for several values of y are given in table V. The corresponding values of a_0t are also given in table V to indicate the rate of change of the flow variables. The full transition from unsteady to steady flow is represented in the subsonic range of the plot of a/a_0 against v/a_0 by y = Constant lines starting at the point $\left(\frac{a}{a_0} = 1, \frac{v}{a_0} = 0\right)$, since, as may be seen from the plot of y against a_0t, y = Constant lines first intersect the disturbance $\frac{v-a}{a_0} = -1$. In the supersonic range, the y = Constant lines will start at the point $\left(\frac{v}{a_0} = 5, \frac{a}{a_0} = 0\right)$, since the y = Constant lines will first intersect the disturbances $\frac{v-a}{a_0} = \frac{v+a}{a_0} = 5$. For the purpose of identification of the intersections of the $\frac{v+a}{a_0}$ lines and the $\frac{v-a}{a_0}$ lines in figure 13, the lines are denoted in the following manner: The $\frac{v+a}{a_0}$ lines carry numbers ranging from −0.1 to −2.4; the numbers refer to their initial conditions on the line $\frac{v-a}{a_0} = 1$. The $\frac{v-a}{a_0}$ lines carry angles β varying from −45° to 78°41'; the angles β represent the starting angles of these lines for the true-scale initial conditions. (See also table IV.) The scarcity of calculated points (see table V) near the maximum values in the supersonic region has the following reason: In the region where the maximum speed of the fluid $\frac{v}{a_0} = 5$ is obtained, the maximum speeds of the small (characteristic) disturbances $\frac{v-a}{a_0} = 5$ and $\frac{v+a}{a_0} = 5$ will also be obtained. They are indicated by the line denoted $\beta = 78°41'$ in figure 13. (See also table IV.) Since the method of calculation is based on the intersection of time histories of $\frac{v+a}{a_0}$ and $\frac{v-a}{a_0}$ disturbances, the size of the step of the step-by-step process will become excessive in the region near the maximum speed unless a very fine net of characteristics is used in this region. This fine net will also be necessary if higher iterations than those in the present calculations are used. The use of a not-too-fine net (used in the present calculations) will cause the $\frac{v + a}{a_0}$ lines and the $\frac{v - a}{a_0}$ lines to become parallel before the physical limit for parallelity ($\tan^{-1} 5 = 78^{\circ}41'$) is reached, namely, at $\tan^{-1} 2.3558 = 67^{\circ}$. No efforts were made, however, to increase the fineness of the net in this region since for the phenomenon to be investigated it was of no particular interest; and in spite of the use of the Bell computer, the use of an extremely fine net in every part of the flow field would cause the time required for the calculations to increase by factors up to 20. (The calculations of one point required about 6 minutes.) It should also be mentioned here that only a small number of the calculated points are indicated in the diagram of y against a_0t (fig. 13).

It appears that the statement concerning the fact that the very-high-speed region was of no particular interest could bear a more detailed explanation. Comparison of figure 13 with figure 10, the final steady-flow picture, shows that most of the supersonic expansion disturbances will take no part in forming the steady-flow picture; they will just disappear and the steady-flow picture will be formed without their help. The question arises now concerning the behavior of the reflected $\frac{v + a}{a_0}$ disturbances. Figure 13 shows that the reflected disturbances in the supersonic region move even at a greater speed in the same direction (+y) as the expansion disturbances. This behavior of the expansion disturbances and their reflections is based on the previously discussed fact that the waves are produced in a coordinate system moving with the speed of the fluid. When the speed of the fluid is supersonic, both the $\frac{v - a}{a_0}$ and the $\frac{v + a}{a_0}$ disturbances move downstream and the $\frac{v + a}{a_0}$ disturbances have greater speed.

A closer investigation of figure 13 and table V also shows that the $y = \text{Constant}$ lines do not completely reach the required values on the steady-flow ellipse. The reason lies in the following facts: The nozzle chosen for the calculations was a typical supersonic nozzle designed for parallel flow at the test section. In such nozzles the large variations in cross section have to occur near the minimum section; thus, large changes in Mach number also occur in this region. This again means that in the region near the minimum cross section large changes in the speed of the $\frac{v - a}{a_0}$ and the $\frac{v + a}{a_0}$ disturbances also occur. In consequence, the intersections of the $\frac{v - a}{a_0}$ and $\frac{v + a}{a_0}$ lines yield inaccurate results unless very small steps are taken. The fact that steps taken too large could have such an effect was first noted during the calculations in the results obtained for the $\frac{v + a}{a_0}$ time histories. The excessively large steps caused the $\frac{v + a}{a_0}$ lines beyond those with the starting point of $y = -1$ and $a_0t = 1$ $\left(\frac{v + a}{a_0}\text{ line } -1\right)$ to become parallel and even to intersect. As pointed out previously in the discussion of figure 10, parallelity of the time histories of the disturbances would indicate that steady flow had been reached. It was found, however, that the reaching of parallelity was premature since the values of v/a_0 and a/a_0 attained in this region were still far from the steady-flow values for the given nozzle shape.

A much smaller division of the $\frac{v - a}{a_0}$ lines was then taken from this region on until similar inaccuracies were noted again for the $\frac{v + a}{a_0}$ disturbance starting at $y = -2.4$ $\left(\frac{v + a}{a_0}\text{ line } -2.4\right)$. Since steady flow conditions corresponding to the shape of the nozzle had almost been reached by then, a further reduction in step size was not attempted. In this slightly premature stoppage of the calculations the subsonic flow part was especially affected, since as may be seen from figure 13 the $\frac{v + a}{a_0}$ line -2.3 still cuts through low values of a_0t especially in the low-subsonic region. Inaccuracies can also be noted in the reaching of such higher steady-flow supersonic Mach numbers as correspond to values of $y = 2$ and $y = 4$. This larger error is due to a growth of the error introduced by the rapid changes in cross section near the minimum section. Since the error increased for increasing values of y, the y = Constant lines for higher values of y (up to $y = 16.5887$) were not plotted in figure 14. The line $M_0 = 5$ is, however, included in figure 14 (dashed line) in order to indicate the range of the final values for the flow through the nozzle designed for a steady-flow Mach number of 5.

For lower values of y (0.5 and 1) the error introduced by rapid variation of cross section was still small and the y = Constant lines came very close to reaching the required steady-flow Mach numbers. (See fig. 14 and table V.) The gradual approach to the steady-flow values (indicated in table V) is the result not only of the approximate nature of the calculations but also of the fact that steady flow actually presents an asymptotic condition which it would take an infinite time to reach. An important conclusion that can be drawn from the calculations is that the main changes in flow variables occur very quickly and that the remaining changes in the infinite time interval are very small. The small number of decimal places used in table V compared to the high number of decimal places used in table IV is adequate for the accuracy of the calculations based on the use of relatively large steps. Further useful information for the transition from unsteady to steady flow may also be obtained by observing the path of the lines $y = 0.5$ and $y = 1$ in the diagram of a/a_0 against v/a_0. The extension of these lines into the area of the final steady-flow ellipse signifies that the final steady flow at first tries to establish itself at a similar lower speed range, that is, along a similar smaller steady-flow ellipse, before it is boosted up to the final steady flow which has the critical pressure corresponding to the tank pressure at the minimum cross section of the nozzle. The boosting process is the reason for the consequent zigzagging of the $y = \text{Constant}$ lines. Another interesting effect that occurs in the near-to-maximum speed region concerns the speed of the $\frac{v - a}{a_0}$ waves in this region; namely, that they have to reduce their speed in the transition from unsteady to steady flow, since the unsteady-flow maximum speed is higher $\left(5a_0\right)$ than the steady-flow maximum speed $\left(\sqrt{5}a_0\right)$.

In order to facilitate the interpretation of figure 14, the steady-flow Mach numbers corresponding to y values of -0.2325, 0, 0.5, 1, and 2 were indicated by the $M = \text{Constant}$ lines. This auxiliary construction is especially useful for $y = 0$ (minimum cross section) since for that case both the initial unsteady flow condition and the final steady flow lie on the $M = 1$ line.

Finally, the time required for the transition from unsteady to steady flow is discussed. Table V shows that the values of a_0t closest to the steady-flow Mach numbers to be reached lie roughly around $a_0t = 4.5$. From the previous discussion it can be concluded that this value will be only very slightly too low in spite of the fact that for an inviscid fluid steady flow is reached asymtotically after an infinite time. (For the calculations the rounded-off value $a_0t = 5$ is used.) This behavior is in agreement with the fact that unsteady disturbances once created in an inviscid fluid do not disappear. From a practical viewpoint, though, after a very short time the unsteady disturbances become sufficiently small so that they are completely dissipated by viscous effects.

Since the dimension of a_0t is the same as y, that is, a length, it can be conveniently expressed in terms of the diameter of the minimum cross section of the nozzle. For a nozzle of circular cross section, both a_0t and y have to be multiplied by $\sqrt{\pi/4}$ since $d = \sqrt{4/\pi}$ for $F_{min} = 1$. An a_0t of 5 then corresponds to $5\,d\sqrt{\pi/4}$. The time it takes to reach steady flow is given by $d_{min} \frac{5\sqrt{\pi/4}}{a_0}$. Since a_0 is a constant, the time it takes to reach steady flow increases with the nozzle size for a given shape. For a d_{min} of 1 foot and room temperature in the tank, for example, the time it takes to reach steady flow will be $\frac{5\sqrt{\pi/4}}{1116} \approx \frac{4}{1000}$ second.

The general expression for the transition time of the flow in a nozzle with given shape but varying size may be extended to include nozzles of affine shapes (stretched or shortened shapes). This extension seems easily understandable inasmuch as the length of a nozzle rather than the diameter is the determining factor in the calculation of the transition time, since the length directly affects the running time of the disturbances. The similarity of unsteady-flow phenomena for nozzles with the same or affine shape but varying size can also be said to signify that a "similarity rule" exists for unsteady flows through tubes with variable cross section. This fact is of importance to the test engineer.

Through a process of affine shortening of a given nozzle shape, a shape with sudden change in cross section is finally obtained, as indicated in figure 15. For this sudden change in cross section, steady flow should establish itself after zero transition time. (These considerations are based on one-dimensional theory.) On the basis of such considerations, the motion of a disturbance of large amplitude through variable cross section may be substituted by that through a series of cylindric tubes with interspaced abrupt changes in cross section. Such a scheme was used in reference 15 without, however, giving thorough explanation as to why it should be permissible to use the scheme. The nature of the difference between the effects of tubes with gradual changes in cross section and sudden changes is brought out to some degree in figure 15. The modifications of the original disturbance in this case consist of a weakened disturbance continuing to move in the same direction and of a reflected disturbance in the opposite direction. Since all the changes of the original disturbance take place at the same location y, the continuing weakened disturbance as well as the reflected disturbance may be substituted by large sudden disturbances (group of dashed centered lines) originating at this location. Each centered disturbance may, furthermore, be substituted by a single line. From the discussion of the small values of transition time it can also be surmised that the scheme here discussed converges to the characteristic method.

Finally, a few practical aspects of the short-duration tests are discussed. It is important to emphasize that, although practically steady flow conditions will be reached in these very-short-duration tests, for most materials commonly used for the construction of the test model and the tunnel walls, conditions of temperature equilibrium between models, walls, and air flow will not be reached. Because the model and the walls will remain close to their initial temperature, the amounts of heat transfer through the boundary layer will be different from those encountered in the customary steady-flow tests. The herewith-connected changes in boundary-layer behavior (transition and others) have to be studied before drawing quantitative conclusions from such tests. It should be realized that the constancy of initial temperature of the model and the walls considered from a different viewpoint also makes it possible to conduct tests with arbitrarily chosen model and wall temperatures at a given steady-flow Mach number. Conclusively, it should be added that the main difficulty of such very-short-duration tests will be the problem of instrumentation.

PART III.- STUDY OF INVARIANT INTEGRALS OF UNSTEADY FLOW DISTURBANCES OF LARGE AND SMALL AMPLITUDES

GENERAL CONSIDERATIONS

In part III the behavior of unsteady flow disturbances is treated from a viewpoint different from that in parts I and II where a detailed study of the complicated pattern of local growths, reflections, and re-reflections within the disturbance was made. In part III integrals over the whole disturbance invariant with respect to time are developed. The value of such invariant integrals is doubtful as long as the disturbance, its reflections, and re-reflections belong to the integrand. Thus, the main purpose of the present paper is to find integrands which permit a separation of the integral effects of the growth of a disturbance and its reflections. The logical choice of an integral for the separation is one which uses the characteristic parameters of the disturbance as the integrand. These characteristic parameters consist of two summands, the velocity increment due to the disturbance, and another quantity of the dimension of a velocity which represents the pressure increment in the disturbance. Depending on whether the sum or the difference of these two velocity increments is used, these parameters are associated with disturbances traveling with the speed of sound to the right and to the left relative to the fluid. (For more detailed statement see section entitled "General Considerations" of part I.)

The integrals with the characteristic parameters as integrands were introduced in reference 5 under the name of pulse areas. In the language of the engineer and the physicist, the invariance of this integral is also called its conservation. This word causes the question to arise immediately as to what relation the conservation of the pulse areas would have to one of the three conservation laws, the mass, energy, and momentum laws. The relation can be quickly found out from the previous definition of the characteristic parameters as a linear combination of two velocity increments, one of them representative of the pressure increment in the disturbance. It can be easily seen that the sum of these two parameters P and Q (or λ and $-\mu$, respectively) will be twice the increment of the flow velocity in the disturbance.

Since the integral contains now a velocity increment as integrand, it is related to the conservation law for the momentum. It has to be emphasized, though, that only a relation exists, but no identity, since the integral of the velocity increment is actually the velocity potential. The conservation of the potential here replaces the conservation of the momentum which, in contrast to the tube with constant cross section, is not conserved because of unknown pressures at the inclined walls. Since the invariant integral for the pulse area has played such a useful part in expressing an invariance of an unsteady disturbance, it is useful to express the invariant integrals for mass and energy in terms of the pulse area. These expressions have been given in reference 6 without introducing, however, the pulse area and applying the restriction of the short disturbance which is necessary for the complete separation of the growths (positive and negative) and reflections of a disturbance. The reasons for the introduction of the short disturbance are given subsequently.

So far the first step in the process of separating the growth in the disturbance and the reflected disturbances has been shown which consists in the introduction of the pulse area. This introduction, however, still does not eliminate the re-reflections from the integrand – it only labels them and discriminates odd and even numbers of reflections. The final step in the separating process consists in uncoupling the growth and the reflections of the disturbance and its reflections by neglecting the re-reflections during the time a single reflection is being produced. This separation (or uncoupling) of the original disturbance and its reflection is achieved by restricting the length and the amplitude of the original disturbance and the inclination of the tube walls to small values (short disturbance).

ANALYSIS

Conservation of Potential

It is shown in references 10 and 14 that for a fluid assumed to be inviscid a velocity potential exists for the motion of an unsteady flow disturbance of large or small amplitude in a steady flow or in a fluid at rest through variable cross section. It is also shown in references 10 and 14 that the potential consists of two parts:

$$d\phi = v\ dy - \left(\frac{v^2}{2} + i\right)dt \tag{45}$$

The next step is to show the invariance of the potential with respect to time. This problem is conveniently investigated in the coordinate system of y against a_0t (see fig. 16). For the sake of discussion a disturbance of large amplitude is assumed to be produced at the time $t = 0$ in such a manner that behind the disturbance its amplitude is zero. The potential ϕ is indicated by the area of the cross-hatched rectangle. The height of the rectangle presents the amplitude of the disturbance, which is, as is shown subsequently, the velocity increment through the disturbance. The height, of course, does not lie in the y,a_0t plane, but it is conveniently indicated in the figure in this plane. The unsteady flow disturbance under consideration is assumed to be an isentropic expansion of an arbitrary gas produced at $t = 0$ and to move through an increasing cross section. In the course of its motion the amplitude and the length of the disturbance change; at the same time reflected disturbances with given amplitude and length are given off. In figure 16 the reflected potentials are indicated by the long, thin cross-hatched areas; whereas the potentials of the growing disturbance are shown by short, thick cross-hatched areas. In order to show the invariance of the potential ϕ with respect to time, equation (45) has to be integrated along a path which does not enclose the region in which the disturbance is produced, the production of the disturbance being an extraneous process (for example, unsteady motion of piston, bursting of diaphragm, unsteady motion of side walls, combustion or condensation front). The integration along such a path is of the following form:

$$\oint d\phi = \int_A^B v\ dy - \int_D^C v\ dy + \int_B^C \left(\frac{v^2}{2} + i\right)dt - \int_A^D \left(\frac{v^2}{2} + i\right)dt = 0 \qquad (46)$$

Since the disturbance is assumed to move in a steady stream, the difference of the integrals with the Bernoulli constant $\left(i_0 = \frac{v^2}{2} + i\right.$ as the integrand$\left.\right)$ is zero. The remaining integrals are equal to each other. The significance of this result is as follows:

When an unsteady isentropic disturbance of large amplitude travels in a steady flow of an arbitrary gas through variable cross section or through a gas at rest, the potential of the growing (positively or negatively) disturbance itself with the addition of the potential due to all its reflections remains constant with respect to time; that is, $\int v\ dy = \text{Constant}$.

