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METHODS OF DESIGNING CASCADE BLADES WITH PRESCRIBED VELOCITY DISTRIBUTIONS IN COMPRESSIBLE POTENTIAL FLOWS

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METHOD OF DESIGNING CASCADE BLADES WITH PRESCRIBED VELOCITY DISTRIBUTIONS IN COMPRESSIBLE POTENTIAL FLOWS

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SUMMARY

By use of the assumption that the pressure-volume relation is linear, a solution to the problem of designing a cascade for a given turning and with a prescribed velocity distribution along the blade in a potential flow of a compressible perfect fluid was obtained by a method of correspondence between potential flows of compressible and incompressible fluids.

If the prescribed velocity distribution is not theoretically attainable, the method gives a way of modifying the distribution so as to obtain a physically significant blade shape.

INTRODUCTION

In order to control boundary-layer growth, transition, and separation in the design of a cascade for a given turning, it is advantageous to prescribe the velocity along the blade as a function of the arc length along the blade and then to compute the blade shape. For the case of an incompressible fluid, several solutions to this problem have been obtained. (See references 1 to 3.)

A similar solution for the two-dimensional potential flow of a compressible perfect fluid has been developed at the NACA Lewis laboratory. This solution is based on the assumption that the pressure-volume relation is given by a linear approximation to the isentropic curve instead of the true curve. The flow pattern of the compressible fluid is obtained by a transformation from a corresponding flow of an incompressible fluid using the transformation developed by Lin (reference 4).

The method of solution consists in using the free-stream velocities upstream and downstream of the cascade and the prescribed dimensionless velocity distribution along the blade to select a suitable incompressible potential flow about the unit circle and then to determine the mapping function that transforms this incompressible flow into a compressible flow about a cascade of airfoils. The image of the unit circle under this mapping gives the cascade with the prescribed velocity distribution along the blade, provided the velocity distribution is theoretically attainable. If the velocity distribution is unattainable, methods are given for modifying the distribution so that a physically significant profile is obtained.

SYMBOLS

The following symbols are used in this report:

A, B, C_1, C_2, D	complex constants
a_1, a_2	location of complex sources in ζ-plane
$C(\zeta)$	function of ζ defined by equation (44)
d	spacing of cascade
$F(\zeta)$	complex potential function (incompressible flow)
$f(\zeta)$	regular function of ζ
$g(\zeta)$	regular function of ζ
$H(\zeta)$	regular function of ζ
$\operatorname{Re} \tilde{H}(\zeta)$	function of ζ defined by equation (59)
Im	imaginary part
$K(\theta)$	function of θ defined by equation (49)
K_1	constant equal to $\dfrac{2q_1}{1 + \sqrt{1 + q_1^2}}$
K_2	constant equal to $\dfrac{2q_2}{1 + \sqrt{1 + q_2^2}}$
k	constant defined by equation (33)
n	number determined by included trailing-edge angle of blade

p pressure

$Q(s)$ auxiliary function of s

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q magnitude of dimensionless velocity in compressible-flow plane (ratio of actual velocity to stagnation velocity of sound)

$q_1 e^{i\alpha_1}$ dimensionless velocity upstream of cascade

$q_2 e^{i\alpha_2}$ dimensionless velocity downstream of cascade

R region in ζ-plane defined by $|\zeta| \geq 1$

Re real part

r number defined by equation (23)

s arc length along blade

$v(\theta)$ velocity on unit circle (incompressible flow)

$z = x + iy$ complex variable (compressible-flow plane)

α angle of velocity in compressible flow (measured from positive x-axis)

Γ circulation (positive counterclockwise)

γ ratio of specific heats

δ included trailing-edge angle of blade

$\zeta = \xi + i\eta$ complex variable (incompressible-flow plane)

θ circle angle (incompressible-flow plane)

λ auxiliary variable defined by equation (37)

ρ density

τ variable of integration

φ velocity potential

ψ stream function

Subscripts:

c	compressible flow
i	incompressible flow
n	leading edge
t	trailing edge

Prime indicates a derivative.

THEORY OF METHOD

In reference 4, Lin has shown that if the pressure-density relation is

$$p = C_1 - \frac{C_2}{\rho} \tag{1}$$

then the compressible potential flow about a cascade of blades can be obtained by transforming the incompressible flow about the unit circle in the following manner. The complex potential function $F(\zeta)$ for the incompressible flow due to two complex sources at $\zeta = a_1$ and $\zeta = a_2$ outside the unit circle $|\zeta| = 1$ is

$$F(\zeta) = A \log_e(\zeta - a_1) + \bar{A} \log_e\left(\zeta - \frac{1}{\bar{a}_1}\right) + B \log_e(\zeta - a_2) + \bar{B} \log_e\left(\zeta - \frac{1}{\bar{a}_2}\right) + D \tag{2}$$

where A and B are complex constants with Re A ≥ 0 and Re A = - Re B, and D is an arbitrary complex constant. The bar indicates the complex conjugate. The mapping between the z-plane and the ζ-plane defined by

$$dz = g(\zeta)\,(\zeta - a_1)^{-1}\,(\zeta - a_2)^{-1}\,d\zeta - \frac{1}{4}\overline{\left[F'(\zeta)\right]^2\,\left[g(\zeta)\right]^{-1}\,(\zeta - a_1)\,(\zeta - a_2)\,d\zeta} \tag{3}$$

gives a compressible flow with the linear pressure-volume relation past a straight cascade of identical blades in the z-plane with the velocity potential φ_c and stream function ψ_c given by

$$\varphi_c + i\psi_c = F(\zeta) \tag{4}$$

1183 provided that $g(\zeta)$ is chosen to satisfy the following requirements:

(a) The function $g(\zeta)$ is regular in closed region R defined by $|\zeta| \geq 1$.

