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METHOD OF DESIGNING AIRFOILS WITH PRESCRIBED VELOCITY DISTRIBUTIONS IN COMPRESSIBLE POTENTIAL FLOWS

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SUMMARY

With the assumption of the linearized pressure-volume relation, a solution of the problem of designing an airfoil with any theoretically attainable prescribed dimensionless velocity distribution in a potential flow of a compressible perfect fluid was obtained by a method of correspondence between potential flows of compressible and incompressible fluids.

If the prescribed velocity distribution was not theoretically realizable, that is, the distribution would result in an open profile, the method gave an easy way of modifying the velocity distribution in order to obtain a closed profile. Numerical examples are included.

INTRODUCTION

In order to avoid flow separation and excessively high local velocities in designing an airfoil or profile, it is advantageous to prescribe the velocity distribution as a function of the arc length along the airfoil and then compute the airfoil shape. For a two-dimensional potential flow of an incompressible nonviscous fluid, several methods based on conformal-mapping theory are available for obtaining a profile with a prescribed velocity distribution in a uniform stream. (See references 1 to 3.)

A similar solution for the two-dimensional potential flow of a compressible nonviscous fluid was obtained at the NACA Lewis laboratory. This solution is based on the assumption that the pressure-volume relation is given by the tangent to the isentropic curve instead of the true curve. The flow pattern of the compressible fluid is obtained by transformation from corresponding flow of an incompressible fluid in the manner developed by Lin in his extension (reference 4) of the method developed by von Kármán and Tsien (reference 5) for obtaining the flow of a compressible fluid past a given profile. An approach similar to that of reference 4 is given by Gelbart in reference 6.

The method presented herein consists in using the prescribed dimensionless velocity distribution about the airfoil in compressible flow (1) to select the proper potential flow of an incompressible fluid about the unit circle, and (2) to determine the mapping function that transforms this incompressible flow into a compressible flow. The image of the unit circle under this mapping gives the airfoil with the prescribed velocity distribution provided the velocity distribution is theoretically attainable. For velocity distributions that are not realizable, a method is given for so modifying the velocity distribution that a closed profile is obtained.

SYMBOLS

The following symbols are used herein:

A,B,C,D, a,b	constants
$F(\zeta)$	complex potential in ζ plane
$f(\zeta)$	regular function of ζ
g	real part of $f(\zeta)$
$H(\zeta)$	analytic function of ζ
h	imaginary part of $f(\zeta)$
Im	imaginary part
k	regular function of ζ
n	number determined by included tail angle of airfoil
P	profile
p	pressure
Q(s)	auxiliary function of s
q	magnitude of dimensionless velocity in compressible plane (ratio of actual velocity to stagnation velocity of sound)
Re	real part

s	arc length on profile
W	free-stream velocity in incompressible flow (See fig. 1.)
w	complex velocity in incompressible flow $\frac{dF(\zeta)}{d\zeta}$
$z = x+iy$	complex variable (compressible-flow plane)
α	angle velocity makes with x-axis (compressible flow)
Γ	circulation (See fig. 1.)
γ	angle determined by trailing-edge stagnation point on unit circle (See fig. 1.)
δ	tail angle of airfoil
$\zeta = \xi+i\eta$	complex variable (incompressible-flow plane)
$\tilde{\zeta}$	auxiliary complex variable
θ	circle angle
ρ	density
φ	velocity potential
Ψ	stream function

Subscripts:

c	compressible
i	incompressible
min	minimum
n	nose

Prime indicates a related function.

THEORY OF METHOD

In reference 4, Lin shows that if the pressure-density relation for a compressible fluid is expressed as

$$p = A - \frac{B}{\rho} \tag{1}$$

where A and B are constants, then given a potential flow of an incompressible nonviscous fluid past a profile P_i in the ζ plane described by the complex potential function $F(\zeta)$ and the complex velocity $w(\zeta)$, the potential flow of a nonviscous compressible fluid about some profile P_c in the z plane can be obtained by choosing a function $k(\zeta)$, regular and nonzero in the exterior of the profile P_i and satisfying the conditions

$$\left|\frac{1}{2}w(\zeta)\right| < \left|k(\zeta)\right| < \infty \tag{2}$$

on P_i, and

$$\oint k(\zeta)d\zeta - \frac{1}{4}\oint \overline{\frac{w^2(\zeta)}{k(\zeta)}\,d\zeta} = 0 \tag{3}$$

along any path enclosing P_i. (The bar denotes the conjugate complex quantity.) Then the equations

$$z = x + iy = \int k(\zeta)d\zeta - \frac{1}{4}\int \overline{\frac{w^2(\zeta)}{k(\zeta)}\,d\zeta} \tag{4}$$

$$\frac{2q}{1 + \sqrt{1 + q^2}}\, e^{-i\alpha} = \frac{w(\zeta)}{k(\zeta)} \tag{5}$$

$$\varphi_c + i\Psi_c = F(\zeta) \tag{6}$$

give the parametric representation of the compressible flow past a profile P_c in the z plane with ζ as a parameter and where P_c has the same general analytic nature as the original profile P_i. In these equations φ_c, Ψ_c, q, and α denote for the compressible flow, the velocity potential, the stream function, the magnitude of the dimensionless velocity, and the angle the velocity makes with the x-axis, respectively. Equation (4) gives the correspondence between the two planes and equations (5) and (6) give the relation between the velocities and the potentials at corresponding points.