The application of this general law to the motion of strong shocks through variable cross section seems doubtful, since when entropy variations appear in the flow the potential ceases to exist. It would seem possible, perhaps, to enforce the existence of a potential even for such a case by choosing a path of integration which encloses positive and negative gradients in entropy. The situation is analogous to that for vortices in steady flow where, at least for skillfully chosen cases (Prandtl's starting vortex), the proper path of integration exists for the maintenance of the concept of potential.

Although the potential $\int v\, dy$ has yielded useful conservation laws for the behavior of a large disturbance moving through variable cross section, the information obtained from this law is incomplete since it only concerns one flow variable and, consequently, does not permit a separation of the integral effects of the disturbances. As indicated in the section entitled "General Considerations," the introduction of the pulse areas $\int P\, dy$ and $\int Q\, dy$ is a first step toward the separation. The parameters P and Q are given for unsteady flow of an arbitrary gas through a variable cross section by

$$\left.\begin{aligned} P &= v + f(a) \\ Q &= v - f(a) \end{aligned}\right\} \tag{47a}$$

or for an ideal gas

$$\left.\begin{aligned} P &= v + \frac{2}{\gamma - 1} a \\ Q &= v - \frac{2}{\gamma - 1} a \end{aligned}\right\} \tag{47b}$$

Conservation of Pulse Areas

A conservation law for the pulse areas may be obtained from the conservation law $\phi = \int v\, dy = \text{Constant}$ for the potential ϕ by expressing the velocity v in terms of the amplitudes P and Q given by equations (47). The following conservation law for the pulse areas results:

$$\int (P + Q) dy = \text{Constant} \tag{48a}$$

or

$$\int P\,dy + \int Q\,dy = \text{Constant} \tag{48b}$$

This conservation law for the combination of the pulse areas for the disturbance of large amplitude, however, has no direct value for the calculation of the behavior of the large disturbance since P and Q (or λ and μ) are interrelated by the system of two nonlinear differential equations shown, for example, by equations (6) and (7) of part I. As stated previously, two cases exist for which the integral effects of the disturbances can be fully separated. One case is that of short disturbances which is discussed in other sections of part III; the other is the case of motion of a large-amplitude disturbance in a steady flow without Mach number gradient through constant cross section. The specification "without Mach number gradient" is made to exclude the case of constant cross section at a steady-flow Mach number around one.

As previously stated, the motion of a large disturbance through constant cross section is characterized by the fact that P is constant for a disturbance with the amplitude Q, and Q is constant for a disturbance with the amplitude P. Thus, the fact is at once established that for a large disturbance the individual pulse areas are conserved; that is,

$$\int P\,dy = \text{Constant} \tag{49a}$$

$$\int Q\,dy = \text{Constant} \tag{49b}$$

In order to avoid confusion it is well to point out the difference between the symbols P and Q used in reference 5 and the symbols λ and μ used in references 4 and 10. The parameter λ and μ are given for arbitrary gases by

$$\left.\begin{aligned} \lambda &= f\left(\frac{a}{a_0}\right) + \frac{v}{a_0} \\ \mu &= f\left(\frac{a}{a_0}\right) - \frac{v}{a_0} \end{aligned}\right\} \tag{50a}$$

or for an ideal gas by

$$\left.\begin{aligned}\lambda &= \frac{2}{\gamma - 1}\left(\frac{a}{a_0} - 1\right) + \frac{v}{a_0}\\ \mu &= \frac{2}{\gamma - 1}\left(\frac{a}{a_0} - 1\right) - \frac{v}{a_0}\end{aligned}\right\} \tag{50b}$$

The equations are the same as equations (4) and (5) but are repeated here for convenience. The difference in presentation between P and Q (equations (47)) and λ and μ (equations (50)), respectively, may be stated directly in terms of the flow variables a and v for ideal gases. Expressed in terms of P and Q (reference 5)

$$v = \frac{P + Q}{2} \tag{51a}$$

and

$$a = \frac{\gamma - 1}{2}\,\frac{P - Q}{2} \tag{51b}$$

whereas in terms of λ and μ

$$\frac{v}{a_0} = \frac{\lambda - \mu}{2} \tag{52a}$$

and

$$\frac{a}{a_0} = 1 + \frac{\gamma - 1}{2}\,\frac{\lambda + \mu}{2} \tag{52b}$$

The convention of writing the velocity as the difference of two flow parameters λ and μ is taken from steady two-dimensional supersonic flow (see reference 16). The convention of presenting the velocity in terms of the sum of the two flow parameters P and Q is taken from reference 17.

Finally, a few remarks seem in place concerning the statement made in this section that only for the cases of a short disturbance and of constant cross section can the integral effects of the disturbances be fully separated. It is shown in reference 15 (see, also, discussion in part II) that an approximate method of separation exists for a large-amplitude disturbance based on substituting the continuous change in cross section by a discontinuous change. This approximate method makes skillful use of the behavior of a large-amplitude disturbance in constant cross section.

Relative Growth (Positive and Negative) and Reflection of the Amplitude of a Short Disturbance at a Given Steady-Flow Mach Number, Due to a Given Cross-Sectional Gradient or Mach Number Gradient

In the process of determining the growth and the reflection of the pulse area of a disturbance the behavior of their amplitudes has to be investigated. The equations for the behavior of the amplitudes of a small disturbance are given in references 5 and 6. As mentioned in the introduction, however, neither of the reference papers fully deal with the growth and the reflection of the pulse area or the amplitude of a short pulse. For this reason the derivation in the present paper is started with the basic equations for the amplitudes of small disturbances moving through a steady flow in variable cross section. The equations for air ($\gamma = 1.4$) (see reference 5, equations (14) and (15)) are as follows:

$$\frac{\partial P}{\partial t} + \left(v_o + a_o + \frac{3P + 2Q}{5}\right)\frac{\partial P}{\partial y} + \left(1 - M_o\right)\left(\frac{3P + 2Q}{5}\right)\frac{dv_o}{dy} + \left(M_o^2 - 1\right)\left(\frac{P^2 - Q^2}{20v_o} + \frac{P + Q}{2M_o} + \frac{P - Q}{10}\right)\frac{dv_o}{dy} = 0 \tag{53a}$$

$$\frac{\partial Q}{\partial t} + \left(v_o - a_o + \frac{2P + 3Q}{5}\right)\frac{\partial Q}{\partial y} + \left(1 + M_o\right)\left(\frac{2P + 3Q}{5}\right)\frac{dv_o}{dy} - \left(M_o^2 - 1\right)\left(\frac{P^2 - Q^2}{20v_o} + \frac{P + Q}{2M_o} + \frac{P - Q}{10}\right)\frac{dv_o}{dy} = 0 \tag{53b}$$

For the case of an arbitrary gas the multiples 2 and 3 of P and Q, respectively, are given by the expressions $\frac{3-\gamma}{2(\gamma-1)}$ and $\frac{1+\gamma}{2(\gamma-1)}$, respectively, derived from equations (51a) and (51b). The denominators 5, 10, and 20 are given by the expressions $\frac{2}{\gamma-1}$, $\frac{4}{\gamma-1}$, and $\frac{8}{\gamma-1}$. For the sake of brevity, equations (53a) and (53b) for air are used in the derivation; the quantity γ is only introduced into the final equations. The sums $\frac{3P+2Q}{5}$ and $\frac{2P+3Q}{5}$ are equal to the sums $v_o + a_o$ and $v_o - a_o$, respectively. In the case of small disturbances the amplitudes P and Q are given by a linear combination of a velocity increment and a pressure increment (expressed in the form of an increment of the velocity of sound) with respect to steady-flow conditions. Since in the expressions for the P and Q pulse (equations (47a) and (47b)) the pressure function f(a) has opposite signs, the alternative of compression or expansion enters into P and Q with opposite signs. The flow velocity (the significant parameter for potential, momentum, and pulse areas) enters into P and Q with the same sign. For the parameters λ and μ the situation is reversed. The motion of the P pulse is designated in the downstream direction by making the arbitrary down-tube direction coincide with the downstream direction.

Equations (53a) and (53b) are identical with equations (14) and (15) in reference 5, which can be derived from the equations given for the variation of the flow variables along characteristic lines for isentropic unsteady disturbances of large amplitude given in part I and in references 4, 5, and 10. For the case of small-amplitude disturbances, P and Q are small and are of the first order. Furthermore, as is indicated by equations (53a) and (53b), the first derivatives of P and Q are also small and of the first order.

For these conditions the following approximations can be made:

(1) The term $\frac{P^2 - Q^2}{20v_o}$ may be considered negligible when compared with the expression $\frac{P+Q}{2M_o} + \frac{P-Q}{10}$.

(2) The term $\frac{3P+2Q}{5}$ may be considered negligible when compared with $v_o + a_o$.

(3) The term $\frac{2P+3Q}{5}$ may be considered negligible when compared with $v_o - a_o$, with certain qualifications.

The qualifications mentioned in approximation (3) are due to the fact that as M_o approaches one, $v_o - a_o$ approaches zero. If $v_o - a_o$, however, approaches zero, the term $\frac{2P + 3Q}{5}$ may not be neglected. In view of these conditions, the behavior of the Q pulse and its reflections has to be investigated separately for the case of $M_o = 1$. The assurance whether or not the term $\frac{2P + 3Q}{5}$ may be actually neglected near $M_o = 1$ may be obtained by comparing the results of the separate investigation at $M_o = 1$ with the results obtained by neglecting $\frac{2P + 3Q}{5}$ at $M_o = 1$.

The approximations made so far in equations (53a) and (53b) are all a direct consequence of the use of disturbances of small amplitude. The approximations, however, still do not simplify the simultaneous differential equations (53a) and (53b) sufficiently to make the solution easily interpretable. For a simple solution of the simultaneous equations, a type of uncoupling of P and Q would be desirable. More specifically, the uncoupling would signify that in the equations (53a) and (53b) P and Q could be alternately neglected depending on whether one is attempting to solve for the growths of the pulse or for the reflected pulses. The "uncoupling" process was substantially used in reference 5 without explaining its full meaning. For a clear understanding of all approximations made in the calculations, the concept of uncoupling is explained briefly: For example, as a Q pulse moves through a tube with variable cross section, its amplitude Q grows to $Q \pm dQ$ and gives off a reflected pulse $\pm dP$. The building-up process of the reflected pulse P (traveling with the speed $v_o + a_o$), during its motion through the growing Q pulse to its full strength dP, starts from zero immediately behind the head of the Q pulse and ends when the reflected pulse at its full strength dP leaves the pulse Q. Because of the building-up process, P is at first very small compared with the average Q. The smallness of P compared with Q can be kept forever when the Q pulse has a small length dy and when the gradient of cross section dF/dy or, consequently, $dF = \frac{dF}{dy}\,dy$, the change in cross section of the tube occupied by the disturbance Q, is small. A disturbance for which amplitude, length, and cross-sectional gradient are subject to these restrictions is called a short disturbance (or pulse) in the present paper. For the same reason, for the growing of the pulses to $Q \pm dQ$, P may be neglected compared with Q. The analogous considerations are true for $P \pm dP$ and the reflections dQ, where Q may be neglected compared with P.

Before attempting to solve equations (53a) and (53b) for the desired quantities, the equations are written in different form by performing the following transformations:

$$\frac{\partial P}{\partial t} + \left(v_o + a_o + \frac{3P + 2Q}{5}\right)\frac{\partial P}{\partial y} = \left(v_o + a_o + \frac{3P + 2Q}{5}\right)\frac{dP}{dy} \tag{54a}$$

and

$$\frac{\partial Q}{\partial t} + \left(v_o - a_o + \frac{2P + 3Q}{5}\right)\frac{\partial Q}{\partial y} = \left(v_o - a_o + \frac{2P + 3Q}{5}\right)\frac{dQ}{dy} \tag{54b}$$

At first equations (54a) and (54b) are substituted in equations (53a) and (53b); use is made of approximation (3) without qualifying assumptions for $M_o = 1$, that is, with neglect of $\frac{2P + 3Q}{5}$ compared with $v_o - a_o$.

Since presenting the deformation and the reflection of pulses in terms of the variation in cross section is desirable, the following substitution obtained from the continuity equation for steady flow is made additionally:

$$\frac{dv_o}{dy} = \frac{v_o}{M_o^2 - 1}\frac{1}{F}\frac{dF}{dy} \tag{55}$$

The substitutions transform equations (53a) and (53b) into

$$(v_o + a_o)\frac{dP}{dy} + \frac{v_o}{M_o^2 - 1}\frac{1}{F}\frac{dF}{dy}\left[(1 - M_o)\left(\frac{3P + 2Q}{5}\right) + \left(M_o^2 - 1\right)\left(\frac{P + Q}{2M_o} + \frac{P - Q}{10}\right)\right] = 0 \tag{56a}$$

and

$$(v_o - a_o)\frac{dQ}{dy} + \frac{v_o}{M_o^2 - 1}\frac{1}{F}\frac{dF}{dy}\left[(1 + M_o)\left(\frac{2P + 3Q}{5}\right) - \left(M_o^2 - 1\right)\left(\frac{P + Q}{2M_o} + \frac{P - Q}{10}\right)\right] = 0 \tag{56b}$$

The fact that dy may be canceled in equations (56a) and (56b) has the significance that the equations are independent of the scale of y. In other words, the equations for short disturbances may also be interpreted to apply to small-amplitude disturbances of arbitrary length traveling through a small discontinuous change in cross section. As for the case of short disturbances no interference exists while the reflection is being produced; the lack of interference is due to the fact that the reflection is produced in the zero length of the discontinuity and that before and behind the small discontinuity the cross section is constant. This last interpretation of the approximation of short disturbances establishes a connection with the approach given in reference 15. The behavior of a small-amplitude disturbance of arbitrary length traveling through a small continuous variation in cross section may be obtained by pairing small discontinuous changes together in such a manner that the distances between them equal the lengths of the growth or the reflection of the small disturbances. The variation of the lengths of disturbances with Mach number is discussed in a subsequent section.

The following relations can now be determined:

dP/P represents the relative growth dP of the pulse P.

dQ/P represents the relative reflection dQ produced by the pulse P.

dQ/Q represents the relative growth dQ of the pulse Q.

dP/Q represents the relative reflection dP produced by the pulse Q.

The following expressions for an arbitrary gas are obtained from equations (56a) and (56b) by making use of equations (40) to (42) of part II and the previously developed "uncoupling" effects, that is, alternately neglecting P and Q:

$$\frac{dP}{P} = -\frac{1}{2} \frac{1 + \frac{\gamma - 1}{2} M_o^2}{(M_o + 1)^2} \frac{dF}{F} \tag{57a}$$

$$\frac{dQ}{P} = -\frac{1}{2} \frac{1 - \frac{\gamma - 1}{2} M_o^2}{(M_o - 1)^2} \frac{dF}{F} \tag{57b}$$

$$\frac{dQ}{Q} = -\frac{1}{2} \frac{1 + \frac{\gamma - 1}{2} M_o^2}{(M_o - 1)^2} \frac{dF}{F} \tag{57c}$$

$$\frac{dP}{Q} = -\frac{1}{2} \frac{1 - \frac{\gamma - 1}{2} M_o^2}{(M_o + 1)^2} \frac{dF}{F} \tag{57d}$$

Equations (57) reduce to the acoustic case for $M_0 = 0$. Expressing equations (57) also in terms of the steady-flow Mach number gradient is useful for the remaining investigations in this paper. This is done by the substitution of

$$-\frac{dF}{F} = \frac{1 - M_0^2}{1 + \frac{\gamma - 1}{2} M_0^2} \frac{dM_0}{M_0}$$

obtained from the steady-flow continuity equation into equations (57). The following equations result

$$\frac{dP}{P} = \frac{1 - M_0}{1 + M_0} \frac{dM_0}{2M_0} \tag{58a}$$

$$\frac{dQ}{P} = -\frac{M_0 + 1}{M_0 - 1} \frac{1 - \frac{\gamma - 1}{2} M_0^2}{1 + \frac{\gamma - 1}{2} M_0^2} \frac{dM_0}{2M_0} \tag{58b}$$

$$\frac{dQ}{Q} = -\frac{M_0 + 1}{M_0 - 1} \frac{dM_0}{2M_0} \tag{58c}$$

$$\frac{dP}{Q} = \frac{1 - M_0}{1 + M_0} \frac{1 - \frac{\gamma - 1}{2} M_0^2}{1 + \frac{\gamma - 1}{2} M_0^2} \frac{dM_0}{2M_0} \tag{58d}$$

A closer study of equations (57) and (58) reveals first of all the interesting fact that the amplitudes of the relative reflections expressed by the ratios dP/Q and dQ/P change their signs at the supersonic steady-flow Mach number $M_0 = \sqrt{\frac{2}{\gamma - 1}}$ which equals $\sqrt{5}$ for air; that is, at $M_0 = \sqrt{\frac{2}{\gamma - 1}}$ both the P and the Q pulses give off no reflections. A more detailed study of equations (57) and (58) is given subsequently.

Equation (57d) indicates that, for example, for the upstream travel of a Q pulse in the subsonic part of a diffuser $(0 < M_0 < 1)$ the reflected pulse dP is negative since it moves in the downstream direction $\left(\frac{dF}{F} > 0\right)$ from zero (while it is being produced from zero).