(b) The function $g(\zeta) \neq 0$ in R, except possibly at one point on the circle where $F'(\zeta) = 0$. (The order of the zero not to exceed 1.)

(c) Along the circle $|\zeta| = 1$, $\oint dz = 0$.

(d) The function $g(\zeta)$ satisfies the inequality $\left| \left[F'(\zeta)\right] \left[g(\zeta)\right]^{-1} (\zeta - a_1)\ (\zeta - a_2) \right| < 2$ in R.

(5)

The magnitude q and the direction α of the dimensionless velocity at any point in the z-plane are given by

$$\frac{2q}{1 + \sqrt{1 + q^2}}\, e^{-i\alpha} = \frac{F'(\zeta)(\zeta - a_1)(\zeta - a_2)}{g(\zeta)} \tag{6}$$

In order to use this transformation in designing a blade with a prescribed dimensionless velocity distribution along the blade in a cascade, the prescribed conditions are used to select a suitable incompressible flow about the unit circle and to determine the function $g(\zeta)$.

The prescribed conditions are the velocity distribution on the airfoil, the upstream velocity $q_1 e^{i\alpha_1}$ and the downstream velocity $q_2 e^{i\alpha_2}$. The upstream and downstream velocities are related by the isentropic-flow equations with $\gamma = -1$. This relation is

$$\frac{q_2^{\,2} \cos^2 \alpha_2}{1 + q_2^{\,2}} = \frac{q_1^{\,2} \cos^2 \alpha_1}{1 + q_1^{\,2}} \tag{7}$$

where the axis of the cascade has been taken along the y-axis for convenience and the flow is from left to right. (See fig. 1.)

Flow in Circle Plane

The flow of an incompressible fluid about the unit circle is selected by determining the constants A, B, a_1, and a_2 in the complex potential 1183

$$F(\zeta) = A \log_e(\zeta - a_1) + \bar{A} \log_e\left(\zeta - \frac{1}{\bar{a}_1}\right) + B \log_e(\zeta - a_2) + \bar{B} \log_e\left(\zeta - \frac{1}{\bar{a}_2}\right) + D \tag{8}$$

from the given conditions. The constants A and B are obtained from the upstream and downstream velocities and the circulation and then a_1 and a_2 are determined by the range of the potential on the airfoil.

<u>Circulation and cascade spacing.</u> - The magnitude of the prescribed dimensionless velocity q along the airfoil is given as a function of the arc lengths $[q = q(s)]$ where the total arc length is taken to be 2π and is measured from the trailing edge along the lower surface. If $Q(s)$ is defined by

$$\left.\begin{aligned} Q(s) &= -\,q(s) \qquad 0 \le s \le s_n \\ Q(s) &= q(s) \qquad s_n \le s \le 2\pi \end{aligned}\right\} \tag{9}$$

where s_n is the leading-edge stagnation point, then

$$\varphi_c(s) = \int_0^s Q(s)\, ds \tag{10}$$

$$\Gamma_c = \int_0^{2\pi} Q(s)\, ds \tag{11}$$

The circulation and the spacing of the cascade are related by

$$d = \frac{\Gamma_c}{q_1 \sin \alpha_1 - q_2 \sin \alpha_2} \tag{12}$$

where d is the spacing. The quantities Γ_c, q_1, q_2, α_1, and α_2 are known so that the spacing is determined.

Determination of A. - The value of d from equation (12) is used to evaluate A and B because the spacing is also given by the absolute value of $\oint dz$ taken along a path around a_1 or a_2. (See fig. 2.) The axis of the cascade has been taken along the y-axis so that

$$id = \oint_{a_1} dz = - \oint_{a_2} dz \tag{13}$$

The second equality comes from the fact that the residues at infinity of

$$g(\zeta)(\zeta-a_1)^{-1} (\zeta-a_2)^{-1}$$

and

$$\left[F'(\zeta)\right]^2 \left[g(\zeta)\right]^{-1} (\zeta-a_1)(\zeta-a_2)$$

in the expression for dz in equation (3) are zero and consequently

$$\oint_c dz + \oint_{a_1} dz + \oint_{a_2} dz = 0 \tag{14}$$

where c is the unit circle. But by equation 5(c)

$$\oint_c dz = 0$$

so that

$$\oint_{a_1} dz = - \oint_{a_2} dz \tag{15}$$

The evaluation of equation (13) in terms of the potential $F(\zeta)$ is

$$id = \oint_{a_1} \frac{g(\zeta)}{(\zeta-a_1)(\zeta-a_2)} d\zeta - \frac{1}{4} \oint_{a_1} \frac{\overline{\left[F'(\zeta)\right]^2 (\zeta-a_1)(\zeta-a_2)}}{g(\zeta)} d\zeta \tag{16}$$

But

$$F'(\zeta) = \frac{A}{\zeta - a_1} + \frac{\bar{A}}{\zeta - \frac{1}{\bar{a}_1}} + \frac{B}{\zeta - a_2} + \frac{\bar{B}}{\zeta - \frac{1}{\bar{a}_2}} \tag{17}$$

so that equation (16) reduces to

$$id = 2\pi i \frac{g(a_1)}{(a_1 - a_2)} - \frac{1}{4} \overline{\left[\frac{A^2(a_1 - a_2)}{g(a_1)}\right] 2\pi i} \tag{18}$$

$$= 2\pi i \left[\frac{g(a_1)}{(a_1 - a_2)} + \frac{\bar{A}^2}{4} \frac{\overline{(a_1 - a_2)}}{\bar{g}\ (\bar{a}_1)}\right] \tag{19}$$

