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PRESSURE DISTRIBUTION AND DAMPING IN STEADY ROLL AT SUPERSONIC MACH NUMBERS OF FLAT SWEPT-BACK WINGS WITH SUBSONIC EDGES

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SUMMARY

A method, which is based upon the concept of the superposition of conical and quasi-conical flows, is presented for calculating the pressure distribution and damping in steady roll at supersonic Mach numbers of thin, flat, swept-back wings having all edges straight and subsonic. Although it can be adapted to wings having negative rake at the tips, the method is developed only for wings with streamwise tips. As outlined, the analysis is rigorous and somewhat complicated; however, several possible simplifications are suggested which considerably reduce the amount of computation without introducing significant error. The method then closely parallels a previously published method for calculating the pressure distribution corresponding to a condition of steady lift.

To illustrate the application of the method, calculations of the pressure distribution and the damping derivative of an untapered swept-back wing and of the damping derivative of a tapered swept-back wing are included. From a comparison of the results obtained by both the rigorous and the shortened methods for the untapered wing, it is concluded that for most practical purposes the shortened method will yield sufficiently accurate results. A comparison of the values for the untapered wing with those obtained by means of a method based upon strip theory is also included; and, although it is shown to be considerably in error in predicting the pressure distribution in the regions adjacent to a subsonic trailing edge or tip, the strip method gives a good approximation of the damping derivative.

In general, the method is formulated in accordance with the usual assumptions and limitations of the linearized potential theory.

INTRODUCTION

The pressure distribution and damping in steady roll at supersonic Mach numbers have been calculated for many of the commonly used wing configurations. These include triangular wings (references 1 and 2), rectangular and trapezoidal wings (references 3 and 4), various forms of swept-back wings (references 3, 5, and 6), and wings of somewhat arbitrary shape but having supersonic leading edges (reference 7). The particular class of swept-back wings in which all the edges are subsonic is the principal subject of this report. This class has not been fully treated heretofore, although an approximate method of analysis for wings with streamwise (subsonic) tips is given in reference 6.

Where subsonic trailing edges (or tips) exist, the effects of changes in flow occurring in the vicinity of those edges are propagated over the surface of the wing and therefore influence the pressure at points within the region defined by the wing boundaries and the Mach lines from the apex of the trailing edge (or from the tips of the leading edge). Hence, whereas the pressure distribution over the wing is in general governed principally by the sweep of the leading edge, the variation of pressure in the vicinity of a subsonic trailing edge (or tip) is additionally influenced by the flow around the trailing edge (or tip). In the following report, a procedure for calculating the pressure distribution near a subsonic edge is presented which consists first, in the evaluation of a basic distribution associated with an infinite triangular wing having leading edges which coincide with those of the swept-back wing, and second, in the correction of these basic values for the effect of having introduced excess pressures in the wake and outboard of the tips of the swept-back wing. The corrections to the basic pressure in the second step are calculated by superposing sectors of conical and quasi-conical pressure along the wing boundaries in order to cancel the excess pressure in the wake and outboard of the tips. The derivative for the damping in steady roll is calculated in an analogous manner. In general, the method follows closely that developed in reference 8 for calculating the lift and pitching moment of a swept-back wing, but with the difference that, where the basic pressures corresponding to lift are conically distributed, those corresponding to roll are quasi-conically distributed (i.e., vary linearly along rays passing through the apex of the leading edge).

The analysis is otherwise confined to the usual assumptions and limitations of the linearized potential theory for supersonic flow.

NOTATION

The quantities listed in the following are assumed to have consistent units and all angles are given in radians:

a slope of any ray through origin divided by slope of Mach lines $\left(\beta \frac{y}{x}\right)$

a_l value of a corresponding to $m_a = 1$

$$\left[a_l = \frac{1-(\beta s/c_o m)(1-m)}{1-(\beta s/m_t c_o m)(1-m)} \right]$$

a_o upper limit of a corresponding to a specified point x,y

$$\left[\text{on tip,} \quad a_o = \frac{\beta s}{x+\beta(y-s)} \text{; on trailing edge,} \right.$$

$$\left. a_o = m_t \frac{\beta y+c_o-x}{\beta y+c_o m_t-x} \right]$$

a_t slope of ray through the tip of the trailing edge divided by slope of Mach lines $\left(\frac{\beta s}{c_o + \frac{\beta s}{m_t}} \right)$

b wing span

c_o root chord of wing

c_t tip chord of wing

C constant factor used in derivation of pressure canceling function

C_l rolling-moment coefficient $\left(\frac{L}{qSb} \right)$

C_{l_p} damping-in-roll derivative $\left[\frac{\partial C_l}{\partial(pb/2V)} \right]$

ΔC_{l_p} corrective term for basic damping derivative

F pressure canceling function (F_1+F_2)

F_1,F_2 auxiliary functions corresponding to the coordinates ξ_1,η_1 and ξ_2,η_2, respectively

L basic rolling moment

m slope of leading edge divided by slope of Mach lines $(\beta \cot \Lambda)$

m_a	slope of ray through point $A(x_A,y_A)$ and tip of leading edge divided by slope of Mach lines $\left(\beta \frac{s-y_A}{(\beta s/m)-x_A}\right)$
m_s	slope of tip divided by slope of Mach lines
m_t	slope of trailing edge divided by slope of Mach lines
M	free-stream Mach number
p	steady rate of roll, radians per unit of time
Δp	pressure difference between upper and lower surfaces of wing
P	pressure coefficient $\left(\frac{\Delta p}{q}\right)$
P_L	pressure coefficient corresponding to steady lift
P_{L_A}	pressure coefficient for steady lift at point $A(x_A,y_A)$
ΔP_L	correction term for basic pressure coefficient for steady lift
P_R	pressure coefficient corresponding to steady roll
P_{R_A}	pressure coefficient for steady roll at point $A(x_A,y_A)$
ΔP_R	correction term for basic pressure coefficient for steady roll
q	dynamic pressure $\left(\frac{1}{2}\rho V^2\right)$
Q	$\frac{2-m^2}{1-m^2} E\left(\sqrt{1-m^2}\right) - \frac{m^2}{1-m^2} K\left(\sqrt{1-m^2}\right)$
s	semispan of wing
S	wing area
t	slope of ray through points x,y and x_A,y_A divided by slope of Mach lines $\left(\beta \frac{y-y_A}{x-x_A}\right)$
t_1,t_2	slope of ray through points x',y' and ξ_1,η_1 and points $x' y'$ and ξ_2,η_2, respectively, divided by slope of Mach lines $\left(\beta \frac{y'-\eta_1}{x'-\xi_1}, \beta \frac{y'-\eta_2}{x'-\xi_2}\right)$

V	free-stream velocity
x,y,z	rectangular coordinates (fig. 1)
x',y'	coordinates defined by $x-x_A$ and $y-y_A$, respectively
x_A,y_A	coordinates of point A on trailing edge or tip $\left[\text{on trailing edge, } x_A = \frac{m_t c_o}{m_t - a},\ y_A = \frac{m_t c_o a}{\beta(m_t - a)};\ \text{on tip, } x_A = \frac{\beta s}{a},\ y_A = s\right]$
$\overline{y_1},\overline{y_2}$	lever arms of increments of force due to conical and quasi-conical pressures, respectively, exerted on an element of wing area
y_m	y coordinate of intersection of trailing edge and center line of an element of area at tip $\left\{y_m = s\left[1 - \frac{m_t t}{m_t - t}\left(\frac{1}{a} - \frac{1}{a_t}\right)\right]\right\}$
Z	force due to basic pressures in z direction
ΔZ	correction term for Z
$\frac{pb}{2V}$	helix angle of wing tip in roll
$\frac{b^2}{S}$	aspect ratio
$H(a),H'(a)$; $R(a),R'(a)$; $g(a),g'(a)$; $h(a),h'(a)$; $r(a),r'(a)$; $s(a),s'(a)$	functions defined in text
α	angle of attack, radians
β	$\sqrt{M^2-1}$
δ	factor used in strip-theory calculations
ξ_1,η_1 and ξ_2,η_2	coordinates in x and y directions, respectively, defining the origins of a series of superposed canceling sectors

η_{1_o},η_{2_o}	upper limits of η_1 and η_2, respectively, corresponding to a specific point x,y $\left[\eta_{1_o} = \frac{a(x'-\beta y')}{\beta(1-a)}, \eta_{2_o} = \frac{m_t(x'-\beta y')}{\beta(1-m_t)}\right]$
Λ	angle of sweep of leading edge
ρ	mass density of air
σ	$\frac{d}{d\eta}\left(\frac{dP_{R_A}}{da}\,da\right)$
χ	argument of inverse-cosine terms in canceling functions
χ_1,χ_2	values of χ corresponding to t_1 and t_2, respectively

Superscripts

$'$	trailing-edge functions (except as noted in text)
$''$	tip functions (except as noted in text)

Subscripts

1	conical pressure canceling sectors
2	quasi-conical pressure canceling sectors

Elliptic Functions

$K(\sqrt{1-m^2})$	complete elliptic integral of first kind, modulus $\sqrt{1-m^2}$
$E(\sqrt{1-m^2})$	complete elliptic integral of second kind, modulus $\sqrt{1-m^2}$
$K(k)$	complete elliptic integral of first kind, modulus k
$E(k)$	complete elliptic integral of second kind, modulus k
$K(k')$	complete elliptic integral of first kind, modulus k'
$E(k')$	complete elliptic integral of second kind, modulus k'
$F(\psi,k')$	incomplete elliptic integral of first kind, modulus $k', am^{-1}\psi$

$E(\psi,k')$ incomplete elliptic integral of second kind, modulus k', $am^{-1}\psi$

k $\sqrt{\dfrac{(m-a_o)(1-m)}{2m(1+a_o)}}$

k' $\sqrt{1-k^2}$

ψ $\sin^{-1}\sqrt{\dfrac{a_o(mx+\beta y)}{\beta s(a_o+m)}}$

METHOD OF ANALYSIS

In reference 1, it is shown that the pressure distribution[1] in steady roll of flat triangular wings is quasi-conical (i.e., varies linearly along rays passing through the apex of the leading edge). Although confined to triangular plan forms, the results of the analysis can be applied readily to swept-back wings which have supersonic trailing edges and tips. Where the wing edges are all subsonic, as shown in figure 1, the method of reference 1 cannot be directly applied since within regions I, II, and III, the pressure distributions are no longer quasi-conical. The configuration in figure 1 has been treated previously for the conditions of steady lift in reference 8. Since the cases of steady lift and steady roll differ only in boundary conditions, the present method of analysis is fundamentally the same as that in reference 8.

The pressure distribution of the swept-back wing illustrated in figure 1 may be derived from that of an infinite triangular wing having coincident leading edges. A basic pressure distribution is first calculated, using the simple expression for the triangular wing; and then several terms, representing primary and secondary corrections of the basic pressure on the wing resulting from the cancellation of the excess pressure not contained within the boundaries of the swept-back wing (shaded areas), are added to the pressure in regions I, II, and III. The cancellation of the excess pressure is accomplished by superposing various sectors of conical and quasi-conical pressure along the trailing edge and tips.

The orthogonal coordinate system shown in figure 2 is chosen. The plane of the wing lies in the x,y plane and is therefore fixed at zero angle of attack[2]. The x axis, which is the axis of roll, is then a principal body axis with an origin at the leading-edge apex. For this

[1]Throughout the text the terms "pressure" and "pressure coefficient" (i.e., $\Delta p/q$) are used interchangeably. All quantities in the analysis are therefore dimensionless.

[2]Wings at other than zero angle of attack may be treated by superposing the individual results for the cases of steady lift and steady roll.

system, the pressure at a point x,y on the surface of the triangular wing in steady roll and having subsonic leading edges is given in reference 1 as

$$P_R = \left(\frac{\Delta p}{q}\right)_{roll} = \left(\frac{pb}{2V}\right) \frac{4m^2}{\beta Q} \frac{y}{s} \frac{1}{\sqrt{m^2-a^2}} \tag{1}$$

where

$$Q = \frac{2-m^2}{1-m^2} E(\sqrt{1-m^2}) - \frac{m^2}{1-m^2} K(\sqrt{1-m^2})$$

$$a = \beta \frac{y}{x}$$

The variation of pressure along any ray a is seen to be directly proportional to the spanwise location y/s, and, as such, conforms with the definition of the term quasi-conical. For plan forms having supersonic trailing edges and tips (except tips with positive rake), the damping moment corresponding to the pressure distribution given by equation (1) can be readily calculated since the flow over the wing is completely independent of that in the wake and outboard of the tips. On the other hand, where these edges are subsonic such that the various regions of flow are interdependent, the effect on the damping of canceling the pressure in the wake and outboard of the tips must also be taken into account.

Cancellation of Basic Pressure in the Wake

In figure 1, regions I and III on the surface of the wing are seen to lie within the Mach cones of pressure disturbances occurring on and behind the trailing edges, and therefore are the areas of pressure that will be influenced by the cancellation of pressure in the wake. The function for a field of pressure which may be superposed on the wake to cancel the basic pressure must fulfill the following conditions:

1. Cancel the pressure at the trailing edge in order to comply with the Kutta condition.

2. Represent a field of quasi-conical pressure that conforms with the pressure in the wake given by equation (1).

3. Be zero outside the Mach cone enclosing the region of pressure to be canceled.

4. Have a complementary function for downwash velocity that reduces to zero on the surface of the wing in order that the plane of the wing remain flat.

5. Satisfy the equations for linearized potential flow and irrotationality.

The function, which fulfills each of these conditions, is composed of both conical and quasi-conical components which can be developed independently.

<u>Conical component of canceling function.</u>- The following function given in reference 8 for the case of a swept-back wing in a steady lifting attitude, namely (r.p. to indicate real part),

$$d\Delta P_L = -\text{r.p. } \frac{1}{\pi}\left(\frac{dP_{L_A}}{da}\right) da \cos^{-1} \chi$$

$$\chi = \frac{(1-a)(t-m_t)-(m_t-a)(1-t)}{(1-m_t)(t-a)}$$

satisfies each of the prescribed boundary conditions except the second. As illustrated in reference 8, this function, in which the term t defines a ray similar to a but having its origin at a point $A(x_A,y_A)$ on the trailing edge, represents a sector of pressure (fig. 3) in which the pressure is conically distributed rather than quasi-conically distributed. The function, as shown in figure 4, in effect cancels a field of pressure which has the constant value $(dP_{L_A}/da)da$ within a sector having an apex at the point A and sides along the ray a and the trailing edge m_t (i.e., $a \leq t \leq m_t$), and which diminishes on the wing from $(dP_{L_A}/da)da$ along the trailing edge to zero along the Mach line from A (i.e., $m_t \leq t \leq 1$). The downwash associated with each sector is shown in reference 8 to be zero in the range $m_t \leq t \leq 1$ and finite in the range $-1 \leq t \leq m_t$, thus complying with condition 4. A series of these pressure fields can be distributed along the trailing edge such that their integrated magnitude will cancel the portion of the pressure in the wake of the wing in steady lift in excess of the pressure at the trailing-edge apex.

If the term $(dP_{L_A}/da)da$ is replaced by $(dP_{R_A}/da)da$ and ΔP_L by $\Delta P_{R_1}'$ in the above equation,[3] that is,

$$d\Delta P_{R_1}' = \frac{-1}{\pi}\left(\frac{dP_{R_A}}{da}\right) da \cos^{-1} \chi' \tag{2}$$

[3]It will be understood that only the real part of the equation applies.

where

$$\chi' = \frac{(1-a)(t-m_t)-(m_t-a)(1-t)}{(1-m_t)(t-a)}$$

then a field of pressure conically distributed in the wake from points along the trailing edge of a rolling wing will be canceled. Equation (2) alone, however, will not cancel all the pressure in the wake, since there still remains to be canceled an additional component field of pressure associated with the linear increase of pressure along rays originating at the trailing edge. The function for this additional quasi-conical component may be developed through a further application of the principle of superposition of conical flows.

Quasi-conical component of canceling function.- The function for a quasi-conical field of pressure, which complies with all the prescribed boundary conditions, can be derived by superposing the conical fields given by equation (2). Thus, two independent auxiliary functions, which define fields of pressure that vary linearly with respect to both x and y, can be obtained first, by integrating a series of sectors of infinitesimal conical pressure with their origins spaced along the ray $t = a$, and second, by integrating these sectors spaced along the ray $t = m_t$. These two new functions will not individually satisfy the two known boundary conditions: (1) that the function define a field of pressure which varies linearly along the ray $t = a$ with respect to y only in order to be consistent with equation (1), and (2) that the function reduce to zero along the ray $t = m_t$ along which all the pressure is canceled by the fields of conical pressure alone. However, they can be employed simultaneously to deduce a single resultant function for a field of pressure which does fulfill these two conditions, and therefore cancels the remaining quasi-conical pressure in the wake.

