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TECHNICAL NOTE 2006

A THEORETICAL INVESTIGATION OF THE EFFECT ON THE LATERAL  
OSCILLATIONS OF AN AIRPLANE OF AN AUTOMATIC CONTROL  
SENSITIVE TO YAWING ACCELERATIONS

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## SUMMARY

A theoretical investigation was made to determine the effect on the lateral oscillations of an airplane of an automatic control sensitive to yawing accelerations. The investigation included calculations of the effect of time lag in a control of this type on the damping of the lateral oscillations of a typical high-speed airplane and also calculations of the effect on the damping of varying the ratio of the rudder deflection to the yawing acceleration. The inadequacy of the approximate lag-operator method as a means of treating time lag is also discussed.

The results indicate that a control of this type can successfully damp the lateral oscillations through a reasonable range of time lag. The presence of the automatic control introduces a higher-frequency mode of motion which becomes unstable with increasing time lag in addition to the existing lower-frequency Dutch roll mode of motion which becomes more stable with increasing time lag. Increasing the ratio of the rudder deflection to the yawing acceleration improved the damping of the lower-frequency mode slightly but, at the same time, reduced rapidly the damping of the higher-frequency mode.

## INTRODUCTION

With the increased speed range, altitude range, and density of current airplanes, the problems of constant-amplitude lateral oscillations and of poorly damped lateral oscillations have become more acute. The familiar snaking oscillation is one example of this type of oscillation which can occur. This type of oscillation may be very annoying to the pilot on a long flight and may also considerably reduce the accuracy of gunfire or of rockets or bombs released from the airplane. Even in the

landing condition certain high-speed airplanes have developed poorly damped Dutch roll oscillations which would be very critical during a blind landing. In a fighter airplane these oscillations usually have small amplitudes and short periods. The oscillations are objectionable even if the period is of the order of magnitude of 5 or 6 seconds such as might be the case in a heavy bomber airplane, because the oscillations still require close pilot attention for control.

In the past, satisfactory damping of the lateral oscillations has been inherently provided in the airplane. It may not be possible to provide enough inherent stability in future high-speed airplanes because of the effect of factors such as Mach number and high airplane density; therefore, some type of automatic control may be required in order to damp these lateral oscillations.

Several types of automatic controls have been used recently. Controls which are sensitive to displacement or velocity have been used in conventional automatic pilots, but these controls have two disadvantages when used for controlling short-period oscillations. These controls oppose the forces applied by the pilot in steady maneuvers and, in addition, large values of time lag in the automatic control reduce the effectiveness of the control in producing the desired damping.

An investigation of the response of an airplane controlled by an automatic control which would be sensitive to yawing accelerations is considered worthwhile. Such a control would not oppose the forces applied by the pilot in steady maneuvers and the time lag in such an automatic-control system may not be as critical a factor as in other types of controls. A theoretical investigation therefore has been made to determine the effect of time lag on the stability produced by this type of control. The effect of varying the ratio between the rudder deflection and the yawing acceleration has also been investigated.

#### SYMBOLS AND COEFFICIENTS

$\phi$	angle of roll, radians
$\psi$	angle of yaw, radians
$\beta$	angle of sideslip, radians
$\rho$	mass density of air, slugs per cubic foot
$V$	airspeed, feet per second

- $q$  dynamic pressure, pounds per square foot  $\left(\frac{1}{2}\rho V^2\right)$   
 $b$  wing span, feet  
 $S$  wing area, square feet  
 $W$  weight of airplane, pounds  
 $g$  acceleration due to gravity, feet per second per second  
 $m$  mass of airplane, slugs  $(W/g)$   
 $I_X$  moment of inertia about the longitudinal principal axis, slug-feet<sup>2</sup>  
 $I_Z$  moment of inertia about the vertical principal axis, slug-feet<sup>2</sup>  
 $I_{XZ}$  product of inertia with respect to the longitudinal and vertical principal axes, slug-feet<sup>2</sup>  $((I_Z - I_X)\sin \eta \cos \eta)$   
 $I'_X = \frac{I_X}{qSb}$   
 $I'_Z = \frac{I_Z}{qSb}$   
 $I'_{XZ} = \frac{I_{XZ}}{qSb}$   
 $\mu$  relative density factor  $(m/\rho S b)$   
 $\eta$  angle of attack of principal longitudinal axis of airplane, positive when principal axis is above flight path at the nose, degrees  
 $\gamma$  angle of flight path to horizontal, positive in a climb, degrees  
 $\delta_r$  rudder deflection, radians  
 $C_L$  lift coefficient  $(W/qS)$   
 $C_{L_p} = \frac{\partial C_L}{\partial \frac{pb}{2V}}$

$$C_{l_r} = \frac{\partial C_l}{\partial \frac{rb}{2V}}$$

$$C_{l_\beta} = \frac{\partial C_l}{\partial \beta}$$

$$C_{n_p} = \frac{\partial C_n}{\partial \frac{pb}{2V}}$$

$$C_{n_r} = \frac{\partial C_n}{\partial \frac{rb}{2V}}$$

$$C_{n_\beta} = \frac{\partial C_n}{\partial \beta}$$

$$C_{Y_\beta} = \frac{\partial C_Y}{\partial \beta}$$

$$C_{Y_p} = \frac{\partial C_Y}{\partial \frac{pb}{2V}}$$

$$C_{Y_r} = \frac{\partial C_Y}{\partial \frac{rb}{2V}}$$

$$C_{n\delta_r} = \frac{\partial C_n}{\partial \delta_r}$$

$\tau$  time lag, seconds

D differential operator (d/dt)

P period of oscillation, seconds ( $2\pi/\omega$ )

$T_{1/2}$  time for oscillation to reach half-amplitude, seconds

$T_2$  time for oscillation to reach double amplitude, seconds

K	gearing ratio between the rudder deflection and yawing acceleration, radians per radian per second <sup>2</sup> $\left(\delta_r/D^2\psi\right)$
$\omega$	circular frequency, radians per second
$\omega_n$	natural undamped circular frequency, radians per second
$\omega'$	ratio of circular frequency to natural undamped circular frequency $\left(\omega/\omega_n\right)$
a	real part of complex stability root
$\zeta$	ratio of damping coefficient to critical damping coefficient
$\theta$	phase angle between $\delta_r$ and $D^2\psi$ , degrees
L	rolling moment, foot-pounds
N	yawing moment, foot-pounds
N'	yawing moment produced by deflecting rudder, foot-pounds
Y	side force, pounds

#### AUTOMATIC CONTROL

The function of the automatic control considered in this paper is to damp the short-period lateral oscillation. It is desirable that this automatic control should not interfere with the normal operation of the airplane by the pilot. For an airplane with an irreversible type of control system, this automatic control could be connected to the existing rudder. If the control system were reversible, the automatic control could be used to oscillate a part of the existing rudder or a small additional vertical surface. Inasmuch as the oscillations under consideration are usually of small amplitude, a comparatively small vertical area would be necessary to provide enough force to damp the oscillations.

An automatic control sensitive to yawing accelerations has an advantage over other types of controls for this purpose because it does not oppose the forces applied by the pilot in steady maneuvers. The operation of this control can be pictured by considering the case of an airplane performing a constant-amplitude oscillation in yaw. The yawing velocity and the yawing acceleration are 90° and 180°, respectively, out of phase with the yawing displacement. If the rudder motion were controlled to oppose the yawing acceleration and with no time lag, the rudder deflection

would be in phase with the displacement and hence would not apply a damping force. With a finite time lag, a component of the rudder deflection would oppose the yawing velocity and provide a damping force.