At $\zeta = a_1$, equation (6) becomes

$$\frac{2q_1}{1 + \sqrt{1 + q_1^2}} e^{-i\alpha_1} = \frac{A(a_1 - a_2)}{g(a_1)}$$

which on writing

$$K_1 = \frac{2q_1}{1 + \sqrt{1 + q_1^2}}$$

reduces to

$$\frac{g(a_1)}{(a_1 - a_2)} = \frac{A}{K_1} e^{i\alpha_1} \tag{20}$$

Substitution of the values from equation (20) in equation (19) gives

$$id = 2\pi i \left(\frac{A}{K_1} e^{i\alpha_1} + \frac{\bar{A}K_1}{4} e^{i\alpha_1}\right) \tag{21}$$

Hence, the bracketed expression in equation (21) must be a real number and

$$\frac{4A + \bar{A}K_1^2}{4K_1} \equiv re^{-i\alpha_1} \tag{22}$$

where

$$r^2 = \left(\frac{4+K_1^{\ 2}}{4K_1} \text{ Re } A\right)^2 + \left(\frac{4-K_1^{\ 2}}{4K_1} \text{ Im } A\right)^2 \tag{23}$$

From equation (22)

$$\frac{(4-K_1^{\ 2}) \text{ Im } A}{(4+K_1^{\ 2}) \text{ Re } A} = - \tan \alpha_1$$

or

$$\text{Im } A = - \frac{4+K_1^{\ 2}}{4-K_1^{\ 2}} \text{ Re } A \tan \alpha_1 \tag{24}$$

$$\text{Im } A = - \sqrt{1+q_1^{\ 2}} \text{ Re } A \tan \alpha_1 \tag{25}$$

which gives the relation between Re A and Im A.

Substitution of the value of Im A from equation (24) in equation (23) yields

$$r^2 = \left(\frac{4+K_1^{\ 2}}{4K_1}\right)^2 \text{Re}^2 A + \left(\frac{4+K_1^{\ 2}}{4K_1}\right)^2 \text{Re}^2 A \tan^2 \alpha_1$$

$$= \left(\frac{4+K_1^{\ 2}}{4K_1}\right)^2 \text{Re}^2 A \sec^2 \alpha_1$$

or

$$r = \frac{4+K_1^{\ 2}}{4K_1} \text{ Re } A \left|\sec \alpha_1\right|$$

Hence

$$d = 2\pi \left(\frac{4+K_1^{\ 2}}{4K_1} \text{ Re } A \left|\sec \alpha_1\right|\right) \tag{26}$$

Substitution of the value of d from equation (26) in equation (12) gives

$$2\pi\left(\frac{4+K_1{}^2}{4K_1}\ \text{Re}\ A\left|\sec\alpha_1\right|\right) = \frac{\Gamma_c'}{q_1\ \sin\alpha_1 - q_2\ \sin\alpha_2}$$

or

$$\text{Re}\ A = \frac{4K_1\ \Gamma_c\ \left|\cos\alpha_1\right|}{2\pi(4+K_1{}^2)(q_1\ \sin\alpha_1 - q_2\ \sin\alpha_2)} \tag{27}$$

and Re A is now determined. By use of this value of Re A in equation (25), Im A is obtained. Hence A is completely given by equations (27) and (25).

Determination of B. - From equation (13)

$$id = -\oint_{a_2} dz$$

$$= -\oint_{a_2} g(\zeta)(\zeta-a_1)^{-1}(\zeta-a_2)^{-1}\ d\zeta + \frac{1}{4}\oint_{a_2}\overline{\left[F'(\zeta)\right]^2\left[g(\zeta)\right]^{-1}(\zeta-a_1)(\zeta-a_2)d\zeta}$$

$$id = -\ 2\pi i\ \frac{g(a_2)}{(a_2-a_1)} + \frac{1}{4}\ \frac{\overline{B^2}\ \overline{(a_2-a_1)}\ \overline{2\pi i}}{\overline{g}\ (a_2)}$$

$$= -\ 2\pi i\ \frac{Be^{i\alpha_2}}{K_2} - \frac{2\pi_1}{4}\ \overline{B}K_2 e^{i\alpha_2}$$

$$= -\ 2\pi i\left[\left(\frac{4B+K_2{}^2\ \overline{B}}{4K_2}\right)e^{i\alpha_2}\right]$$

where

$$K_2 = \frac{2q_2}{1+\sqrt{1+q_2{}^2}}$$

The bracketed expression must be real, so that

$$\frac{(4-K_2{}^2)\ \mathrm{Im}\ B}{(4+K_2{}^2)\ \mathrm{Re}\ B} = -\tan \alpha_2 \tag{28}$$

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But

$$\mathrm{Re}\ B = -\ \mathrm{Re}\ A \tag{29}$$

and equation (28) can be written

$$\mathrm{Im}\ B = \frac{4+K_2{}^2}{4-K_2{}^2}\ \mathrm{Re}\ A \tan \alpha_2$$

$$= \sqrt{1+q_2{}^2}\ \mathrm{Re}\ A \tan \alpha_2 \tag{30}$$

Consequently, B is determined by equations (29) and (30) because Re A is known from equation (27).