Referring to figure 5, let ξ_1,η_1 be the coordinates, originating at A, of the apexes of the conical sectors to be integrated along the ray a. Defining a constant linear rate of increase of pressure in the η direction as

$$\sigma = \frac{d}{d\eta}\left[\left(\frac{dP_{R_A}}{da}\right) da\right]$$

where P_{R_A} is given by equation (1) at the point A (i.e., $y = y_A$), then the expression for a single element of the series of conical pressure fields to be superposed may be written

$$dF_1 = \frac{\sigma}{\pi}\, d\eta_1 \cos^{-1}\chi_1$$

$$\chi_1 = \frac{(1-a)(t_1-m_t)-(m_t-a)(1-t_1)}{(1-m_t)(t_1-a)}$$

$$t_1 = \beta\,\frac{y'-\eta_1}{x'-\xi_1}$$

$$a = \beta\,\frac{\eta_1}{\xi_1}$$

$$y' = y-y_A, \quad x' = x-x_A$$

$$y_A = \frac{m_t c_o a}{\beta(m_t-a)}, \quad x_A = \frac{m_t c_o}{(m_t-a)}$$

The rearwardmost sector, identified by the ordinate η_{1_o}, that can influence the point $P(x,y)$ on the wing corresponds to the ray $t_1 = 1$, that is,

$$t_1 = 1 = \beta\,\frac{y'-\eta_{1_o}}{x'-\frac{\beta}{a}\,\eta_{1_o}}$$

$$\eta_{1_o} = \frac{a(x'-\beta y')}{\beta(1-a)}$$

The first quasi-conical function is then found by integrating dF_1 between the limits 0 and η_{1_o}, that is,

$$F_1 = \frac{\sigma}{\pi}\int_0^{\eta_{1_o}} \cos^{-1}\chi_1\,d\eta_1$$

$$= -\frac{\sigma}{\pi}\left[\frac{a(1-m_t)(\beta y'-ax')}{2\beta(m_t-a)(1-a)}\left(\chi'\cos^{-1}\chi'-\sqrt{1-\chi'^2}\right)\right] \tag{3}$$

where

$$\chi' = \frac{(1-a)(t-m_t)-(m_t-a)(1-t)}{(1-m_t)(t-a)}$$

$$t = \beta\,\frac{y'}{x'} = \beta\,\frac{y-y_A}{x-x_A}$$

In a similar manner, the second function F_2 can be obtained by integrating fields of infinitesimal conical pressure having the apex coordinates ξ_2,η_2 along the line $t = m_t$. Thus,

$$F_2 = \frac{\sigma}{\pi}\int_0^{\eta_{2_0}} \cos^{-1} \chi_2 \, d\eta_2$$

$$= \frac{\sigma}{\pi} \frac{m_t(\beta y'-ax')}{\beta(m_t-a)} \left(\cos^{-1} \chi' - \sqrt{1-\chi'^2}\right) \qquad (4)$$

where

$$\chi_2 = \frac{(1-a)(t_2-m_t)-(m_t-a)(1-t_2)}{(1-m_t)(t_2-a)}$$

$$t_2 = \beta \frac{y'-\eta_2}{x'-\xi_2}$$

$$m_t = \beta \frac{\eta_2}{\xi_2}$$

$$\eta_{2_0} = \frac{m_t(x'-\beta y')}{\beta(1-m_t)}$$

Equations (3) and (4) can now be combined in the following manner to give the desired resultant function F, which reduces to zero along the trailing edge, $t = m_t$. Let F be defined as

$$F = F_1 + CF_2 \qquad (C = \text{constant})$$

Then along the ray $t = m_t$, the factor C becomes[4]

$$C = -\frac{(F_1)_{t=m_t}}{(F_2)_{t=m_t}} = -\frac{a(1-m_t)}{2m_t(1-a)}$$

and the function F,

$$F = -\frac{\sigma}{\pi} y' \frac{a}{t} \frac{m_t-t}{m_t-a} \left(\cos^{-1} \chi' - \frac{t-a}{t-m_t} \frac{1-m_t}{1-a} \sqrt{1-\chi'^2}\right)$$

[4]In the region $a \leq t \leq m_t$, $\cos^{-1} \chi' = \pi$.

then, along the ray a this equation reduces to

$$(F)_{t=a} = -\sigma y' = -y' \frac{d}{dy'}\left[\left(\frac{dP_{R_A}}{da}\right) da\right]$$

As required in boundary condition 2 for the rolling wing, the following proportion can be written:

$$\frac{y' \dfrac{d}{dy'}\left[\left(\dfrac{dP_{R_A}}{da}\right) da\right]}{\left(\dfrac{dP_{R_A}}{da}\right) da} = \frac{y-y_A}{y_A}$$

and therefore the desired quasi-conical form,

$$(F)_{t=a} = -\frac{y-y_A}{y_A}\left(\frac{dP_{R_A}}{da}\right) da$$

is obtained along the ray a.

Hence the quasi-conical component of the function for canceling the basic pressure in the wake may be expressed as

$$d\Delta P_{R_2}' = -\frac{1}{\pi}\left(\frac{dP_{R_A}}{da}\right) da \; \frac{y-y_A}{y_A} \; \frac{a}{t} \; \frac{m_t-t}{m_t-a}\left(\cos^{-1}\chi' - \frac{t-a}{t-m_t}\,\frac{1-m_t}{1-a}\sqrt{1-\chi'^2}\right) \qquad (5)$$

A sketch of the sector of pressure represented by equation (5) is shown in figure 6.

<u>Complete function for canceling basic pressure in wake</u>.— The complete function $d\Delta P_R'$ for canceling the basic pressure in the wake can now be formed by combining the individual functions for the conical and quasi-conical components. Thus,

$$d\Delta P_R' = d\Delta P_{R_1}' + d\Delta P_{R_2}'$$

$$= -\frac{1}{\pi}\left(\frac{dP_{R_A}}{da}\right) da \cos^{-1}\chi' - \frac{1}{\pi}\left(\frac{dP_{R_A}}{da}\right) da \; \frac{y-y_A}{y_A} \; \frac{a}{t} \; \frac{m_t-t}{m_t-a}$$

$$\left(\cos^{-1}\chi' - \frac{t-a}{t-m_t}\,\frac{1-m_t}{1-a}\sqrt{1-\chi'^2}\right) \qquad (6)$$

A sketch of the combined fields of pressure represented in equation (6) is shown in figure 7.

The primary correction for the basic pressure in region I and region III in part (fig. 1) can now be found by integrating equation (6) along the trailing edge. If a_o (defined by $t=1$) denotes the ray to the outermost sector that can affect the pressure at the point $P(x,y)$, the correction at P becomes

$$\Delta P_R{}' = -\frac{1}{\pi}\int_0^{a_o}\left(\frac{dP_{R_A}}{da}\right)\left[\cos^{-1}\chi' + \frac{y-y_A}{y_A}\,\frac{a}{t}\,\frac{m_t-t}{m_t-a}\left(\cos^{-1}\chi' - \frac{t-a}{t-m_t}\,\frac{1-m_t}{1-a}\sqrt{1-\chi'^2}\right)\right]da \tag{7}$$

where

$$\frac{dP_{R_A}}{da} = \left(\frac{pb}{2V}\right)\frac{4m^2}{\beta Q}\,\frac{y_A}{s}\,\frac{1}{\sqrt{m^2-a^2}}\left[\frac{a}{m^2-a^2} + \frac{m_t}{a(m_t-a)}\right]$$

and

$$a_o = m_t\,\frac{\beta y+c_o-x}{\beta y+m_tc_o-x}$$

A graphical method of integration is suggested. Separate terms representing the conical and quasi-conical components are retained in equations (6) and (7) for the purpose of showing their relative magnitudes later in the report. It will be shown then that the quasi-conical terms may be dropped in most practical applications without introducing significant error.

Cancellation of Basic Pressure Outboard of the Tips

A suitable function for canceling the basic pressure outboard from the tips can be deduced directly from equation (6). It is observed in figure 8 that the tips lie in the negative range of t of each superposed sector, in which case it is necessary to substitute $-a$, $-t$, and $-m_s$ for a, t, and m_t, respectively. For example, in the case of a wing with zero rake at the tips (i.e., $m_s=0$), equation (6) transforms to

$$d\Delta P_R'' = d\Delta P_{R_1}'' + d\Delta P_{R_2}''$$

$$= -\frac{1}{\pi}\left(\frac{dP_{R_A}}{da}\right) da \cos^{-1}\chi'' - \frac{1}{\pi}\left(\frac{dP_{R_A}}{da}\right) da \frac{y-s}{s}\left[\cos^{-1}\chi'' - \frac{t-a}{t(1+a)}\sqrt{1-\chi''^2}\right] \tag{8}$$

where

$$\chi'' = \frac{a+t+2at}{t-a}$$

$$x_A = \frac{\beta s}{a}, \quad y_A = s$$

The field of pressure defined by equation (8) is schematically shown in figure 8.

The expression for a tip with negative rake can also be deduced directly from equation (6). Tips with positive rake, on the other hand, do not come within the scope of this report, although they may be treated through an adaptation of the method given in reference 9 for the case of steady lift. In application, however, the present analysis will be limited to wings with zero rake at the tips.

The primary correction for the pressure at a point $P(x,y)$ near the tip (with zero rake) due to cancellation of basic pressure outboard of the tip can be found by integrating equation (8) along the tip between the limits $a=m$ and $a=a_o$, where

$$a_o = \frac{\beta s}{x+\beta(y-s)}$$

In performing such an integration, it is noted that the term dP_{R_A}/da, appearing after equation (7), becomes infinite at the leading edge $(a=m)$. Reference 8 presents a method for treating the singularity at this upper limit. The method leads to the following expression in which the conical and quasi-conical parts are again retained as separate components for comparison of their magnitudes:

$$\Delta P_R'' = \left(\frac{pb}{2V}\right)\frac{4m^2}{\beta Q}\frac{1}{\pi}(\beta y+x)\sqrt{\frac{a_o(s-y)}{s}}\left[\int_{a_o}^{m}\frac{da}{(\beta y-ax)\sqrt{(m^2-a^2)(1+a)(a-a_o)}} + \int_{a_o}^{m}\frac{a-a_o}{a_o(1+a)}\frac{da}{(\beta y-ax)\sqrt{(m^2-a^2)(1+a)(a-a_o)}}\right] \tag{9}$$

An analytical solution to the conical part (first integral) is given in reference 8 as

$$\Delta P_{R_1}'' = \left(\frac{pb}{2V}\right) \frac{4m^2}{\beta Q} \frac{1}{\pi}\left[\sqrt{\frac{2\beta(s-y)}{m(x+\beta y)}}\, K(k) - \right.$$

$$\left. 2x\sqrt{\frac{1}{m^2x^2-\beta^2y^2}}\left\{\frac{F(\psi,k')}{K(k')}\left[\frac{\pi}{2} - K(k)E(k')\right] + K(k)E(\psi,k')\right\}\right] \qquad (10)$$

where

$$k = \sqrt{\frac{(m-a_o)(1-m)}{2m(1+a_o)}}$$

$$k' = \sqrt{1-k^2}$$

$$\psi = \sin^{-1}\sqrt{\frac{a_o(mx+\beta y)}{\beta s(a_o+m)}}$$

The quasi-conical part has been solved in a similar manner, resulting in

$$\Delta P_{R_2}'' = \left(\frac{y-s}{s}\right)\Delta P_{R_1}'' - \left(\frac{pb}{2V}\right)\frac{4m^2}{\beta Q}\frac{1}{\pi}\left[\frac{1+a_o}{a_o}\sqrt{\frac{a_o(s-y)}{s}}\right.$$

$$\left.\left\{\frac{1}{1-m^2}\sqrt{\frac{2}{m(1+a_o)}}\left[(1+m)K(k)-2mE(k)\right]\right\}\right] \qquad (11)$$

Equations (10) and (11) together, therefore, constitute the primary correction to be added to the basic pressure at each point in regions II and III in figure 1.

An interesting result is obtained in evaluating the primary correction for the pressure at points lying along the tip Mach cone. For these points, $a_o=m$ and $k=0$, and equations (10) and (11) reduce, respectively, to

$$\Delta P_{R_1}'' = -\left(\frac{pb}{2V}\right)\frac{2m}{\beta Q}\sqrt{\frac{2s}{(1+m)(s-y)}} \qquad (12)$$

and

$$\Delta P_{R_2}" = 0$$

Thus, an abrupt reduction of pressure resulting from the cancellation of the conical portion of the basic pressure outboard of the tips is observed to occur along the Mach line from the tip of the leading edge. No such change, on the other hand, is introduced along the Mach lines from the apex of the trailing edge by either the conical or the quasi-conical components of the correction due to the cancellation of the basic pressure in the wake (see equation (7)), since both components diminish continuously to zero on the Mach line. The abrupt reduction of pressure at the tip was encountered also in the case of steady lift in reference 8.

Secondary Corrections

Cancellation of secondary pressure.- Although the basic pressure outside of the boundaries of the swept-back wing is completely canceled by means of equations (7), (10), and (11), some residual secondary pressure still remains to be canceled if a complete solution is desired in certain regions. Reference to the right-hand half of the wing shown in figure 9(a) indicates that, as a result of superposing canceling pressures along the trailing edge, a small amount of pressure is again introduced in the shaded region beyond the tip. Similar pressures exist along the trailing edge, as shown in the shaded region on the left-hand half, as a result of canceling the basic pressure outboard of the tip. For wings of very low aspect ratio, the region of secondary pressure in the wake may even extend onto the surface of the opposite half of the wing. (See fig. 9(b).) These secondary pressures are no longer conically or quasi-conically distributed with respect to a fixed point, hence, in order to cancel them, it would be necessary to superpose a series of sectors having different origins. Such secondary canceling sectors would, in turn, add still further but smaller extraneous pressures, which for a more precise analysis would likewise have to be canceled, and so on. The numerical method of canceling these additional pressures is more fully described in references 8 and 9; but, because the process is complicated and the effects are relatively small in comparison with the effects of canceling the basic pressure, only an approximate method of canceling the secondary pressures will be considered in this report, the method being sufficiently accurate for most practical applications. Thus, where these uncanceled pressures exist, the calculated pressure on the trailing edge or tip will have been reduced to a negative value instead of zero. Knowing the magnitude of this error (namely, the amount of the tip correction along the trailing edge, and the amount of the trailing-edge correction along the tip, both in region III) and also the extent to which the adjacent region of the wing is influenced by the uncanceled pressures (defined by reflected Mach lines, see left half of wing in fig. 10), the correct distribution of pressure near the boundary can be easily estimated using as a guide the trends of the primary corrections. This is most conveniently accomplished by sketching one or two chordwise pressure distributions near the tip. This procedure is illustrated in the examples which follow. In some cases, a more accurate estimate of the loading at the tip is desirable in calculating the rolling moment, since the moment arm is greatest at the tip.

In wing configurations having large angles of sweep and high aspect ratio, some of the Mach lines from the trailing edge may intersect the leading edge such that a portion of the sectors of pressure used to cancel the pressure in the wake near the center of the wing will lie upstream from the leading edge as shown in figure 9(b). The pressure fields required to cancel these minor pressures in the lifting case are developed in reference 9; but, since they produce only slight reductions of the basic pressures on the surface of the wing, their inclusion here is believed to be unwarranted in view of the amount of additional computation that would be required in a practical application.

Secondary downwash effects.- One of the boundary conditions that has been imposed on the function chosen for the canceling procedure is that of producing no downwash velocity components on the surface of the wing. Equation (6) when applied to either the trailing edge or tip does, in general, comply with this condition, since each superposed sector has zero downwash between the wing boundary and the Mach line extending across the wing as shown in reference 8. However, between the wing boundary and the opposite Mach line ($t=-1$ at points on the trailing edge) the downwash is finite. It will be observed that these Mach lines of the trailing-edge sectors at small values of a will intersect the opposite half of the wing, thus introducing some overlapping downwash on the wing, and therefore violating the flat-plate boundary condition. Since only small regions of the wing are affected, and since the downwash components approach zero when the canceling sectors lie close to the root chord ($a=0$), it is concluded that the resultant error would be insignificant and may therefore be neglected.

Illustrative Examples

To illustrate the application of equations (7), (10), and (11), the pressure distributions along two chordwise and three spanwise sections of the configuration shown in figure 10 (also used in reference 8) have been computed and are plotted in figure 11. Separate plots of the conical and quasi-conical terms are shown for comparison. The secondary corrections were estimated where necessary, as previously pointed out, in order to fulfill the boundary conditions at the edges. It is immediately apparent that the quasi-conical components are for the most part insignificant, and in most applications may be neglected entirely. The maximum error involved in the above example is less than 7 percent. By omitting the quasi-conical terms, the method then closely parallels that of reference 8, except that the term dP_{R_A}/da is used in place of dP_{L_A}/da.

A good approximation of the pressure distribution in roll can therefore be made in the following four steps:

1. Calculation of the basic pressure distribution (equation (1)).

2. Addition to basic pressure between a subsonic trailing edge and the Mach line from the trailing-edge apex of the primary correction, $\Delta P_{R_1}'$ (equation (2)).

3. Addition to the basic pressure between a subsonic tip and the Mach line from the tip of the leading edge of the primary correction, $\Delta P_{R_1}''$ (equation (10)).

4. Estimation of the smaller secondary corrections for the pressure between edges adjoining regions of uncanceled secondary pressures and the secondary reflected Mach lines, using as guides the trends of the corrections in steps 2 and 3.