In operation, an automatic control inherently introduces some time lag between the yawing acceleration in an oscillation and the corresponding rudder motion. This time lag usually is a function of the frequency of the oscillation, but, for the lower frequencies, the lag is often approximately constant and independent of frequency. The assumption of a constant time lag has therefore been made in this analysis.

#### METHOD OF ANALYSIS

The effect of time lag in an automatic control sensitive to yawing accelerations on an airplane equipped with an automatic control which is free to yaw, to roll, and to move laterally was first investigated by the use of the lag operator mentioned in references 1 and 2. This method of treating time lag implies that the amount of rudder deflection applied at a given instant is proportional to the amount of yawing acceleration which existed at a fixed time previous to the given instant. Because use of the exact time-lag operator  $e^{-\tau D}$  would result in a transcendental equation, the expression  $1 - \tau D + \frac{\tau^2 D^2}{2}$ , which is equal to the first three terms of the power series that represents  $e^{-\tau D}$ , was used to represent time lag in these calculations.

Results obtained from a step-by-step analysis of the problem were found to be in conflict with results obtained by use of this approximate method. The development of a better method for determining the effect of time lag was therefore considered necessary. The frequency-response method described in reference 3 was extended to give an exact method of analysis. A more complete analysis of the problem of accounting for time lag by frequency-response methods is presented in reference 4.

The values of the stability derivatives and mass characteristics used in the calculations are given in table I. These values are representative of a high-speed research or fighter airplane. A Mach number of 0.80 at an altitude of 30,000 feet was assumed in the calculations.

Preliminary calculations indicated that for this airplane the frequency responses for one degree of freedom and for three degrees of freedom were essentially the same. As a further simplification, therefore, the analysis was restricted to the one-degree-of-freedom system. This simplification is possible because of the low dihedral

effect and the small product of inertia of this configuration. A complete discussion of the different methods of analysis is presented in the appendix.

In determining the effect of time lag, values were assigned to all parameters except  $\tau$ . For the normal airplane configuration, a rudder deflection of approximately  $\pm 1^\circ$  will provide sufficient force to damp rapidly an oscillation with a yawing displacement of  $\pm 1^\circ$ , which corresponds to a value of  $K$  of 0.0427 radian per radian per second<sup>2</sup> for the typical airplane used in these calculations. This value of  $K$  was used in these calculations.

In order to determine the effect of variation in the ratio of the rudder deflection to the yawing acceleration, calculations were repeated for values of  $K$  ranging from 0.01 to 0.0427 radian per radian per second<sup>2</sup>.

#### RESULTS AND DISCUSSION

The results of the investigation considering an airplane free to move laterally, to roll, and to yaw compared with the case of an airplane free only to yaw are presented in figure 1. This figure is a plot of the damping and period of the lateral oscillation calculated with the lag operator  $1 - \tau D + \frac{\tau^2 D^2}{2}$  as a function of time lag. For

these calculations a fixed gearing constant  $K = \frac{\delta_r}{D^2 \psi}$  of 0.0427 radian per radian per second<sup>2</sup> was used. These calculations are in good agreement and demonstrate that for the configuration chosen, which has a low dihedral effect and a small product of inertia, further analysis can be simplified by consideration of the single-degree-of-freedom system.

With no automatic control, this configuration has a time to damp to half-amplitude of approximately 2.50 seconds and a period of approximately 1.30 seconds. With the automatic control in operation and no time lag, the lateral oscillation damps to half-amplitude in approximately 3.40 seconds and has a period of about 1.65 seconds. Use of the approximate lag operator  $1 - \tau D + \frac{\tau^2 D^2}{2}$  to represent time lag in the automatic-control system gives results which show that the damping improves with time lag in the range considered and that the period of the lateral oscillation increases slightly with increasing time lag.



The results of the exact frequency-response analysis for the case of an airplane free only to yaw are presented in figure 2, which is a plot of the damping and period of the lateral oscillations as a function of time lag. These calculations were also made for a fixed gearing constant of 0.0427 radian per radian per second<sup>2</sup>. The frequency-response calculations predict instability of the system if the time lag is increased to large values. The presence of the automatic control introduces a higher-frequency mode of motion that becomes unstable with increasing time lag superposed on the lower-frequency stable mode. The approximate method fails to predict this mode of motion. The damping curve obtained with the approximate lag-operator calculations is in good agreement with the damping of the lower-frequency mode of motion as predicted by the frequency-response analysis. (Compare figs. 1 and 2.) The period of the lower-frequency mode of motion increases slightly with increasing values of time lag. The period of the higher-frequency mode of motion is proportional to the amount of time lag.

The most significant fact that figure 2 demonstrates is that, with this type of control, the time lag is not a very critical factor. If the actual automatic control has a time lag ranging from about 0.10 to 0.28 second, which is not at all unusual for an automatic-control installation in an airplane, good damping will result. If the automatic control were sensitive to an angular displacement or to an angular velocity, satisfactory damping might not be obtainable for this same range of time lag. By making the control sensitive to yawing acceleration, effectively 90° more phase lag can be tolerated and still produce good damping with the control.

The effect of varying the gearing constant  $K = \frac{\delta_r}{D^2\psi}$  is shown in figure 3 as a plot of  $\frac{1}{T_{1/2}}$ , which is proportional to the damping, against time lag. The plot is presented in this manner to avoid having the damping curves approach infinity when the oscillation becomes neutrally stable.

Mathematically, the phase relationship between the rudder deflection and the yawing acceleration may be expressed as a lead or a lag. The negative values of time lag are shown as long-dash lines on the curves in order to illustrate that negative time lag would result in a reduction in damping of the oscillation. In the actual operation of this automatic control the rudder deflection can never lead the yawing acceleration because the rudder cannot be operated by a yawing acceleration occurring at a later time.

In the time-lag range considered to give good damping, from about 0.10 second to 0.28 second, increasing  $K$  from 0.01 to 0.0427 radian per radian per second<sup>2</sup> increased slightly the damping of the lower-frequency mode of motion and, at the same time, rapidly decreased the damping of the higher-frequency mode of motion.

References 5 and 6 specify that for lateral oscillations with periods equal to or less than 2 seconds, the damping of the oscillation is considered satisfactory if the time to damp to half-amplitude is equal to or less than 1.50 seconds. By cross-plotting the results of figure 3 in the  $K\tau$ -plane (reference 4) the combinations of time lag and control gearings for this configuration, which satisfy this criterion, can be obtained. These results are presented in figure 4, which demonstrates that the criterion can be satisfied with a large range of gearing ratios in addition to the values of 0.0427 radian per radian per second<sup>2</sup> used in these calculations.

#### CONCLUDING REMARKS

A theoretical investigation, which was made to determine the effect on the lateral oscillations of an airplane of an automatic-control system sensitive to yawing accelerations, indicated that lateral oscillations, such as snaking or Dutch roll, can be satisfactorily damped through the use of this control. The main advantages of a control of this type are that the time lag in the automatic control is not a very critical factor and the control does not oppose the forces applied by the pilot in any steady maneuvers.