Determination of a_1 and a_2. - After A and B are known, the points a_1 and a_2 are to be selected to satisfy the single condition that the range of potential on the circle must equal the range of potential on the airfoil, that is,

$$\varphi_c\ (2\pi) - \varphi_c(s_n) = F\left[e^{i(\theta_t+2\pi)}\right] - F(e^{i\theta_n}) \tag{31}$$

where θ_t and θ_n are the trailing-edge and leading-edge stagnation angles, respectively. This condition is only one equation in two unknowns, a_1 and a_2; consequently, the values of a_1 and a_2 are not uniquely determined. By imposing an additional restriction that a_1 and a_2 are real and

$$a_1 = -\ a_2 \tag{32}$$

unique values are obtained for a_1 and a_2 in all cases. In particular problems, however, some other restriction may be more useful, such as assigning a definite value for a_1 and computing a_2.

With the restriction given in equation (32), it is possible to express θ_t and θ_n in terms of a_1 and substitute these values in equation (31), but the resulting equation cannot be solved explicitly for a_1. 1183

One method for obtaining a_1 is as follows: Let

$$\left.\begin{aligned} a_1 &= e^k \\ a_2 &= -e^k \end{aligned}\right\} \qquad (33)$$

where $k > 0$. Then equation (8) becomes

$$F(\zeta) = A \log_e(\zeta - e^k) + \bar{A} \log_e(\zeta - e^{-k}) + B \log_e(\zeta + e^k) + \bar{B} \log_e(\zeta - e^{-k}) + D$$

or, for points on the unit circle $\zeta = e^{i\theta}$

$$F(e^{i\theta}) = \varphi_i(\theta) + i\psi_i(\theta) = \text{Re } A \log_e \frac{(e^{i\theta}-e^k)(e^{i\theta}-e^{-k})}{(e^{i\theta}+e^k)(e^{i\theta}+e^{-k})} +$$

$$i \text{ Im } A \log_e \frac{e^{i\theta}-e^k}{e^{i\theta}-e^{-k}} + i \text{ Im } B \log_e \frac{e^{i\theta}+e^k}{e^{i\theta}+e^{-k}} + D \qquad (34)$$

Hence, $\varphi(\theta)$ may be written in the form

$$\varphi(\theta) = -2 \text{ Re } A \tanh^{-1} \frac{\cos\theta}{\cosh k} + (\text{Im } A + \text{Im } B) \tan^{-1} \frac{\tan\theta}{\tanh k} +$$

$$(\text{Im } A - \text{Im } B) \tan^{-1} \frac{\sin\theta}{\sinh k} + 2 \text{ Re } A \tanh^{-1} \frac{\cos\theta_t}{\cosh k} -$$

$$(\text{Im } A + \text{Im } B) \tan^{-1} \frac{\tan\theta_t}{\tanh k} - (\text{Im } A - \text{Im } B) \tan^{-1} \frac{\sin\theta_t}{\sinh k} \qquad (35)$$

where D has been chosen to make $\varphi_i(\theta_t) = 0$ and the angle convention is

$$-\frac{\pi}{2} < \tan^{-1} \frac{\sin\theta}{\sinh k} < \frac{\pi}{2}$$

1183 and $\tan^{-1} \frac{\tan \theta}{\tanh k}$ is taken in the same quadrant and same direction as θ.

The velocity on the circle $v(\theta)$ is

$$v(\theta) = \frac{2\text{Re } A \sin \theta \cosh k}{\cosh^2 k - \cos^2 \theta} + \frac{(\text{Im } A + \text{Im } B) \tanh k \sec^2 \theta}{\tanh^2 k + \tan^2 \theta} +$$

$$(\text{Im } A - \text{Im } B) \frac{\sinh k \cos \theta}{\sinh^2 k + \sin^2 \theta}$$

$$= \frac{2}{(\cosh 2k - \cos 2\theta)} \Big[2\text{Re } A \sin \theta \cosh k +$$

$$(\text{Im } A - \text{Im } B) \cos \theta \sinh k + (\text{Im } A + \text{Im } B) \sinh k \cosh k \Big] \tag{36}$$

Equation (36) can be further simplified by defining λ as

$$\tan \lambda = \frac{(\text{Im } A - \text{Im } B) \sinh k}{2\text{Re } A \cosh k}\, .$$

$$-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$$

Then

$$v(\theta) = \frac{2\sqrt{4\text{Re}^2 A \cosh^2 k + (\text{Im } A - \text{Im } B)^2 \sinh^2 k}}{\cosh 2k - \cos 2\theta} \Bigg[\sin \theta \cos \lambda +$$

$$\cos \theta \sin \lambda + \frac{(\text{Im } A + \text{Im } B) \sinh k \cosh k}{\sqrt{4\text{Re}^2 A \cosh^2 k + (\text{Im } A - \text{Im } B)^2 \sinh^2 k}} \Bigg]$$

$$= \frac{2\sqrt{4\text{Re}^2 A \cosh^2 k + (\text{Im } A - \text{Im } B)^2 \sinh^2 k}}{\cosh 2k - \cos 2\theta} \Bigg[\sin (\theta + \lambda) +$$

$$\frac{(\text{Im } A + \text{Im } B) \cosh k \sinh k}{\sqrt{4\, \text{Re}^2 A \cosh^2 k + (\text{Im } A - \text{Im } B)^2 \sinh^2 k}} \Bigg] \tag{38}$$

The stagnation angles θ_t and θ_n are therefore the roots of the equation

$$\sin(\theta+\lambda) = \frac{-(\text{Im } A + \text{Im } B) \sinh k \cosh k}{\sqrt{4 \, \text{Re}^2 A \cosh^2 k + (\text{Im } A - \text{Im } B)^2 \sinh^2 k}} \tag{39}$$

The desired value of k is obtained as follows:

(1) Assume a value of k.