Some indication of the magnitude and extent of the estimated secondary corrections are also revealed in the above example, the regions affected in general being small.

Calculation of the Damping Derivative

The corrected damping moment, or the damping derivative, in steady roll may likewise be calculated, first, by determing the basic value for the over-all swept-back plan form, and then, by adding corrective terms to account for the effect in regions I, II, and III of canceling pressures beyond the trailing edges and tips. As before, a comparison of the magnitudes of the conical and quasi-conical terms can be made by determining their effects separately.

Basic derivative for damping in roll.- The basic rolling moment is readily obtained from the known distribution of pressure. The increment of force dZ (refer to fig. 12), exerted on an element of wing area $(dS/da)da$ and based on the average pressure given by equation (1) is

$$dZ = q\,\frac{2}{3}\,(P_R)_{max}\left(\frac{dS}{da}\right) da$$

$$= q\,\frac{2}{3}\left(\frac{pb}{2V}\right)\frac{4m^2}{\beta Q}\,\frac{y_A}{s}\,\frac{1}{\sqrt{m^2-a^2}}\left(\frac{dS}{da}\right) da$$

where $(P_R)_{max}$ is the maximum pressure along the ray a at the point $A(x_A,y_A)$. With the center of pressure located at $(3/4)y_A$, the corresponding increment of rolling moment becomes

$$dL = \frac{3}{4} y_A \, dZ$$

The damping derivative, which by definition is

$$C_{l_p} = \frac{\partial C_l}{\partial (pb/2V)}$$

may be written for a single element of wing area as

$$dC_{l_p} = \frac{1}{pb/2V} \frac{dL}{qSb}$$

since the pressure coefficient and the parameter $(pb/2V)$ are linearly related in the linearized potential theory. From figure 12 it is evident that in order to obtain the moment on half the wing an integration of dL must be made over two ranges: first, $0 \leq a \leq a_t$, for which

$$y_A = \frac{m_t c_o a}{\beta(m_t - a)}; \qquad \frac{dS}{da} = \frac{m_t^2 c_o^2}{2\beta(m_t - a)^2}$$

and, second, $a_t \leq a \leq m$, for which

$$y_A = s; \qquad \frac{dS}{da} = \frac{\beta s^2}{2a^2}$$

The following expressions for the basic damping derivative, after substitution and integration, have been determined for the two cases $m \neq m_t$ and $m = m_t$:

$m \neq m_t$

$$C_{l_p} = \frac{-1}{4} \frac{m^2}{Q} \left(\frac{b^2}{S}\right) \left[\frac{\sqrt{m^2 - a_t^2}}{m^2 a_t} + \frac{1}{3} \left(\frac{m_t c_o}{\beta s}\right)^4 \left\{ - \frac{m^3(2m^2 + 13m_t^2)}{2m_t(m^2 - m_t^2)^3} + \right. \right.$$

$$\frac{\sqrt{m^2-a_t^2}}{2(m^2-m_t^2)^3(m_t-a_t)^3}\left[2m_t^2(m^2-m_t^2)^2+m_t(m_t^2-6m^2)(m^2-m_t^2)(m_t-a) + \right.$$

$$\left.(m_t-a_t)^2(6m^4+10m^2m_t^2-m_t^4)\right] +$$

$$\left.\left.\frac{3m_tm^2(4m^2+m_t^2)}{2(m^2-m_t^2)^3\sqrt{m_t^2-m^2}}\left[\cos^{-1}\frac{m}{m_t} - \cos^{-1}\frac{m^2-a_tm_t}{m(m_t-a_t)}\right]\right\}\right] \qquad (13)$$

$m = m_t$

$$C_{l_p} = \frac{-1}{4}\frac{1}{Q}\left(\frac{b^2}{S}\right)\left\{\frac{\sqrt{m^2-a_t^2}}{a_t} + \left(\frac{mc_o}{\beta s}\right)^4\left[\frac{\sqrt{m^2-a_t^2}}{105(m-a_t)^4}(8m^3-32m^2a_t + \right.\right.$$

$$\left.\left.52ma_t^2-13a_t^3) - \frac{8}{105}\right]\right\} \qquad (14)$$

Where C_{l_p} is to be computed for more than one Mach number, it is simpler to substitute in equations (13) and (14) $\overline{m}$, $\overline{a_t}$, and $\overline{m_t}$ for m, a_t, and m_t, respectively, where

$$\overline{m} = \frac{m}{\beta}$$

$$\overline{a_t} = \frac{a_t}{\beta}$$

$$\overline{m_t} = \frac{m_t}{\beta}$$

With the exception that the term $m_tc_o/\beta s$ must be replaced by $\overline{m_t}c_o/s$, the transformed equations are otherwise identical to equations (13) and (14).

<u>Primary correction resulting from cancellation of basic pressure in wake</u>.- The effect on the damping of canceling pressures in the wake can be calculated by a twofold integration: one, to determine the effect in the region of the wing between the trailing edge and Mach line of a single canceling sector; and the second, the combined effect of all the sectors. Thus, in figure 13, the increments of force corresponding to

the conical and quasi-conical pressures exerted on an element of wing area $(dS/dt)dt$ are (from equation (6))

$$d\Delta Z_1' = \frac{q}{\pi}\left(\frac{dP_{R_A}}{da}\right) da \cos^{-1}\chi' \left(\frac{dS}{dt}\right) dt$$

$$d\Delta Z_2' = \frac{2}{3}\frac{q}{\pi}\left(\frac{dP_{R_A}}{da}\right) da \frac{s-y_A}{y_A}\frac{a}{t}\frac{m_t-t}{m_t-a}\left(\cos^{-1}\chi' - \frac{t-a}{t-m_t}\frac{1-m_t}{1-a}\sqrt{1-\chi'^2}\right)\left(\frac{dS}{dt}\right) dt$$

in which the average quasi-conical pressure is two-thirds of the maximum value at the point $y_A = s$. Noting that the two respective moment arms are

$$\overline{y_1} = y_A + \frac{2}{3}(s-y_A)$$

$$\overline{y_2} = y_A + \frac{3}{4}(s-y_A)$$

the correction due to the cancellation of basic pressure in the wake of both halves of the wing may be written as

$$d\Delta C_{l_p}' = d\Delta C_{l_{p_1}}' + d\Delta C_{l_{p_2}}'$$

$$= \frac{2}{q(pb/2V)Sb}\left[\int_0^{a_t}\int_{m_t}^{1}\overline{y_1}\,(d\Delta Z_1')_{t,a} + \int_0^{a_t}\int_{m_t}^{1}\overline{y_2}\,(d\Delta Z_2')_{t,a}\right]$$

Substituting

$$\frac{dS}{dt} = \frac{\beta m_t^2 s^2}{2a_t^2}\frac{(a_t-a)^2}{(m_t-a)^2}\frac{1}{t^2}$$

and integrating with respect to t gives

$$\Delta C_{l_p}' = \frac{(b^2/S)}{(pb/2V)}\frac{\beta}{4b}\left(\frac{m_t}{a_t}\right)^2\int_0^{a_t}\left(\frac{dP_{R_A}}{da}\right)\left(\frac{y_A}{3}+\frac{2s}{3}\right)\left(\frac{a_t-a}{m_t-a}\right)^2\frac{1}{a}\left[\sqrt{\frac{(m_t-a)(1-a)}{m_t}} - \frac{m_t-a}{m_t}\right]da + \frac{(b^2/S)}{(pb/2V)}\frac{\beta}{4b}\left(\frac{m_t}{a_t}\right)^2\int_0^{a_t}\left(\frac{dP_{R_A}}{da}\right)\left(\frac{y_A}{4}+\frac{3s}{4}\right)\left(\frac{a_t-a}{m_t-a}\right)^2\frac{s-y_A}{y_A}\frac{1}{3a}\left[-\frac{m_t-a}{m_t}+\frac{2m_t-am_t-a}{2m_t(1-a)}\sqrt{\frac{(m_t-a)(1-a)}{m_t}}\right]da \qquad (15)$$

where, as before,

$$\frac{dP_{R_A}}{da} = \left(\frac{pb}{2V}\right) \frac{4m^2}{\beta Q} \frac{y_A}{s} \frac{1}{\sqrt{m^2-a^2}} \left[\frac{a}{m^2-a^2} + \frac{m_t}{a(m_t-a)}\right]$$

A graphical method of integration with respect to a is required as before in calculating the pressure distribution. The integrands of equation (15) contain indeterminate factors at $a = 0$, for which the following expression may be used:

$$(\Delta C_{l_p}')_{a=0} = \frac{1}{3} \frac{m}{\beta Q} \left(\frac{b^2}{S}\right) \frac{c_o}{b} \frac{1-m_t}{m_t} + \frac{1}{64} \frac{m}{Q} \left(\frac{b^2}{S}\right)\left(\frac{1-m_t}{m_t}\right)^2$$

In those cases in which the Mach lines from the pressure fields superposed along the trailing edge near the root chord intersect the leading edge, the rolling-moment integration of each sector must then be made over the two regions $m_t \leq t \leq m_a$ and $m_a \leq t \leq 1$, where m_a designates the ray passing through $A(x_A,y_A)$ and the tip of the leading edge, that is

$$m_a = \beta \frac{s-y_A}{(\beta s/m)-x_A}$$

When a exceeds the value a_l (determined for $m_a = 1$), where

$$a_l = \frac{1-(\beta s/c_o m)(1-m)}{1-(\beta s/m_t m c_o)(1-m)}$$

then the range $m_a \leq t \leq 1$ no longer exists, and only equation (15) is required. If, for $0 \leq a \leq a_l$, subscript r identifies the region $m_t \leq t \leq m_a$ and subscript f the region $m_a \leq t \leq 1$, then

$$\left(\frac{dS}{dt}\right)_f = \frac{{x_A}^2(m-a)^2}{2\beta(t-m)^2}$$

$$(\overline{y_1})_f = y_A + \frac{2}{3}\left(\frac{my_A}{a} \frac{a-t}{m-t} - y_A\right) = y_A\left(1+ \frac{2}{3} \frac{t}{a} \frac{m-a}{t-m}\right)$$

(The terms $(dS/dt)_r$ and $(\overline{y_1})_r$ were previously given in the derivation of equation (15).) Since, as will be shown later, the effects of the quasi-conical component of the pressure in the wake in most cases can be neglected, only the correction corresponding to the conical component will be considered. Then, for $a \leq a_l$

$$(\Delta C_{l_p}{}')_{a \le a_l} = \frac{2}{(pb/2V)qSb} \int_0^{a_l} \left[\int_{m_t}^{m_a} \frac{q}{\pi} (\overline{y_1})_r \left(\frac{dP_{R_A}}{da}\right) \cos^{-1} \chi' \left(\frac{dS}{dt}\right)_r dt + \right.$$

$$\left. \int_{m_a}^{1} \frac{q}{\pi} (\overline{y}_1)_f \left(\frac{dP_{R_A}}{da}\right) \cos^{-1} \chi' \left(\frac{dS}{dt}\right)_f dt \right] da \qquad (16)$$

while the correction for $a_l \le a \le a_t$ is given by equation (15). An analytical integration of $(\Delta C_{l_p}{}')_{a \le a_l}$ with respect to t is given for the cases $m \ne m_t$ and $m = m_t$ in Appendix A, but, as in equation (15), a graphical method is required for the integration with respect to a.

Primary correction resulting from cancellation of basic pressure outboard of tips.- An analogous procedure may be used to calculate the damping corrections due to the cancellation of pressure outboard of the tips. The increments of force (see fig. 14 and equation (8)) associated with the conical and quasi-conical canceling sectors are, respectively,

$$d\Delta Z_1'' = \frac{q}{\pi} \left(\frac{dP_{R_A}}{da}\right) da \cos^{-1} \chi'' \left(\frac{dS}{dt}\right) dt$$

$$d\Delta Z_2'' = \frac{2}{3} \frac{q}{\pi} \left(\frac{dP_{R_A}}{da}\right) da \frac{y_m - s}{s} \left[\cos^{-1} \chi'' - \frac{t-a}{t(1+a)} \sqrt{1-\chi''^2}\right] \left(\frac{dS}{dt}\right) dt$$

in which

$$\frac{y_m - s}{s} = \frac{-m_t t}{m_t - t} \left(\frac{1}{a} - \frac{1}{a_t}\right)$$

$$\frac{dS}{dt} = \frac{\beta m_t^2 s^2}{2(m_t - t)^2} \left(\frac{1}{a_t} - \frac{1}{a}\right)^2$$

The corresponding moment arms are

$$\overline{y_1} = s + \frac{2}{3} (y_m - s)$$

$$\overline{y_2} = s + \frac{3}{4} (y_m - s)$$

such that the resultant correction for both tips after integrating over the region $-1 \le t \le 0$ becomes

$$\Delta C_{l_p}{}'' = \Delta C_{l_{p_1}}{}'' + \Delta C_{l_{p_2}}{}''$$

$$= \frac{(b^2/S)}{(pb/2V)} \frac{\beta m_t{}^2}{8} \int_{a_t}^{m} \left(\frac{dP_{R_A}}{da}\right) H(a)da + \frac{(b^2/S)}{(pb/2V)} \frac{\beta m_t{}^3}{24} \int_{a_t}^{m} \left(\frac{dP_{R_A}}{da}\right) R(a)da$$

where $H(a)$ and $R(a)$ are listed in Appendix B. Since the term $(dP_{R_A}/da)da$ is infinite at the upper limit of integration, it is desirable to rewrite the above equation, as outlined in reference 8, in the following more suitable form for a graphical solution:

$$\Delta C_{l_p}{}'' = \left(\frac{b^2}{S}\right) \frac{m^2 m_t{}^2}{2Q} \left[H'(m) \cos^{-1}\frac{a_t}{m} + \int_{a_t}^{m} \frac{H'(a)-H'(m)}{\sqrt{m^2-a^2}} da\right] +$$

$$\left(\frac{b^2}{S}\right) \frac{m^2 m_t{}^3}{6Q} \left[R'(m) \cos^{-1}\frac{a_t}{m} + \int_{a_t}^{m} \frac{R'(a)-R'(m)}{\sqrt{m^2-a^2}} da\right] \qquad (17)$$

where $H'(a) = dH(a)/da$ and $R'(a) = dR(a)/da$. The expressions for $H'(a)$ and $R'(a)$, together with the particular forms of $H'(m)$ and $R'(m)$ for the untapered wing $(m = m_t)$, are given in Appendix B.

Secondary corrections.– Equations (15) and (17) (and equation (A1) or (A2), if required, in Appendix A) therefore comprise the primary corrections to be added to the basic damping derivative. A complete analysis, however, would next require an evaluation of the effects of canceling the secondary and smaller pressures previously described. As before, the calculation of these effects are not warranted in most applications in view of their mathematical complexity, but, where necessary, approximate secondary corrections to the damping derivative can be deduced from the secondary pressure corrections discussed previously. It is observed that, although they are small and are effective over only small portions of the wing, the secondary pressures in some cases may give rise to rolling moments which are not trivial, since they are distributed at large distances from the rolling axis. In such cases, the net change in loading near the tips associated with the estimated secondary corrections for the pressure together with the corresponding moment arm to the center of gravity of the load correction can be easily approximated.

Illustrative examples: The damping-in-roll derivative for the untapered- and tapered-wing configurations shown in figures 10 and 15, respectively, has been computed by the above method, and the results, which for the untapered wing include the separate corrections resulting from the cancellation of the conical and quasi-conical components of pressure, are listed in tables I and II. As before, in the calculation of the pressure distribution, it is evident from table I that the effects of the quasi-conical terms are so small that their computation would not, in general, be justified.

An estimate of the correction resulting from the cancellation of the secondary pressures along the trailing edge and tips of the untapered wing of figure 10 is also included in table I. The 7.3-percent magnitude of this correction may be of significance in some applications. Inclusion of the secondary corrections for the tapered wing (fig. 15) at the lower Mach numbers, at which the Mach line from the trailing-edge apex intercepts the leading edge, is not within the scope of this report; however, on the basis of the results given in reference 9 for the case of lift, the correction that would result from the cancellation of the secondary pressure upstream from the leading edge is believed to be negligible. At all the Mach numbers, the uncanceled secondary pressure outboard of the tips and in the wake was also neglected, since for the tapered plan form only small portions of pressure distribution on the wing are affected by their presence.

The quantities listed in table II for the tapered wing are shown plotted against Mach number in figure 16. The spacing between the curves for the basic and the corrected values gives an indication of the over-all reduction of the damping moment due to the subsonic trailing edges and tips, particularly for high angles of sweep and low Mach numbers. It is noted that the corrected damping derivative increases with Mach number until the trailing edge becomes supersonic (about M=1.65), and then decreases.

Thus, by retaining only the conical terms in the corrections, the method for calculating the damping derivative is reduced to the following steps:

1. Calculation of the basic damping-in-roll derivative (equation (13) or (14)).

2. Correction of the basic value for the effect of a subsonic trailing edge (conical terms $(\Delta C_{l_{p_1}}{}')$ in equation (15) and in equation (16), if required).

3. Correction of the basic value for the effect of both subsonic tips (conical term $(\Delta C_{l_{p_2}}{}')$ in equation (17)).