It is also shown in reference 4 that if the ζ plane is conformally transformed into the $\tilde{\zeta}$ plane by the analytical function $\zeta = H(\tilde{\zeta})$ (taking the profile P_1 into a profile $\tilde{P}_1$) and this resulting flow about $\tilde{P}_1$ is mapped into a compressible flow in the z plane by

$$z = \int \tilde{k}(\tilde{\zeta})\, d\tilde{\zeta} - \frac{1}{4}\int \overline{\frac{\tilde{w}^2(\tilde{\zeta})}{\tilde{k}(\tilde{\zeta})}\, d\tilde{\zeta}}$$

$$\frac{2q}{1 + \sqrt{1 - q^2}}\, e^{-i\alpha} = \frac{\tilde{w}(\tilde{\zeta})}{\tilde{k}(\tilde{\zeta})}$$

$$\varphi_c + i\Psi_c = \tilde{F}(\tilde{\zeta})$$

where

$$\tilde{F}(\tilde{\zeta}) = F(\zeta)$$

$$\tilde{w}(\tilde{\zeta}) = \frac{d}{d\tilde{\zeta}}\tilde{F}(\tilde{\zeta}) = w(\zeta)\,\frac{d\zeta}{d\tilde{\zeta}}$$

$$\tilde{k}(\tilde{\zeta}) = k(\zeta)\,\frac{d\zeta}{d\tilde{\zeta}}$$

then the profile obtained is the same as that given by equations (4) to (6). Consequently, the unit circle in the ζ plane can be taken as P_1. When the unit circle is taken as P_1, a profile with a pointed tail can be obtained as indicated in reference 4 by allowing $k(\zeta)$ to vanish at the trailing-edge stagnation point where $w(\zeta) = 0$.

In order to apply this transformation to the problem of obtaining the profile with a prescribed dimensionless velocity distribution in a compressible potential flow, the complex potential about the circle in the ζ plane must be determined and the function $k(\zeta)$ obtained. For the actual computation of the profile, only the values of these functions for points on the circle are needed.

Flow in Circle Plane

The complex potential in the ζ plane for the incompressible flow about the unit circle $\zeta = e^{i\theta}$ is

$$F(\zeta) = \varphi_i + i\Psi_i = -W\left(\zeta + \frac{1}{\zeta}\right) + \frac{\Gamma_i}{2\pi i} \log \zeta + C \tag{7}$$

where W is the free-stream velocity and is in the direction of the negative ξ axis, Γ_i is the circulation, and C is an arbitrary constant. For points on the circle, equation (7) becomes

$$\varphi_i(\theta) = -2W(\cos\theta + \cos\gamma) + \frac{\Gamma_i}{2\pi}(\theta + \pi - \gamma) \tag{8}$$

when C is so chosen that $\varphi_i = 0$ at the trailing-edge stagnation point $\zeta = -e^{i\gamma}$. (See fig. 1.) The velocity is

$$w(\theta) = -ie^{-i\theta}\left(2W \sin\theta + \frac{\Gamma_i}{2\pi}\right) \tag{9}$$

Both $\varphi_i(\theta)$ and $W(\theta)$ are completely determined when Γ_i, W, and γ are known and these quantities are obtained from the prescribed velocity distribution in the following manner: The magnitude of the prescribed dimensionless velocity along the airfoil is given as a function of arc length $q = q(s)$ (fig. 2) where the total arc length is taken as 2π and is measured from the tail along the lower surface. If $Q(s)$ is defined as

$$Q(s) = -q(s) \qquad \text{when} \qquad 0 \leqq s \leqq s_n$$

$$Q(s) = q(s) \qquad \text{when} \qquad s_n \leqq s \leqq 2\pi$$

where s_n is the nose stagnation point (fig. 2), then

$$\varphi_c(s) = \int_0^s Q(s)\, ds \tag{10}$$

$$\Gamma_c = \int_0^{2\pi} Q(s)\,ds \tag{11}$$

$$\varphi_{c,min} = \int_0^{s_n} Q(s)\,ds \tag{12}$$

By equation (6) the potentials are equal at corresponding points so that

$$\varphi_{i,min} = \varphi_{c,min}$$

$$\Gamma_i = \Gamma_c \tag{13}$$

Then γ is given by (reference 2)

$$\cot \gamma + \gamma = \frac{\pi}{2} - \frac{\pi\varphi_{i,min}}{\Gamma_i} = \frac{\pi}{2} - \frac{\pi\varphi_{c,min}}{\Gamma_c} \tag{14}$$

and W is given by

$$W = \frac{\Gamma_i}{4\pi \sin \gamma} = \frac{\Gamma_c}{4\pi \sin \gamma} \tag{15}$$

The flow about the circle is obtained by using these values of Γ_i, W, and γ in equations (8) and (9)

Function $k(\zeta)$

The function $k(\zeta)$ is computed for points on the unit circle by using the prescribed velocity distribution on the airfoil and the velocity on the circle to determine the real part of $k(\zeta)$ on the circle. Then Poisson's integral gives the imaginary part of $k(\zeta)$ on the circle. The values of $k(\zeta)$ for any point in the exterior of the circle can be obtained from these values on the circle by further use of Poisson's integral. For the computation of the airfoil, however, only the values on the circle are needed.

Airfoil with pointed tail. - In order that the airfoil have a pointed tail, it is necessary for $k(\zeta)$ to vanish at the trailing-edge stagnation point $(\zeta = -e^{i\gamma})$ of the unit circle. Hence $k(\zeta)$ can be written

$$k(\zeta) = \left(1 + \frac{e^{i\gamma}}{\zeta}\right)^n e^{g + ih} \tag{16}$$

where $g + ih$ is a regular function of ζ on and outside the circle. The exponent n and the included tail angle δ of the airfoil are related by

$$n = 1 - \frac{\delta}{\pi} \tag{17}$$

as shown in appendix A. For points on the circle $\zeta = e^{i\theta}$, equation (16) becomes

$$k(\theta) = \left[1 + e^{i(\gamma-\theta)}\right]^n e^{g(\theta) + ih(\theta)} \tag{18}$$