(2) Compute λ by equation (37).

(3) Obtain θ_t and θ_n from equation (39).

(4) Compute $\varphi_i(\theta_t+2\pi) - \varphi_i(\theta_n)$.

(5) Repeat (1) to (4) several times to obtain a plot of $\varphi_i(\theta_t+2\pi) - \varphi_i(\theta_n)$ as a function of k.

(6) Interpolate to obtain k such that

$$\varphi_i(\theta_t+2\pi) - \varphi_i(\theta_n) = \varphi_c(2\pi) - \varphi_c(s_n)$$

With k determined the flow about the circle is known. The potential $\varphi_i(\theta)$ and velocity $v(\theta)$ for points on the circle are given by equations (35) and (38), respectively.

Function $g(\zeta)$

The function $g(\zeta)$ can be computed for points on the unit circle by using the prescribed velocity on the airfoil and the velocity on the unit circle to determine the real part of $g(\zeta)$. The imaginary part of $g(\zeta)$ can then be computed by Poisson's integral. Because of the restrictions imposed by the given conditions, however, $g(\zeta)$ is actually obtained in a slightly different manner, as shown in the following sections.

Airfoil with pointed trailing edge. - If an airfoil with a pointed trailing edge is desired, then $g(\zeta)$ must vanish at the trailing-edge stagnation point $\zeta = e^{i\theta_t}$. Hence, $g(\zeta)$ can be written in the form

$$g(\zeta) = \left(1 - \frac{e^{i\theta_t}}{\zeta}\right)^n e^{f(\zeta)} \tag{40}$$

where $f(\zeta)$ is regular in the exterior of the unit circle and

$$n = 1 - \frac{\delta}{\pi} \tag{41}$$

where δ is the included trailing-edge angle of the airfoil. (See reference 4.)

Values of $g(\pm e^k)$. - Because the velocities are given for the compressible flow upstream and downstream of the cascade, the value of $g(\zeta)$ at $\zeta = \pm e^k$ is determined from equation (6),

$$\left.\begin{aligned} g(e^k) &= \frac{2e^k A e^{i\alpha_1}}{K_1} \\ g(-e^k) &= \frac{-2e^k B e^{i\alpha_2}}{K_2} \end{aligned}\right\} \tag{42}$$

In order that $g(\zeta)$ have these values, $f(\zeta)$ is written in the form

$$f(\zeta) = C(\zeta) + \frac{(e^{2k} - \zeta^2)(\zeta^2 - e^{-2k})}{\zeta^2} H(\zeta) \tag{43}$$

where

$$C(\zeta) = \frac{1}{2}\left(1 + \frac{e^k}{\zeta}\right) \log_e \left[\frac{2Ae^{i\alpha_1 + k}}{K_1 \ (1 - e^{i\theta_t - k})^n}\right] +$$

$$\frac{1}{2}\left(1 - \frac{e^k}{\zeta}\right) \log_e \left[\frac{-2Be^{i\alpha_2 + k}}{K_2 \ (1 + e^{i\theta_t - k})^n}\right] \tag{44}$$

and $H(\zeta)$ is regular in the exterior of the unit circle with

$$\lim_{\zeta \to \infty} \zeta H(\zeta) = 0 \tag{45}$$

The restriction on $H(\zeta)$ imposed by equation (45) is necessary so that $f(\zeta)$ (equation (43)) will be regular. By use of equation (43), $g(\zeta)$ is expressed as

$$g(\zeta) = \left(1 - \frac{e^{i\theta_t}}{\zeta}\right)^n e^{C(\zeta)} + \frac{(e^{2k}-\zeta^2)(\zeta^2-e^{-2k})}{\zeta^2} H(\zeta) \tag{46}$$

and $g(\zeta)$ will be known when $H(\zeta)$ is determined. For the actual computation of the blade shape, only the values of $g(\zeta)$ on the unit circle are needed. Hence, it is only necessary to compute $H(\zeta)$ for points on the circle. If desired, the values of $H(\zeta)$ for any point in the exterior of the circle can be obtained from the values on the circle by Poisson's integral.

Determination of Re H on the circle. - By equation (4), the potentials $\varphi_c(s)$ and $\varphi_i(\theta)$ are equal at corresponding points. Thus, by matching these potentials a correspondence is established between points along the airfoil arc and the circle angles; that is, $s = s(\theta)$. By use of this correspondence, the magnitude of the prescribed velocity along the airfoil is obtained as a function of the circle angle $q = q(\theta)$. Hence, by taking absolute values of equation (6)

$$\frac{2q(\theta)}{1 + \sqrt{1+q(\theta)^2}} = \frac{\left|F'(e^{i\theta})(e^{i\theta}-e^k)(e^{i\theta}+e^k)\right|}{\left|g(e^{i\theta})\right|} \tag{47}$$

for points on the circle. Substitution of the value of $g(\zeta)$ from equation (46) with $\zeta = e^{i\theta}$ and replacing $F'(e^{i\theta})$ by the velocity $v(\theta)$ (equation (38)) on the circle give