4. Estimation of the secondary corrections based on the amount and location of uncanceled pressure existing on the wing after canceling the basic pressure in the wake and outboard of the tips.

Calculation of the Pressure Distribution and Damping in Roll by Strip Theory

A direct correlation between the pressure distributions due to steady lift and to steady roll can be made along chordwise strips in some regions of a swept-back wing. Thus, in the basic equations for the two cases,

$$P_L = \alpha \frac{4m^2}{\beta\, E\left(\sqrt{1-m^2}\right)} \frac{1}{\sqrt{m^2-a^2}}$$

and

$$P_R = \left(\frac{pb}{2V}\right) \frac{4m^2}{\beta Q} \frac{y}{s} \frac{1}{\sqrt{m^2-a^2}}$$

it is seen that the pressure due to roll is related to that due to lift in the following manner:

$$P_R = \delta \frac{y}{s} P_L$$

where

$$\delta = \frac{(pb/2V)}{\alpha} \frac{E\left(\sqrt{1-m^2}\right)}{Q}$$

This relationship is valid in those regions that are not affected by the existence of pressure in the wake and outboard of the tips, but elsewhere (regions I, II, and III in fig. 1) the strip theory will not yield correct values, since from equations (6) and (8)

$$\frac{dP_{R_A}}{da} \neq \frac{y_A}{s} \frac{dP_L}{da}$$

where y_A is a function of a. The extent of the error introduced by the strip method is shown in figure 17, which includes the pressure distributions as calculated by both the strip and the conical-flow methods along sections A-A and B-B of the wing in figure 10. Since the magnitudes given by the two methods differ by as much as 40 percent near the subsonic boundaries, it is concluded that the strip theory is generally not satisfactory for calculating the pressure distribution near a subsonic edge. These differences in pressure, however, when integrated to determine their moment about the rolling axis, are much less significant, the value as calculated by strip theory being only about 2-1/2 percent less than the value given in table I. A good approximation of the damping-in-roll derivative therefore may often be calculated by means of strip theory.

CONCLUDING REMARKS

A method of analysis based upon the usual assumptions made in the linearized theory of potential flow has been presented for calculating in a supersonic flow the pressure distribution and damping in steady roll of flat swept-back wings having all edges straight and subsonic. Although restricted to wings with zero rake at the tips in the applications considered herein, the analysis can be readily adapted to the treatment of negatively raked tips. Tips with positive rake, although not within the scope of this report, can be analyzed through an adaptation of the method given in reference 9 for the case of steady lift. Although complete solutions for configurations in which the Mach lines from the apex of the trailing edge intersect the leading edge are not included, it is believed, on the basis of the results given in reference 9, that the method will yield values which are sufficiently accurate for most practical applications.

It is demonstrated that from a practical standpoint the analysis, as outlined, can be simplified considerably by omitting several minor terms in the complete solutions. The procedure for calculating the pressure distribution and damping in roll is thereby shortened to the extent that it closely parallels the method presented in reference 8 for the case of steady lift. The maximum error involved in applying the simplified analysis to an untapered wing of low aspect ratio is shown to be less than 7 percent of the total pressure at any point and less than 3 percent of the damping derivative. A parallel computation of the pressure distribution in roll can therefore be made when the lift distribution is being evaluated.

Although strip theory can be used to correlate the pressure distribution in roll and the lift distribution over a swept-back wing having trailing edges and tips which are supersonic, it is found to be inaccurate in the vicinity of a subsonic boundary. However, a close approximation of the damping-in-roll derivative can be calculated by means of strip theory.

Ames Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Moffett Field, Calif., Jan. 6, 1950.

APPENDIX A

CALCULATION OF PRIMARY CORRECTION FOR DAMPING DERIVATIVE – DUE TO CANCELLATION OF EXCESS PRESSURE IN WAKE

The damping correction $(\Delta C_{l_p}{}')_{a \leq a_l}$ for the range $0 \leq a \leq a_l$ due to the cancellation of pressure in the wake of a swept-back wing (from equation (16)) is as follows:

$m \neq m_t$

$$(\Delta C_{l_p}{}')_{a\leq a_l} = \frac{(b^2/S)}{(pb/2V)} \frac{\beta}{4b} \left(\frac{m_t}{a_t}\right)^2 \int_0^{a_l} \Bigg[\left(\frac{dP_{R_A}}{da}\right)\left(\frac{y_A}{3} + \frac{2s}{3}\right)\left(\frac{a_t-a}{m_t-a}\right)^2 \frac{1}{a}\Bigg[-\frac{m_t-a}{m_t} +$$

$$\frac{1}{\pi}\left(\frac{m_a-a}{m_a}\right) \cos^{-1} \frac{(1-a)(m_a-m_t)-(m_t-a)(1-m_a)}{(1-m_t)(m_a-a)} +$$

$$\frac{1}{\pi}\sqrt{\frac{(m_t-a)(1-a)}{m_t}} \cos^{-1} \frac{2m_t-m_a(1+m_t)}{m_a(1-m_t)}\Bigg] +$$

$$\left(\frac{dP_{R_A}}{da}\right)\frac{y_A}{3}\frac{{}^3(m-a)^2 a_t{}^2}{s^2 m_t{}^2}\frac{1}{a^3}\Bigg\{\frac{m\sqrt{(1-a)(m_t-a)(m_a-m_t)(1-m_a)}}{\pi(m_a-m)(m_t-m)(1-m)} -$$

$$\frac{1}{m-a}\sqrt{\frac{(1-a)(m_t-a)}{(1-m)(m_t-m)}}\left[a + \frac{m(1-a)(m_t-m)+m(1-m)(m_t-a)}{2(m_t-m)(1-m)}\right]$$

$$\frac{1}{\pi}\cos^{-1}\frac{(1-m)(m_a-m_t)-(m_t-m)(1-m_a)}{(1-m_t)(m_a-m)} +$$

$$\frac{(m_a-a)[m(m_a-a)+a(m_a-m)]}{(m-a)(m_a-m)^2}\frac{1}{\pi}\cos^{-1}\frac{(1-a)(m_a-m_t)-(m_t-a)(1-m_a)}{(1-m_t)(m_a-a)}\Bigg\}\Bigg] da \tag{A1}$$

The correction for the entire trailing edge is

$$\Delta C_{l_p}{}' = (\Delta C_{l_p}{}')_{a\leq a_l} + (\Delta C_{l_p}{}')_{a_l\leq a\leq a_t}$$

$m = m_t$

$$(\Delta C_{l_p}{}')_{a \leq a_l} = \frac{(b^2/S)}{(pb/2V)} \int_0^{a_l} \left[\frac{\beta}{4b} \left(\frac{m_t}{a_t}\right)^2 \left(\frac{dP_{R_A}}{da}\right) \left(\frac{y_A}{3} + \frac{2s}{3}\right) \left(\frac{a_t - a}{m_t - a}\right)^2 \frac{1}{a} \left[- \frac{m_t - a}{m_t} + \right.\right.$$

$$\frac{1}{\pi} \frac{m_a - a}{m_a} \cos^{-1} \frac{(1-a)(m_a - m_t) - (m_t - a)(1 - m_a)}{(1 - m_t)(m_a - a)} +$$

$$\left. \frac{1}{\pi} \sqrt{\frac{(m_t - a)(1-a)}{m_t}} \cos^{-1} \frac{2m_t - m_a(1 + m_t)}{m_a(1 - m_t)} \right] +$$

$$\left(\frac{y_A x_A{}^2}{b^3}\right) \frac{(m-a)^2}{\pi\beta} \left(\frac{dP_{R_A}}{da}\right) \left\{ \left[\frac{m_a - a}{(m_a - m)(m - a)} + \right.\right.$$

$$\left. \frac{(m-a)(2m_a - m)}{3a(m_a - m)^2} + \frac{m - 2a}{3a(m-a)} \right] \cos^{-1} \frac{(m_a - m)(1-a) - (m-a)(1 - m_a)}{(m_a - a)(1 - m)} -$$

$$2 \left[1 + \frac{m - 2a}{3a} + \frac{m(m-a)}{9a(m_a - m)} + \right.$$

$$\left.\left.\left. \frac{2m(m-a)}{9a(1-m)} \right] \frac{\sqrt{(m_a - m)(1 - m_a)(m-a)(1-a)}}{(m-a)(1-m)(m_a - m)} \right\} \right] da \qquad \text{(A2)}$$

APPENDIX B

CALCULATION OF PRIMARY CORRECTION FOR DAMPING DERIVATIVE – DUE TO CANCELLATION OF EXCESS PRESSURE OUTBOARD OF TIPS

The expressions for $H(a)$ and $R(a)$ in the tip corrections to the damping derivative (p. 25) are as follows:

$$H(a) = \left(\frac{1}{a} - \frac{1}{a_t}\right)^2 \left\{ \frac{1}{m_t} + g(a) + \frac{m_t}{3}\left(\frac{1}{a} - \frac{1}{a_t}\right)\left[\frac{1}{m_t} + h(a)\right] \right\}$$

$$R(a) = \left(\frac{1}{a} - \frac{1}{a_t}\right)^3 \left\{ \frac{1}{m_t} + h(a) - \frac{1}{2m_t(1+m_t)(1+a)}\sqrt{\frac{a(1+a)}{m_t(1+m_t)}} + \right.$$

$$\left. \frac{3m_t}{2}\left(\frac{1}{a} - \frac{1}{a_t}\right)\left[\frac{1}{3m_t} - s(a) - \frac{1}{8m_t(1+m_t)^2(1+a)}\sqrt{\frac{a(1+a)}{m_t(1+m_t)}}\right] \right\}$$

where

$$g(a) = \frac{1}{m_t-a}\left[\sqrt{\frac{a(1+a)}{m_t(1+m_t)}} - 1\right]$$

$$h(a) = \frac{1}{m_t-a}\left[(m_t-2a)\ g(a) - \frac{1+2m_t}{2(1+m_t)}\sqrt{\frac{a(1+a)}{m_t(1+m_t)}}\right]$$

$$s(a) = \frac{1}{m_t-a}\left[r(a) - \frac{3+8m_t+8m_t^2}{24(1+m_t)^2}\sqrt{\frac{a(1+a)}{m_t(1+m_t)}}\right]$$

$$r(a) = \frac{1}{m_t-a}\left[-\frac{3a^2-3am_t+m_t^2}{3}\ g(a) - \frac{(3a-2m_t)(1+2m_t)}{6(1+m_t)}\sqrt{\frac{a(1+a)}{m_t(1+m_t)}}\right]$$

The terms $H'(a)$ and $R'(a)$ in equation (17) are

$$H'(a) = -\frac{2}{a^2}\left(\frac{1}{a}-\frac{1}{a_t}\right)\left[\frac{1}{m_t}+g(a)\right]+\left(\frac{1}{a}-\frac{1}{a_t}\right)^2\left\{g'(a)-\frac{m_t}{a^2}\left[\frac{1}{m_t}+h(a)\right]+\frac{m_t}{3}\left(\frac{1}{a}-\frac{1}{a_t}\right)h'(a)\right\}$$

$$R'(a) = -\left(\frac{1}{a}-\frac{1}{a_t}\right)^3\left[-h'(a)+\frac{1}{4am_t(1+m_t)(1+a)^2}\sqrt{\frac{a(1+a)}{m_t(1+m_t)}}\right]+$$

$$\frac{3}{a^2}\left(\frac{1}{a}-\frac{1}{a_t}\right)^2\left[-\frac{1}{m_t}-h(a)+\frac{1}{2m_t(1+m_t)(1+a)}\sqrt{\frac{a(1+a)}{m_t(1+m_t)}}\right]-$$

$$\frac{3m_t}{2}\left(\frac{1}{a}-\frac{1}{a_t}\right)^4\left[s'(a)+\frac{1}{16am_t(1+m_t)^2(1+a)^2}\sqrt{\frac{a(1+a)}{m_t(1+m_t)}}\right]-$$

$$\frac{6m_t}{a^2}\left(\frac{1}{a}-\frac{1}{a_t}\right)^3\left[\frac{1}{3m_t}-s(a)-\frac{1}{8m_t(1+m_t)^2(1+a)}\sqrt{\frac{a(1+a)}{m_t(1+m_t)}}\right]$$

where

$$g'(a) = \frac{1}{m_t-a}\left[\frac{2a+1}{2\sqrt{am_t(1+a)(1+m_t)}}+g(a)\right]$$

$$h'(a) = \frac{1}{m_t-a}\left[(m_t-2a)g'(a)-2g(a)+h(a)-\frac{(1+2m_t)(1+2a)}{4(1+m_t)\sqrt{am_t(1+a)(1+m_t)}}\right]$$

$$s'(a) = \frac{1}{m_t-a}\left[s(a)+r'(a)-\frac{(3+8m_t+8m_t^2)(2a+1)}{48(1+m_t)^2\sqrt{am_t(1+a)(1+m_t)}}\right]$$

$$r'(a) = \frac{1}{m_t-a}\left[r(a)-\frac{3a^2-3am_t+m_t^2}{3}g'(a)-(2a-m_t)\,g(a)-\frac{(2a+1)(3a-2m_t)(1+2m_t)}{12(1+m_t)\sqrt{am_t(1+m_t)(1+a)}}-\frac{1+2m_t}{2(1+m_t)}\sqrt{\frac{a(1+a)}{m_t(1+m_t)}}\right]$$

Expressions for $H'(m)$ and $R'(m)$ for the untapered wing are as follows:

$$H'(m) = -\frac{2}{m^2}\left(\frac{1}{m} - \frac{1}{a_t}\right)\left[\frac{1}{m} - \frac{2m+1}{2m(1+m)}\right] +$$

$$\left(\frac{1}{m} - \frac{1}{a_t}\right)^2\left[-\frac{1}{4m^2(1+m)^2} + \frac{1}{3}\left(\frac{1}{m} - \frac{1}{a_t}\right)\frac{1}{8m(1+m)^3}\right]$$

$$R'(m) = -\frac{3m}{2}\left(\frac{1}{m} - \frac{1}{a_t}\right)^4\frac{3}{128m^2(1+m)^4} + \frac{3}{m^2}\left(\frac{1}{m} - \frac{1}{a_t}\right)^2\frac{1}{8m(1+m)^2}$$

where

$$g(m) = -\frac{2m+1}{2m(1+m)}$$

$$h(m) = -\frac{1}{m} + \frac{3}{8m(1+m)^2}$$

$$s(m) = \frac{1}{3m} - \frac{5}{48m(1+m)^3}$$

$$r(m) = \frac{3+8m+8m^2}{24(1+m)^2}$$

$$g'(m) = \frac{1}{8m^2(1+m)^2}$$

$$h'(m) = \frac{1}{8m^2(1+m)^3}$$

$$s'(m) = -\frac{5}{128m^2(1+m)^4}$$

$$r'(m) = -\frac{4+17m+12m^2}{24m(1+m)^3}$$

REFERENCES

1. Brown, Clinton E., and Adams, Mac C.: Damping in Pitch and Roll of Triangular Wings at Supersonic Speeds. NACA TN 1566, 1948.

2. Ribner, Herbert S., and Malvestuto, Frank S., Jr.: Stability Derivatives of Triangular Wings at Supersonic Speeds. NACA TN 1572, 1948.

3. Jones, Arthur L., and Alksne, Alberta: The Damping Due to Roll of Triangular, Trapezoidal, and Related Plan Forms in Supersonic Flow. NACA TN 1548, 1948.

4. Harmon, Sidney M.: Stability Derivatives of Thin Rectangular Wings at Supersonic Speeds. Wing Diagonals Ahead of Tip Mach Lines. NACA TN 1706, 1948.

5. Malvestuto, Frank S., Jr., and Margolis, Kenneth: Theoretical Stability Derivatives of Thin Sweptback Wings Tapered to a Point With Sweptback or Sweptforward Trailing Edges for a Limited Range of Supersonic Speeds. NACA TN 1761, 1949.

6. Malvestuto, Frank S., Jr., Margolis, Kenneth, and Ribner, Herbert S.: Theoretical Lift and Damping in Roll of Thin Sweptback Wings at Arbitrary Taper and Sweep at Supersonic Trailing Edges. NACA TN 1860, 1949.

7. Moeckel, W. E., and Evvard, J. C.: Load Distributions Due to Steady Roll and Pitch for Thin Wings at Supersonic Speeds. NACA TN 1689, 1948.

8. Cohen, Doris: The Theoretical Lift of Flat Swept-Back Wings at Supersonic Speeds. NACA TN 1555, 1948.

9. Cohen, Doris: Theoretical Loading at Supersonic Speeds of Flat Swept-Back Wings with Interacting Trailing and Leading Edges. NACA TN 1991, 1949.