The presence of the automatic control introduces a higher-frequency mode of motion, which becomes unstable with increasing time lag, in addition to the existing lower-frequency Dutch roll mode of motion, which becomes more stable with increasing time lag. Increasing the ratio of the rudder deflection to the yawing acceleration from 0.01 to 0.0427 radian per radian per second<sup>2</sup> improved the damping of the lower-frequency mode slightly but, at the same time, reduced rapidly the damping of the higher-frequency mode.

The approximate lag-operator method of analysis was shown to be inadequate for treating time lag. The frequency-response method is the most suitable means of handling time lag at present. The results of these calculations show that an investigation of the possibilities of designing and using a control of this type to control such lateral oscillations as snaking or Dutch roll would be worthwhile.

Langley Aeronautical Laboratory  
National Advisory Committee for Aeronautics  
Langley Air Force Base, Va., October 4, 1949

## APPENDIX

## METHOD OF ANALYSIS

## Three Degrees of Freedom with Lag Operator

The equations of motion of an airplane which is free to move laterally, to roll, and to yaw, as given in reference 7, have been modified to account for a yawing-moment term produced by deflecting the rudder. These equations then become:

Lateral motion

$$m(VD\beta + VD\psi) = Y + mg \sin \phi$$

Rolling motion

$$I_X D^2 \phi - I_{XZ} D^2 \psi = L$$

Yawing motion

$$I_Z D^2 \psi - I_{XZ} D^2 \phi = N + N'$$

(1)

where the operator  $D$  indicates differentiation with respect to time and  $N'$  is equal to the yawing moment produced by deflecting the rudder. The effect on the rolling motion and on the lateral motion of deflecting the rudder is neglected.

In order to investigate the effect of lag in the automatic-control system, the time-lag operator  $e^{-\tau D}$  as given in references 1 and 2 could be used. This method of treating time lag implies that the amount of rudder deflection applied at a given instant is proportional to the yawing acceleration which existed at a fixed time previous to the given instant. Because use of the time-lag operator directly results in a transcendental equation, the expression  $1 - \tau D + \frac{\tau^2 D^2}{2}$ , which is equal to the first three terms of the power series that represents  $e^{-\tau D}$ , was used to represent time lag in these calculations.

The expression for an automatic control sensitive to yawing accelerations with time lag is

$$\delta_r = K D^2 \psi e^{-\tau D} \approx K D^2 \psi \left( 1 - \tau D + \frac{\tau^2 D^2}{2} \right) \quad (2)$$

where  $K$  equals the gearing ratio between the rudder deflection occurring at some time  $t$  and the yawing acceleration which occurred at the time  $t - \tau$ .

When the equations of motion (1) are expressed in terms of coefficients and equation (2) and the following substitutions are made:

$$\begin{aligned} \mu &= \frac{m}{\rho S b} & \gamma &= 0 \\ C_L &= \frac{W}{qS} & I'_X &= \frac{I_X}{qSb} \\ C_{Y_p} &= 0 & I'_Z &= \frac{I_Z}{qSb} \\ C_{Y_r} &= 0 & I'_{XZ} &= \frac{I_{XZ}}{qSb} \end{aligned}$$

and if small motions of the airplane are assumed, the equations become:

$$\left. \begin{aligned} \left( \frac{2\mu b_D}{V} - C_{Y_\beta} \right) \beta + \left( \frac{2\mu b_D}{V} \right) \psi - C_L \phi &= 0 \\ - C_{L_\beta} \beta - \left( I'_{XZ} D^2 + DC_{L_r} \frac{b}{2V} \right) \psi + \left( I'_X D^2 - DC_{L_p} \frac{b}{2V} \right) \phi &= 0 \\ - C_{n_\beta} \beta + \left[ I'_Z D^2 - DC_{n_r} \frac{b}{2V} - KD^2 \left( 1 - \tau D + \frac{\tau^2 D^2}{2} \right) C_{n_{\delta_r}} \right] \psi & \\ - \left( I'_{XZ} D^2 + DC_{n_p} \frac{b}{2V} \right) \phi &= 0 \end{aligned} \right\} (3)$$

The determinant of the coefficients of the three variables is then set equal to zero and expanded. This equation yields the characteristic equation of the system in the following form:

$$aD^6 + bD^5 + cD^4 + dD^3 + eD^2 + fD + g = 0 \quad (4)$$

where

$$a = I'_X \frac{2\mu b}{V} \frac{K\tau^2}{2} C_{n\delta_r}$$

$$b = -\frac{K\tau^2}{2} \frac{\mu b^2}{V^2} C_{l_p} C_{n\delta_r} - \frac{K\tau^2}{2} I'_X C_{Y_\beta} C_{n\delta_r} - I'_X \frac{2\mu b}{V} K\tau C_{n\delta_r}$$

$$c = \frac{K\tau^2}{2} \frac{b}{2V} C_{l_p} C_{Y_\beta} C_{n\delta_r} + K\tau \frac{\mu b^2}{V^2} C_{l_p} C_{n\delta_r} + K\tau I'_X C_{Y_\beta} C_{n\delta_r} -$$

$$I'_Z I'_X \frac{2\mu b}{V} + K I'_X \frac{2\mu b}{V} C_{n\delta_r} + \frac{2\mu b}{V} I'^2_{XZ}$$

$$d = -I'^2_{XZ} C_{Y_\beta} + \frac{\mu b^2}{V^2} I'^2_{XZ} C_{l_r} - \frac{K\tau^2}{2} C_{l_p} C_{l_\beta} C_{n\delta_r} -$$

$$K\tau \frac{b}{2V} C_{l_p} C_{Y_\beta} C_{n\delta_r} + I'_Z \frac{\mu b^2}{V^2} C_{l_p} + I'_Z I'_X C_{Y_\beta} -$$

$$K \frac{\mu b^2}{V^2} C_{l_p} C_{n\delta_r} - K I'_X C_{Y_\beta} C_{n\delta_r} + \frac{\mu b^2}{V^2} I'_X C_{n_r} + \frac{\mu b^2}{V^2} I'^2_{XZ} C_{n_p}$$

$$e = -\frac{b}{2V} I'^2_{XZ} C_{Y_\beta} C_{l_r} - \frac{2\mu b}{V} I'_X C_{n_\beta} + K\tau C_{l_p} C_{l_\beta} C_{n\delta_r} -$$

$$I'_Z \frac{b}{2V} C_{l_p} C_{Y_\beta} - I'^2_{XZ} \frac{b}{2V} C_{Y_\beta} C_{n_p} + \frac{\mu b^2}{V^2} \frac{b}{2V} C_{l_r} C_{n_p} +$$

$$K \frac{b}{2V} C_{l_p} C_{Y_\beta} C_{n\delta_r} - \frac{b}{2V} \frac{\mu b^2}{V^2} C_{n_r} C_{l_p} - \frac{b}{2V} I'_X C_{n_r} C_{Y_\beta} - \frac{2\mu b}{V} I'^2_{XZ} C_{l_\beta}$$

$$f = -\left(\frac{b}{2V}\right)^2 C_{Y\beta} C_{l_r} C_{n_p} + \frac{\mu b^2}{V^2} C_{n\beta} C_{l_p} + I'_{XZ} C_L C_{l_\beta} - K C_L C_{l_\beta} C_{n\delta_r} +$$

$$I'_{XZ} C_L C_{n\beta} + \left(\frac{b}{2V}\right)^2 C_{n_r} C_{l_p} C_{Y\beta} - \frac{\mu b^2}{V^2} C_{l_\beta} C_{n_p}$$

$$g = -\frac{b}{2V} C_L C_{l_\beta} C_{n_r} + \frac{b}{2V} C_L C_{n\beta} C_{l_r}$$