$$\frac{2q(\theta)}{1 + \sqrt{1+q(\theta)^2}} = \frac{e^k \left|v(\theta)\right| (2\cosh 2k - 2\cos 2\theta)^{\frac{1}{2}}}{\left[2 - 2\cos(\theta_t-\theta)\right]^{\frac{n}{2}} e^{\left[\mathrm{Re}\, C(e^{i\theta}) + (2\cosh 2k - 2\cos 2\theta)\,\mathrm{Re}\, H(e^{i\theta})\right]}}$$

or, with the equation solved for $\mathrm{Re}\, H(\theta)$,

$$\mathrm{Re}\, H(e^{i\theta}) = \frac{\log_e \left[\dfrac{\left|v(\theta)\right| (2\cosh 2k-2\cos 2\theta)^{\frac{1}{2}}}{K(\theta)\left[2 - 2\cos(\theta_t - \theta)\right]^{\frac{n}{2}}}\right] - \mathrm{Re}\, C(e^{i\theta}) + k}{2\cosh 2k - 2\cos 2\theta} \tag{48}$$

where

$$K(\theta) = \frac{2q(\theta)}{1 + \sqrt{1+q(\theta)^2}} \tag{49}$$

Restrictions on Re $H(e^{i\theta})$. - Equation (45) imposes restrictions on the values of Re $H(e^{i\theta})$, as shown by writing $H(\zeta)$ in the form

$$H(\zeta) = h_0 + \frac{h_1}{\zeta} + \frac{h_2}{\zeta^2} + \ldots \tag{50}$$

For points on the circle, equation (50) becomes

$$\begin{aligned} H(e^{i\theta}) &= \text{Re } H(e^{i\theta}) + i \text{ Im } H(e^{i\theta}) \\ &= \text{Re } h_0 + \sum_{j=1} (\text{Re } h_j \cos j\theta + \text{Im } h_j \sin j\theta) + \\ &\quad i \left[\text{Im } h_0 + \sum_{j=1} (\text{Im } h_j \cos j\theta - \text{Re } h_j \sin j\theta \right] \end{aligned} \tag{51}$$

Equation (51) is a Fourier expansion and

$$\text{Re } h_0 = \frac{1}{\pi} \int_0^{2\pi} \text{Re } H(e^{i\theta})\, d\theta \tag{52}$$

$$\text{Re } h_1 = \frac{1}{2\pi} \int_0^{2\pi} \text{Re } H(e^{i\theta}) \cos\theta\, d\theta \tag{53}$$

$$\text{Im } h_1 = \frac{1}{2\pi} \int_0^{2\pi} \text{Re } H(e^{i\theta}) \sin\theta\, d\theta \tag{54}$$

But equation (45) requires that

$$\text{Re } h_0 = \text{Im } h_0 = \text{Re } h_1 = \text{Im } h_1 = 0 \tag{55}$$

Consequently, $\text{Re } H(e^{i\theta})$ must satisfy the equations

$$\int_0^{2\pi} \text{Re } H(e^{i\theta})d\theta = 0 \tag{56}$$

$$\int_0^{2\pi} \text{Re } H(e^{i\theta})\cos\theta \, d\theta = 0 \tag{57}$$

$$\int_0^{2\pi} \text{Re } H(e^{i\theta})\sin\theta \, d\theta = 0 \tag{58}$$

Adjustment of $\text{Re } H(e^{i\theta})$. - If the values of $\text{Re } H(e^{i\theta})$ from equation (48) do not satisfy equations (56), (57), and (58), the values must be adjusted until the conditions are satisfied. One method for adjusting the function is to define $\text{Re } \tilde{H}(e^{i\theta})$ by

$$\text{Re } \tilde{H}(e^{i\theta}) = \text{Re } H(e^{i\theta}) - \frac{1}{2}\text{Re } h_0 - 2 \text{ Re } h_1 \cos\theta - 2 \text{ Im } h_1 \sin\theta \tag{59}$$

where $\text{Re } h_0$, $\text{Re } h_1$, and $\text{Im } h_1$ are given by equations (52), (53), and (54), respectively. The modified function $\text{Re } \tilde{H}(e^{i\theta})$ will then satisfy equations (56), (57), and (58). This method of modifying $\text{Re } H(e^{i\theta})$, however, changes the velocity distribution all along the profile and, if the correction terms in equation (59) are not small, these changes in the velocity may be extensive because

$$\frac{2\tilde{q}}{1 + \sqrt{1+\tilde{q}^2}} = \frac{2q}{1 + \sqrt{1+q^2}} e^{\left(\frac{1}{2}\text{Re } h_0 + 2 \text{ Re } h_1 \cos\theta + 2 \text{ Im } h_1 \sin\theta\right)}$$

where $\tilde{q}$ denotes the new velocity. In some cases, consequently, $\text{Re } H(e^{i\theta})$ can best be adjusted to satisfy the requirements by adding to $\text{Re } H(e^{i\theta})$ odd and even functions that have nonzero values only in small neighborhoods of the points $\theta = 0$ and $\theta = -\pi$. The particular functions to be added to $\text{Re } H(e^{i\theta})$ and their range of values depends on the specific problem; no general method can be given for determining the functions.