TABLE I.— CALCULATED DAMPING DERIVATIVE FOR
UNTAPERED WING SHOWN IN FIGURE 10

Quantity	Magnitude	Percent of total
C_{l_p} (basic uncorrected value)	−0.2978	171.0
$\Delta C_{l_p}{}'$ (T.E. conical)	.0235	−13.5
$\Delta C_{l_p}{}'$ (T.E. quasi-conical)	.0010	−.6
$\Delta C_{l_p}{}''$ (tip conical)	.1076	−61.8
$\Delta C_{l_p}{}''$ (tip quasi-conical)	.0042	−2.4
Estimated secondary corrections	−.0127	7.3
C_{l_p} (corrected)	−.1742	100.0

TABLE II.— CALCULATED DAMPING DERIVATIVE FOR
TAPERED WING SHOWN IN FIGURE 15

M	C_{l_p} (basic)	C_{l_p} (corrected)	$\Delta C_{l_p}{}'$ (T.E. conical)		$\Delta C_{l_p}{}''$ (tip conical)	
			Magnitude	Percent of total	Magnitude	Percent of total
1.2	−0.3238	−0.2521	0.0413	−16.4	0.0304	−12.1
1.3	−.3196	−.2614	.0308	−11.8	.0274	−10.5
1.4	−.3154	−.2690	.0213	−7.9	.0251	−9.3
1.5	−.3112	−.2761	.0115	−4.2	.0237	−8.6
1.6	−.3070	−.2828	.0020	−.7	.0222	−7.8
1.7	−.3028	−.2817	0	0	.0211	−7.5

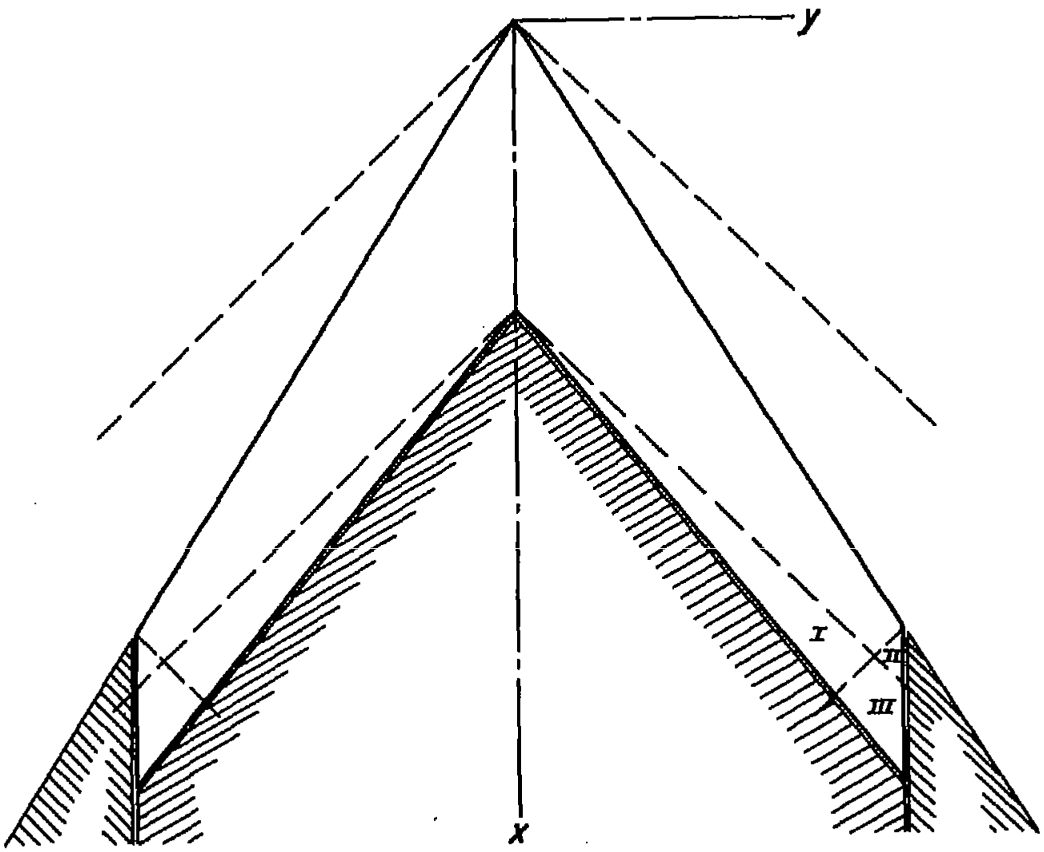


Figure 1.— Regions of pressure (shaded areas) to be canceled on infinite triangular wing in calculation of pressure distribution on swept-back wing, and regions (I, II, and III) of swept-back wing affected by cancellation process.

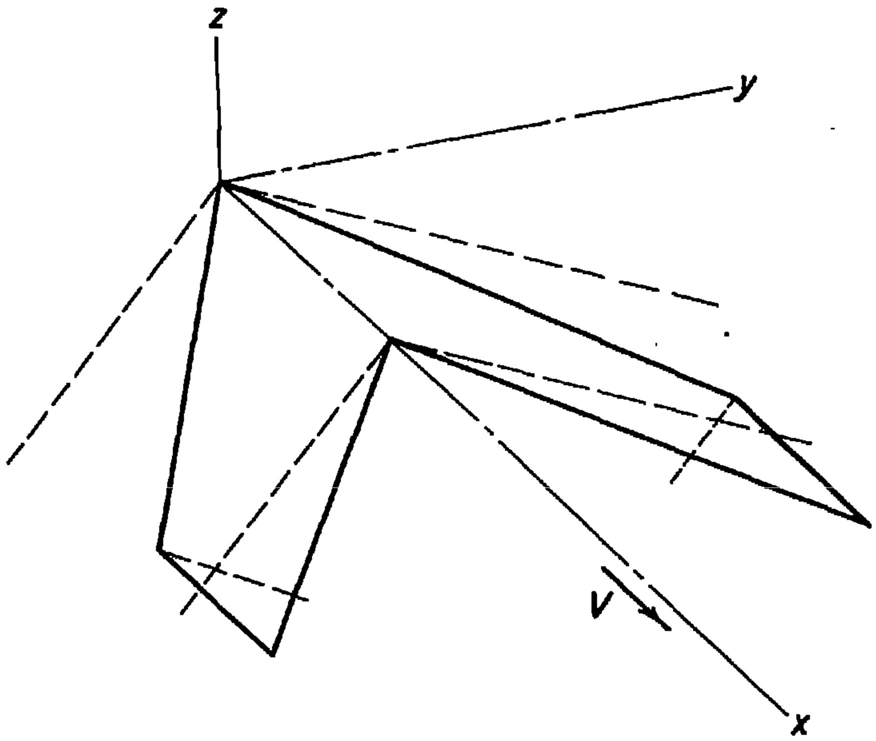


Figure 2.– Coordinate system and Mach line configuration.

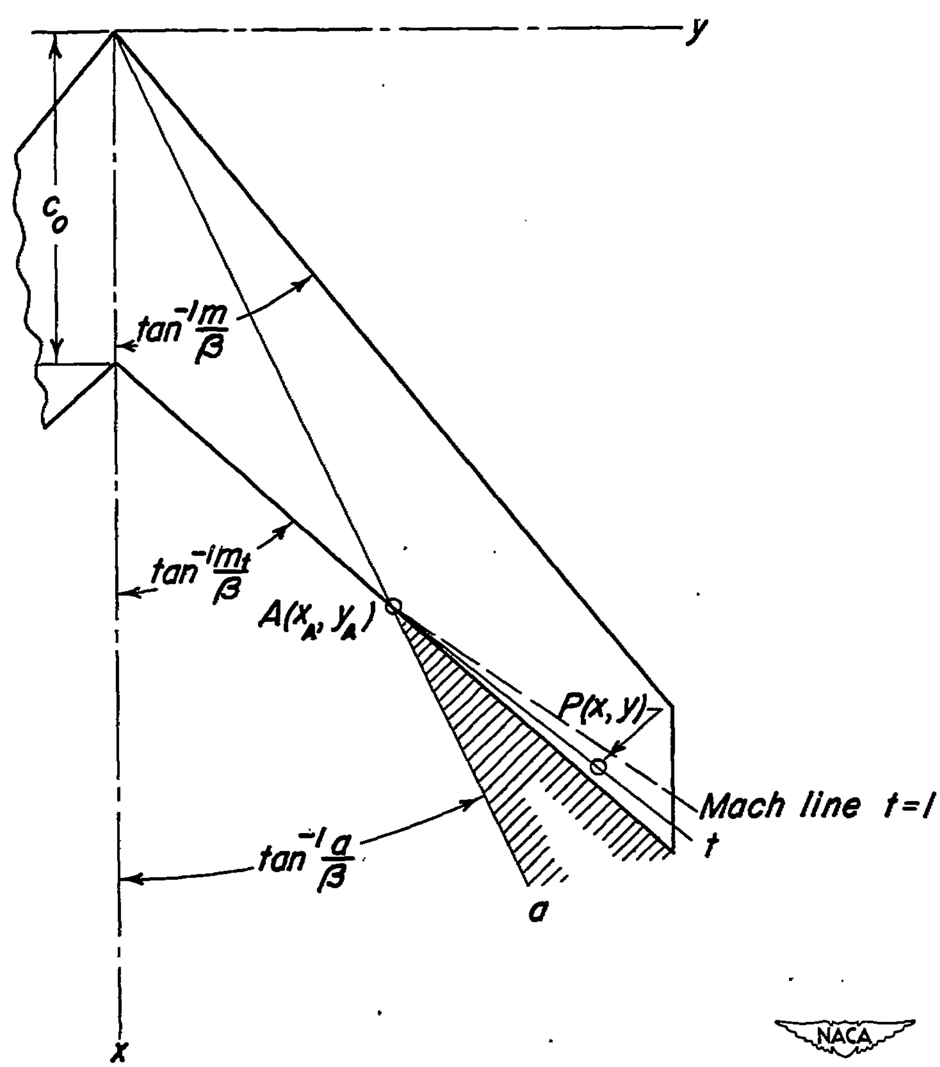


Figure 3.- A sector of infinitesimal canceling pressure, corresponding to a ray a, to be superposed on field of excess pressure in wake of wing in steady lift.

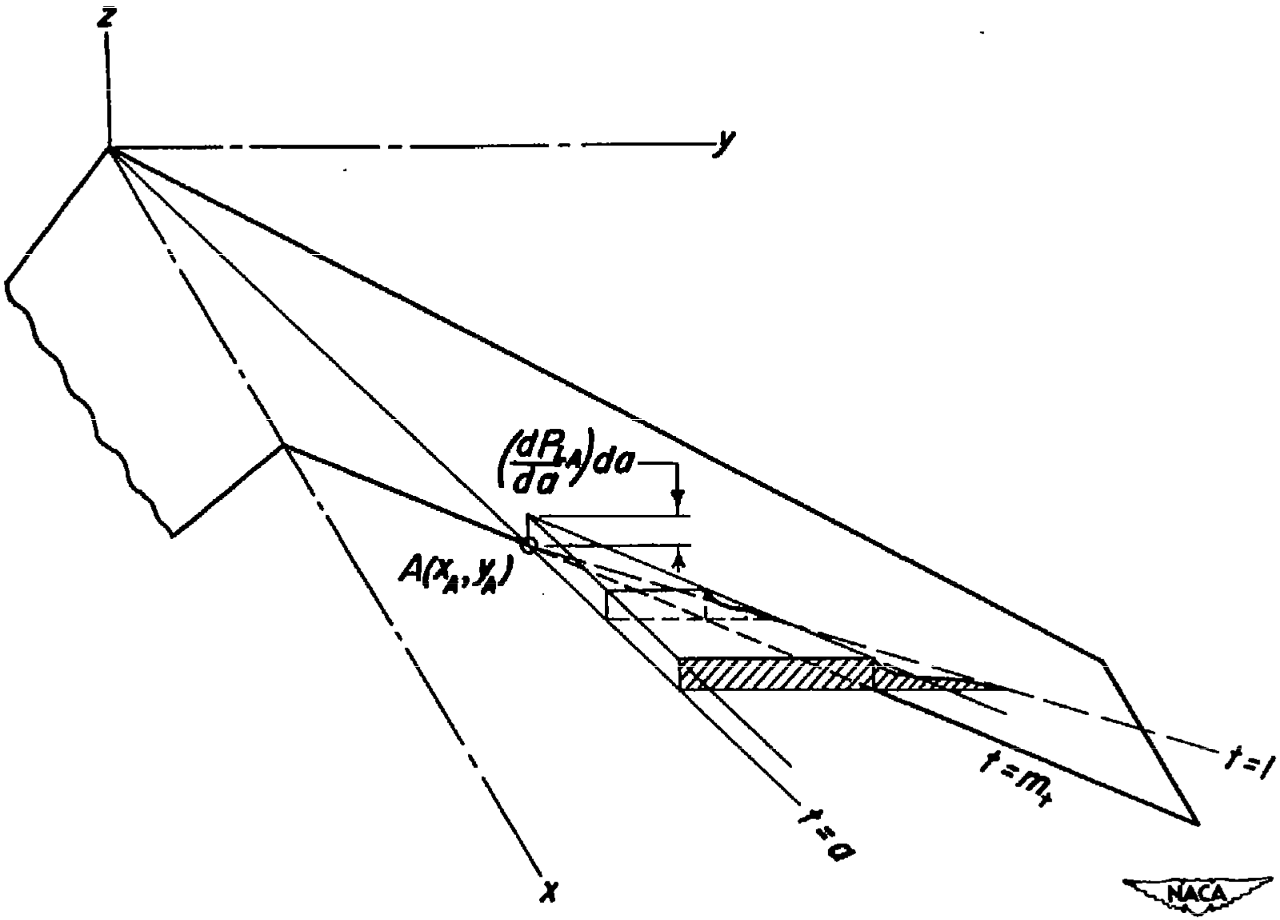


Figure 4.- One sector of field of conical pressure used to cancel basic pressure in wake of wing in steady lift.

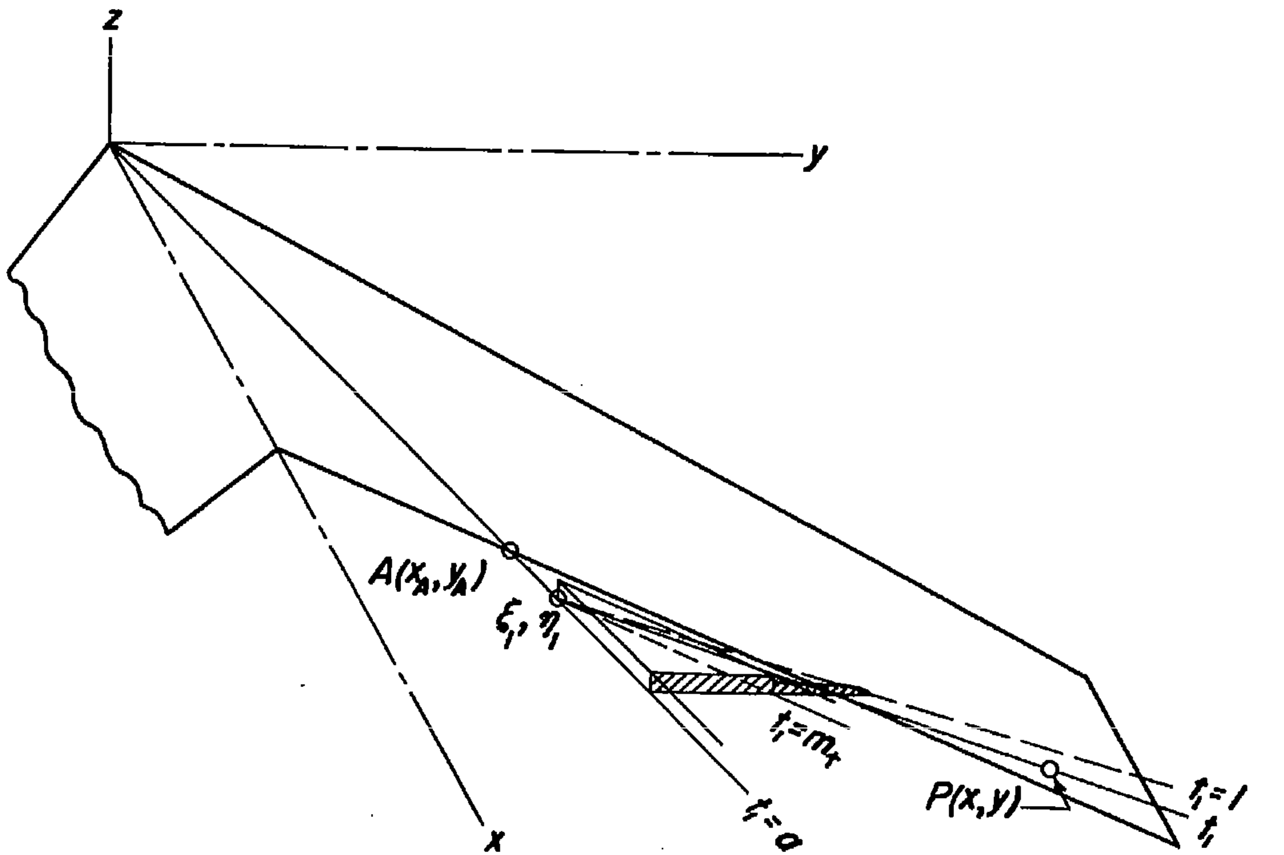


Figure 5.- One sector of series of fields of conical pressure to be superposed along ray a in derivation of quasi-conical pressure field.