Determination of $g(\theta)$ and $h(\theta)$. - From equation (6) the potentials $\varphi_i(\theta)$ and $\varphi_c(s)$ are equal at corresponding points. Thus, by matching these potentials a correspondence is established between points along the airfoil arc and the circle angle; that is, $s = s(\theta)$. By this relation, the magnitude of the dimensionless prescribed velocity along the airfoil is obtained as a function of the circle angle $q = q(\theta)$. Hence, by taking the absolute values of equation (5)

$$\frac{2q(\theta)}{1 + \sqrt{1 + q^2(\theta)}} = \frac{|w(\theta)|}{|k(\theta)|} \tag{19}$$

for points on the circle. By use of equations (9) and (18), equation (19) reduces to

$$\frac{2q(\theta)}{1 + \sqrt{1 + q^2(\theta)}} = \frac{2W \sin\theta + \frac{\Gamma}{2\pi}}{\left[2 + 2\cos(\gamma - \theta)\right]^{\frac{n}{2}} e^{g(\theta)}} \tag{20}$$

or, on solving for $g(\theta)$,

$$g(\theta) = \log_e\left(2W \sin\theta + \frac{\Gamma}{2\pi}\right) + \log_e\left[\frac{1 + \sqrt{1 + q^2(\theta)}}{2q(\theta)}\right] - \frac{n}{2}\log_e\left[2 + 2\cos(\gamma-\theta)\right] \tag{21}$$

The function $h(\theta)$, the harmonic conjugate of $g(\theta)$, is obtained for points on the circle by using Poisson's integral

$$h(\theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\tau) \cot\left(\frac{\tau - \theta}{2}\right) d\tau + D \tag{22}$$

where D is a constant (reference 7). For convenience, this constant is taken as zero because a change in D merely rotates the coordinates in the z plane, as shown in appendix B. Hence $k(\zeta)$ is now completely determined on the circle by using the values of $g(\theta)$ and $h(\theta)$ (equations (21) and (22), respectively) in equation (18).

<u>Closure conditions.</u> - In order that the airfoil P_c be closed, $k(\zeta)$ must satisfy the condition

$$\oint k(\zeta)\, d\zeta - \frac{1}{4}\oint \frac{\overline{w^2(\zeta)}}{k(\zeta)}\, d\zeta = 0 \tag{23}$$

along any path enclosing the unit circle. If these functions are expanded about infinity and residues considered, this condition can be expressed in a more useful form, which permits the adjustment of the values of $k(\theta)$ as given by equation (18) so that equation (23) is satisfied.

Now

$$k(\zeta) = \left(1 + \frac{e^{i\gamma}}{\zeta}\right)^n e^{f(\zeta)}$$

where $f(\zeta)$ is regular on and outside the circle. By writing

$$f(\zeta) = a_0 + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \ldots$$

then

$$k(\zeta) = e^{a_0}\left[1 + \frac{ne^{i\gamma} + a_1}{\zeta}\left(+ \text{ terms in } \frac{1}{\zeta^j},\ j \geqq 2\right)\right] \tag{24}$$

Hence,

$$\oint k(\zeta)\, d\zeta = 2\pi i e^{a_0}(ne^{i\gamma} + a_1) \tag{25}$$

From the incompressible flow about the circle,

$$w(\zeta) = \frac{dF(\zeta)}{d\zeta} = -W - \frac{2Wi\ \sin\gamma}{\zeta} + \frac{W}{\zeta^2}$$

and

$$\frac{w^2(\zeta)}{k(\zeta)} = e^{-a_0}\left[W^2 + \frac{4W^2\ i\ \sin\gamma - mW^2e^{i\gamma} - a_1W^2}{\zeta} + \left(\text{terms in } \frac{1}{\zeta^j},\ j \geqq 2\right)\right]$$

Therefore

$$\oint \overline{\frac{w^2(\zeta)}{k(\zeta)}}\, d\zeta = 2\pi i\, \overline{e^{-a_0}(4W^2 i\ \sin\gamma - nW^2e^{i\gamma} - a_1W^2)} \tag{26}$$

Then by using equations (25) and (26), equation (23) becomes

$$2\pi i e^{a_0}(ne^{i\gamma} + a_1) - \frac{1}{4}\overline{\left[2\pi i e^{-a_0}\, W^2\ (4i\ \sin\gamma - ne^{i\gamma} - a_1)\right]} = 0$$

or

$$2\pi i e^{i\ \mathrm{Im}(a_0)}e^{-\mathrm{Re}(a_0)}\left\{e^{2\mathrm{Re}(a_0)}\left[ne^{i\gamma} + \mathrm{Re}(a_1) + i\ \mathrm{Im}(a_1)\right] + \frac{W^2}{4}\overline{\left[4i\ \sin\gamma - ne^{i\gamma} - \mathrm{Re}(a_1) + i\ \mathrm{Im}(a_1)\right]}\right\} = 0 \tag{27}$$

where Re and Im indicate the real and imaginary parts, respectively. Equating the real part of equation (27) to zero gives

$$\left[e^{2\ \mathrm{Re}(a_0)} - \frac{W^2}{4}\right]\left[n\ \cos\gamma + \mathrm{Re}(a_1)\right] = 0 \tag{28}$$

But

$$e^{2\ \mathrm{Re}(a_0)} - \frac{W^2}{4}$$

cannot be zero except for zero free-stream velocity; hence

$$n \cos \gamma + \mathrm{Re}(a_1) = 0 \tag{29}$$

The imaginary part of equation (27) gives

$$e^{2\ \mathrm{Re}(a_0)}\left[n \sin \gamma + \mathrm{Im}(a_1)\right] + \frac{W^2}{4}\left[-4 \sin \gamma + \mathrm{Im}(a_1) + n \sin \gamma\right] = 0$$

or

$$\mathrm{Im}(a_1) + n \sin \gamma = \frac{W^2 \sin \gamma}{e^{2\ \mathrm{Re}(a_0)} + \frac{W^2}{4}} \tag{30}$$

Consequently, $k(\zeta)$ will satisfy equation (23) if $\mathrm{Re}(a_1)$ and $\mathrm{Im}(a_1)$ satisfy equations (29) and (30), respectively.