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<u>Determination of $\operatorname{Im} H(e^{i\theta})$.</u> - After $\operatorname{Re} H(e^{i\theta})$ satisfying equations (56), (57), and (58) is obtained, the function $\operatorname{Im} H(e^{i\theta})$ is given by Poisson's integral (reference 5)

$$\operatorname{Im} H(e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} H(e^{i\tau}) \cot \left(\frac{\tau-\theta}{2}\right) d\tau \tag{60}$$

where the constant term in Poisson's integral has been taken as zero so that

$$\operatorname{Im} h_0 = \frac{1}{\pi} \int_0^{2\pi} \operatorname{Im} H(e^{i\theta})\, d\theta = 0$$

as required by equation (55). Hence, $H(e^{i\theta})$ is determined for points on the unit circle by

$$H(e^{i\theta}) = \operatorname{Re} H(e^{i\theta}) + i \operatorname{Im} H(e^{i\theta})$$

<u>Adjustment of $g(e^{i\theta})$.</u> - By use of these values of $H(e^{i\theta})$ in equation (46), $g(\zeta)$ is determined for points on the circle,

$$g(e^{i\theta}) = \left[1-e^{i(\theta_t-\theta)}\right]^n e^{\left[C(e^{i\theta}) + (2 \cosh 2k - 2 \cos 2\theta) H(e^{i\theta})\right]} \tag{61}$$

Because of the adjustments in $H(e^{i\theta})$, $g(e^{i\theta})$ may no longer satisfy condition (5d). If $g(e^{i\theta})$ does not satisfy the inequality for points on the circle, then the values of $g(\zeta)$ can be adjusted to satisfy the inequality by changing the second or higher harmonic terms in $H(\zeta)$ or by other methods. It should be noted that if the velocity along the profile is finite then $g(\zeta)$ satisfies the inequality. In fact, if the prescribed conditions are theoretically attainable, then no modification is necessary, not even in $\operatorname{Re} H(e^{i\theta})$.

Blade Coordinates

By use of the values of $g(e^{i\theta})$ that satisfy all conditions, the blade coordinates are obtained from integration of equation (3), that is

$$z = \int g(\zeta)(\zeta-e^{k})^{-1}(\zeta+e^{k})^{-1}\, d\zeta - \frac{1}{4}\int \overline{\frac{F'(\zeta)^{2}(\zeta-e^{k})(\zeta+e^{k})d\zeta}{g(\zeta)}}$$

which on replacing $F'(e^{i\theta})$ by $v(\theta)e^{-i(\theta+\frac{\pi}{2})}$ and writing

$$g(e^{i\theta}) = g_1(\theta)\ e^{ig_2(\theta)}$$

reduces to

$$z = \int \left[g_1(\theta)\ (e^{2i\theta}-e^{2k})^{-1} - \frac{v(\theta)^2}{4g_1(\theta)}\ \overline{(e^{2i\theta}-e^{2k})}\right]\ e^{i\left[\theta+\frac{\pi}{2}+g_2(\theta)\right]}\, d\theta \tag{62}$$

COMPUTATIONAL PROCEDURE

An outline of the procedure for computing the blade shape is as follows:

(1) Obtain $\varphi_c(s)$ and Γ_c from equations (10) and (11), respectively.

(2) Compute Re A, Im A, and Im B by equations (27), (25), and (30), respectively.

(3) Obtain k as outlined in the text. Compute $\varphi_i(\theta)$ and $v(\theta)$ by equations (35) and (38), respectively.

(4) Plot $\varphi_c(s)$ and $\varphi_i(\theta)$. By comparing the abscissas for equal values of these potentials, obtain s as a function of θ, which permits writing the prescribed velocity q as a function of θ, $q = q(\theta)$.

(5) Compute Re $H(e^{i\theta})$ by equation (48) and determine Re h_0, Re h_1, and Im h_1 by equations (52), (53), and (54), respectively. If these values are not zero, then adjust Re $H(e^{i\theta})$ either by equation (59) or by addition of functions so that Re $H(e^{i\theta})$ satisfies equations (56), (57), and (58).

(6) Obtain Im $H(e^{i\theta})$ by equation (60) using the adjusted values of Re $H(e^{i\theta})$.

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(7) Obtain $g(e^{i\theta})$ by equation (61). The function $g(e^{i\theta})$ must satisfy inequality (5d) for points on the circle. If $g(e^{i\theta})$ does not satisfy the inequality, adjust $g(e^{i\theta})$, as suggested in
1183 the text.

(8) After $g(e^{i\theta})$ has been adjusted to satisfy all conditions, the blade shape is obtained by integrating equation (62).

DISCUSSION

The magnitude of the dimensionless velocity along the blade cannot be entirely prescribed arbitrarily as a function of the arc length, but is subject to some restrictions in addition to the conditions imposed on $H(\zeta)$ previously discussed. The magnitude must be finite everywhere along the profile and by the method given here the velocity can be zero in at most two places - the leading-edge and trailing-edge stagnation points. By a limiting process, however, the method can be extended to provide for additional stagnation points. The zero of q at the trailing edge is of the order $\frac{\delta}{\pi}$, where δ is the included trailing-edge angle of the blade. Thus, for a cusp at the tail, δ is zero and q need not be zero at the trailing edge.

Another restriction is imposed on the velocity distribution when the spacing of the cascade, as well as the turning, is specified in advance because the distribution must be selected so that Γ_c will satisfy equation (12).

If a velocity distribution is selected to satisfy these conditions but otherwise is arbitrary, the resulting profile may not be a physically real blade but may result in a blade with zero or negative thickness in some portions of the blade. The negative or zero thickness is caused by specifying too low velocities along parts of the blade and a physically real blade can be obtained by increasing the prescribed velocity along the blade.