Since the factor $1 - \frac{\gamma - 1}{2} M_0^2 > 0$ for $0 < M_0 < 1$, the sign of dP consisting of the product of one negative quantity and two positive quantities, that is, the product $(-)(+)(+)$ is negative. In order to determine whether the reflection dP is of the expansion or compression type if Q is, for example, a compression, equations (47) must be considered. They state that P and Q are of the opposite sign. Thus, if according to equation (57d) dP has a sign opposite to Q, the additional information of equations (47) indicates that dP and Q are of the same type; that is, if Q is a compression, dP is a compression also. For $M_0 = 0$, this result is in agreement with the well-known acoustic behavior. The result obtained may also be conveniently discussed in terms of equation (58d) which uses the Mach number gradient. For $0 < M_0 < 1$, the product of signs $(+)(+)(-)$ is negative $(dM_0 < 0)$ which naturally agrees with the result obtained from equation (57d). Another especially interesting illustration of equations (57) and (58) concerns the reflections dQ produced by a P pulse traveling downstream in the supersonic part of a nozzle. In this case, both the pulse and its reflection travel down the tube since for supersonic flow an upstream motion with the speed $v_0 - a_0$ is a down-tube motion. The behavior of the reflections of a P pulse is described by equations (57b) and (58b). For $1 < M_0 < \sqrt{\frac{2}{\gamma - 1}}$ the sign product $(-)(+)(+)$ is negative for equation (57b) and $(-)(+)(+)$ is negative for equation (58b). Since P and Q are of the opposite sign, both the original pulse and the reflection are of the same type. For $\sqrt{\frac{2}{\gamma - 1}} < M_0 < \infty$ the sign products for (57b) and (58b) are both $(-)(-)(+)$ which is positive; thus, in this case the amplitude of the original pulse and that of the reflection are of opposite types. With the aid of the sign products the types of the growths and reflections of the downstream-moving P pulses and the upstream-moving Q pulses not mentioned so far may be also determined from equations (57) and (58) with two exceptions. The two exceptions concern the behavior of the reflection dQ produced by P (equations (57b) and (58b)) and the growth dQ produced by Q (equations (57c) and (58c)) at a steady-flow Mach number of $M_0 = 1$. At that Mach number, dQ becomes infinite with respect to both the amplitudes of the pulse P and the pulse Q. This growth toward infinity is in contradiction to the assumption of small disturbances for which this result is obtained. Equations (57) and (58) only hold for a range in which the order of magnitude of "small" is not exceeded for the amplitude dQ. The range can be increased such as to include the closest proximity of $M_0 = 1$ by choosing values close to zero for the amplitudes P and Q of the pulses. The deterioration of dQ at $M = 1$ itself, however, cannot be eliminated. Similar considerations apply to the gradient $\partial Q/\partial y$.

Another case concerning the behavior of the amplitude of a small disturbance near a steady-flow Mach number of one concerns the steady-flow equilibrium conditions involving a shock near $M_o = 1$. A discussion of the equilibrium of shocks, however, is outside the scope of this paper.

The discussion so far gives a very unsatisfactory picture about the singular behavior of a small disturbance at $M_o = 1$. This picture does not seem to show anything significant, for it does not seem reasonable that an important quantity like the energy or the mass of a reflected disturbance dQ produced by a downstream disturbance P should become infinite at $M_o = 1$. The reason for the discrepancy between the physical expectations and the results obtained so far lies in the fact that only amplitudes of the small disturbances have been discussed so far and not the integrals over the whole disturbance extension as required for the mass, energy, and pulse area. A discussion of these integral quantities is given in the following section.

Before going into these problems, the behavior of the amplitude dQ near $M_o = 1$ is explained, without neglect of the term $\frac{2P + 3Q}{5}$ compared with $v_o - a_o$ in equation (53b). For a short disturbance at $M_o = 1$, equation (53b) yields:

$$\frac{dQ}{Q} = -\frac{2}{5} \frac{3}{v_o - a_o + \frac{3Q}{5}} dv_o \tag{59}$$

if P is neglected (uncoupled) with respect to Q. The neglect (uncoupling) of Q with respect to P (equation (53b)) yields

$$\frac{dQ}{P} = -\frac{2}{5} \frac{2}{v_o - a_o + \frac{2P}{5}} dv_o \tag{60}$$

By comparing equations (59) with equations (57c) and (58c) and equation (60) with equations (57b) and (58b), it may be seen that similarly to equations (57c), (58c), (57b), and (58b), equations (59) and (60) have an infinity. The only difference is that the infinity of equations (59) and (60) does not occur at $M_o = 1$, but rather at a Mach number corresponding to a velocity $v_o - a_o + \frac{3Q}{5}$. Similar considerations can be made for equation (61) and equations (57b) and (58b). The physical meaning of these more exact equations is that the growths or the reflections of the Q pulses do not actually accumulate exactly at the minimum cross section of the nozzle.

Conservation of Mass, Energy, and Sum of Pulse Areas for Short Disturbances

The conservation of mass and energy for large and small disturbances traveling through tubes with variable cross section without the benefit of uncoupling between the original disturbance and its reflection is discussed in reference 6. For the case of short (uncoupled) disturbances it is clear that the amount of mass, energy, and pulse area of the growth of the original disturbance equals the amount of mass, energy, and pulse area, respectively, traveling in the reflected disturbance, since no mass, energy, or pulse area can disappear by the interference between the original and the reflected disturbance during the building-up process. The proof of the conservation for short disturbances is given subsequently for the simplest case of the pulse area. For the sake of convenience, equation (48b) is rewritten here:

$$\int P\,dy + \int Q\,dy = \text{Constant}$$

Under the assumption that the growth and the reflection of the pulse area have the shape of a rectangle with the lengths Δy_P and Δy_Q, equation (48b) can be presented in the following form:

$$P\,\Delta y_P + Q\,\Delta y_Q = \text{Constant} \tag{61a}$$

or

$$d\left(P\,\Delta y_P + Q\,\Delta y_Q\right) = 0 \tag{61b}$$

The differentiation in equation (61b) results in the following equation:

$$dP\,\Delta y_P + P\,d\Delta y_P + dQ\,\Delta y_Q + Q\,d\Delta y_Q = 0 \tag{62}$$

If one chooses, for example, to express the conservation law of a relative growth dP/P and the relative reflection dQ/P, for the case of a short P pulse, Q starts from zero and the last term can be neglected. The conservation law thus assumes the following form:

$$dP\,\Delta y_P + P\,d\Delta y_P = -\Delta y_Q dQ \tag{63a}$$

or

$$d(P\,\Delta y_P) = -\Delta y_Q dQ \tag{63b}$$

Equation (63b) indicates that the pulse area of the growth $d(P\,\Delta y_P)$ equals the reflected pulse area, since P and Q are of opposite sign. Equation (63a) may also be expressed in terms of the relative amplitudes given in equations (57) and (58):

$$\frac{dP}{P} + \frac{d\Delta y_P}{\Delta y_P} = -\frac{\Delta y_Q}{\Delta y_P}\frac{dQ}{P} \tag{64}$$

Pulse Length and Pulse Time for Short Disturbances

It was stated previously that the travel of a short disturbance through a small continuous change in cross section is identical with the travel of a small disturbance through a small discontinuous change in cross section. For the case of a small discontinuous change in cross section (see fig. 17), the pulse times Δt_P and Δt_Q are equal and the Mach number gradients are equal and opposite. These conditions are expressed by the following equations:

$$\Delta y_P = \Delta t_P(v_o + a_o) \tag{65}$$

$$\Delta y_Q = \Delta t_Q(v_o - a_o) \tag{66}$$

It should be recalled that without the restriction to a short disturbance, reflections and re-reflections occur which will necessitate a step-by-step integration for the determination of the pulse lengths and pulse times.

Relative Growth and Reflection of the Pulse Area of a Short Disturbance Due to a Given Mach Number Gradient, as a Function of the Steady-Flow Mach Number

The behavior of the relative reflection $d(Q\,\Delta y_Q)/P\,\Delta y_P$ of the pulse area $P\,\Delta y_P$ and of the relative growth $d(Q\,\Delta y_Q)/Q\,\Delta y_Q$ of the pulse area $Q\,\Delta y_Q$ is investigated first since their amplitudes were shown to degenerate for $M_o = 1$. The relative reflection may be written as follows:

$$\frac{d(Q\,\Delta y_Q)}{P\,\Delta y_P} = \frac{dQ}{P}\frac{\Delta y_Q}{\Delta y_P} + \frac{Q}{P}\frac{d\Delta y_Q}{\Delta y_P} \tag{67}$$

The second term is small as Q starts at zero; therefore,

$$\frac{d(Q\,\Delta y_Q)}{P\,\Delta y_P} = \frac{dQ}{P}\frac{\Delta y_Q}{\Delta y_P} \tag{68}$$

The terms dQ/P and $\Delta y_Q/\Delta y_P$ are given in terms of the Mach number. The relative reflection dQ/P is expressed already in equation (58b). The ratio $\Delta y_Q/\Delta y_P$ is formed by substituting equations (42) and (41) of part II into equations (65) and (66), respectively, which gives

$$\Delta y_P = a_o \Delta t_P \frac{M_o + 1}{\sqrt{1 + \frac{\gamma - 1}{2} M_o^2}} \tag{69a}$$

and

$$\Delta y_Q = a_o \Delta t_Q \frac{M_o - 1}{\sqrt{1 + \frac{\gamma - 1}{2} M_o^2}} \tag{69b}$$

The required ratio of Δy_Q and Δy_P is then given by

$$\frac{\Delta y_Q}{\Delta y_P} = \frac{v_o - a_o}{v_o + a_o} = \frac{M_o - 1}{M_o + 1} \tag{70}$$

The value of the relative reflection is then

$$\frac{d(Q\ \Delta y_Q)}{P\ \Delta y_P} = -\frac{M_o + 1}{M_o - 1}\,\frac{1 - \frac{\gamma - 1}{2} M_o^2}{1 + \frac{\gamma - 1}{2} M_o^2}\,\frac{M_o - 1}{M_o + 1}\,\frac{dM_o}{2M_o}$$

$$= -\frac{1 - \frac{\gamma - 1}{2} M_o^2}{1 + \frac{\gamma - 1}{2} M_o^2}\,\frac{dM_o}{2M_o} \tag{71}$$

Thus, the magnitude (not the shape) of the relative reflection of the pulse has no singularity at $M_o = 1$. The relative growth $d(Q\ \Delta y_Q)/Q\ \Delta y_Q$ may be written as follows:

$$\frac{d(Q\ \Delta y_Q)}{Q\ \Delta y_Q} = \frac{dQ}{Q} + \frac{d\Delta y_Q}{\Delta y_Q} \tag{72}$$

The quantity dQ/Q in terms of Mach number is already expressed in equation (58c). The ratio $d\Delta y_Q/\Delta y_Q$ is obtained by logarithmic differentiation of equation (69b)

$$\frac{d\Delta y_Q}{\Delta y_Q} = d\ \log \Delta y_Q = \frac{dM_o}{M_o - 1} - \frac{\frac{\gamma - 1}{2} M_o dM_o}{1 + \frac{\gamma - 1}{2} M_o^2} \tag{73}$$

The value of the relative growth is then

$$\frac{d(Q\ \Delta y_Q)}{Q\ \Delta y_Q} = -\frac{M_o + 1}{M_o - 1}\,\frac{dM_o}{2M_o} + \frac{dM_o}{M_o - 1} - \frac{\frac{\gamma - 1}{2} M_o dM_o}{1 + \frac{\gamma - 1}{2} M_o^2}$$

$$= \frac{dM_o}{2M_o} - \frac{\frac{\gamma - 1}{2} M_o dM_o}{1 + \frac{\gamma - 1}{2} M_o^2} \tag{74}$$

Thus the relative growth of the pulse area has no singularity at $M_o = 1$.

Next, the behavior of the relative reflection $d(P\ \Delta y_P)/Q\ \Delta y_Q$ and of the relative growth $d(P\ \Delta y_P)/P\ \Delta y_P$ of the pulse area is investigated. The relative reflection may be written as follows:

$$\frac{d(P\ \Delta y_P)}{Q\ \Delta y_Q} = \frac{dP}{Q}\frac{\Delta y_P}{\Delta y_Q} + \frac{P}{Q}\frac{d\Delta y_P}{\Delta y_Q} \tag{75}$$

For short disturbances (P neglected) the relative reflection is

$$\frac{d(P\ \Delta y_P)}{Q\ \Delta y_Q} = \frac{dP}{Q}\frac{\Delta y_P}{\Delta y_Q} \tag{76}$$

Equation (76) is written in terms of Mach number by substituting equations (58d) and (70)

$$\frac{d(P\ \Delta y_P)}{Q\ \Delta y_Q} = \frac{1 - M_o}{1 + M_o}\,\frac{1 - \frac{\gamma - 1}{2} M_o^2}{1 + \frac{\gamma - 1}{2} M_o^2}\,\frac{M_o + 1}{M_o - 1}\,\frac{dM_o}{2M_o}$$

$$= -\frac{1 - \frac{\gamma - 1}{2} M_o^2}{1 + \frac{\gamma - 1}{2} M_o^2}\,\frac{dM_o}{2M_o} \tag{77}$$

The interesting fact is indicated by equation (77) that the relative reflection of the pulse area $Q\ \Delta y_Q$ is zero at $M_o = \sqrt{\frac{2}{\gamma - 1}}$ which equals $\sqrt{5}$ for air but will not be zero at $M_o = 1$, in spite of the fact that the amplitude is zero there. In reference 5 it is concluded from the zero amplitude that the relative reflection of the pulse area would have to be zero at $M_o = 1$. In the next section of the present paper a numerical evaluation of this inaccuracy is given. The relative growth $d(P\ \Delta y_P)/P\ \Delta y_P$ may be written as follows:

$$\frac{d(P\ \Delta y_P)}{P\ \Delta y_P} = \frac{dP}{P} + \frac{d\Delta y_P}{\Delta y_P} \tag{78}$$

Equation (78) is expressed in terms of Mach number by substituting equation (58a) and the logarithmically differentiated form of equation (69a). The logarithmic differentiation of equation (69a) results in

$$\frac{d\Delta y_P}{\Delta y_P} = d \log \Delta y_P = \frac{dM_o}{1 + M_o} - \frac{\frac{\gamma - 1}{2} M_o dM_o}{1 + \frac{\gamma - 1}{2} M_o^2} \tag{79}$$

The relative growth then is

$$\frac{d(P \Delta y_P)}{P \Delta y_P} = \frac{1 - M_o}{1 + M_o} \frac{dM_o}{2M_o} + \frac{dM_o}{1 + M_o} - \frac{\frac{\gamma - 1}{2} M_o dM_o}{1 + \frac{\gamma - 1}{2} M_o^2}$$

$$= \frac{dM_o}{2M_o} - \frac{\frac{\gamma - 1}{2} M_o dM_o}{1 + \frac{\gamma - 1}{2} M_o^2} \tag{80}$$

Thus, the relative growths of the pulse areas $P \Delta y_P$ (equation (80)) and $Q \Delta y_Q$ (equation (74)) as well as the relative reflections (equations (77) and (71)) are the same.

Accumulated Growths and Reflections of the Pulse Area of a Short Disturbance as a Function of Steady-Flow Mach Number

The accumulated growths and reflections are obtained by integrating the relative growths and reflections, respectively. The integration of the relative growths $d(P \Delta y_P)/P \Delta y_P$ and $d(Q \Delta y_Q)/Q \Delta y_Q$ can be performed without difficulties. The integration (accumulation) of the relative reflections, however, cannot be presented in closed form for the following reasons: The short disturbance during its motion through variable cross section gives off reflections, the accumulated value of which will soon have a length larger than that required for a short disturbance; furthermore, within this accumulated reflection, re-reflections will occur. Consequently, the integration of the reflections would require the use of a point-by-point method like that given in part I. The fact that the values of the accumulated reflections cannot be represented by an integration in closed form is not as unfortunate in the present case as might be expected since a conservation law exists for the pulse area which permits determination of the accumulated relative reflection readily from the easily integrable relative growth. In the following discussion the variation with Mach number of the accumulated relative growths is given. The integration of the relative growths $d(Q \Delta y_Q)/Q \Delta y_Q$ and $d(P \Delta y_P)/P \Delta y_P$, as given in equations (74) and (80), results in

$$\left.\begin{aligned} \text{or} \qquad \int \frac{d\left(Q\ \Delta y_Q\right)}{Q\ \Delta y_Q} &= \int \frac{d\left(P\ \Delta y_P\right)}{P\ \Delta y_P} \\ \log\left(Q\ \Delta y_Q\right) &= \log\left(P\ \Delta y_P\right) \end{aligned}\right\} = \log\left(C\sqrt{\frac{M_o}{1 + \frac{\gamma - 1}{2} M_o^2}}\right) \tag{81}$$

The definite integration from a reference pulse area $Q_0 \Delta y_{Q_0}$ to $Q\ \Delta y_Q$ and the substitution of the definite integral $\int Q\ dy$ over the pulse length Δy_Q at a given time t for $Q\ \Delta y_Q$ result in

$$\int Q\ dy = C_1\sqrt{\frac{M_o}{1 + \frac{\gamma - 1}{2} M_o^2}} \tag{82}$$

The definite integration from a reference pulse area $P_0 \Delta y_{P_0}$ to $P\ \Delta y_P$ and the substitution of the definite integral $\int P\ dy$ over the pulse length Δy_P at a given time t for $P\ \Delta y_P$ result in

$$\int P\ dy = C_2\sqrt{\frac{M_o}{1 + \frac{\gamma - 1}{2} M_o^2}} \tag{83}$$

Equations (82) and (83) are formally equivalent, the difference in constants of the two expressions is due to the fact that the P pulse moves with the velocity $v_o + a_o$, whereas the Q pulse moves with the velocity $v_o - a_o$. In terms of the Mach numbers appearing in equations (82) and (83) the meaning of this difference is that the Mach number in one equation will have the negative value of that in the other. The constants in equations (82) and (83) are thus related by the equation

$$C_2 = jC_1$$

The significance of the imaginary factor j in the equation is that a real P pulse can never become a real Q pulse.