The results of the calculations are presented as the time to damp to half-amplitude and the period of the oscillation. The values of the real and imaginary parts of the complex stability roots of the characteristic stability equation are related to the damping and the period of the lateral oscillation by the following equations:

$$\left. \begin{aligned} T_{1/2} &= \frac{-0.693}{\text{Real part}} \\ P &= \frac{6.28}{\text{Imaginary part}} \end{aligned} \right\} \quad (5)$$

Without the automatic control, the characteristic stability equation of this three-degree-of-freedom case is a fourth-degree equation. Equation (4) indicates the presence of two additional roots. These additional roots result from the use of the three-term-series expansion of the quantity  $e^{-\tau D}$  to represent the time lag in the system. Calculations were made to show that the additional pair of roots does not satisfy the stability equations when the exact operator  $e^{-\tau D}$  is used to represent the time lag.

If the exact time-lag operator, that is,  $e^{-\tau D}$  is used to represent the time lag of the automatic-control system, the characteristic lateral-stability equation then becomes:

$$aD^4 + bD^3 + cD^2 + dD + e = 0 \quad (6)$$

where

$$a = \frac{2\mu b}{V} I'_{XZ}{}^2 - \frac{2\mu b}{V} I'_{XZ} I'_{XZ} + \frac{2\mu b}{V} K e^{-\tau D} C_{n\delta_r} I'_{XZ}$$

$$\begin{aligned}
b &= \frac{\mu b^2}{v^2} C_{l_r} I'_{XZ} + \frac{\mu b^2}{v^2} C_{n_p} I'_{XZ} - C_{Y_\beta} I'_{XZ}^2 + \frac{\mu b^2}{v^2} C_{n_r} I'_{X} - \\
&\quad C_{Y_\beta} C_{n_{\delta_r}} K e^{-\tau D} I'_{X} + \frac{\mu b^2}{v^2} C_{l_p} I'_{Z} - \frac{\mu b^2}{v^2} C_{l_p} C_{n_{\delta_r}} K e^{-\tau D} + C_{Y_\beta} I'_{X} I'_{Z} \\
c &= \frac{\mu b^2}{v^2} \frac{b}{2V} C_{l_r} C_{n_p} - \frac{b}{2V} C_{Y_\beta} C_{l_r} I'_{XZ} - \frac{b}{2V} C_{Y_\beta} C_{n_p} I'_{XZ} - \frac{2\mu b}{V} C_{n_\beta} I'_{X} - \\
&\quad \frac{b}{2V} C_{n_r} C_{Y_\beta} I'_{X} - \frac{2\mu b}{V} C_{l_\beta} I'_{XZ} - \frac{b}{2V} C_{l_p} C_{Y_\beta} I'_{Z} - \frac{b}{2V} \frac{\mu b^2}{v^2} C_{n_r} C_{l_p} + \\
&\quad \frac{b}{2V} C_{l_p} C_{Y_\beta} C_{n_{\delta_r}} K e^{-\tau D} \\
d &= \frac{\mu b^2}{v^2} C_{n_\beta} C_{l_p} - \left(\frac{b}{2V}\right)^2 C_{Y_\beta} C_{l_r} C_{n_p} + C_L C_{l_\beta} I'_{Z} - C_L C_{l_\beta} C_{n_{\delta_r}} K e^{-\tau D} + \\
&\quad C_L C_{n_\beta} I'_{XZ} - \frac{\mu b^2}{v^2} C_{l_\beta} C_{n_p} + \left(\frac{b}{2V}\right)^2 C_{n_r} C_{l_p} C_{Y_\beta} \\
e &= \frac{b}{2V} C_L C_{n_\beta} C_{l_r} - \frac{b}{2V} C_L C_{l_\beta} C_{n_r}
\end{aligned}$$

The left-hand side of equation (6) was evaluated for the case of a small time lag for which the series approximation still gives good results. The stability roots  $D$ , obtained from the expression  $1 - \tau D + \frac{\tau^2 D^2}{2}$  for  $\tau = 0.10$  second and  $K = 0.0427$  radian per radian per second<sup>2</sup>, were substituted in the equation for  $e^{-\tau D}$ . If the root is correct, the left-hand side of the equation should

equal zero. The results of this substitution are tabulated in the following table:

D	$aD^4 + bD^3 + cD^2 + dD + e$
-0.0115	$(0.02937 \times 10^{-6}) + (10.7847 \times 10^{-6}) + (0.003478) + (-1.09676) + (1.092) = -0.001288$
-3.9085	$(467.483) + (-497.425) + (404.901) + (-376.092) + (1.092) = -0.0410$
$-0.428 \pm 3.7571$	$(328.109 \pm 100.8911) + (75.823 \mp 375.3241) + (-366.715 \mp 82.1871) + (38.211 \pm 358.3851) + (1.092) = 0.092 \pm 1.7651$
$10.478 \pm 19.7851$	$(-134856.9 \mp 191285.41) + (47079.1 \pm 4258.11) + (-7207.1 \pm 10735.21) + (989.2 \pm 1840.31) + (1.092) = -188157.8 \mp 174451.81$

The values obtained for the two negative real roots and for the conjugate roots with negative real parts approximately satisfy the equation for  $e^{-\tau D}$ , but the conjugate roots with positive real parts do not; therefore these roots are extraneous.

#### Single Degree of Freedom with Lag Operator

The equation of motion of an airplane and automatic control free only to yaw may be written as

$$I' Z D^2 \psi - C_{n_r} \frac{b}{2V} D \psi + C_{n_\beta} \psi = K D^2 \psi \left( 1 - \tau D + \frac{\tau^2 D^2}{2} \right) C_{n_{\delta_r}} \quad (7)$$

Numerical values for the known quantities, taken from table I, are substituted to simplify the expression. These substitutions yield

$$D^2 \left[ 0.01024 + 0.163K \left( 1 - \tau D + \frac{\tau^2 D^2}{2} \right) \right] + 0.00704D + 0.250 = 0 \quad (8)$$



which, for fixed values of  $K$  and  $\tau$ , can be written as

$$aD^4 + bD^3 + cD^2 + dD + e = 0 \quad (9)$$

The values of the real and imaginary parts of the complex stability roots of equation (9) are related to the damping and period by equations (5).