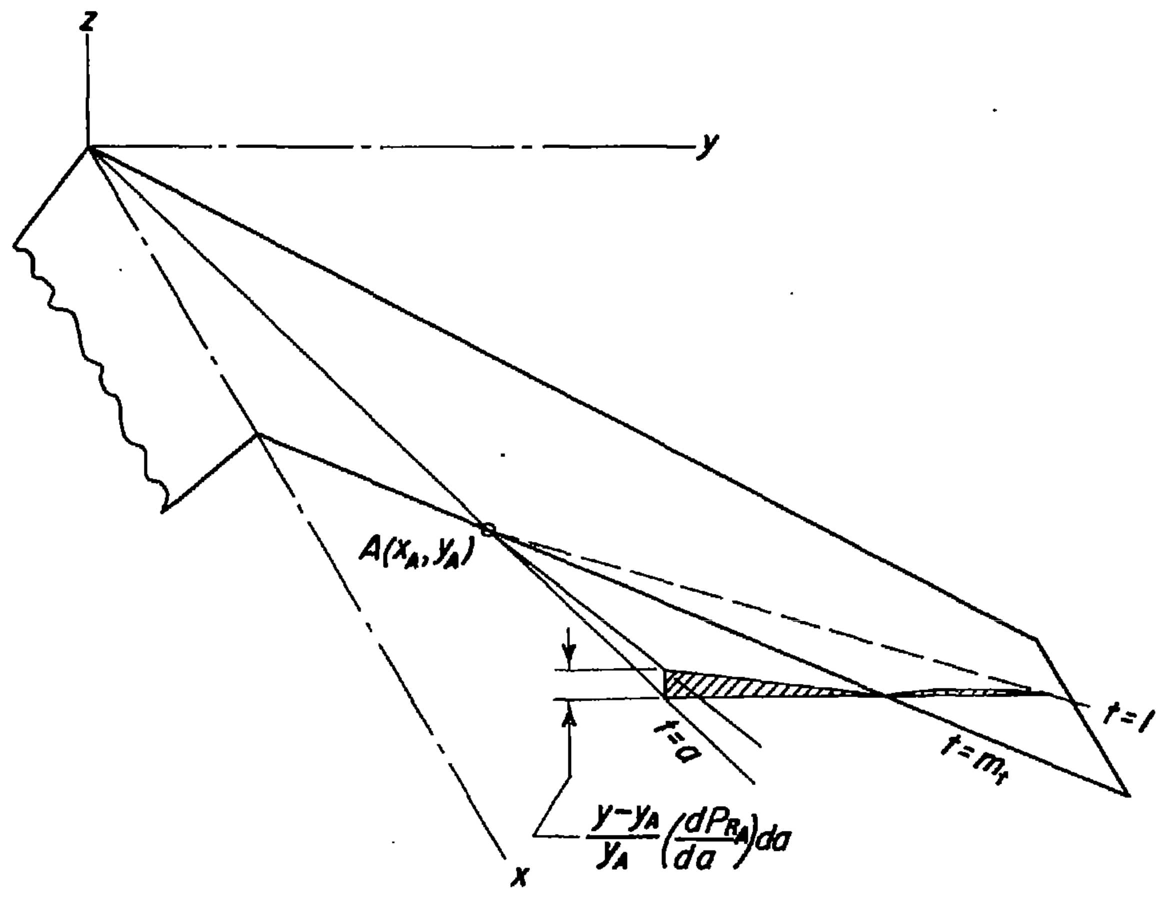


Figure 6.- One sector of field of quasi-conical pressure used to cancel pressure in wake of wing in steady roll.

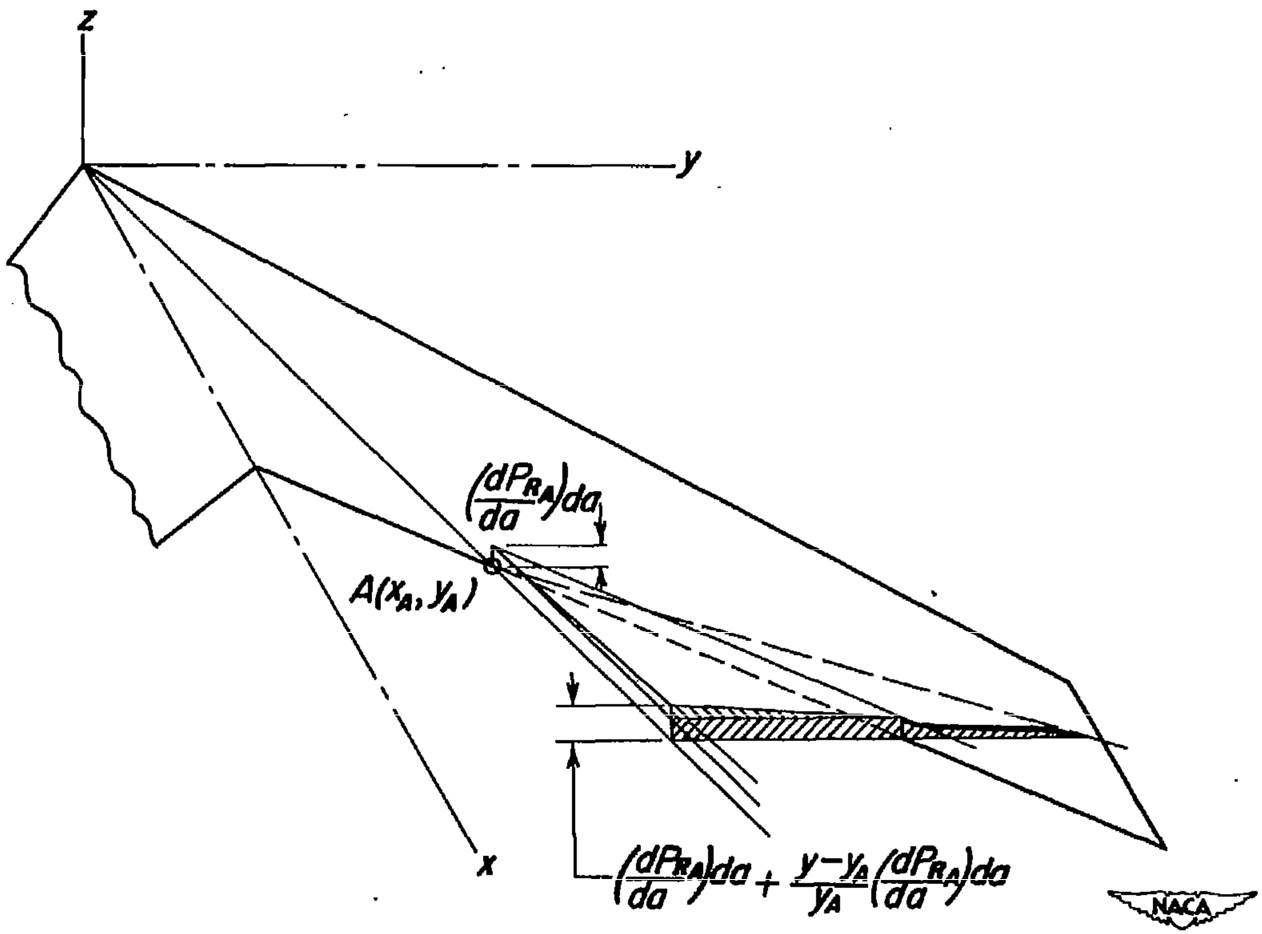


Figure 7.- One sector of combined fields of conical and quasi-conical pressure used to cancel basic pressure in wake of wing in steady roll.

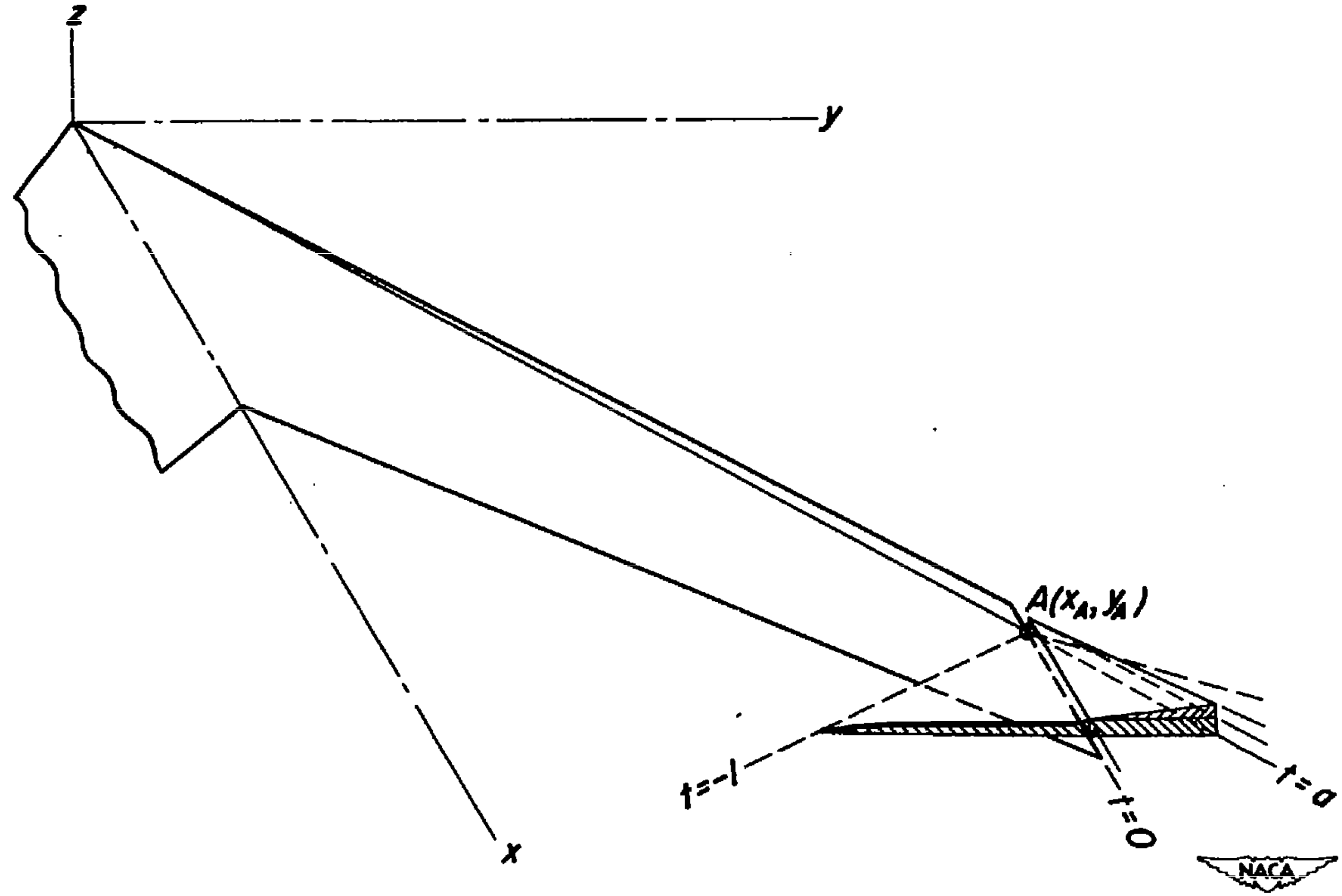
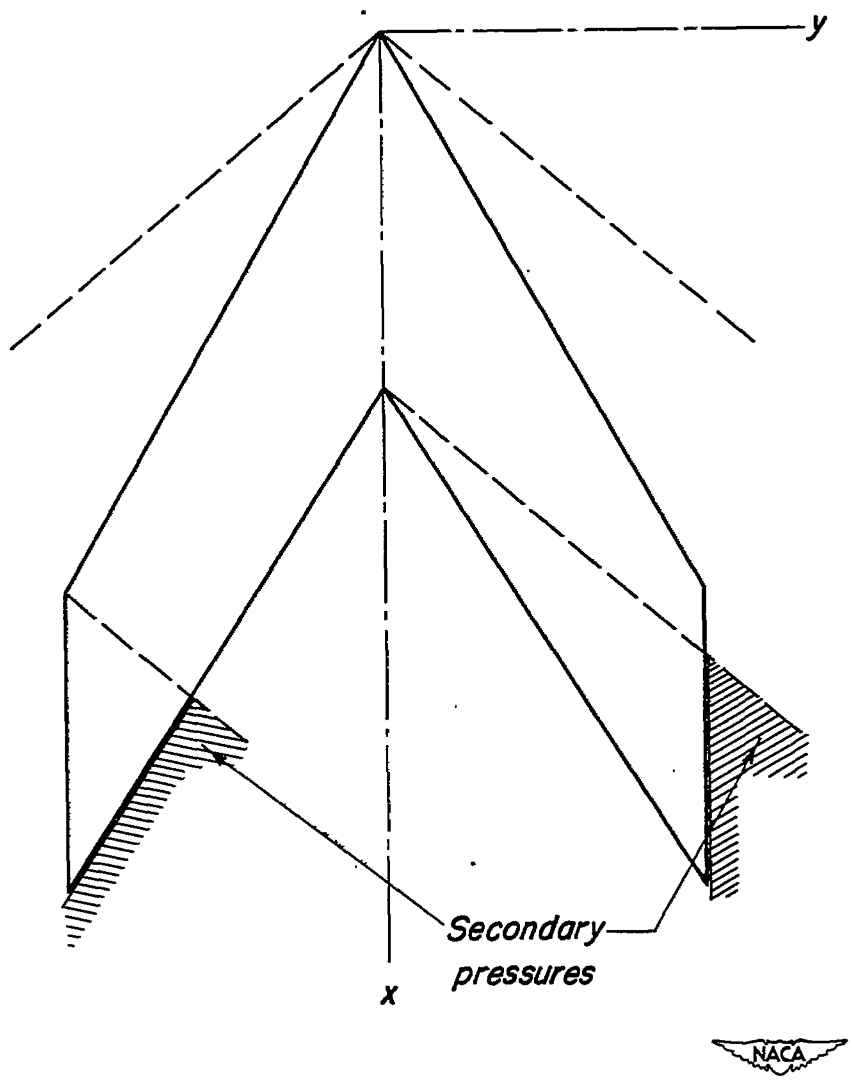
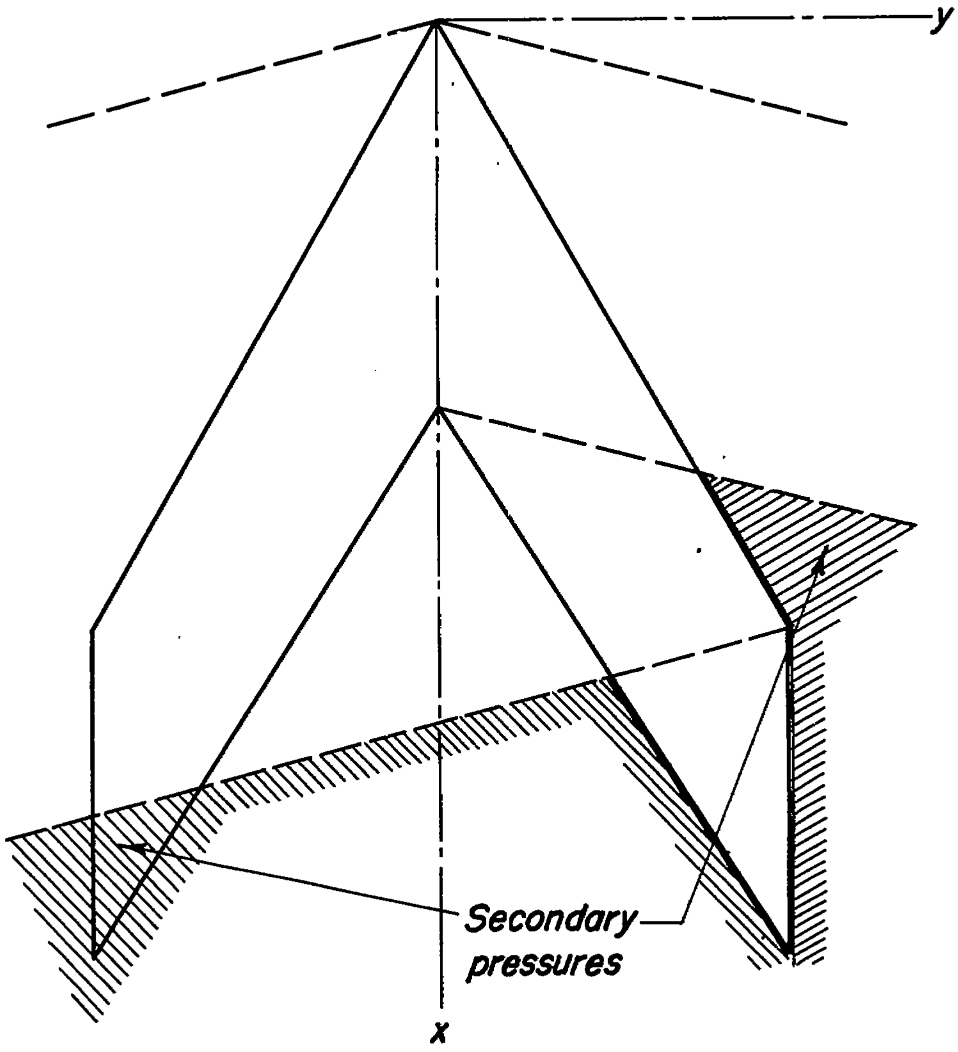


Figure 8.– One sector of combined fields of conical and quasi-conical pressure used to cancel basic pressure outboard of tip of wing in steady roll.