CONCLUSION

By use of the assumption that the pressure-volume relation is linear, a method has been given for computing the blade shape in a straight cascade of identical blades having a prescribed velocity distribution along the blade and given upstream and downstream velocities in a potential flow of a compressible perfect fluid. If the prescribed velocity is not theoretically realizable, the method gives a way of modifying the distribution so as to obtain a blade shape. Whether the resulting blade is practical will depend on other considerations. The applicability of the method is limited only by the accuracy of the linear approximation to the pressure-volume relation.

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Lewis Flight Propulsion Laboratory,
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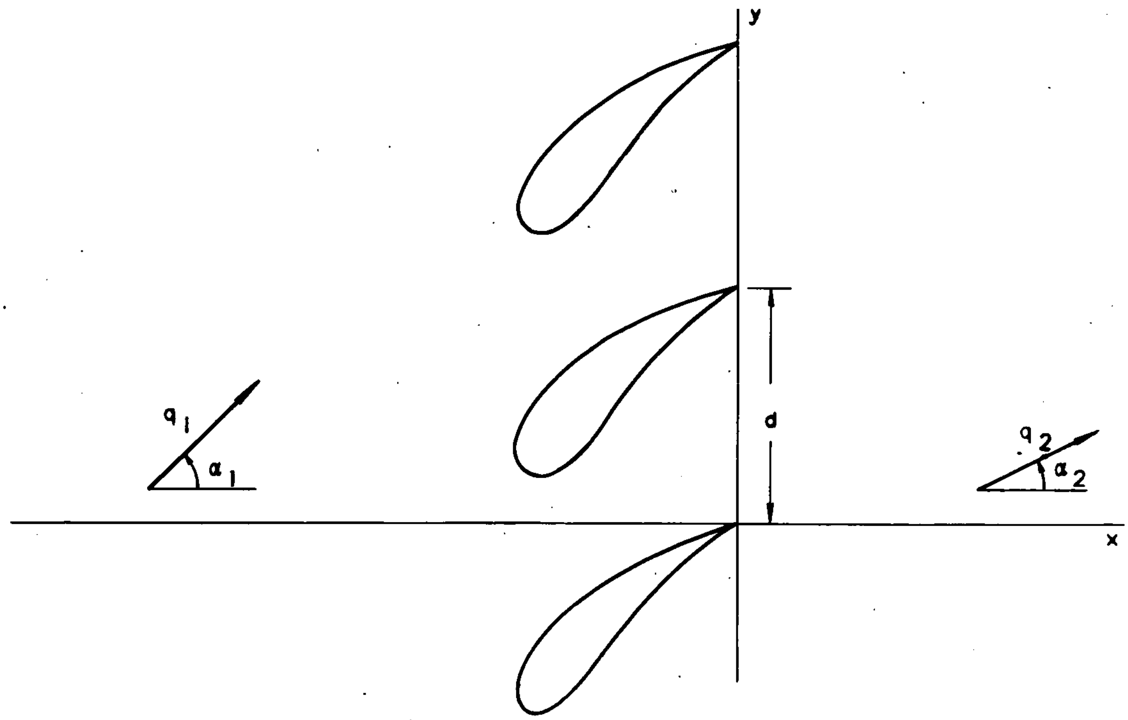

Figure 1. - Cascade in z-plane.

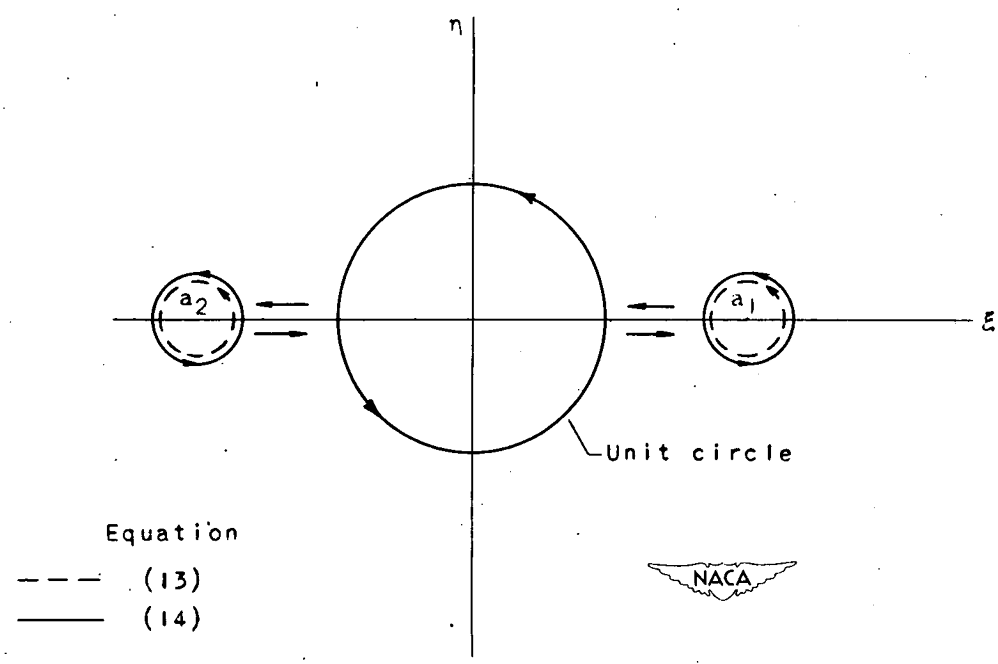

Figure 2. - Paths of integration in ζ-plane.