#### Frequency-Response Calculations

The frequency-response characteristics of an airplane free only to yaw may be calculated by an extension of the methods of reference 3. The equation of motion of the single-degree-of-freedom system (equation (7), without automatic control) may be simplified by the following substitutions:

$$-\frac{C_{nr} \frac{b}{2Y}}{I'Z} = 2\zeta\omega_n$$

$$\frac{C_{n\beta}}{I'Z} = \omega_n^2$$

$$\frac{C_{n\delta_r}}{I'Z} = C$$

where

- $\zeta$            damping ratio
- $\omega_n$         natural circular frequency
- $C$            constant

Equation (7) may then be written in the form

$$\frac{\delta_r}{D^2\psi} = \frac{D^2 + 2\zeta\omega_n D + \omega_n^2}{CD^2} \quad (10)$$

or as

$$\frac{\delta_r}{D^2\psi} = f(D)$$

Reference 3 presents a method for determining the amplitude ratio and phase angle at which a hunting oscillation exists. The critical hunting condition exists when the real part of the complex stability root is equal to zero. The phase and amplitude of the rudder deflection  $\delta_r$  required to maintain a sinusoidal motion of the airplane is obtained by substituting  $i\omega$  for  $D$  in the expression for  $\delta_r/D^2\psi$ .

In order to investigate the phase and amplitude ratio required to maintain an oscillation of increasing or decreasing amplitude by the frequency-response analysis, the complex frequency  $a + i\omega$  can be substituted for the operator  $D$ . (See reference 8.) This substitution allows the determination of the values of  $K$  and  $\tau$  necessary so that the damping represented by the quantity  $a$  exists. If this substitution is made, equation (10) becomes

$$\frac{\delta_r}{D^2\psi} = f\left(\frac{a}{\omega_n} + \frac{\omega}{\omega_n}i\right) = A + Bi$$

and may be written in vector form as

$$\frac{\delta_r}{D^2\psi} = A + Bi = R \angle \theta$$

where

$$R = \sqrt{A^2 + B^2}$$

and

$$\theta = \tan^{-1} \frac{B}{A}$$

The terms  $R$  and  $\theta$  are then plotted as a function of  $\omega/\omega_n$  for different values of  $a$ . The results of these calculations, which are presented in figure 5, show the phase and amplitude of the ratio of rudder deflection to yawing acceleration required to maintain oscillations with different amounts of positive and negative damping.

In order to investigate the phase and amplitude of the ratio of rudder deflection to yawing acceleration supplied by the automatic control in an oscillation with different amounts of time lag and damping, the operator  $D = a + i\omega$  is substituted into the automatic-control equation (equation (2)). Therefore,

$$\frac{\delta_r}{D^2\psi} = Ke^{-\tau a}(\cos \tau\omega - i \sin \tau\omega)$$

The magnitude of the automatic-control amplitude ratio

$$\left| \frac{\delta_r}{D^2\psi} \right| = Ke^{-\tau a}$$

is now a function of time lag  $\tau$  and of the damping term  $a$ .

In order to determine the damping curves presented in figure 2, the airplane frequency-response curves and the automatic-control frequency response were combined by use of the relationship shown in reference 3 that, if at some value of frequency  $\omega'$ , the condition exists that

$$\left| \frac{\delta_r}{D^2\psi} \right|_{\text{Airplane}} = \left| \frac{\delta_r}{D^2\psi} \right|_{\text{Automatic control}}$$

then the airplane will oscillate at that frequency provided that the phase angles of the airplane and automatic control are equal.

These conditions can be satisfied as follows. For a given value of  $K$  and a constant value of  $a$ , values are assigned to  $\tau$  and the

absolute values of  $\left| \frac{\delta_r}{D^2\psi} \right|_{\text{Automatic control}}$  are computed. A plot

of  $\left| \frac{\delta_r}{D^2\psi} \right|_{\text{Automatic control}}$  against time lag is then made. For the

same constant value of  $a$ , the values of airplane amplitude ratio and

phase angles at different frequencies are noted. The phase angles are changed to a dimensional time lag by means of the following relationship:

$$\tau = \frac{\theta}{360} \frac{2\pi}{\omega' \omega_n} = \frac{\theta}{360} \frac{2\pi}{\omega}$$

The phase angle requirement is therefore satisfied and a plot

of  $\left| \frac{\delta_r}{D^2\psi} \right|_{\text{Airplane}}$  against  $\tau$  is made. At values of  $\tau$  for

which  $\left| \frac{\delta_r}{D^2\psi} \right|_{\text{Airplane}} = \left| \frac{\delta_r}{D^2\psi} \right|_{\text{Automatic control}}$  the critical conditions

are satisfied and one point has been determined on the damping curve. This entire process may be repeated for each value of  $a$ , corresponding to a value of  $T_{1/2}$ , to obtain a plot such as figure 2.

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TABLE I  
 STABILITY DERIVATIVES AND MASS CHARACTERISTICS  
 USED IN CALCULATIONS

W, lb . . . . .	8410
S, sq ft . . . . .	130
b, ft . . . . .	28
$I_x$ , slug-ft <sup>2</sup> . . . . .	1981
$I_z$ , slug-ft <sup>2</sup> . . . . .	10519
$I_{xz}$ , slug-ft <sup>2</sup> . . . . .	298
V, ft/sec . . . . .	797
$\rho$ , slug/cu ft . . . . .	0.000889
$\mu$ . . . . .	80.7
$C_L$ . . . . .	0.229
$C_{L_p}$ , per radian . . . . .	-0.400
$C_{L_r}$ , per radian . . . . .	0.0800
$C_{L_\beta}$ , per radian . . . . .	-0.126
$C_{n_p}$ , per radian . . . . .	-0.0155
$C_{n_r}$ , per radian . . . . .	-0.400
$C_{n_\beta}$ , per radian . . . . .	0.250
$C_{Y_p}$ , per radian . . . . .	0
$C_{Y_r}$ , per radian . . . . .	0
$C_{Y_\beta}$ , per radian . . . . .	-1.00
$C_{n_{\delta_r}}$ , per radian . . . . .	-0.163
$\eta$ , deg . . . . .	-2.0
$\gamma$ , deg . . . . .	0