(a) At small sweep angles or at high Mach numbers.

Figure 9.– Regions of secondary pressure (shaded areas) introduced in wake and outboard of tips as result of canceling basic pressure.



(b) At large sweep angles or at low Mach numbers.

Figure 9.— Concluded.

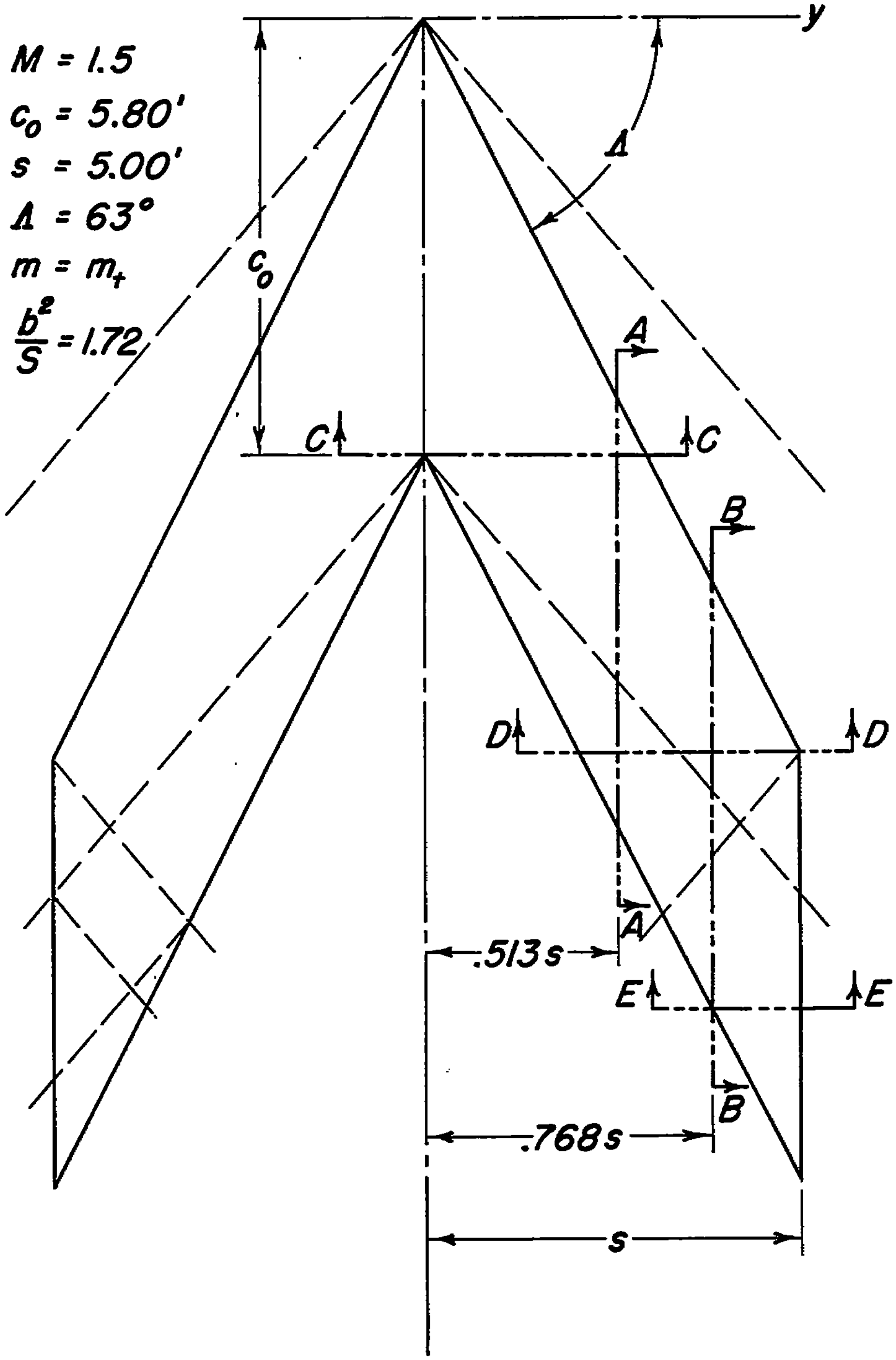
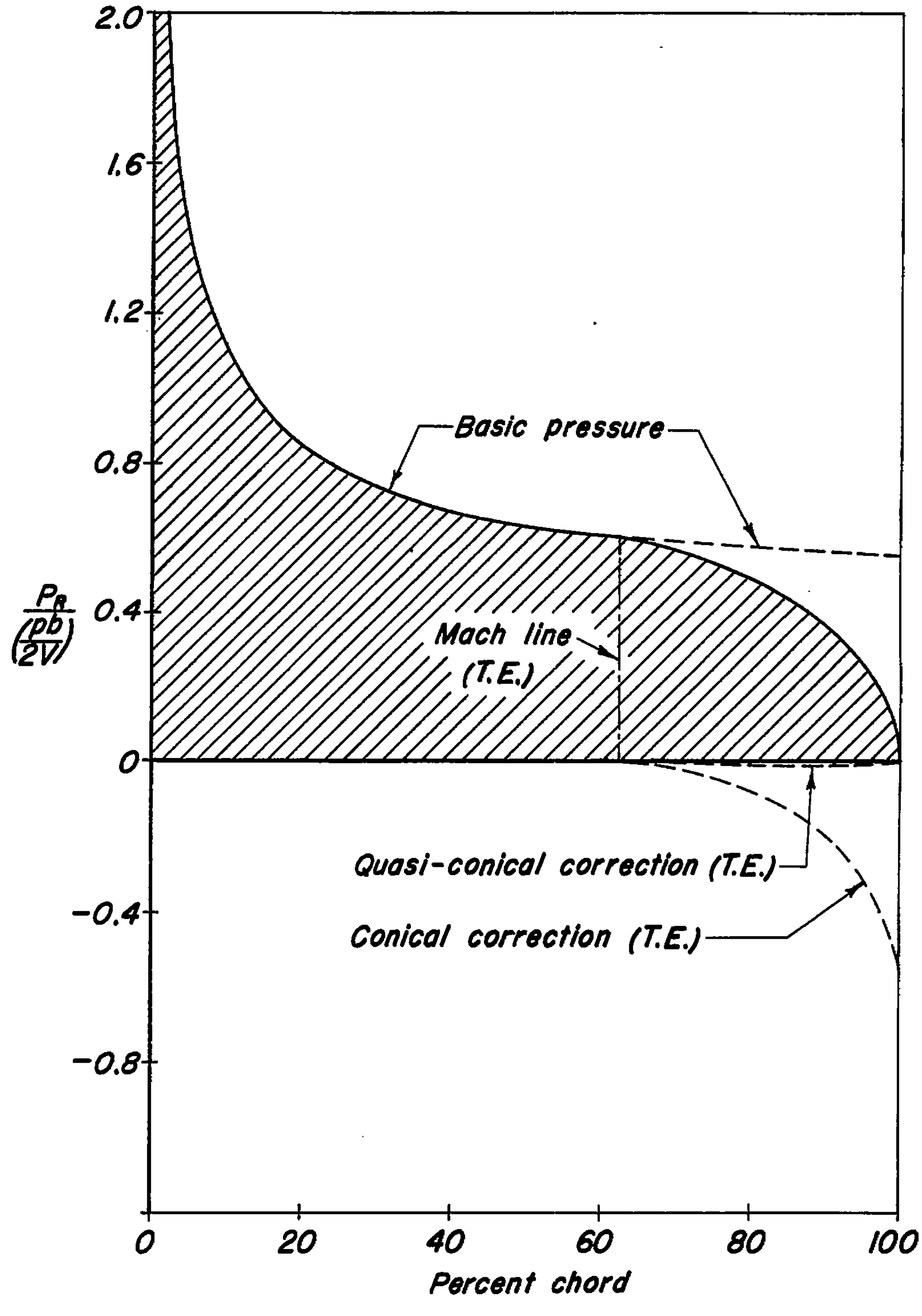
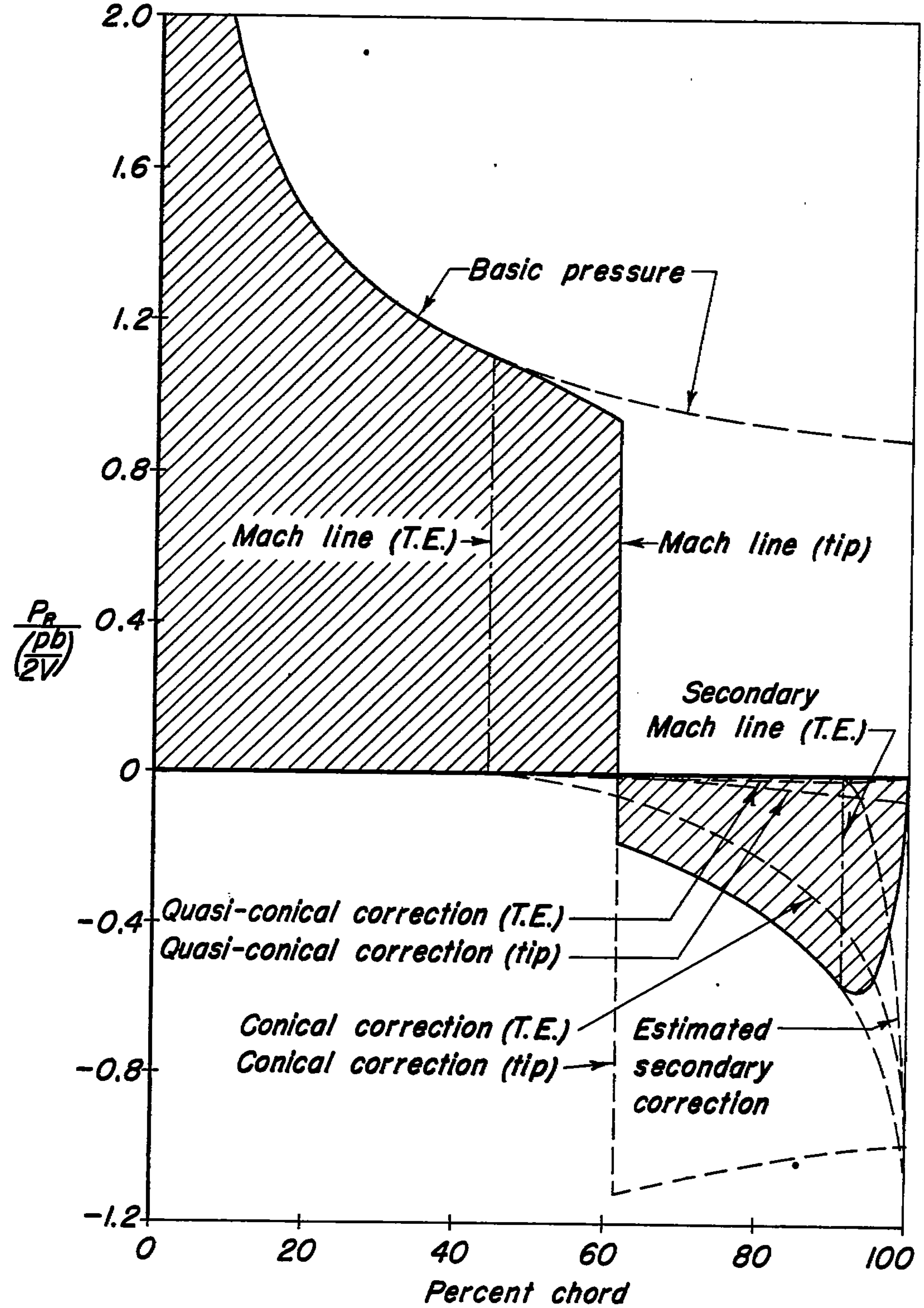


Figure 10.— Illustrative untapered wing plan form and Mach line configuration.



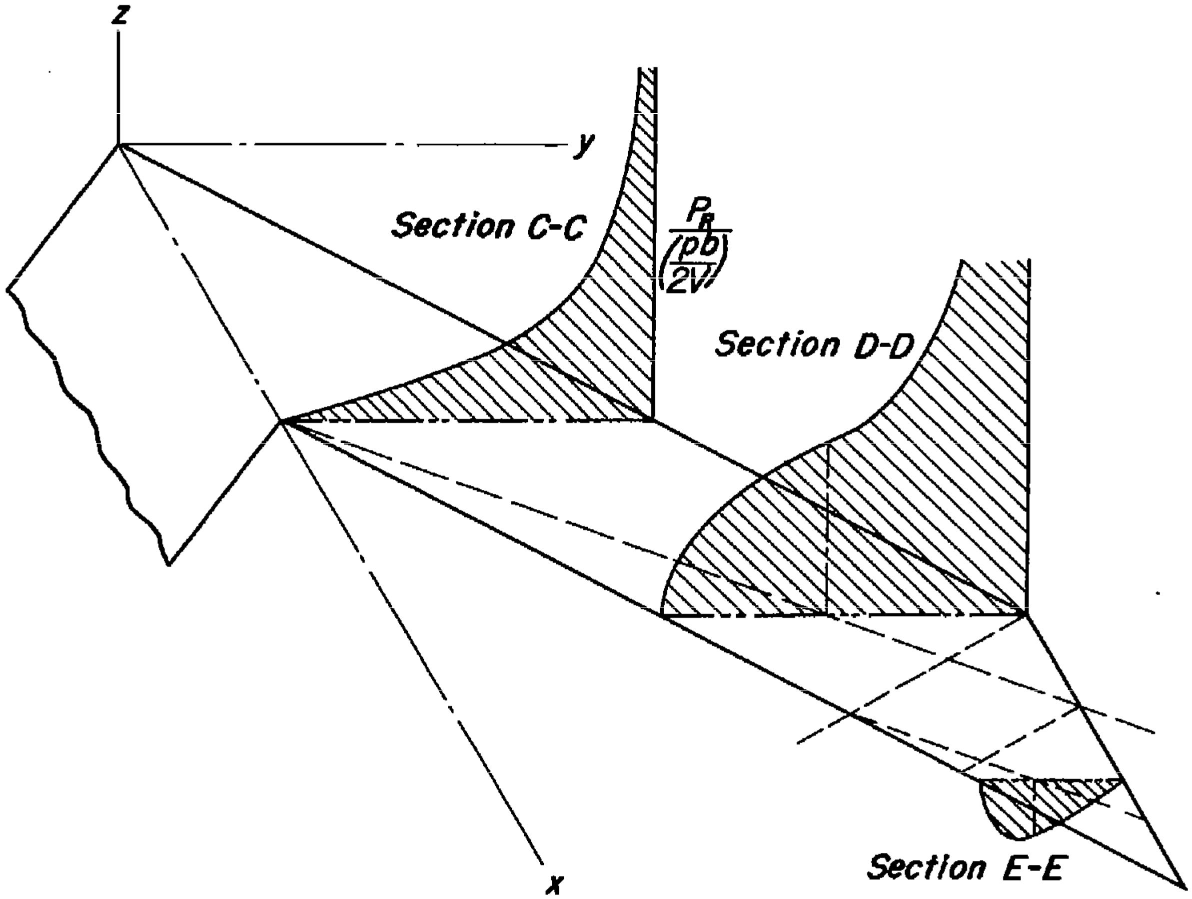
(a) Section A-A.

Figure 11.- Chordwise and spanwise pressure distributions of illustrative untapered plan form (fig. 10) in steady roll.



(b) Section B-B.

Figure 11.— Continued.



(c) Sections C-C, D-D, E-E.

Figure 11.—Concluded.