It may be seen from equations (82) and (83) that the accumulated relative growths of the pulse areas $Q\ \Delta y_Q$ and $P\ \Delta y_P$ have a maximum for $M_o = \sqrt{\frac{2}{\gamma - 1}}$ which equals $\sqrt{5}$ for air. The accumulated reflections thus remain constant only for $M_o = \sqrt{\frac{2}{\gamma - 1}}$ which equals $\sqrt{5}$ for air.

(The relative reflection is zero.) Equations (82) and (83) permit a numerical evaluation of the variation of the accumulated growth with steady-flow Mach number (see table VI). Table VI indicates that the assumption made in reference 5 concerning the conservation of pulse area at $M_0 = 1$ is still fairly accurate, since $f(M_0)$ representing the variation with M_0 of the accumulated growth remains fairly constant in the range of $M_0 = 1$ to $M_0 = 5$. The main changes in $f(M_0)$ occur in the regions of low subsonic and high supersonic Mach numbers.

Accumulated Growths and Reflections of the Mass of a Short

Disturbance as a Function of Steady-Flow Mach Number

For reasons similar to those in the case of the pulse area only the accumulated growths are calculated. The basis for the calculation is the following definite integral over the pulse length Δy at a given time t (see also reference 6):

$$\int \rho F \, dy \tag{84}$$

The quantity ρ represents the excess in density compared with steady flow at a given cross section F. The first step in the separation of the growth and the reflection is obtained by expressing ρ in terms of P and Q (see equations (47))

$$P = v + \frac{2}{\gamma - 1} a$$

$$Q = v - \frac{2}{\gamma - 1} a$$

or

$$P + Q = 2v$$

$$P - Q = \frac{4}{\gamma - 1} a$$

The next step in the separation is obtained by the introduction of the assumption of short disturbances. Since for the pulse area the variations of accumulated growths of pulses P and Q gave equations of the same form (equations (82) and (83)), the behavior of one of the pulses is only treated here. For the growth of, for example, the pulse P, Q is neglected; that is,

$$(P)_{Q=0} = 2v = \frac{4}{\gamma - 1} a \tag{85}$$

The quantity a is expressed in terms of ρ for a small disturbance by the following isentropic relation:

$$\frac{\rho}{\rho_o} = \frac{2}{\gamma - 1} \frac{a}{a_o} \tag{86}$$

The substitution of equation (86) into equation (85) results in

$$\rho = \frac{\rho_o}{2a_o} P \tag{87}$$

Substituting equation (87) into equation (84) gives

$$\int \rho F\, dy = \int \frac{\rho_o F}{2a_o}(P\, dy) = \frac{\rho_o F v_o}{2a_o v_o} \int P\, dy \tag{88}$$

where $\rho_o F v_o/2$ has a constant value, let us say, C_3 according to the continuity equation for steady flow. The equation (88) is written in terms of Mach number by substituting for v_o and a_o the values from equations (40) and (39) in part II and for $\int P\, dy$ the expression from equation (83). The following equation results:

$$\int \rho F\, dy = \frac{C_3 C_2}{a_0{}^2} \sqrt{\frac{1 + \frac{\gamma - 1}{2} M_o{}^2}{M_o}} \tag{89}$$

The function $f(M_o) = \sqrt{\left(1 + \frac{\gamma - 1}{2} M_o{}^2\right)\frac{1}{M_o}}$ is tabulated in table VII. At $M_o = \sqrt{\frac{2}{\gamma - 1}}$ which equals $\sqrt{5}$ for air, $f(M_o)$ has a minimum. The physical significance of this behavior is that the accumulated growth of the mass of a short disturbance and, consequently, the reflection will be zero at that Mach number. No special behavior occurs for $M_o = 1$. The infinity indicated in table VII for $M_o = 0$ (acoustic case) is due to the fact that from the point of view of the steady-flow problem, the acoustic case has infinite cross section. The infinity at $M_0 = \infty$ is due to the same reason.

Accumulated Growth and Reflections of the Energy of a Short Disturbance as a Function of Steady-Flow Mach Number

The basis for the calculation of the accumulated growths is the following definite integral over the pulse length Δy at a given time t:

$$\int (E_0 + E)(\rho_0 + \rho)F\ dy - \int E_0\rho_0 F\ dy$$

which when the second-order terms are negligible compared with first-order terms is equal to

$$\int F\rho_0 E\ dy + \int FE_0\rho\ dy \tag{90}$$

In the above expressions, $E + E_0$ is the total convective energy stored per unit mass of the small disturbance

$$\begin{aligned} E + E_0 &= \frac{V^2}{2} + c_v T - e_0 \\ &= \frac{V^2}{2} + \frac{A^2}{\gamma(\gamma - 1)} - e_0 \end{aligned} \tag{91}$$

and E_0 is the contribution of the steady flow to the convective energy of the small disturbance

$$E_0 = \frac{v_0^2}{2} + c_v T_0 - e_0 \tag{92a}$$

or, since E_0 is a steady-flow contribution, it may also be written in the form

$$E_0 = \frac{a_0^2}{\gamma - 1} - \frac{a_0^2}{\gamma} - e_0 \tag{92b}$$

The quantity e_0 is an arbitrary constant, which enters equation (91) because the convective energy is a potential and, therefore, can be referred to an arbitrary level. The terms V, T, and A are the total values of the velocity, absolute temperature, and velocity of sound, respectively. Neglecting the second-order terms, which are negligible when compared with the first-order terms, in equation (91) gives

$$E = v_0 V + \frac{2a_0 a}{\gamma(\gamma - 1)} \tag{93}$$

The substitution of equation (85), (87), (92b), (93), (39), and (40) into equation (90) results in

$$\int F\rho_0 E\, dy + \int FE_0 \rho\, dy = C_3 \int P\, dy + \frac{1 + \frac{\gamma - 1}{2} M_0^2}{(\gamma - 1)M_0} C_3 \int P\, dy - e_0 \frac{C_3}{a_0^2} \frac{1 + \frac{\gamma - 1}{2} M_0^2}{M_0} \int P\, dy \tag{94}$$

The substitution of equation (83) into equation (94) gives

$$\int F\rho_0 E\, dy + \int FE_0 \rho\, dy = C_3 C_2 \left(\sqrt{\frac{M_0}{1 + \frac{\gamma - 1}{2} M_0^2}} + \frac{1}{\gamma - 1}\sqrt{\frac{1 + \frac{\gamma - 1}{2} M_0^2}{M_0}} - \frac{e_0}{a_0^2}\sqrt{\frac{1 + \frac{\gamma - 1}{2} M_0^2}{M_0}} \right) \tag{95}$$

For reasonable values of the constant e_0, the accumulated growth of the energy of a short disturbance shows a behavior similar to that of the pulse area; namely, it has an extremum at $M_0 = \sqrt{\frac{2}{\gamma - 1}}$ which equals $\sqrt{5}$ for air and shows no special behavior at $M_0 = 1$.

CONCLUSIONS

Part I.- The point-by-point method developed for the calculation of unsteady flows through tubes with variable cross section permits a simple presentation of the interaction of strong shocks and large temperature contact discontinuities, a detailed treatment of which had not been given so far. The point-by-point method permits furthermore a presentation of shock calculations and calculations of flows with initial entropy gradient in a form convenient for computation by use of computing machines. Under certain assumptions the formulas established may also be used for the calculation of flows with continuous heat addition over a large space.

Part II.– The calculations, made of the flow pattern created by bursting into a vacuum of a diaphragm at the minimum section of a supersonic nozzle without a second throat, indicate the following: The transition time from the starting of the flow to the attainment of approximately steady flow conditions is sufficiently short to permit the use of very-short-duration tests. The transition time for the specific nozzle is presented in such a form that a "similarity rule" can be established concerning the transition time for nozzles of different size but of the same or affine shape.

Part III.– The restriction to short disturbances permits a simple presentation in terms of steady-flow Mach number of the growths and reflection of pulse area, mass, and energy of a disturbance traveling through a steady flow in a tube with variable cross section. The calculations show the interesting result that the conditions for zero reflection of mass, energy, and pulse area exist at a steady-flow Mach number $M_o = \sqrt{\frac{2}{\gamma - 1}}$ (which equals $\sqrt{5}$ for air), where γ is the ratio of specific heats, rather than at $M_o = 1$; they also show that for practical purposes for the range of Mach number from 1 to 5 the reflections are small enough so that the mass, energy, and pulse area of the original disturbance may be considered constant. However, at low subsonic and at high supersonic Mach numbers, the reflections may not be neglected.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Air Force Base, Va., January 14, 1949

APPENDIX A

DERIVATION OF ANALYTIC EXPRESSION FOR m

An analytic expression for m is obtained by differentiation of the relation $\frac{a_2}{a_1} = f\left(\Delta v/a_1\right)$. This relation is obtained by elimination of $\Delta u/a_1$ from equations (19) and (20). Thus,

$$\frac{a_2}{a_1} = \sqrt{1 + \left[\frac{\left(\frac{\Delta v}{a_1}\right)^2 + \sqrt{\left(\frac{\Delta v}{a_1}\right)^4 + \left(\frac{4}{\gamma + 1}\right)^2\left(\frac{\Delta v}{a_1}\right)^2}}{\frac{4}{\gamma - 1}}\right]\left[\gamma + \frac{\frac{1}{2}\left(\frac{4}{\gamma + 1}\right)^2}{\left(\frac{\Delta v}{a_1}\right)^2 + \sqrt{\left(\frac{\Delta v}{a_1}\right)^4 + \left(\frac{4}{\gamma + 1}\right)^2\left(\frac{\Delta v}{a_1}\right)^2} + \frac{1}{2}\left(\frac{4}{\gamma + 1}\right)^2}\right]}$$

The differentiation results in

$$m = \frac{d(a_2/a_1)}{d(\Delta v/a_1)}$$

$$= \frac{0.5}{a_2/a_1}\left(\left[\gamma + \frac{\frac{1}{2}\left(\frac{4}{\gamma + 1}\right)^2}{\left(\frac{\Delta v}{a_1}\right)^2 + \sqrt{\left(\frac{\Delta v}{a_1}\right)^4 + \left(\frac{4}{\gamma + 1}\right)^2\left(\frac{\Delta v}{a_1}\right)^2} + \frac{1}{2}\left(\frac{4}{\gamma + 1}\right)^2}\right]\left[\frac{2\,\frac{\Delta v}{a_1} + \frac{2\left(\frac{\Delta v}{a_1}\right)^3 + \left(\frac{4}{\gamma - 1}\right)^2 \frac{\Delta v}{a_1}}{\sqrt{\left(\frac{\Delta v}{a_1}\right)^4 + \left(\frac{4}{\gamma + 1}\right)^2\left(\frac{\Delta v}{a_1}\right)^2}}}{\frac{4}{\gamma - 1}}\right]\right.$$

$$\left. - \left[\frac{\left(\frac{\Delta v}{a_1}\right)^2 + \sqrt{\left(\frac{\Delta v}{a_1}\right)^4 + \left(\frac{4}{\gamma + 1}\right)^2\left(\frac{\Delta v}{a_1}\right)^2}}{\frac{4}{\gamma - 1}}\right]\left\{\frac{\frac{1}{2}\left(\frac{4}{\gamma + 1}\right)^2\left[2\,\frac{\Delta v}{a_1} + \frac{2\left(\frac{\Delta v}{a_1}\right)^3 + \left(\frac{4}{\gamma + 1}\right)^2 \frac{\Delta v}{a_1}}{\sqrt{\left(\frac{\Delta v}{a_1}\right)^4 + \left(\frac{4}{\gamma + 1}\right)^2\left(\frac{\Delta v}{a_1}\right)^2}}\right]}{\left[\left(\frac{\Delta v}{a_1}\right)^2 + \sqrt{\left(\frac{\Delta v}{a_1}\right)^4 + \left(\frac{4}{\gamma + 1}\right)^2\left(\frac{\Delta v}{a_1}\right)^2} + \frac{1}{2}\left(\frac{4}{\gamma + 1}\right)^2\right]^2}\right\}\right)$$

APPENDIX B

DERIVATION OF FORMULA FOR THE SLOPE n

The analytic expression for $\Pi = f(a_2/a_1)$ is obtained by substituting into equation (9) the value for

$$\frac{\rho_2}{\rho_1} = \frac{1}{1 - \frac{2}{\gamma + 1}\left(1 - \frac{a_1}{\Delta u}\right)}$$

and then by expressing $a_1/\Delta u$ as a function of a_2/a_1. Thus,

$$\Pi = \frac{1}{\left(\frac{a_2}{a_1}\right)^{\frac{2}{\gamma-1}}\left\{1 - \frac{2}{\gamma + 1}\left[1 - \frac{\frac{4\gamma(\gamma - 1)}{(\gamma + 1)^2}}{b}\right]\right\}} \tag{B1}$$

where

$$b = -\left[1 - \frac{2(\gamma - 1)^2}{(\gamma + 1)^2} - \left(\frac{a_2}{a_1}\right)^2\right] + \left\{\left[1 - 2\left(\frac{\gamma - 1}{\gamma + 1}\right)^2 - \left(\frac{a_2}{a_1}\right)^2\right]^2 + \frac{16\gamma(\gamma - 1)^2}{(\gamma + 1)^4}\right\}^{1/2}$$

The analytic expression for n, obtained by differentiating equation (B1), is

$$n = \frac{d\Pi}{d(a_2/a_1)}$$

$$= -\Pi^2\left(\frac{2}{\gamma - 1}\left(1 - \frac{2}{\gamma + 1}\right)\left(\frac{a_2}{a_1}\right)^{\frac{3-\gamma}{\gamma-1}} + \frac{\frac{16\gamma}{(\gamma + 1)^3}\left(\frac{a_2}{a_1}\right)^{\frac{3-\gamma}{\gamma-1}} b - \frac{8\gamma(\gamma - 1)}{(\gamma + 1)^3}\left(\frac{a_2}{a_1}\right)^{\frac{2}{\gamma-1}}\left\{2\left(\frac{a_2}{a_1}\right) - \frac{2\frac{a_2}{a_1}\left[1 - \frac{2(\gamma - 1)^2}{(\gamma + 1)^2} - \left(\frac{a_2}{a_1}\right)^2\right]}{C}\right\}}{b^2}\right)$$

where

$$C = \left\{\left[1 - \frac{2(\gamma - 1)^2}{(\gamma + 1)^2} - \left(\frac{a_2}{a_1}\right)^2\right]^2 + \frac{16\gamma(\gamma - 1)^2}{(\gamma + 1)^4}\right\}^{1/2}$$

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TABLE I

VARIATION OF a_2/a_1 WITH $\Delta v/a_1$, $\Delta u/a_1$,

AND Π FOR A SHOCK IN AIR

$[\gamma = 1.4]$

$\Delta v/a_1$	a_2/a_1	$\Delta u/a_1$	Π	$\Delta v/a_1$	a_2/a_1	$\Delta u/a_1$	Π
0	1.0000	1.00	1.00000	0.6944	1.1490	1.50	0.92980
.0330	1.0066	1.02	.99990	.7184	1.1548	1.52	.92336
.0654	1.0131	1.04	.99990	.7422	1.1606	1.54	.91667
.0972	1.0195	1.06	.99970	.7658	1.1664	1.56	.90975
.1284	1.0257	1.08	.99950	.7892	1.1723	1.58	.90253
.1591	1.0320	1.10	.99890	.8125	1.1781	1.60	.89526
.1893	1.0381	1.12	.99830	.8356	1.1840	1.62	.88763
.2190	1.0442	1.14	.99721	.8585	1.1899	1.64	.87989
.2483	1.0502	1.16	.99602	.8813	1.1958	1.66	.87199
.2771	1.0561	1.18	.99453	.9040	1.2017	1.68	.86393
.3056	1.0621	1.20	.99285	.9265	1.2076	1.70	.85572
.3336	1.0680	1.22	.99069	.9488	1.2136	1.72	.84731
.3613	1.0738	1.24	.98834	.9711	1.2195	1.74	.83893
.3886	1.0797	1.26	.98571	.9932	1.2255	1.76	.83029
.4156	1.0855	1.28	.98270	1.0152	1.2315	1.78	.82156
.4423	1.0913	1.30	.97934	1.0370	1.2376	1.80	.81261
.4687	1.0971	1.32	.97570	1.0588	1.2436	1.82	.80379
.4948	1.1028	1.34	.97182	1.0804	1.2497	1.84	.79472
.5206	1.1086	1.36	.96759	1.1020	1.2558	1.86	.78573
.5461	1.1144	1.38	.96302	1.1234	1.2619	1.88	.77658
.5714	1.1201	1.40	.95822	1.1447	1.2680	1.90	.76740
.5965	1.1259	1.42	.95302	1.1660	1.2742	1.92	.75809
.6213	1.1317	1.44	.94760	1.1871	1.2804	1.94	.74884
.6459	1.1374	1.46	.94197	1.2082	1.2866	1.96	.73954
.6703	1.1432	1.48	.93598	1.2291	1.2928	1.98	.73025
				1.2500	1.2990	2.00	.72093