Determination of $\mathrm{Re}(a_0)$, $\mathrm{Re}(a_1)$, and $\mathrm{Im}(a_1)$. - In order to determine $\mathrm{Re}(a_0)$, $\mathrm{Re}(a_1)$, and $\mathrm{Im}(a_1)$ when $k(\theta)$ is given by equation (18), the expansion in equation (24) is written as

$$
\begin{aligned}
k(\theta) = \operatorname{Re} k(\theta) + i \operatorname{Im} k(\theta) = A + \Big\{ A\big[n \cos\gamma + \operatorname{Re}(a_1)\big] - B\big[n \sin\gamma + \operatorname{Im}(a_1)\big] \Big\} \cos\theta \; + \\
\Big\{ A\big[n \sin\gamma + \operatorname{Im}(a_1)\big] + B\big[n \cos\gamma + \operatorname{Re}(a_1)\big] \Big\} \sin\theta \; + \\
(\text{terms in } e^{ij\theta},\ j \geqq 2) \; + \\
i\Big(B + \Big\{ A\big[n \sin + \operatorname{Im}(a_1)\big] + B\big[n \cos\gamma + \operatorname{Re}(a_1)\big] \Big\} \cos\theta \; + \\
\Big\{ - A\big[n \cos\gamma + \operatorname{Re}(a_1)\big] + B\big[n \sin\gamma + \operatorname{Im}(a_1)\big] \Big\} \sin\theta \; + \\
(\text{terms in } e^{ij\theta},\ j \geqq 2) \Big)
\end{aligned}
$$

where $e^{a_0} = A + iB$. This expansion is a Fourier series, and therefore

$$A = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} k(\theta)\, d\theta$$

$$B = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Im} k(\theta)\, d\theta$$

$$\frac{1}{\pi} \int_0^{2\pi} \operatorname{Re} k(\theta) \cos\theta\, d\theta = A\big[n \cos\gamma + \operatorname{Re}(a_1)\big] - B\big[n \sin\gamma + \operatorname{Im}(a_1)\big]$$

and

$$\frac{1}{\pi}\int_0^{2\pi} \text{Re } k(\theta) \sin \theta \, d\theta = A\left[n \sin \gamma + \text{Im}(a_1)\right] + B\left[n \cos \gamma + \text{Re}(a_1)\right]$$

Then

$$e^{\text{Re}(a_0)} = \sqrt{A^2 + B^2} \tag{31}$$

$$n \cos \gamma + \text{Re}(a_1) = \frac{1}{\pi} \frac{A\int_0^{2\pi} \text{Re } k(\theta) \cos \theta \, d\theta + B\int_0^{2\pi} \text{Re } k(\theta) \sin \theta \, d\theta}{A^2 + B^2} \tag{32}$$

$$n \sin \gamma + \text{Im}(a_1) = \frac{1}{\pi} \frac{A\int_0^{2\pi} \text{Re } k(\theta) \sin \theta \, d\theta - B\int_0^{2\pi} \text{Re } k(\theta) \cos \theta \, d\theta}{A^2 + B^2} \tag{33}$$

These equations give $\text{Re}(a_0)$, $\text{Re}(a_1)$, and $\text{Im}(a_1)$, which are required to satisfy equations (29) and (30) for closure.

Adjustment of $k(\zeta)$ for closure. - If these values of a_0 and a_1 do not satisfy the conditions, $k(\zeta)$ must be modified and this modification will change the velocity distribution slightly in most cases. For adjusting $k(\zeta)$, b_1 is defined

$$\text{Re}(b_1) = -\left[n \cos \gamma + \text{Re}(a_1)\right]$$

$$\text{Im}(b_1) = -\left[\text{Im}(a_1) + n \sin \gamma - \frac{W^2 \sin \gamma}{e^{2\text{Re}(a_0)} + \frac{W^2}{4}}\right]$$

where $Re(a_0)$, $Re(a_1)$, and $Im(a_1)$ are given by equations (31) to (33). Then the function $k'(\zeta)$ defined by

$$k'(\zeta) = k(\zeta)\, e^{(b_1/\zeta)} \tag{34}$$

will satisfy equation (22) and the airfoil obtained using $k'(\zeta)$ will close. The velocity distribution q' on the resulting profile will differ slightly from the prescribed velocity because of this adjustment. The ratio of these velocities is

$$\frac{q'}{q} = \frac{\left(4|k|^2 - |w|^2\right) e^{Re(b_1)\cos\theta + Im(b_1)\sin\theta}}{4|k|^2\, e^{2\left[Re(b_1)\cos\theta + Im(b_1)\sin\theta\right]} - |w|^2}$$