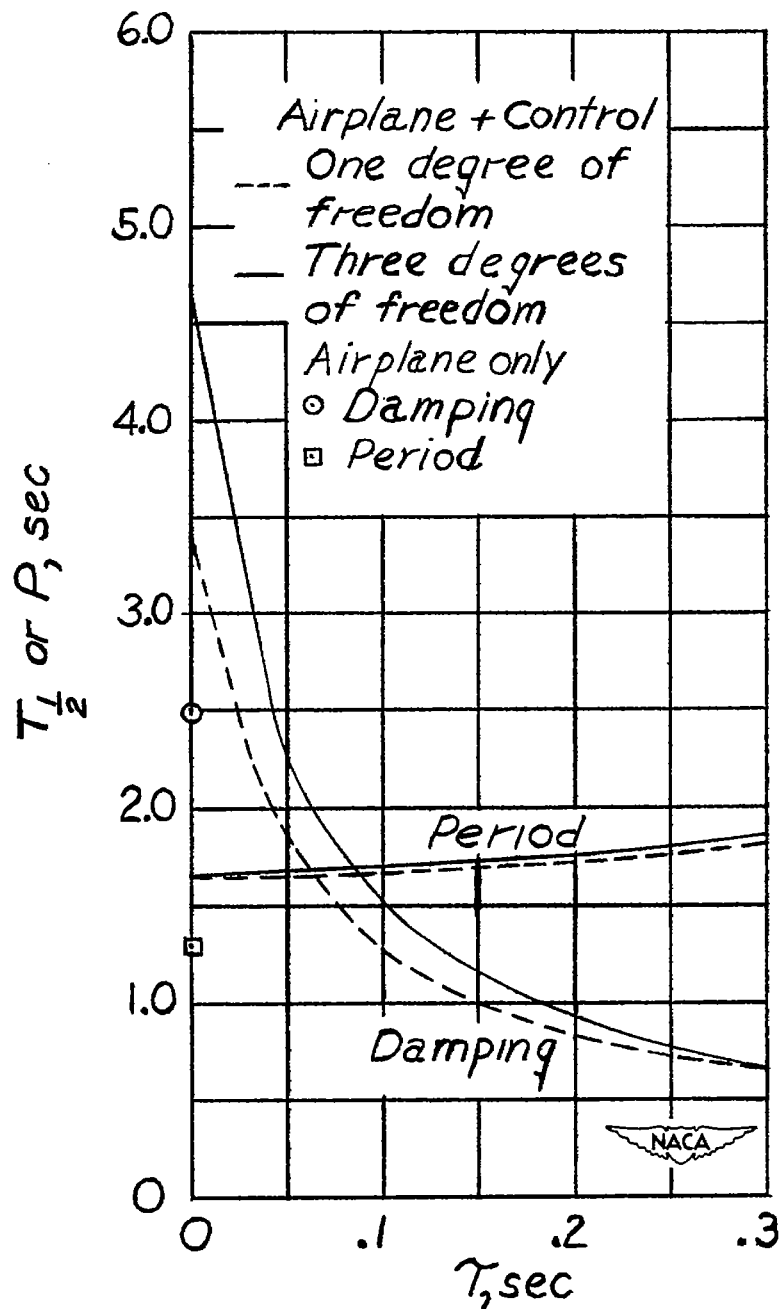


Figure 1.- The variation of the time to damp to half-amplitude and the period of the lateral oscillation with time lag for an automatic control sensitive to yawing accelerations

as calculated by use of the lag operator  $\left(1 - \tau D + \frac{\tau^2 D^2}{2}\right)$ .  
 $K = 0.0427$  radian per radian per second<sup>2</sup>.

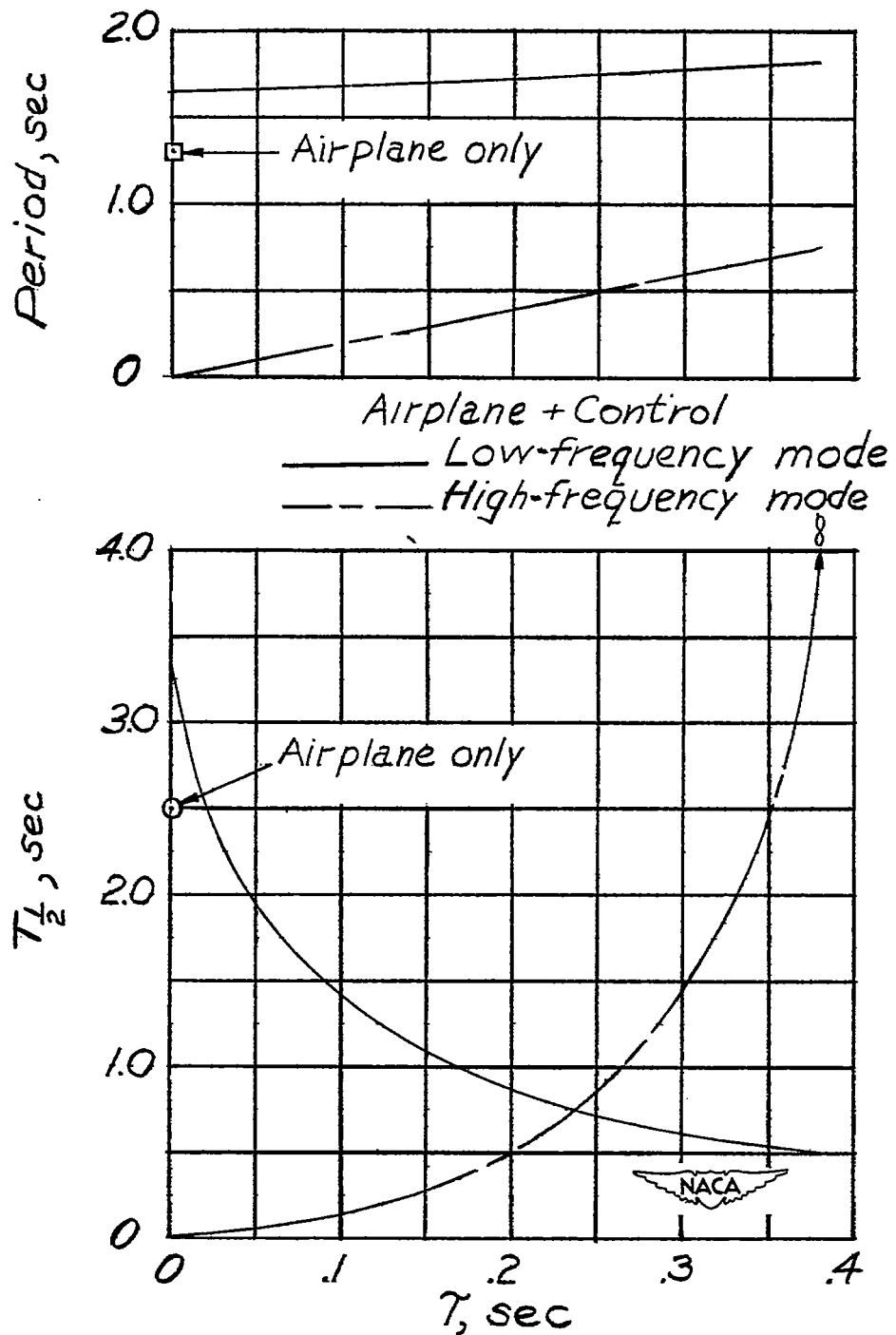


Figure 2.- The variation of the time to damp to half-amplitude and the period of the lateral oscillation with time lag for an automatic control sensitive to yawing accelerations as calculated by the frequency-response analysis.  
 $K = 0.0427$  radian per radian per second<sup>2</sup>.