TABLE II

VARIATION OF $\frac{d(a_2/a_1)}{d(\Delta v/a_1)} = m$ WITH $\Delta v/a_1$

FOR A SHOCK IN AIR

$[\gamma = 1.4]$

$\Delta v/a_1$	$\frac{d(a_2/a_1)}{d(\Delta v/a_1)} = m$
0	0.2000
.05	.2003
.10	.2011
.15	.2024
.20	.2041
.25	.2062
.30	.2089
.35	.2120
.40	.2152
.45	.2189
.50	.2229
.55	.2270
.60	.2316
.65	.2363
.70	.2412
.75	.2462
.80	.2514
.85	.2567
.90	.2620
.95	.2674
1.00	.2729
1.05	.2839
1.10	.2890
1.15	.2948
1.20	.3008
1.25	.3073
8.00	.5106

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TABLE III

VARIATION OF $\frac{d\Pi}{d(a_2/a_1)} = n$ WITH a_2/a_1

FOR A SHOCK IN AIR

$[\gamma = 1.4]$

a_2/a_1	$\frac{d\Pi}{d(a_2/a_1)} = n$	a_2/a_1	$\frac{d\Pi}{d(a_2/a_1)} = n$
0	0	1.1490	-1.0944
1.0066	-.0041	1.1548	-1.1347
1.0131	-.0176	1.1606	-1.1720
1.0195	-.0380	1.1664	-1.2076
1.0257	-.0652	1.1723	-1.2413
1.0320	-.0980	1.1781	-1.2720
1.0381	-.1367	1.1840	-1.3015
1.0442	-.1786	1.1899	-1.3286
1.0502	-.2243	1.1958	-1.3528
1.0561	-.2724	1.2017	-1.3756
1.0621	-.3254	1.2076	-1.3959
1.0680	-.3777	1.2136	-1.4145
1.0738	-.4316	1.2195	-1.4311
1.0797	-.4883	1.2255	-1.4458
1.0855	-.5430	1.2315	-1.4583
1.0913	-.5982	1.2376	-1.4696
1.0971	-.6537	1.2436	-1.4785
1.1028	-.7079	1.2497	-1.4862
1.1086	-.7619	1.2558	-1.4923
1.1144	-.8145	1.2619	-1.4967
1.1201	-.8649	1.2680	-1.4993
1.1259	-.9151	1.2742	-1.5009
1.1317	-.9625	1.2804	-1.5012
1.1374	-1.0081	1.2866	-1.5000
1.1432	-1.0522	1.2928	-1.4977
		1.2990	-1.4940

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TABLE IV

INITIAL CONDITIONS FOR BURSTING OF DIAPHRAGM INTO VACUUM

$[\lambda = 0]$

β	v/a_0	a/a_0	μ	$\frac{v+a}{a_0}$	$\tan\beta$	$\frac{2}{5}\tan\beta$	$\tan^{-1}\left(\frac{2}{5}\tan\beta\right)$
-45°	0	1.0000000	0	1.000000	-1.0000000	-0.4000000	-21°48'
-40°	.1340837	.9731833	-.2681674	1.107267	-.8390996	-.3356398	-18°33'
-35°	.2498272	.9500346	-.4996544	1.199862	-.7002074	-.2800830	-15°39'
-30°	.3522081	.9295584	-.7044162	1.281767	-.5773503	-.2309401	-13°00'
-25°	.4447437	.9110513	-.8894874	1.355795	-.4663076	-.1865230	-10°34'
-20°	.5300248	.8939950	-1.060050	1.424020	-.3639702	-.1455881	-8°17'
-15°	.6100424	.8779915	-1.220085	1.488034	-.2679491	-.1071796	-6°07'
-10°	.6863941	.8627212	-1.372788	1.549115	-.1763271	-.0705308	-4°02'
-9°	.7013462	.8597308	-1.402692	1.561077	-.1583846	-.0633538	-3°37'
-8°	.7162161	.8567568	-1.432432	1.572973	-.1405407	-.0562163	-3°13'
-7°	.7310130	.8537974	-1.462026	1.584810	-.1227844	-.0491138	-2°49'
-6°	.7457465	.8508507	-1.491493	1.596597	-.1051042	-.0420417	-2°24'
-5°	.7604262	.8479148	-1.520852	1.608341	-.08748860	-.0349954	-2°00'
-4°	.7750611	.8449878	-1.550122	1.620049	-.06992670	-.0279703	-1°36'
-3°	.7896602	.8420680	-1.579320	1.631728	-.05240780	-.0209631	-1°12'
-2°	.8042327	.8391535	-1.608465	1.643386	-.03492080	-.0139683	-0°48'
-1°	.8187875	.8362425	-1.637575	1.655030	-.01745500	-.0069820	-0°24'
0°	.8333333	.8333333	-1.666667	1.666667	.00000000	.0000000	0°
1°	.8478791	.8304242	-1.695758	1.678303	.0174549	.0069820	0°24'
2°	.8624341	.8275132	-1.724868	1.689947	.0349209	.0139684	0°48'
3°	.8770066	.8245987	-1.754013	1.701605	.0524079	.0209632	1°12'
4°	.8916057	.8216789	-1.783211	1.713285	.0699268	.0279707	1°36'
5°	.9062407	.8187519	-1.812481	1.724993	.0874888	.0349955	2°00'
6°	.9209199	.8158160	-1.841840	1.736736	.1051039	.0420416	2°24'
7°	.9356532	.8128694	-1.871306	1.748523	.1227838	.0491135	2°49'
8°	.9504507	.8099099	-1.900901	1.760361	.1405408	.0562163	3°13'
9°	.9653207	.8069359	-1.930641	1.772257	.1583848	.0633539	3°37'
10°	.9802724	.8039455	-1.960545	1.784218	.1763269	.0705308	4°03'
13°	1.025723	.7948554	-2.051446	1.820578	.2308676	.0923470	5°17'
16°	1.072287	.7855426	-2.144574	1.857830	.2867444	.1146978	6°33'
19°	1.120273	.7759454	-2.240546	1.896218	.3443276	.1377310	7°51'
22°	1.170022	.7659956	-2.340044	1.936018	.4040264	.1616106	9°11'
25°	1.221923	.7556154	-2.443846	1.977538	.4663076	.1865230	10°34'
28°	1.276425	.7447150	-2.552850	2.021140	.5317100	.2126840	12°00'
31°	1.334051	.7331898	-2.668102	2.067241	.6008612	.2403445	13°31'
34°	1.395424	.7209152	-2.790848	2.116339	.6745088	.2698035	15°06'
37°	1.461295	.7077410	-2.922590	2.169036	.7535540	.3014216	16°46'
40°	1.532583	.6934834	-3.065166	2.226066	.8390996	.3356398	18°33'
43°	1.610429	.6779142	-3.220858	2.288343	.9325148	.3730059	20°27'
46°	1.696275	.6607450	-3.392550	2.357020	1.035530	.4142120	22°30'
49°	1.791973	.6416054	-3.583946	2.433578	1.150368	.4601472	24°43'
52°	1.899951	.6200098	-3.799902	2.519961	1.279941	.5119764	27°07'
55°	2.023457	.5953086	-4.046914	2.618766	1.428148	.5712592	29°44'
58°	2.166946	.5666108	-4.333892	2.733557	1.600335	.6401340	32°37'
61°	2.336705	.5326590	-4.673410	2.869364	1.804046	.7216184	35°49'
64°	2.541919	.4916162	-5.083838	3.033535	2.050303	.8201212	39°21'
67°	2.796543	.4406914	-5.593086	3.237234	2.355852	.9423408	43°18'
70°	3.122897	.3754206	-6.245794	3.498318	2.747476	1.0989904	47°42'
73°	3.559046	.2881908	-7.118092	3.847237	3.270855	1.3083420	52°37'
76°	4.175647	.1648706	-8.351294	4.340518	4.010776	1.6043104	58°04'
78°41'	5.000000	0	-10.000000	5.000000	5.000000	1.9987768	63°25'

TABLE V

NUMERICAL RESULTS OF CALCULATIONS FOR TRANSITION FROM UNSTEADY TO STEADY FLOW

$\frac{v+a}{a_0}$ line	a_0t	λ	μ	v/a_0	a/a_0	M
$y = -0.2325$						
$F = 1.340$; $M_o = 0.500$; $\cot^{-1}M_o = 63^\circ 26'$; $\frac{v_o}{a_0} = 0.48$; $\frac{a_o}{a_0} = 0.976$						
0.1	-----	----	----	----	-----	-----
.2	-----	----	----	----	-----	-----
.3	0.363	0.01	0.49	0.25	0.951	0.267
.4	.548	.05	.73	.39	.932	.423
.5	.731	.10	.79	.45	.930	.483
.6	.915	.14	.81	.47	.934	.509
.7	1.100	.18	.78	.48	.940	.515
.8	1.286	.21	.75	.48	.947	.513
.9	1.473	.25	.72	.49	.952	.510
1.0	1.662	.27	.69	.48	.957	.505
1.1	-----	----	----	----	-----	-----
1.2	-----	----	----	----	-----	-----
1.3	-----	----	----	----	-----	-----
1.4	-----	----	----	----	-----	-----
1.5	-----	----	----	----	-----	-----
1.6	-----	----	----	----	-----	-----
1.7	3.034	.29	.61	.45	.968	.468
1.8	3.229	.30	.60	.45	.969	.469
1.9	3.425	.31	.60	.46	.970	.469
2.0	3.621	.31	.59	.45	.971	.467
2.1	3.819	.31	.59	.45	.971	.466
2.2	4.018	.30	.61	.46	.969	.475
$y = 0$						
$F = 1.0$; $M_o = 1.0$; $\cot^{-1}M_o = 45^\circ$; $\frac{v_o}{a_0} = 0.913$; $\frac{a_o}{a_0} = 0.913$						
0.1	0.170	0.01	1.67	0.84	0.834	1.007
.2	.349	.05	1.66	.86	.839	1.025
.3	.522	.10	1.65	.88	.845	1.041
.4	.716	.15	1.63	.89	.852	1.044
.5	.881	.20	1.61	.91	.859	1.059
.6	1.058	.25	1.58	.92	.867	1.061
.7	1.230	.29	1.54	.92	.875	1.051
.8	1.423	.32	1.52	.92	.880	1.045
.9	1.609	.35	1.49	.92	.886	1.038
1.0	1.815	.37	1.47	.92	.890	1.033
1.1	2.009	.37	1.45	.91	.892	1.020
1.2	2.210	.38	1.43	.91	.895	1.016
1.3	2.400	.38	1.43	.91	.895	1.016
1.4	2.591	.38	1.42	.90	.896	1.004
1.5	2.780	.38	1.42	.90	.896	1.004
1.6	2.980	.39	1.42	.91	.897	1.014
1.7	3.187	.39	1.41	.90	.898	1.002
1.8	3.373	.40	1.41	.91	.899	1.012
1.9	3.570	.40	1.41	.91	.899	1.012
2.0	3.778	.40	1.41	.91	.899	1.012
2.1	3.982	.38	1.42	.90	.896	1.004
2.2	4.185	.35	1.49	.92	.886	1.038
$y = 0.5$						
$F = 2.107$; $M_o = 2.255$; $\cot^{-1}M_o = 23^\circ 55'$; $\frac{v_o}{a_0} = 1.59$; $\frac{a_o}{a_0} = 0.703$						
0.1	0.407	0.32	4.02	1.85	0.568	3.257
.2	.593	.30	3.82	1.76	.588	2.993
.3	.772	.25	3.57	1.66	.618	2.686
.4	.949	.20	3.45	1.63	.635	2.566
.5	1.127	.15	3.38	1.62	.647	2.503
.6	1.306	.10	3.35	1.63	.655	2.488
.7	1.488	.07	3.34	1.64	.659	2.488
.8	1.674	.04	3.31	1.68	.665	2.526
.9	1.860	.01	3.30	1.65	.669	2.466
1.0	2.050	0	3.29	1.65	.671	2.459
1.1	2.245	.01	3.29	1.65	.672	2.455
$y = 0.5$						
$F = 2.107$; $M_o = 2.255$; $\cot^{-1}M_o = 23^\circ 55'$; $\frac{v_o}{a_0} = 1.59$; $\frac{a_o}{a_0} = 0.703$ (Concluded)						
1.2	2.442	.02	3.28	1.65	.674	2.448
1.3	2.639	.01	3.28	1.65	.673	2.451
1.4	2.835	.02	3.25	1.64	.677	2.422
1.5	3.029	.02	3.23	1.63	.679	2.400
1.6	3.224	.02	3.26	1.64	.676	2.426
1.7	3.420	.03	3.26	1.65	.677	2.437
1.8	3.716	.04	3.26	1.65	.678	2.433
1.9	3.817	.05	3.26	1.66	.679	2.444
2.0	4.021	.05	3.26	1.66	.679	2.444
2.1	4.232	.13	3.26	1.70	.687	2.474
2.2	4.445	.12	3.00	1.56	.712	2.191
2.3	4.681	.21	3.05	1.63	.716	2.276
$y = 1.0$						
$F = 3.508$; $M_o = 2.802$; $\cot^{-1}M_o = 19^\circ 38'$; $\frac{v_o}{a_0} = 1.75$; $\frac{a_o}{a_0} = 0.624$						
0.1	0.610	0.54	4.68	2.61	0.478	5.460
.2	.801	.51	4.46	2.49	.503	4.950
.3	.985	.47	4.30	1.92	.523	3.671
.4	1.165	.42	4.10	1.84	.548	3.357
.5	1.343	.39	4.01	1.81	.560	3.232
.6	1.522	.34	3.96	1.81	.570	3.175
.7	1.702	.30	3.92	1.81	.578	3.131
.8	1.886	.27	3.90	1.82	.583	3.121
.9	2.073	.25	3.89	1.82	.586	3.105
1.0	2.263	.24	3.88	1.82	.588	3.095
1.1	2.456	.22	3.86	1.82	.591	3.076
1.2	2.653	.23	3.86	1.82	.591	3.079
1.3	2.851	.23	3.84	1.81	.593	3.052
1.4	3.048	.22	3.86	1.82	.592	3.074
1.5	3.244	.23	3.84	1.81	.593	3.052
1.6	3.437	.22	3.84	1.81	.594	3.047
1.7	3.631	.20	3.83	1.82	.597	3.048
1.8	3.828	.20	3.85	1.83	.595	3.075
1.9	4.027	.19	3.84	1.83	.597	3.065
2.0	4.232	.19	3.84	1.83	.597	3.065
2.1	4.441	.13	3.84	1.86	.603	3.084
2.2	4.671	.16	3.84	1.84	.600	3.066
2.3	4.897	.10	3.67	1.79	.623	2.873
$y = 2.0$						
$F = 6.177$; $M_o = 3.398$; $\cot^{-1}M_o = 16^\circ 24'$; $\frac{v_o}{a_0} = 1.86$; $\frac{a_o}{a_0} = 0.549$						
0.1	0.998	0.75	5.24	2.25	0.401	5.611
.2	1.196	.73	5.07	2.17	.420	5.166
.3	1.387	.69	4.91	2.11	.440	4.795
.4	1.573	.65	4.77	2.06	.458	4.497
.5	1.756	.62	4.64	2.01	.474	4.240
.6	1.936	.58	4.52	1.97	.490	4.020
.7	2.117	.54	4.45	1.96	.501	3.912
.8	2.300	.52	4.41	1.95	.507	3.846
.9	2.485	.50	4.38	1.94	.512	3.789
1.0	2.674	.48	4.37	1.95	.515	3.786
1.1	2.867	.47	4.36	1.94	.517	3.763
1.2	3.064	.47	4.36	1.95	.518	3.756
1.3	-----	----	----	1.94	.519	3.734
1.4	3.459	.47	4.34	1.94	.519	3.738
1.5	3.656	.47	4.34	1.94	.519	3.738
1.6	3.850	.47	4.33	1.93	.520	3.711
1.7	4.043	.46	4.32	1.93	.522	3.697
1.8	4.238	.45	4.32	1.94	.523	3.709
1.9	4.436	.44	4.32	1.94	.524	3.702
2.0	4.641	.43	4.32	1.95	.525	3.714
2.1	4.845	.38	4.32	1.97	.530	3.717
2.2	5.078	.42	4.33	1.96	.525	3.733

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TABLE VI

VARIATION WITH MACH NUMBER OF THE FUNCTION $f(M_o) = \sqrt{\dfrac{M_o}{1 + \dfrac{\gamma - 1}{2} M_o^2}}$

REPRESENTATIVE OF THE ACCUMULATIVE GROWTH

OF PULSE AREAS FOR AIR

$[\gamma = 1.4]$

M_o	$f(M_o)$
0	0
.1	.316
.3	.543
.5	.690
.7	.798
1.0	.913
1.4	1.002
1.5	1.017
1.6	1.029
1.8	1.045
2.0	1.054
$\sqrt{5}$	1.057
2.4	1.056
2.5	1.054
3.0	1.035
4.0	.976
5.0	.913
10.0	.690
100.0	.224
∞	0

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TABLE VII

VARIATION WITH MACH NUMBER OF THE FUNCTION $f(M_o) = \sqrt{\dfrac{1 + \frac{\gamma - 1}{2} M_o^2}{M_o}}$

REPRESENTATIVE OF THE ACCUMULATIVE GROWTH OF MASS FOR AIR

$[\gamma = 1.4]$

M_o	$f(M_o)$
0	∞
.1	3.165
.3	1.842
.5	1.449
.7	1.252
1.0	1.095
1.4	.997
1.5	.983
1.6	.972
1.8	.957
2.0	.949
$\sqrt{5}$	.946
2.4	.947
2.5	.949
3.0	.966
4.0	1.025
5.0	1.095
10.0	1.449
100.0	4.473
∞	∞

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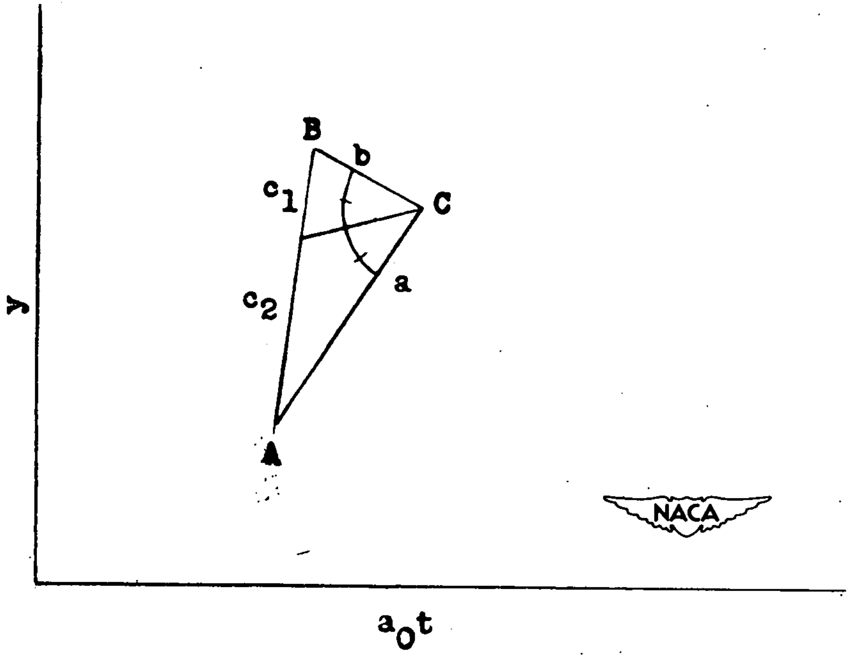

Figure 1.- Diagram for the construction of point C in a flow with entropy gradient through a tube of variable cross section.