Because

$$|k'(\theta)| = |k|\, e^{Re(b_1)\cos\theta + Im(b_1)\sin\theta}$$

it is possible that $k'(\zeta)$ will not satisfy inequality (2). If it doesn't satisfy the inequality, a new funtion $k''(\zeta)$ defined by

$$k''(\zeta) = k(\zeta)\, e^{d_0 + \frac{d_1}{\zeta}} \tag{35}$$

where d_0 is chosen so that $k''(\zeta)$ will satisfy inequality (2) and

$$Re(d_1) = Re(b_1)$$

$$Im(d_1) = Im(b_1) - \frac{W^2 \sin\gamma}{e^{2Re(a_0)} + \frac{W^2}{4}} + \frac{W^2 \sin\gamma}{e^{2Re(a_0 + d_0)} + \frac{W^2}{4}}$$

will satisfy all requirements and yield a closed airfoil. That such a d_0 exists is shown by taking d_0 to be any positive real number such that

$$d_0 > |Re(b_1)| + |Im(b_1)| + \left|\frac{W^2 \sin\gamma}{\frac{W^2}{4}}\right|$$

but

$$\left|k''(\zeta)\right| = \left|k(\zeta)\right| e^{d_0 + \mathrm{Re}(d_1)\cos\theta + \mathrm{Im}(d_1)\sin\theta}$$

and

$$\left|\mathrm{Re}(d_1)\right| + \left|\mathrm{Im}(d_1)\right| \leqq \left|\mathrm{Re}(b_1)\right| + \left|\mathrm{Im}(b_1)\right| + \left|\frac{W^2 \sin\gamma}{e^{2\mathrm{Re}(a_0)} + \frac{W^2}{4}} - \frac{W^2 \sin\gamma}{e^{2\mathrm{Re}(a_0)+2d_0} + \frac{W^2}{4}}\right| \leqq$$

$$\left|\mathrm{Re}(b_1)\right| + \left|\mathrm{Im}(b_1)\right| + \left|\frac{W^2 \sin\gamma}{\frac{W^2}{4}}\right| < d_0$$

Therefore

$$\left|k''(\zeta)\right| > \left|k(\zeta)\right| > \left|\frac{1}{2}\,w(\zeta)\right|$$

which is inequality (2).

If this method is used, d_0 should be chosen as small as possible, so that the velocity on the profile will differ only slightly from the prescribed values. With $k''(\zeta)$, the velocity on the profile is

$$q'' = \frac{4\left|k(\theta)\right|\left|w(\theta)\right| e^{d_0 + \mathrm{Re}(d_1)\cos\theta + \mathrm{Im}(d_1)\sin\theta}}{4\left|k^2(\theta)\right| e^{2\left[d_0 + \mathrm{Re}(d_1)\cos\theta + \mathrm{Im}(d_1)\sin\theta\right]} - \left|w^2(\theta)\right|}$$

In some cases it may be better to modify $k'(\zeta)$ to satisfy inequality (2) by changing the higher harmonic terms which, of course, do not affect the closure; that is, multiply $k'(\zeta)$ by $e^{\frac{b_2}{\zeta^2} - \frac{b_3}{\zeta^3} - \cdots}$. In fact, these terms may be used to reduce the changes induced in the prescribed velocity distribution when modifying $k(\zeta)$ to obtain a closed profile. These terms also may be used to alter the airfoil shape when the computed profile is too thin or has negative thickness. The selection of the coefficients b_2, b_3, . . . depends on the particular problem and the velocity distribution desired and therefore no general method can be given for determining them.

Airfoil Coordinates

The function $k(\theta)$ has been obtained to satisfy all requirements; hence, the airfoil coordinates in the z plane are given by equation (4) on integrating around the unit circle. For convenience let $k(\theta)$ be written

$$k(\theta) = k_1(\theta)\, e^{ik_2(\theta)} \tag{36}$$

Then from equation (4) the airfoil coordinates are

$$x = -\int \frac{4k_1^{\,2} - \left(2W \sin\theta + \frac{\Gamma_1}{2\pi}\right)^2}{4k_1} \sin(k_2 + \theta)\, d\theta \tag{37}$$

$$y = \int \frac{4k_1^{\,2} - \left(2W \sin\theta + \frac{\Gamma_1}{2\pi}\right)^2}{4k_1} \cos(k_2 + \theta)\, d\theta \tag{38}$$

COMPUTATIONAL PROCEDURE

An outline of the procedure for computing the airfoil follows:

1. Obtain $\varphi_c(s)$, Γ_c, and $\varphi_{c,min}$ from equations (10), (11), and (12), respectively.

2. By use of these values in equations (13), (14), and (15), obtain φ_i, γ, and W, respectively. Then calculate $\varphi_i(\theta)$ and $w(\theta)$ from equations (8) and (9).

3. Plot $\varphi_c(s)$ and $\varphi_i(\theta)$. By comparing equal values of these potentials, obtain s as a function of θ, which permits writing the prescribed velocity q as a function of θ, $q = q(\theta)$.

4. Compute $g(\theta)$ by equation (21) and $h(\theta)$ by equation (22). Then $k(\theta)$ is obtained from equation (18).

5. The closure conditions are checked by computing $Re(a_0)$, $Re(a_1)$, and $Im(a_1)$ from equations (31), (32), and (33), respectively, and checking these values in equations (29) and (30). If

these equations are not satisfied, $k(\theta)$ is adjusted by equation (34). This new function $k'(\theta)$ must be checked in inequality (2). If it does not satisfy this condition, $k'(\theta)$ is further modified as indicated in equation (35).

6. By use of the adjusted values of $k(\theta)$ that satisfy all conditions, the airfoil coordinates are obtained from equations (37) and (38) by integration.

ILLUSTRATIVE EXAMPLES

Circular profile. - As a check on the method, the theoretical velocity distribution on a nearly circular profile, as computed in reference 8 from the simplified pressure-density relation, was taken as the prescribed velocity distribution (fig. 2) and the profile was computed. The profile agrees with the one used in reference 8, as shown in figure 3.

Airfoil profile. - In this example, the prescribed velocity q was obtained from the incompressible velocity distribution about a symmetrical Joukowski profile, computed by Lipman Bers at Syracuse University in the form of the ratio of actual velocity to free-stream velocity, by taking the dimensionless free-stream velocity to be 0.538. The resulting distribution is shown in figure 4 together with the final velocity after adjustment for closure. The computed profile (fig. 5) is slightly thicker than the Joukowski profile toward the nose and has some reflex camber. The peaks in the velocity distribution about the computed profile are lower than those that would occur in a compressible flow about the Joukowski profile with the same free-stream velocity and angle of attack because: (1) the circulation was kept the same as for the incompressible flow, which resulted in the reflex camber; and (2) the thickening of the profile reduced the curvatures in the vicinity of the velocity peaks.