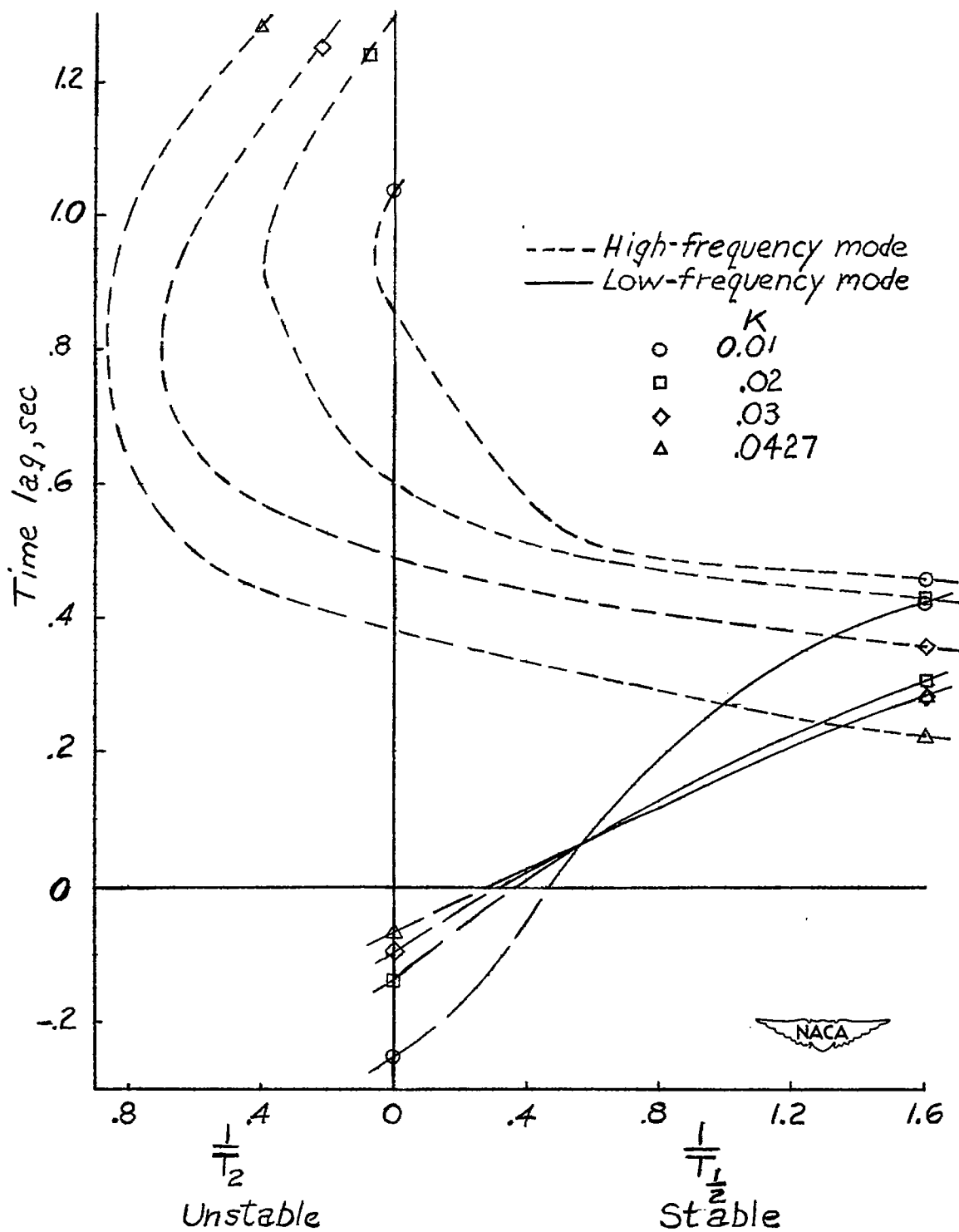


Figure 3.- The variation of the damping of the lateral oscillations with time lag for different values of K for an automatic control sensitive to yawing accelerations.