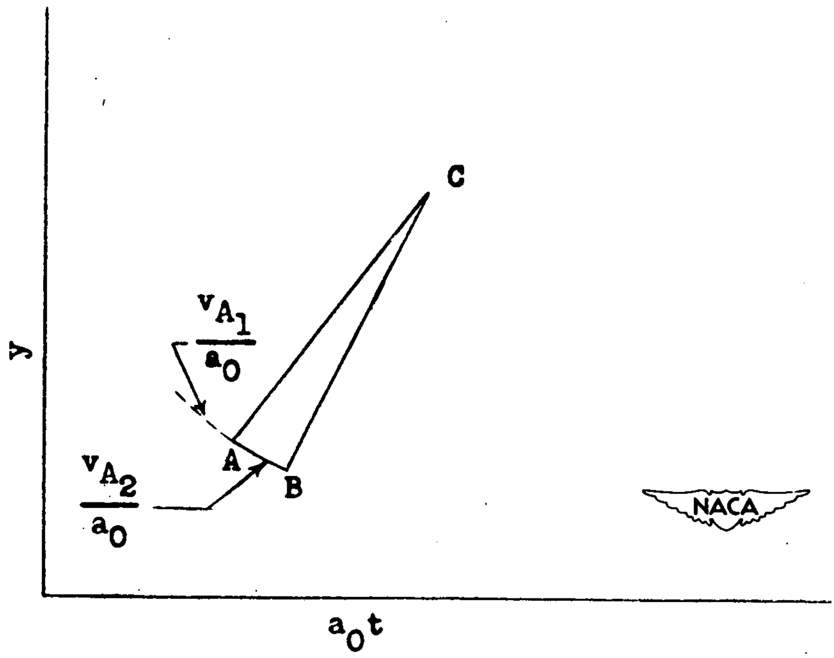

Figure 2.- Diagram for the construction of point C in a flow through a tube of variable cross section containing a shock.

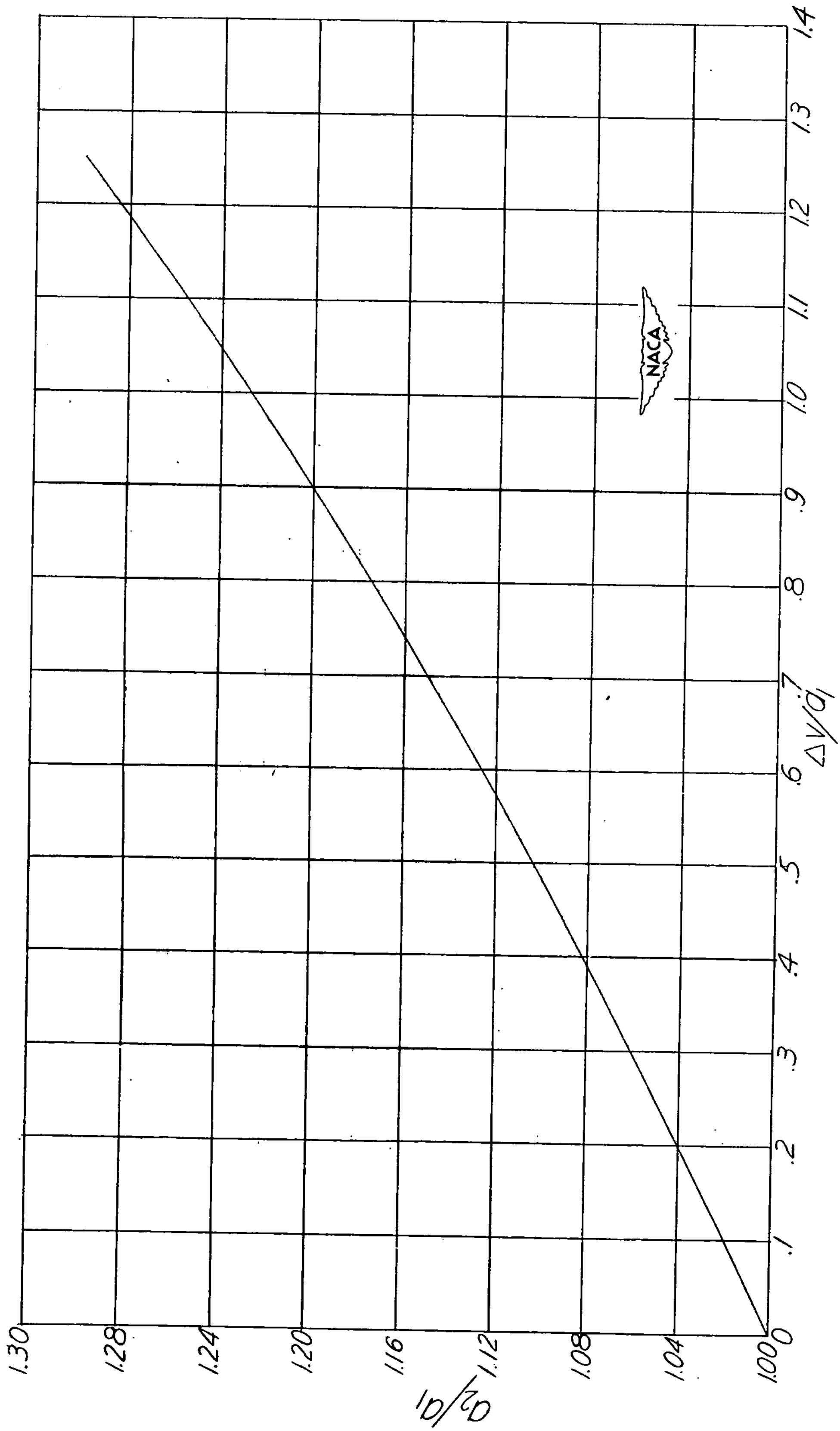

Figure 3.- Variation of a_2/a_1 with $\Delta v/a_1$ for a shock in air. $\gamma = 1.4$.

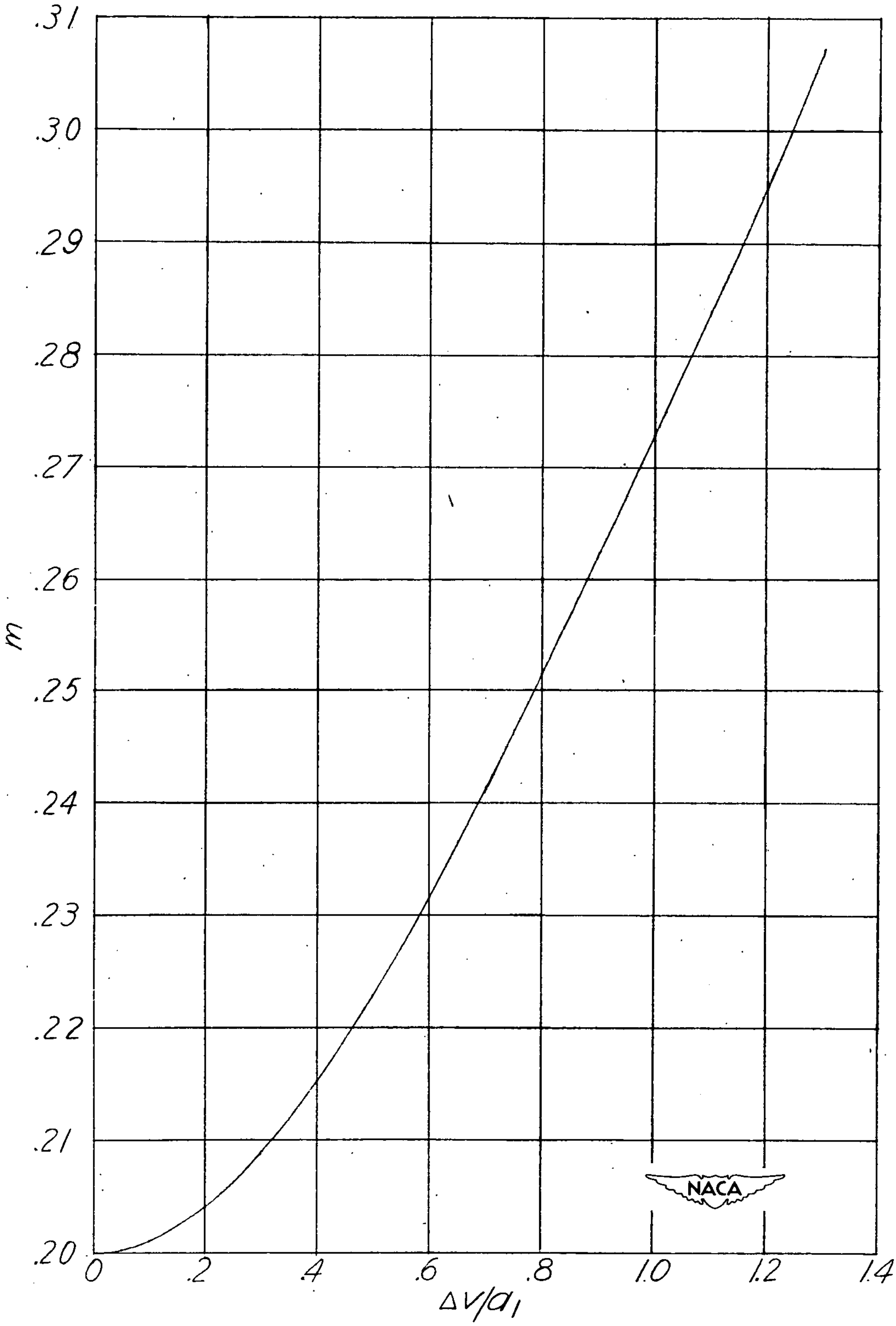

Figure 4.- Variation of $\frac{d\left(a_2/a_1\right)}{d\left(\Delta v/a_1\right)} = m$ with $\Delta v/a_1$ for a shock in air. $\gamma = 1.4$.

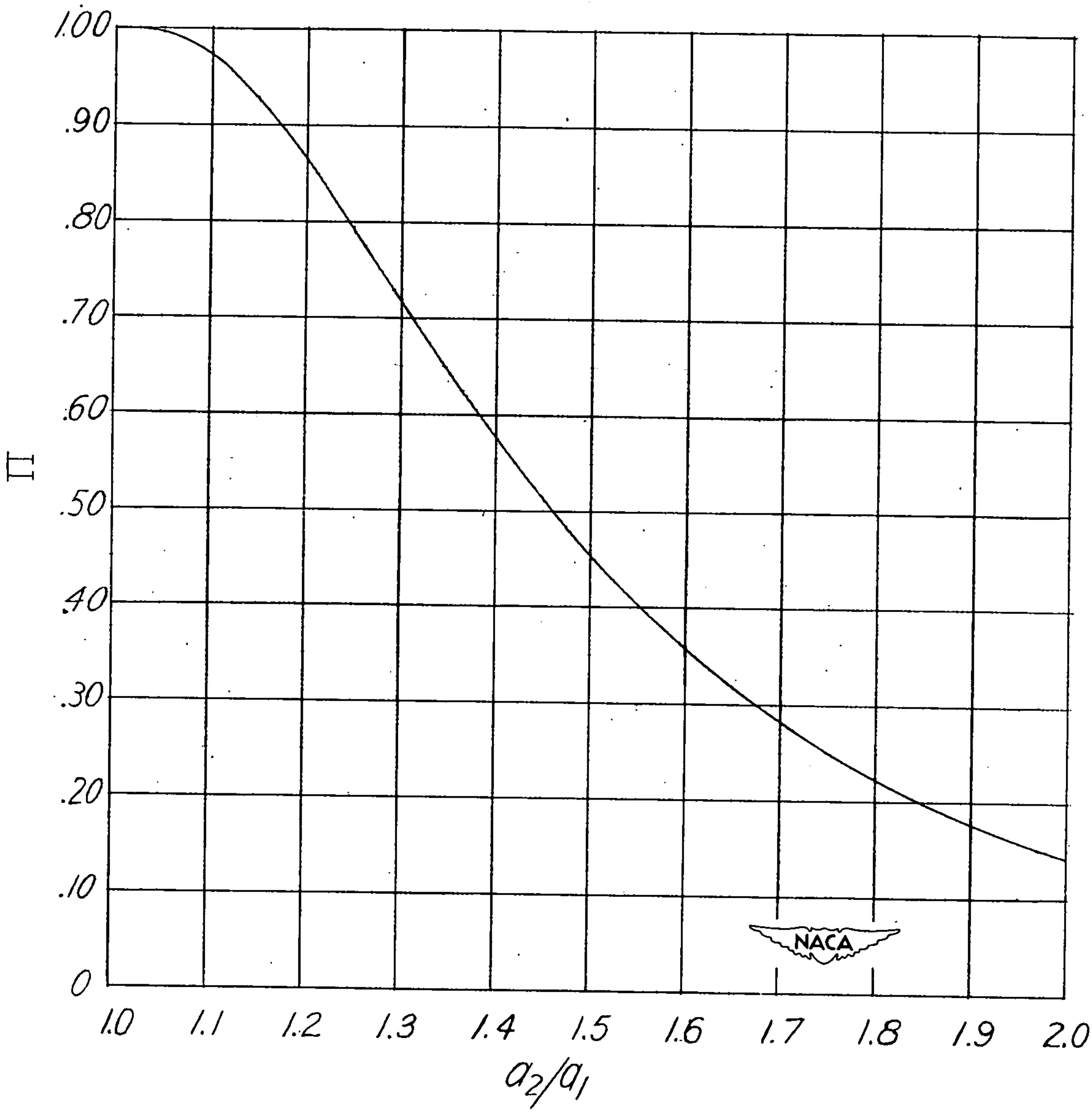

Figure 5.- Variation of Π with a_2/a_1 for a shock in air. $\gamma = 1.4$.