CONCLUSIONS

By the assumption of the linear pressure-volume relation, a method has been given for uniquely obtaining the shape of an airfoil having any prescribed velocity distribution, which is theoretically realizable - that is, yields a closed profile - in a potential flow of a compressible perfect fluid. If the prescribed velocity is not realizable, the method gives a means of modifying it so that a closed profile is obtained. Whether the resulting profile is practical or not will depend on other considerations such as thickness. The applicability of the method is limited only by the accuracy of this linear approximation to the actual pressure-volume relation.

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, May 4, 1949.

APPENDIX A

RELATION BETWEEN n AND δ

On the unit circle, equation (5) becomes

$$\frac{2q}{1+\sqrt{1+q^2}}\, e^{-i\alpha} = \frac{e^{-i\left(\theta+\frac{\pi}{2}\right)}(\sin\theta+\sin\gamma)\,2W}{\left[1+e^{i(\gamma-\theta)}\right]^n e^{g+ih}}$$

or

$$\frac{2q}{1+\sqrt{1+q^2}}\, e^{-i\alpha} = \frac{e^{-i\left(\theta+\frac{\pi}{2}\right)}(\sin\theta+\sin\gamma)\,2\,W}{\left[2+2\cos(\gamma-\theta)\right]^{\frac{n}{2}} e^{i\left[h+n\arctan\frac{\sin(\gamma-\theta)}{1+\cos(\gamma-\theta)}\right]+g}} \tag{A1}$$

For a point p_1 on the upper surface of the airfoil very near the tail of the airfoil, the angle of the velocity in the z plane is given by equation (A1) and

$$\alpha_1 = \left(\theta_1+\frac{\pi}{2}\right) + n\arctan\frac{\sin(\gamma-\theta_1)}{1+\cos(\gamma-\theta_1)} + h(\theta_1)$$

where θ_1 is the circle angle corresponding to p_1 and

$$\pi < \theta_1 < \pi + \gamma$$

Similarly, for a point p_2 on the lower surface

$$\alpha_2 = \left(\theta_2+\frac{3\pi}{2}\right) + n\arctan\frac{\sin(\gamma-\theta_2)}{1+\cos(\gamma-\theta_2)} + h(\theta_2)$$

where

$$-\pi+\gamma < \theta_2 < -\frac{\pi}{2}$$

Hence

$$\alpha_1-\alpha_2 = (\theta_1-\theta_2) - \pi + n\left[\arc \tan\frac{\sin(\gamma-\theta_1)}{1 + \cos(\gamma-\theta_1)} - \arc \tan\frac{\sin(\gamma-\theta_2)}{1 + \cos(\gamma-\theta_2)}\right] +$$

$$h(\theta_1) - h(\theta_2)$$

Then the angle δ is the limit of $(\alpha_1-\alpha_2)$ as p_1 and p_2 approach the tail. But

$$\underset{\theta_1 \to \pi + \gamma}{\text{limit}}\left[\arc \tan \frac{\sin(\gamma-\theta_1)}{1 + \cos(\gamma-\theta_1)}\right] = -\frac{\pi}{2}$$

and

$$\underset{\theta_2 \to -\pi + \gamma}{\text{limit}}\left[\arc \tan \frac{\sin(\gamma-\theta_2)}{1 + \cos(\gamma-\theta_2)}\right] = \frac{\pi}{2}$$

Hence

$$\delta = 2\pi - \pi + n\left(-\frac{\pi}{2} - \frac{\pi}{2}\right)$$

$$\delta = \pi + n(-\pi)$$

or

$$n = 1 - \frac{\delta}{\pi}$$

APPENDIX B

INFLUENCE OF D ON PROFILE

If D is not taken as zero in equation (22), the airfoil coordinates x' and y' are

$$x' = -\int \frac{4k_1^2 - \left(2W \sin \gamma + \frac{\Gamma_1}{2\pi}\right)^2}{4k_1} \sin(k_2 + D + \theta)\, d\theta$$

$$= -\int \frac{4k_1^2 - \left(2W \sin \gamma + \frac{\Gamma_1}{2\pi}\right)^2}{4k_1} \left[\sin(k_2 + \theta) \cos D + \cos(k_2 + \theta)\sin D\right] d\theta$$

$$= x \cos D - y \sin D$$

and

$$y' = \int \frac{4k_1^2 - \left(2W \sin \gamma + \frac{\Gamma_1}{2\pi}\right)^2}{4k_1} \cos(k_2 + D + \theta)\, d\theta$$

$$= \int \frac{4k_1^2 - \left(2W \sin \gamma + \frac{\Gamma_1}{2\pi}\right)^2}{4k_1} \left[\cos(k_2 + \theta) \cos D - \sin(k_2 + \theta) \sin D\right] d\theta$$

$$= y \cos D + x \sin D$$

Consequently, the airfoil given by x' and y' is the same as the one given by x and y rotated through the angle D.