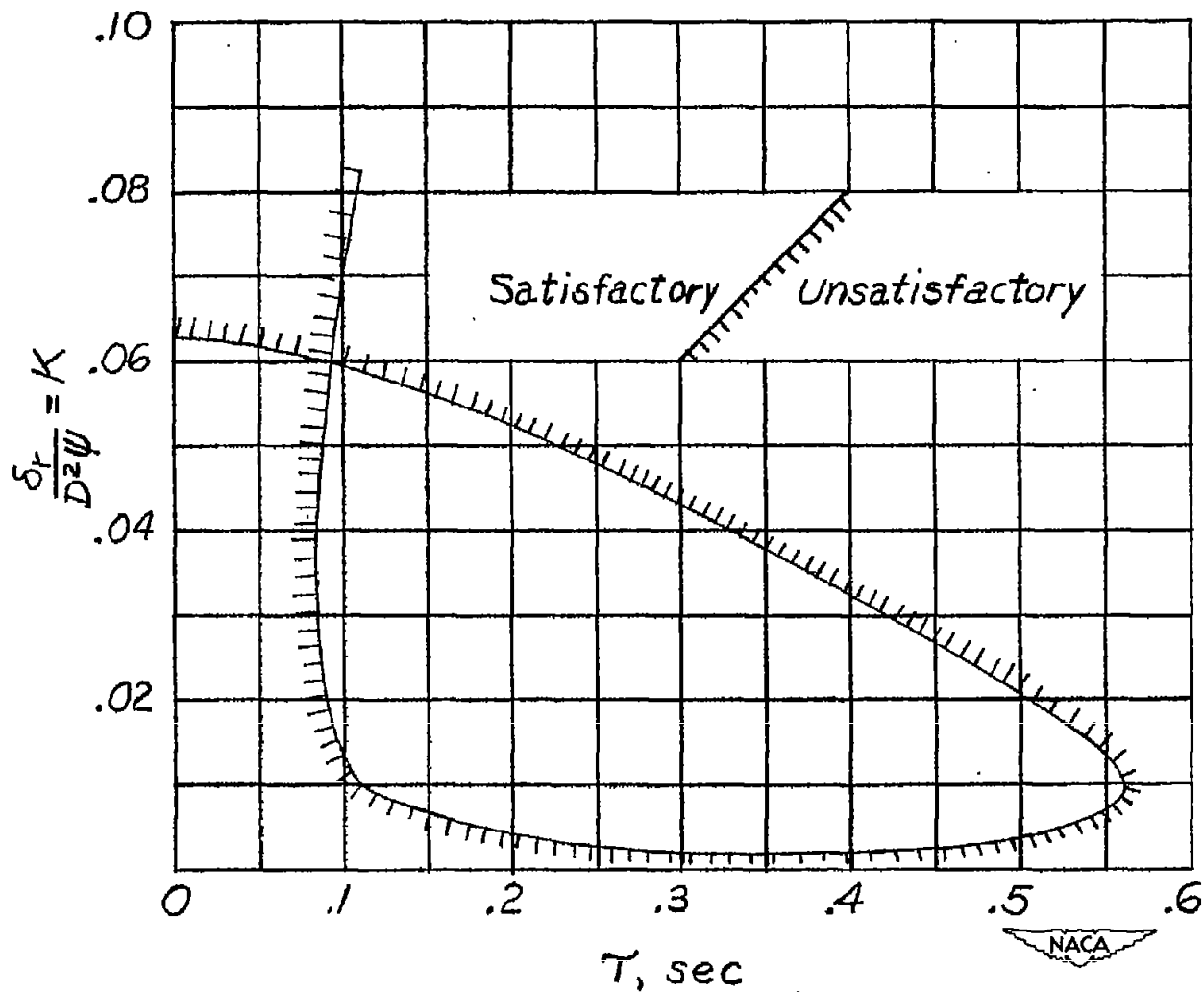
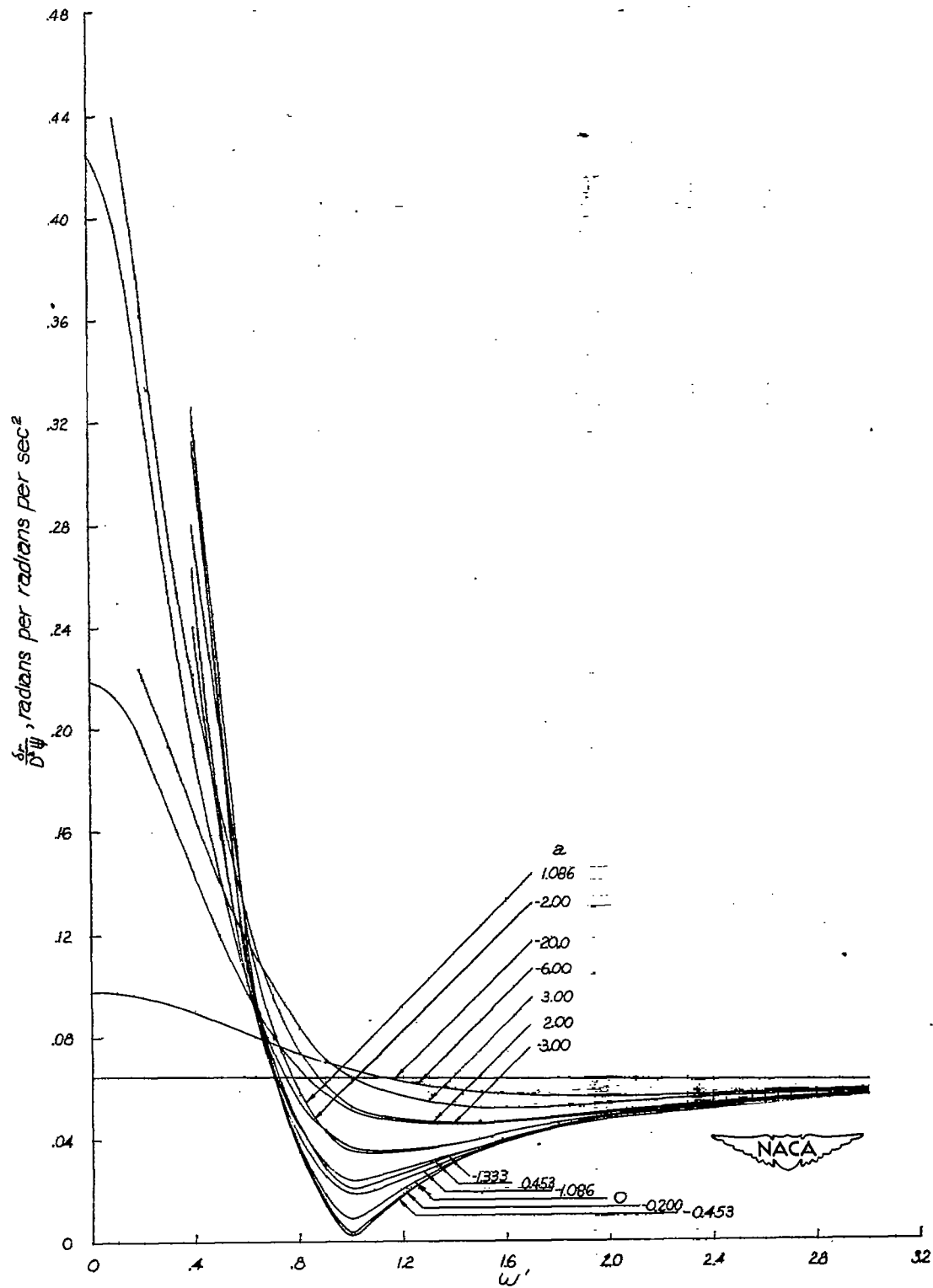
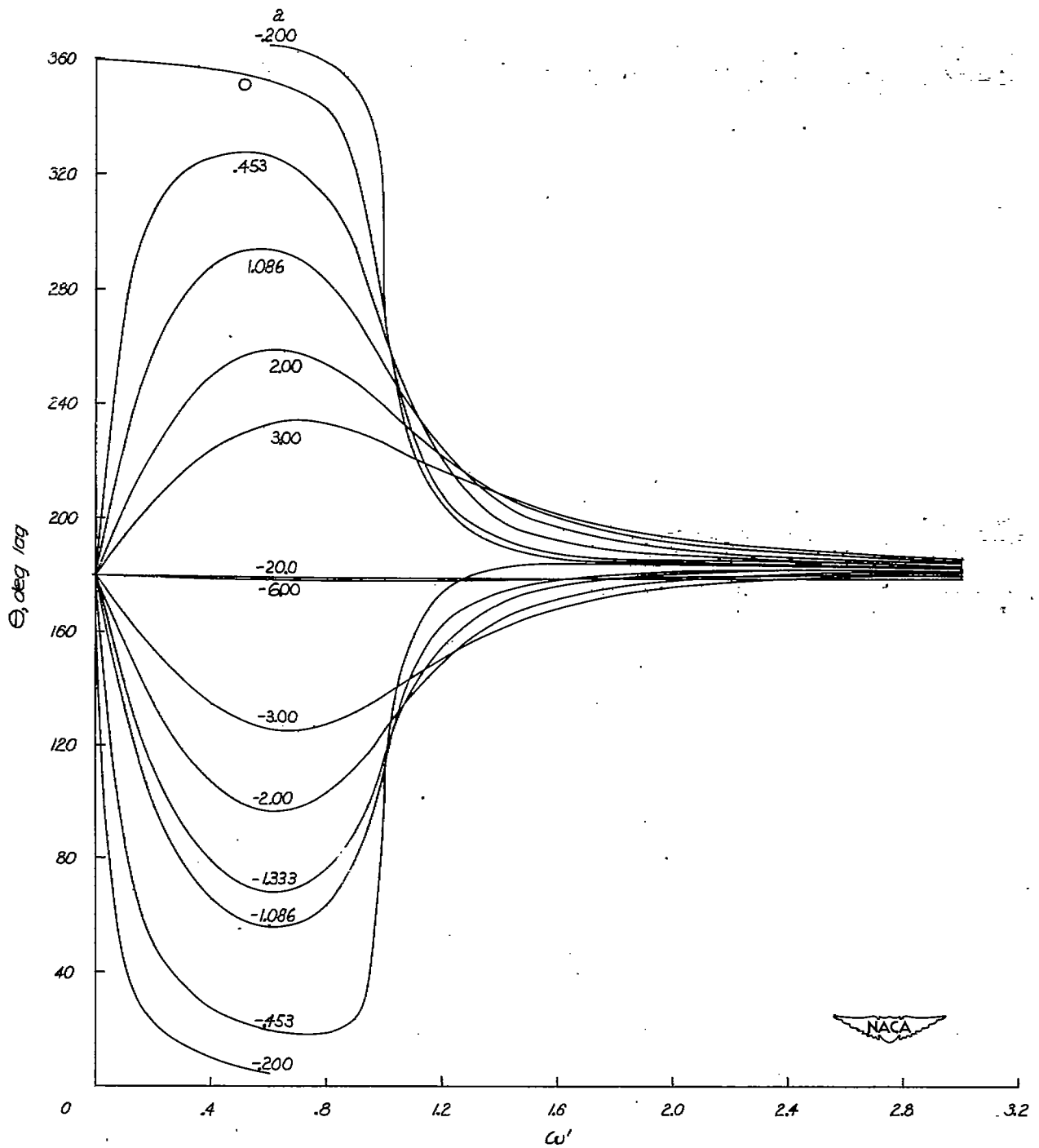


Figure 4.- Combinations of time lag and gearing ratios which satisfy the Army-Navy criterion for satisfactory damping of short-period lateral oscillations.



(a) Amplitude ratio.

Figure 5.- The ratio of rudder deflection to yawing acceleration as a function of frequency required to maintain an oscillation with different amounts of damping.



(b) Phase angle.

Figure 5.- Concluded.