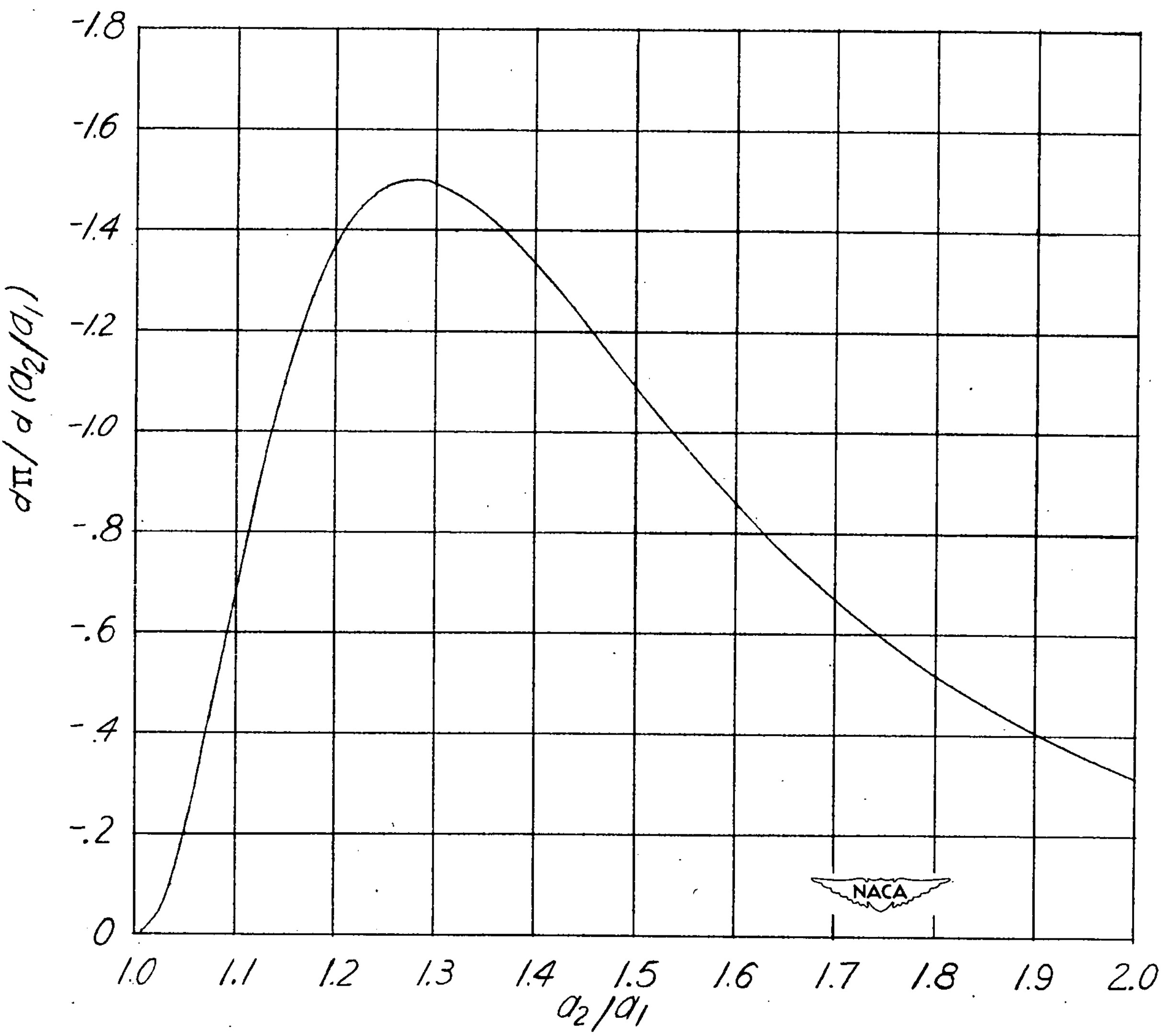

Figure 6.- Variation of $\frac{d\Pi}{d(a_2/a_1)} = n$ with a_2/a_1 for a shock in air. $\gamma = 1.4$.

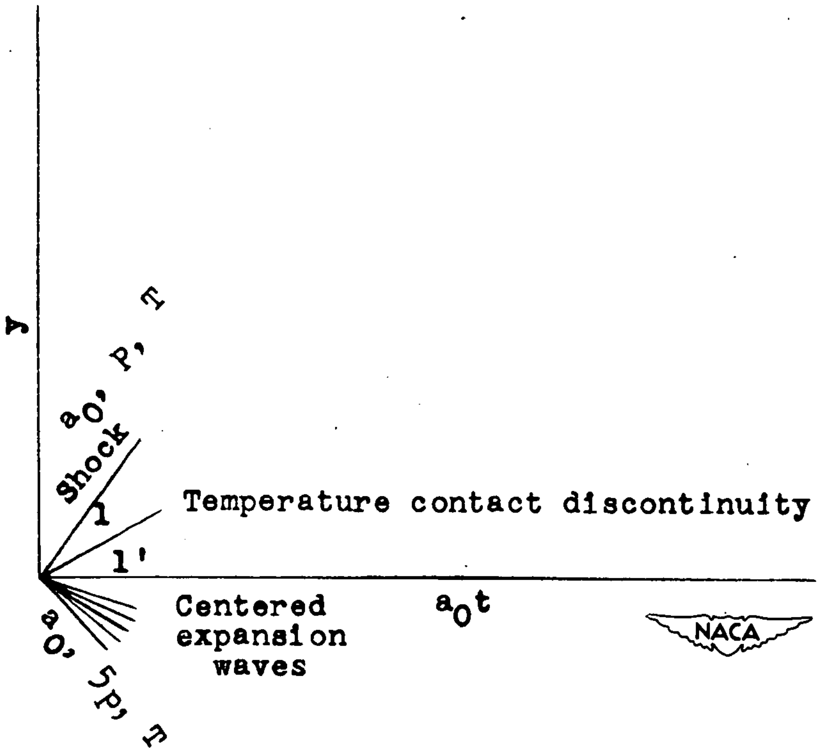

Figure 7.- Sample results of bursting of a diaphragm presented in a diagram of y against a_0t.

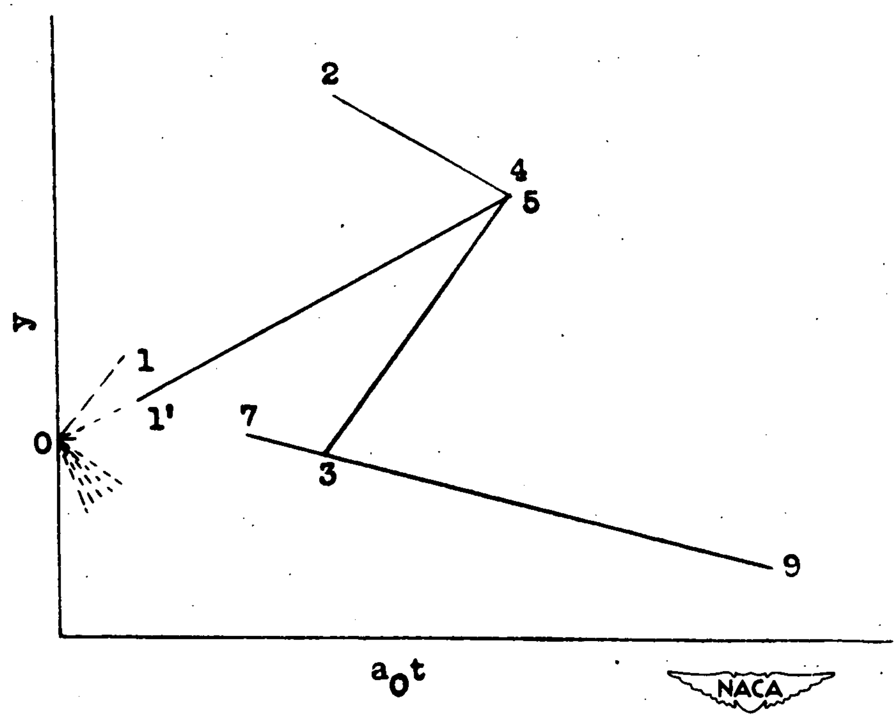

Figure 8.- Diagram for the construction of a point of the time-history curve of the temperature contact discontinuity.

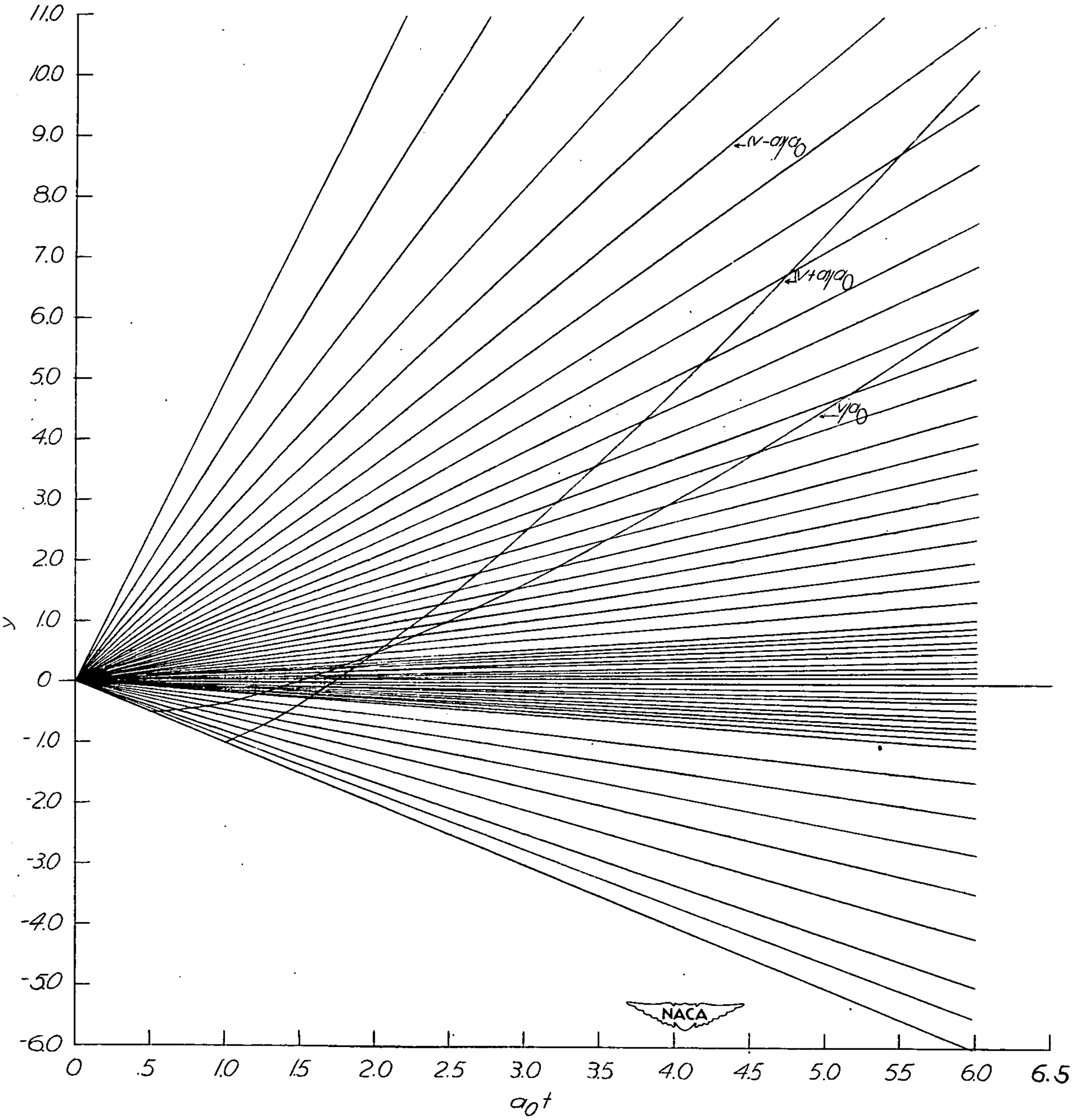

Figure 9.- Results of bursting diaphragm into vacuum for tube with constant cross section presented in the plot of y against a_0t.

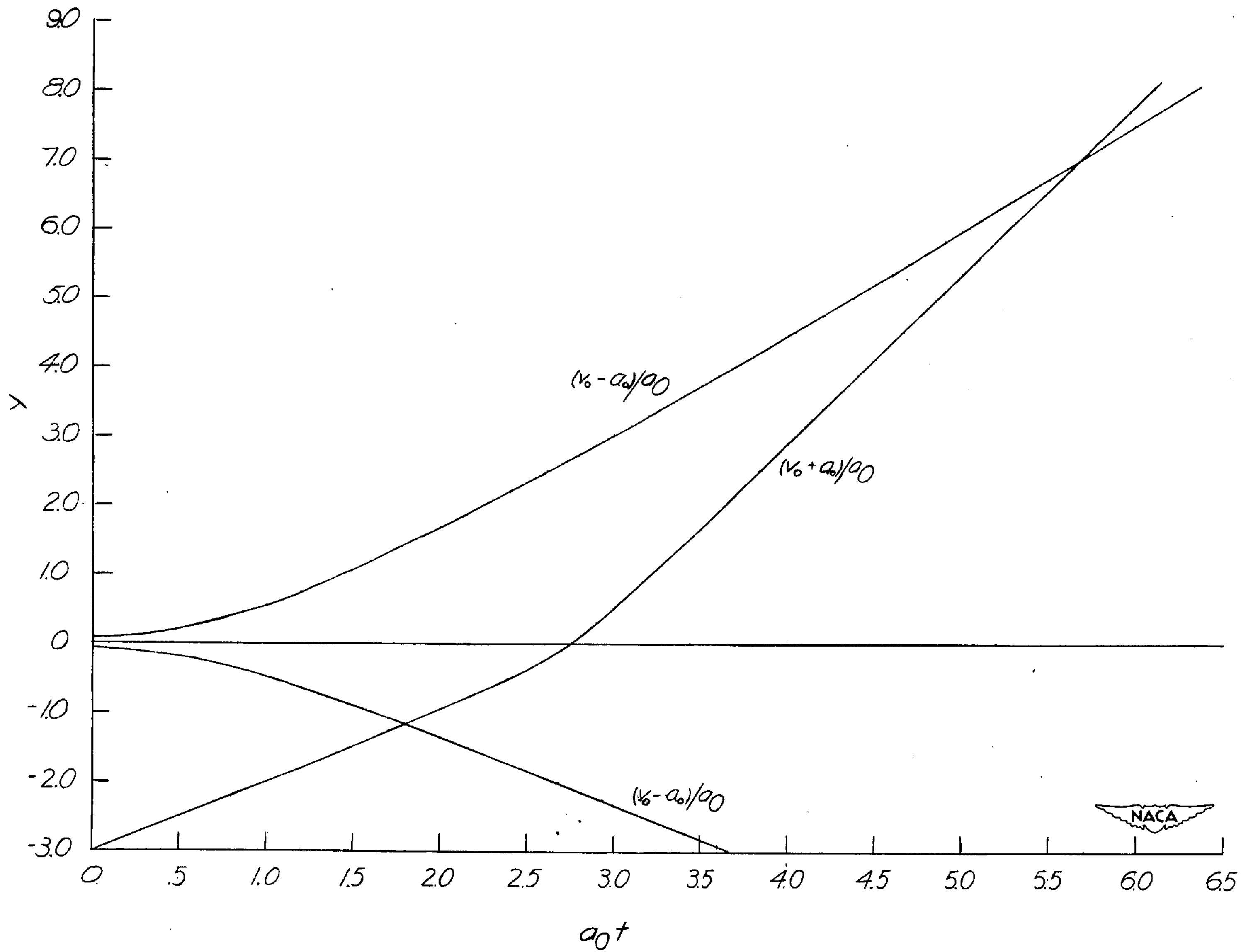

Figure 10.- Time histories of small disturbances moving in a steady flow.

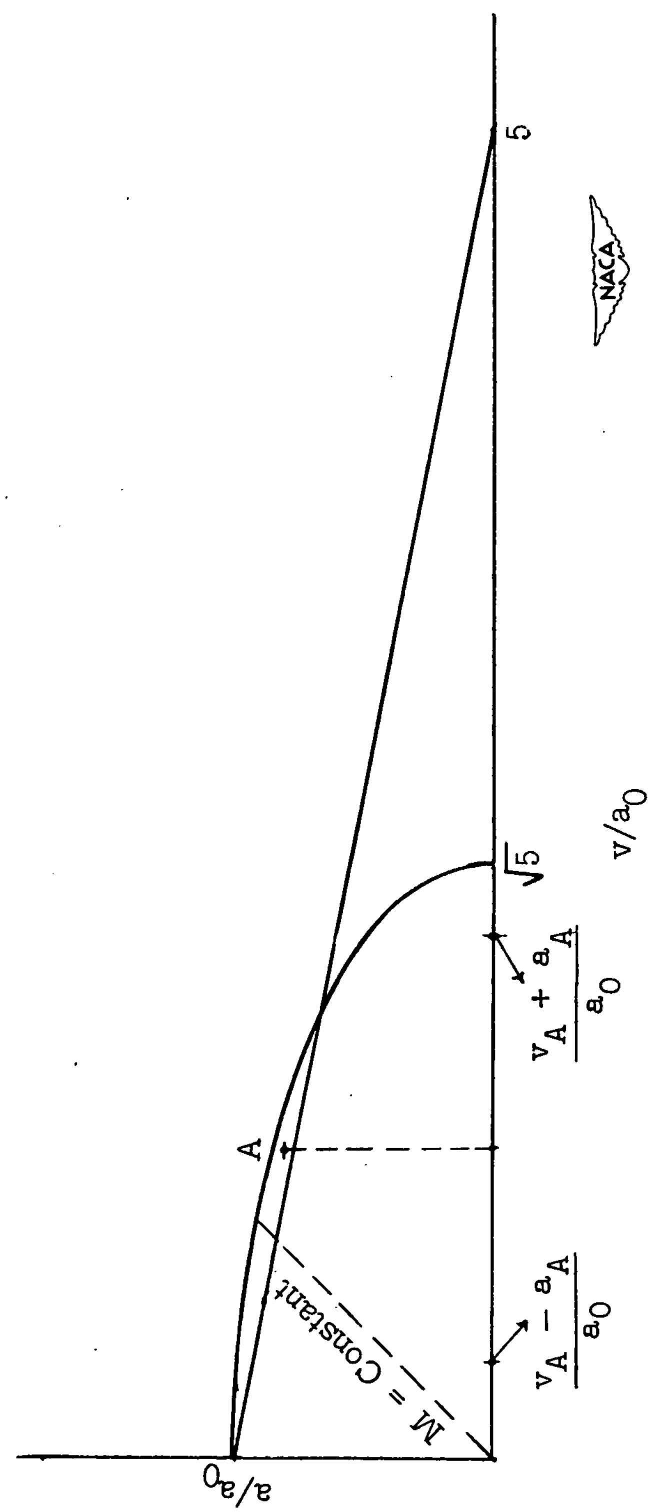

Figure 11.- Presentation of unsteady and steady flow in the plot of a/a_0 against v/a_0.