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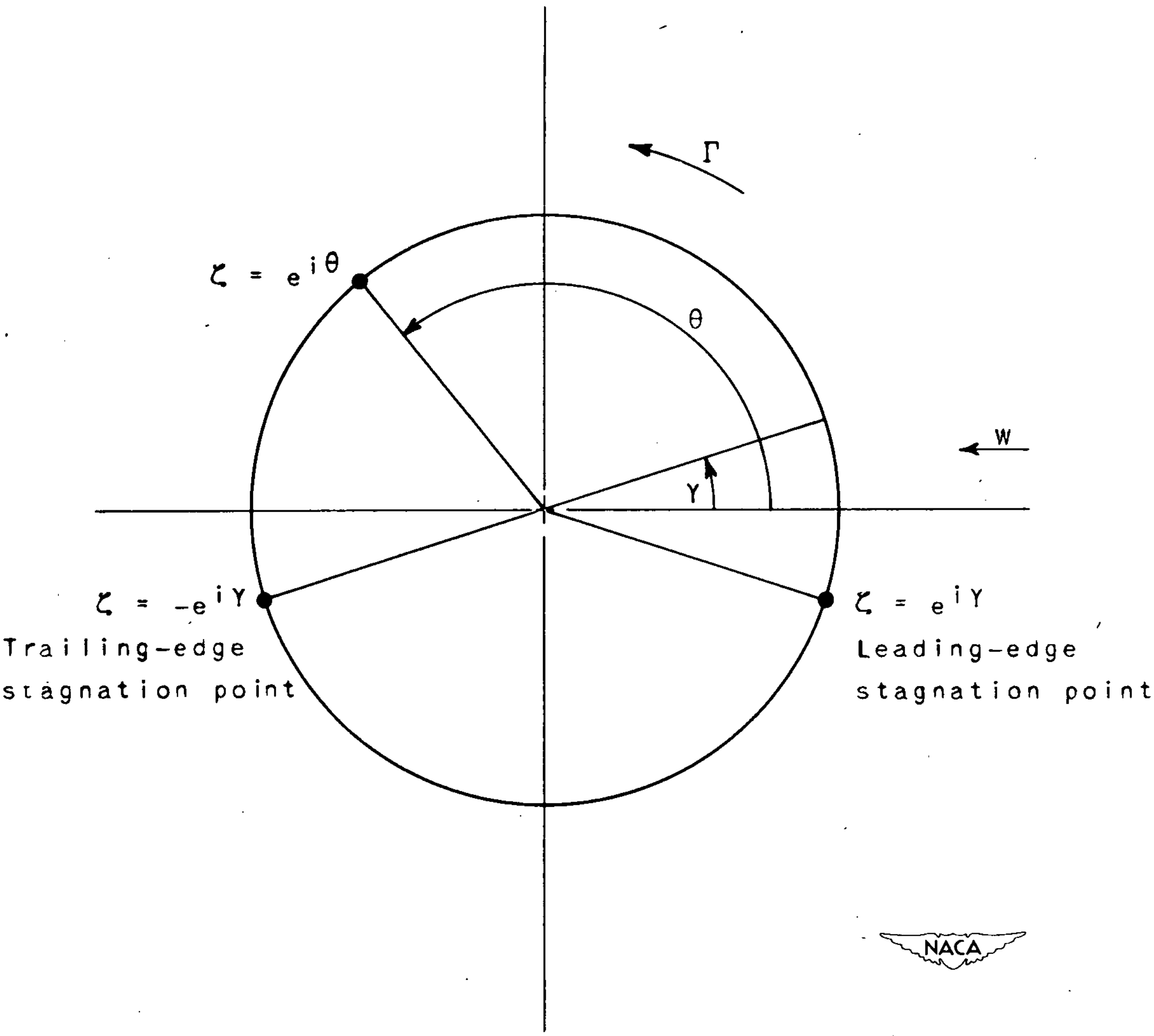

Figure 1. - Variables in circle plane.

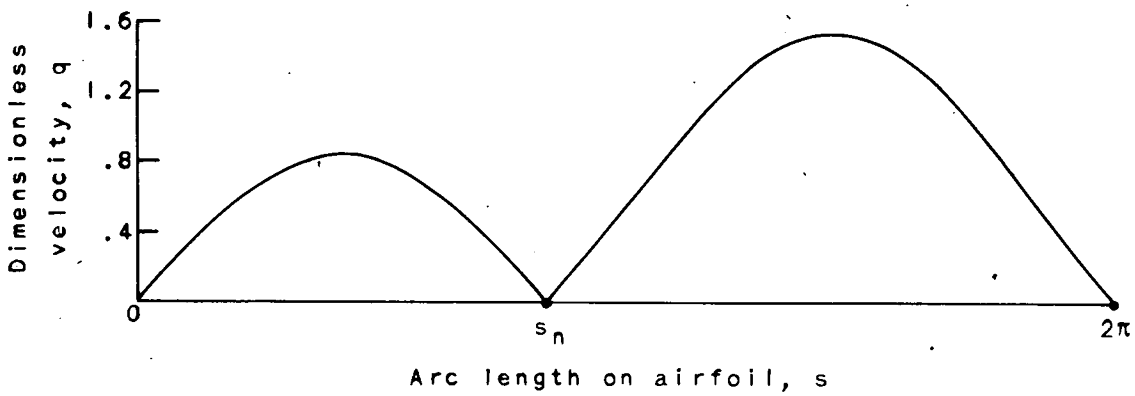

Figure 2. - Prescribed velocity distribution for circular profile.