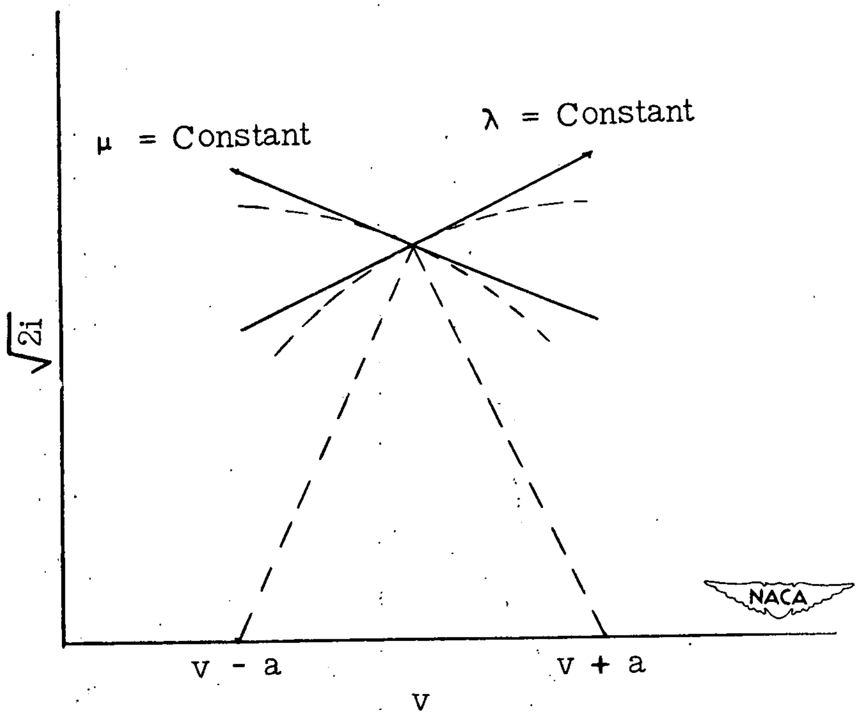

Figure 12.- Presentation of disturbance velocities in the plot of $\sqrt{2i}$ against v.

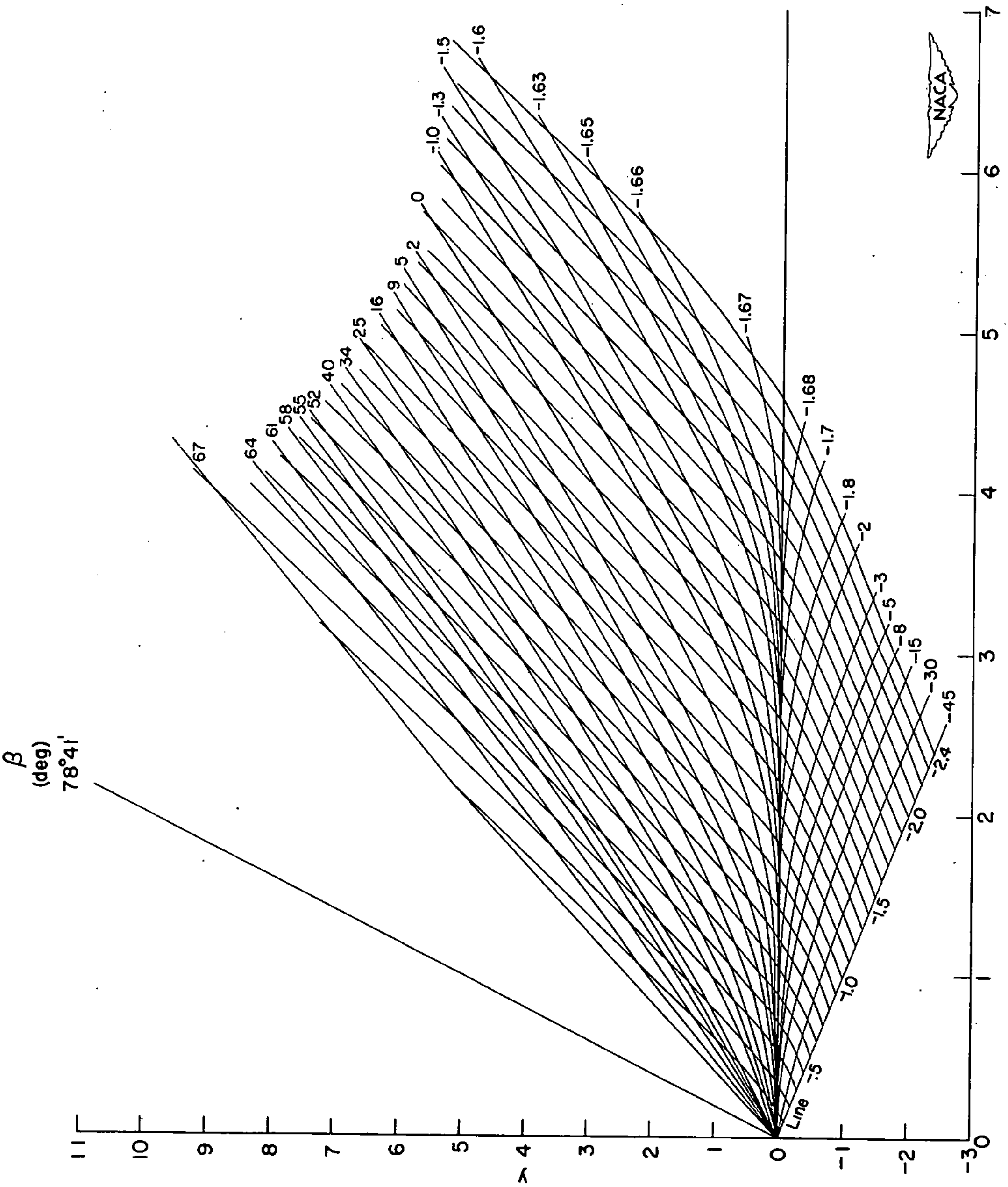

Figure 13.- Results of calculations for transition from unsteady to steady flow in the plot of y against a_0t.

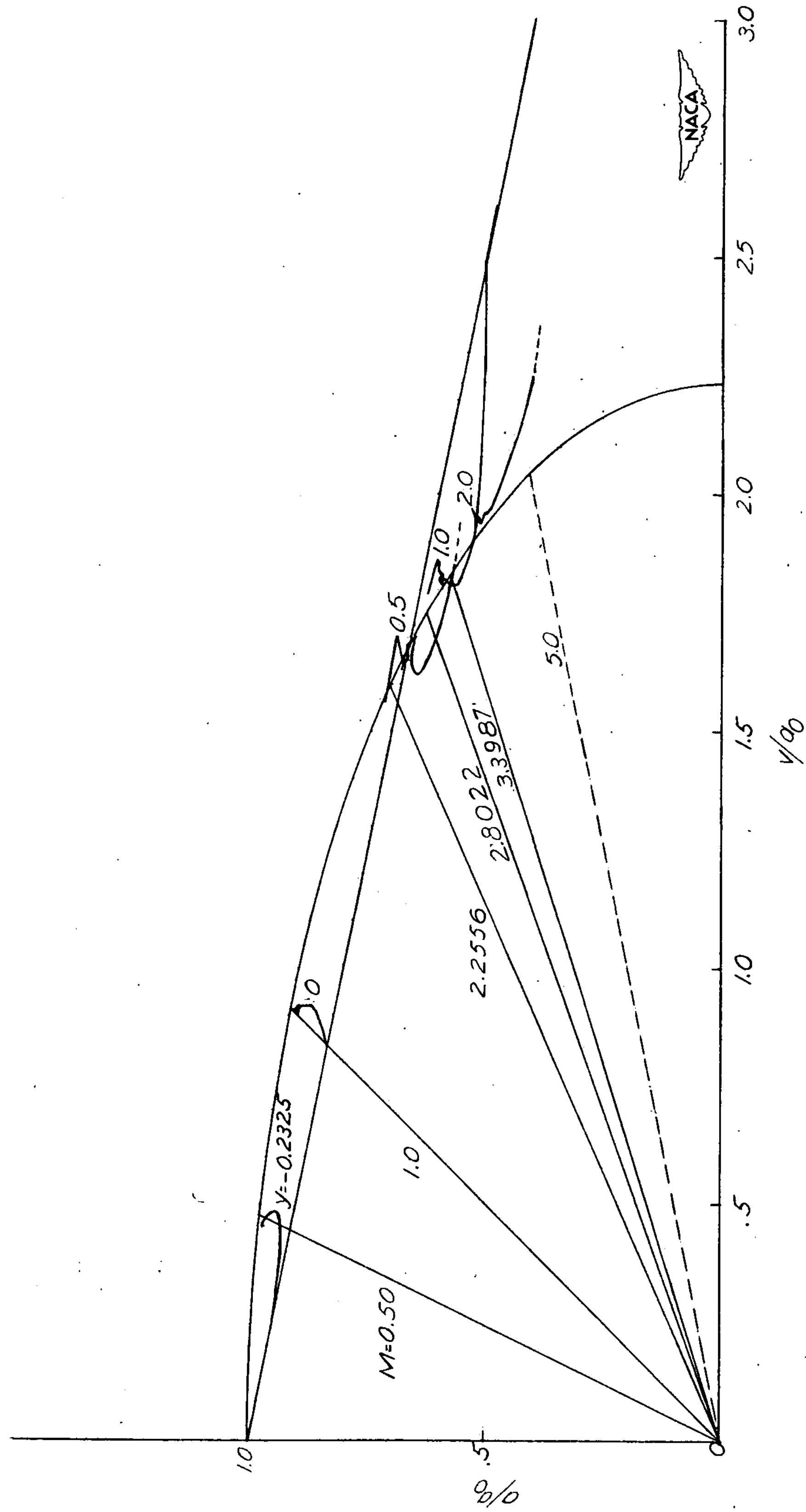

Figure 14.- Results of calculations for transition from unsteady to steady flow in the plot of a/a_0 against v/a_0.

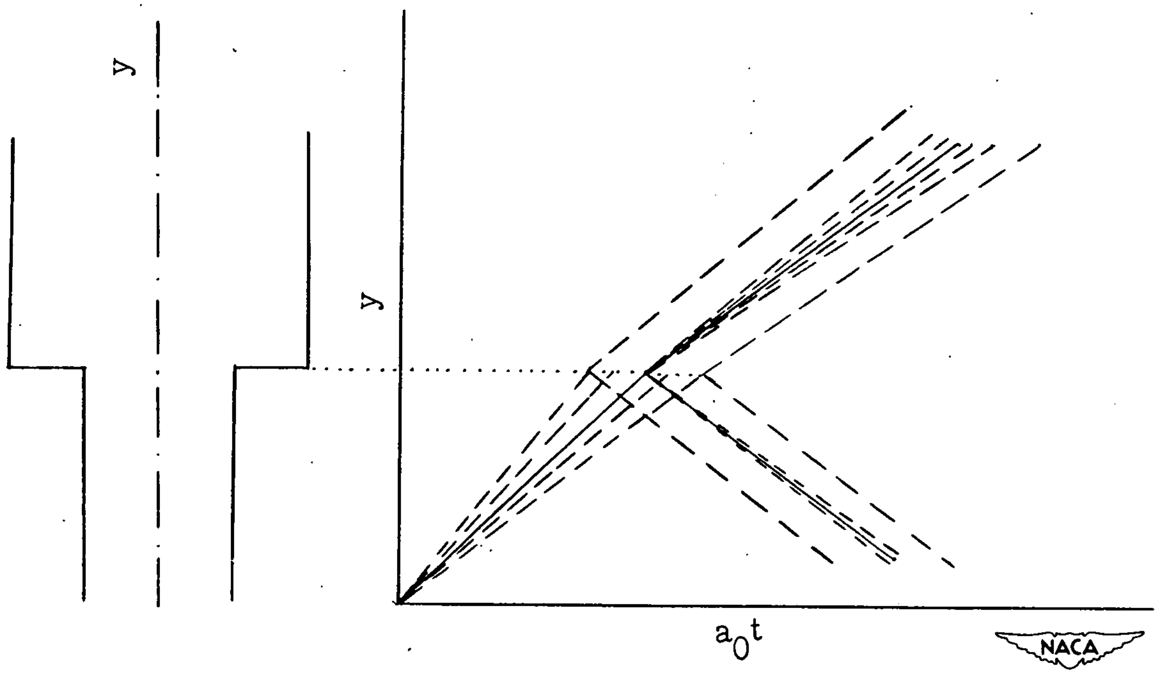

Figure 15.- Large disturbance moving through sudden change in cross section presented in the plot of y against a_0t.

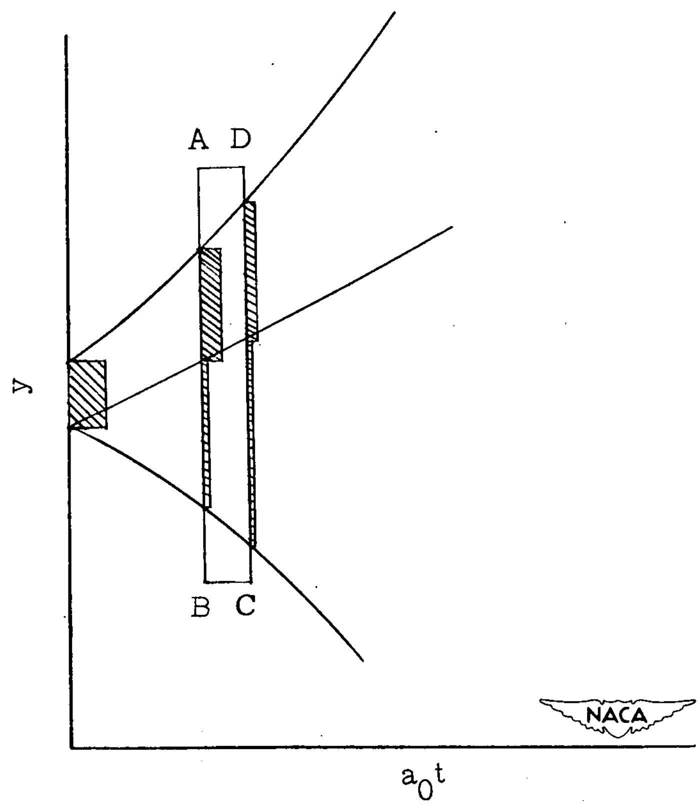

Figure 16.- Conservation of potential presented in the plot of y against a_0t.

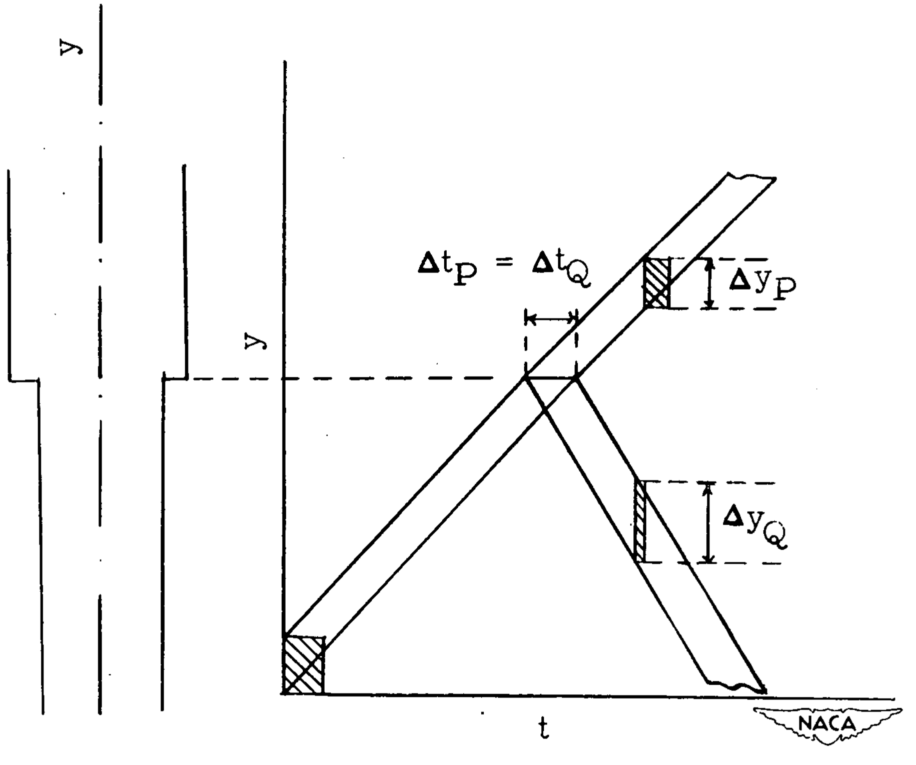

Figure 17.- Motion of a small disturbance through a small discontinuous change in cross section.