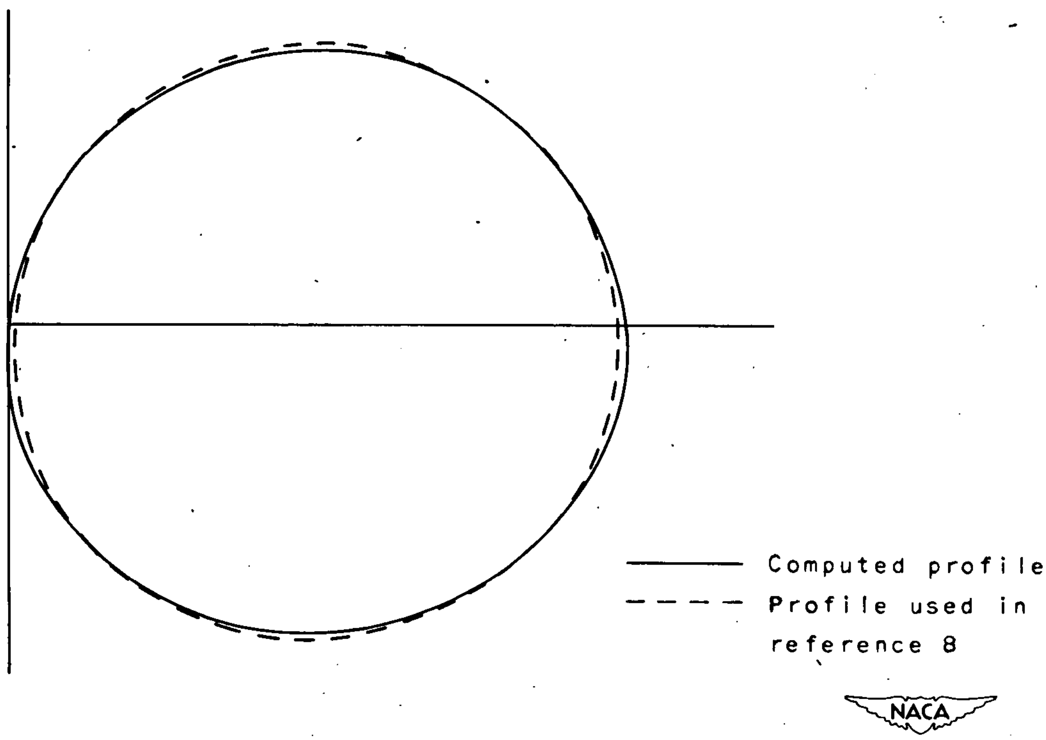

Figure 3. - Comparison of profile used in reference 8 and computed profile having same velocity distribution.

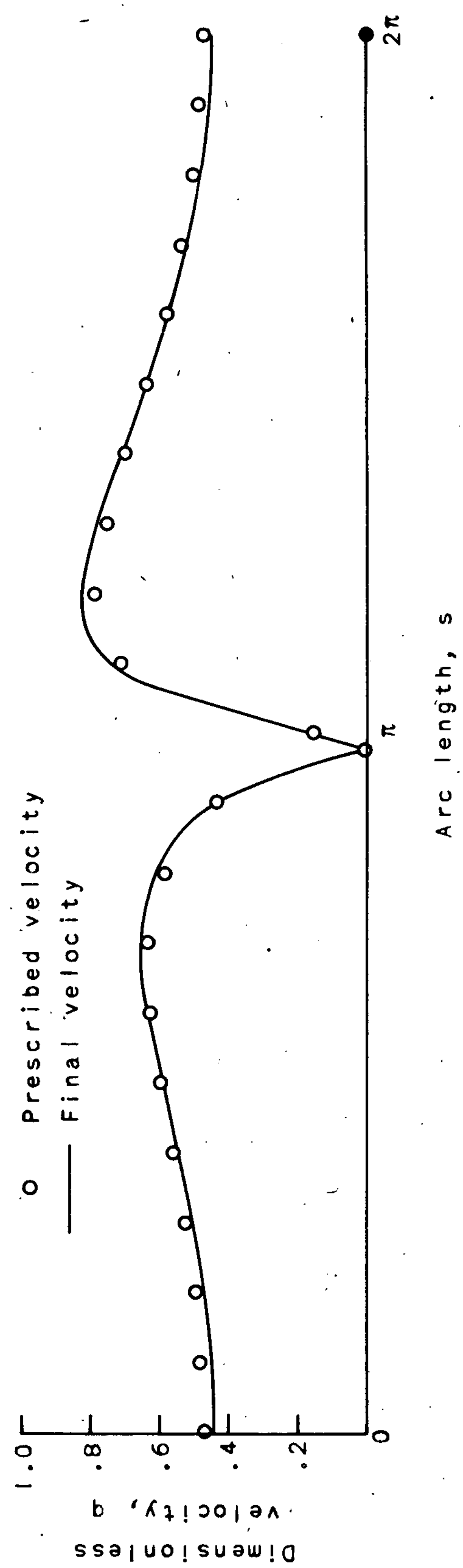

Figure 4. - Comparison of prescribed velocity and final velocity distributions for airfoil.

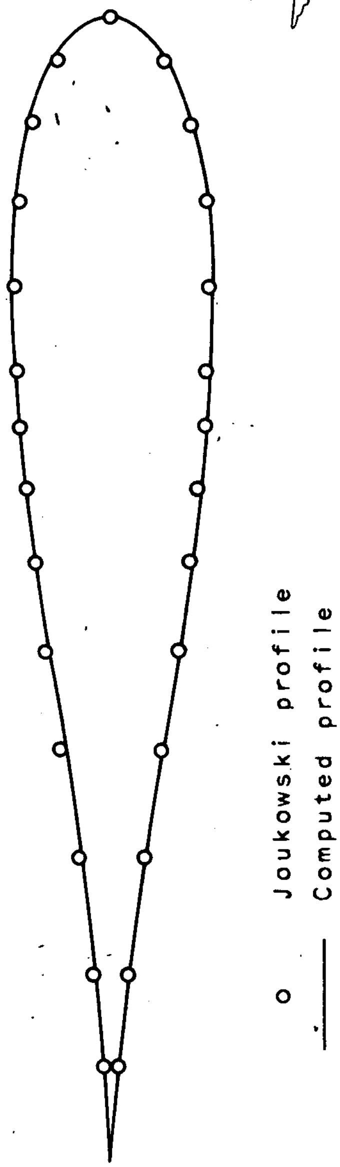

Figure 5. - Comparison of computed profile and Joukowski profile.