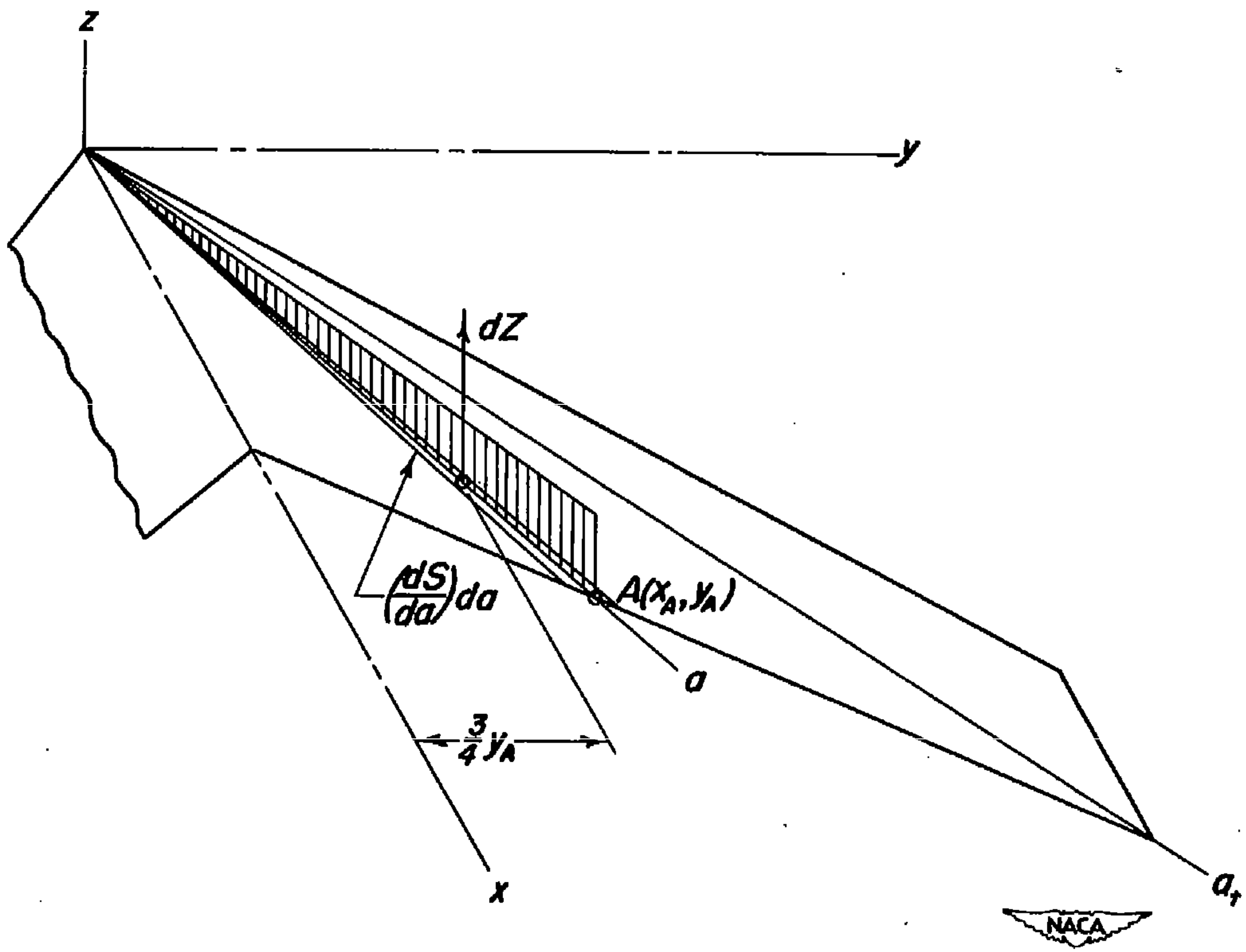


Figure 12.–Force due to basic pressure on element of wing area.

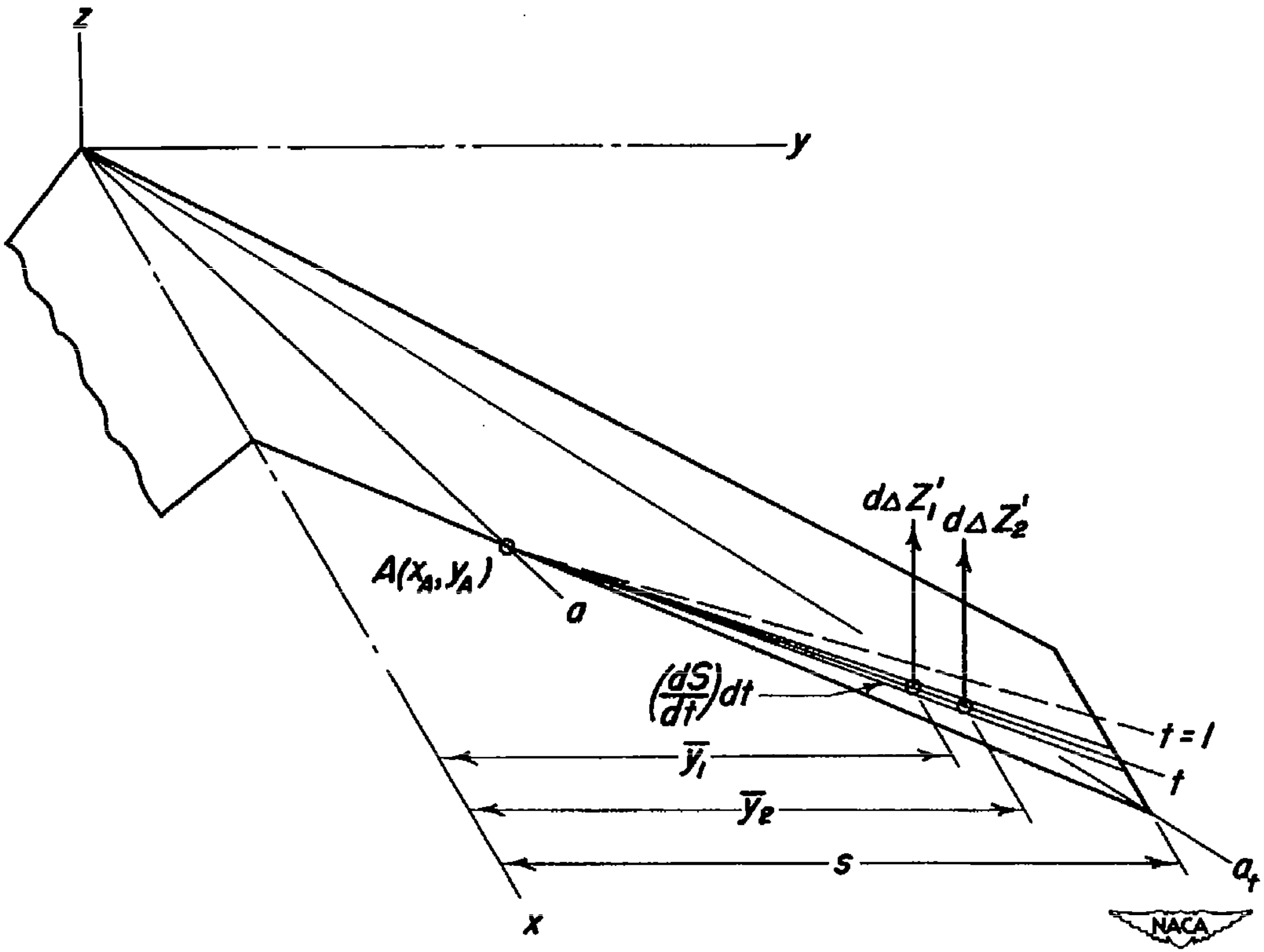


Figure 13.– Forces on element of wing area corresponding to conical and quasi-conical components of single sector of canceling pressure in wake.

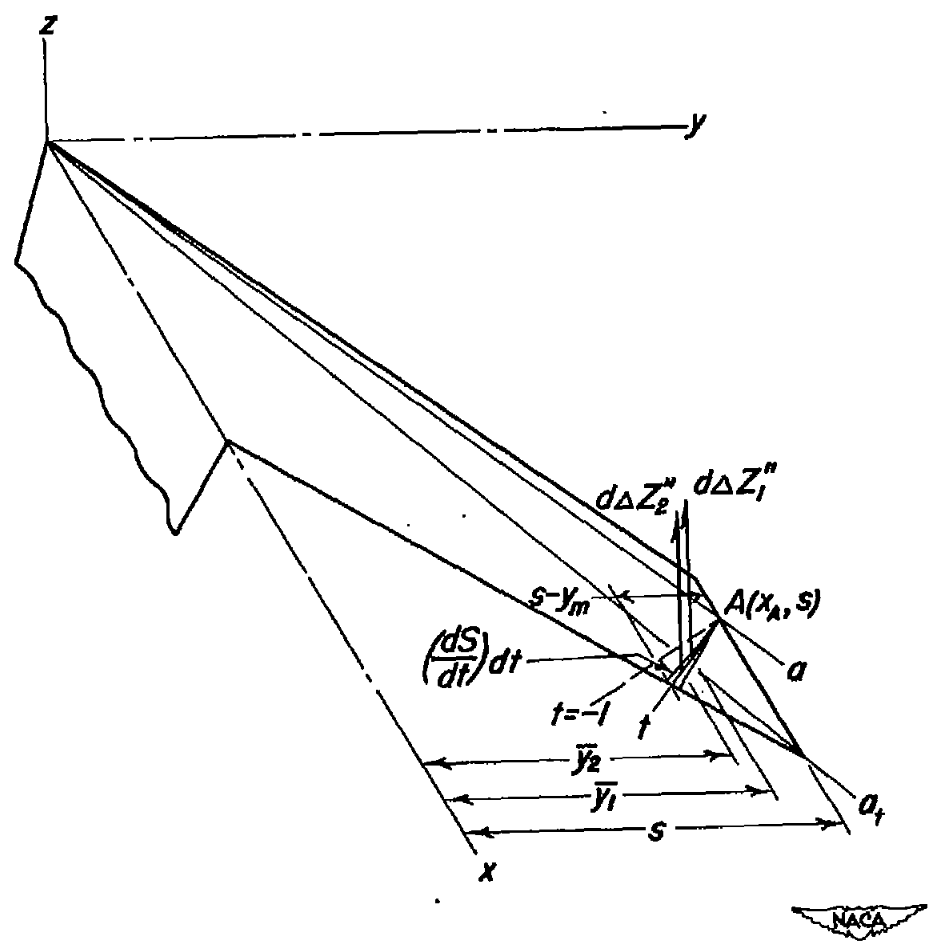


Figure 14.–Forces on element of wing area corresponding to conical and quasi-conical components of single sector of canceling pressure outboard of tip.

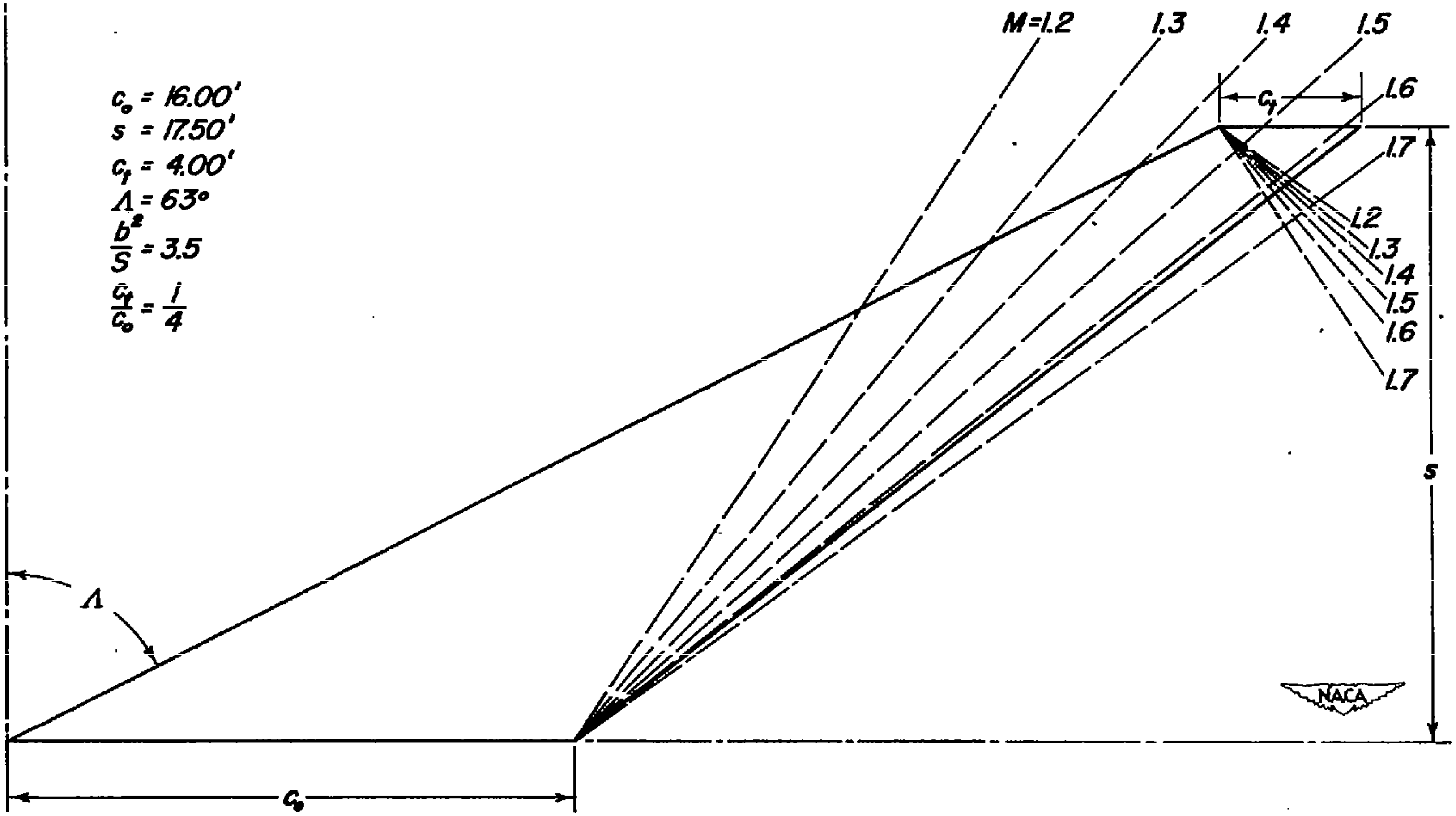


Figure 15.– Illustrative tapered wing plan form and Mach line configuration.

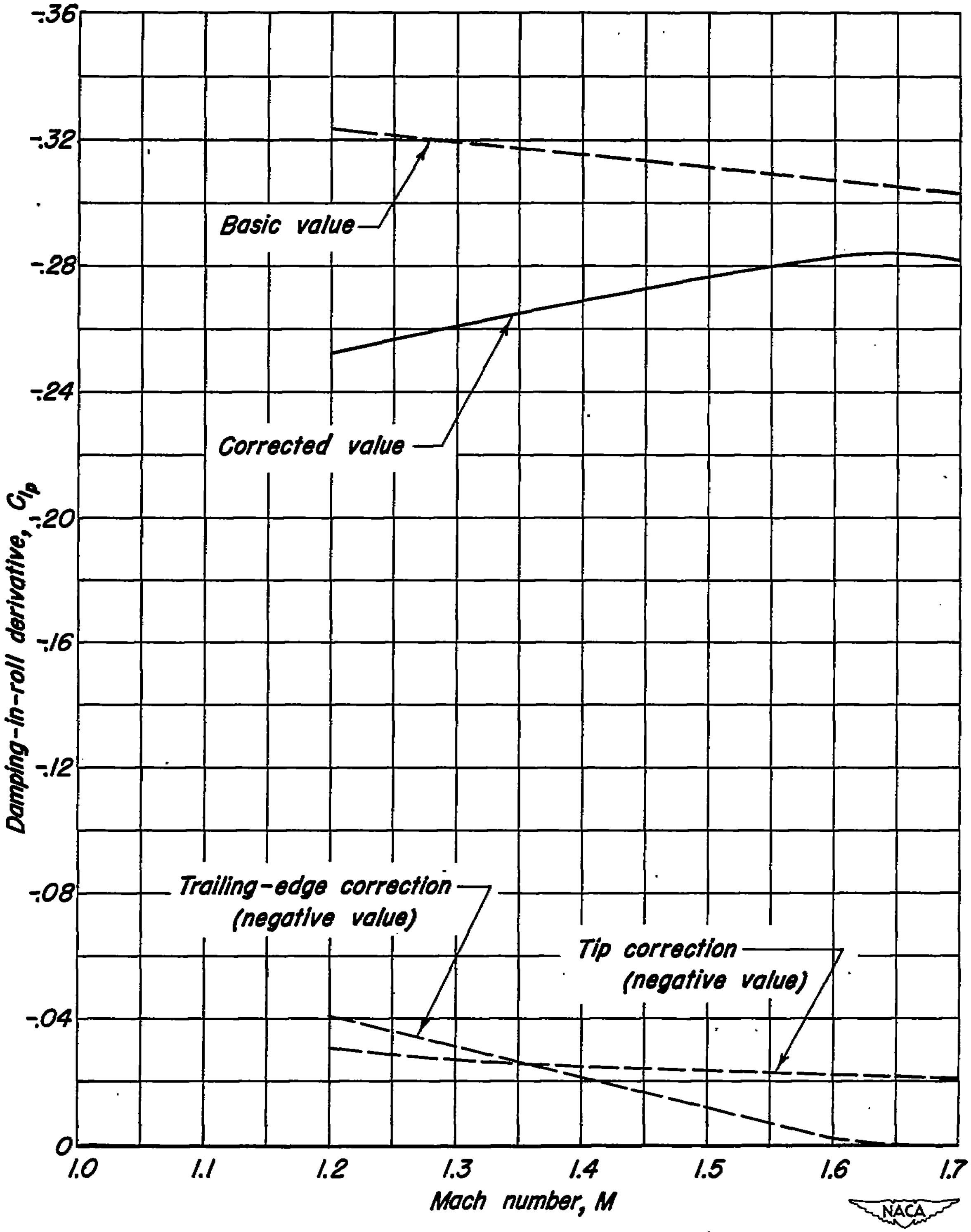
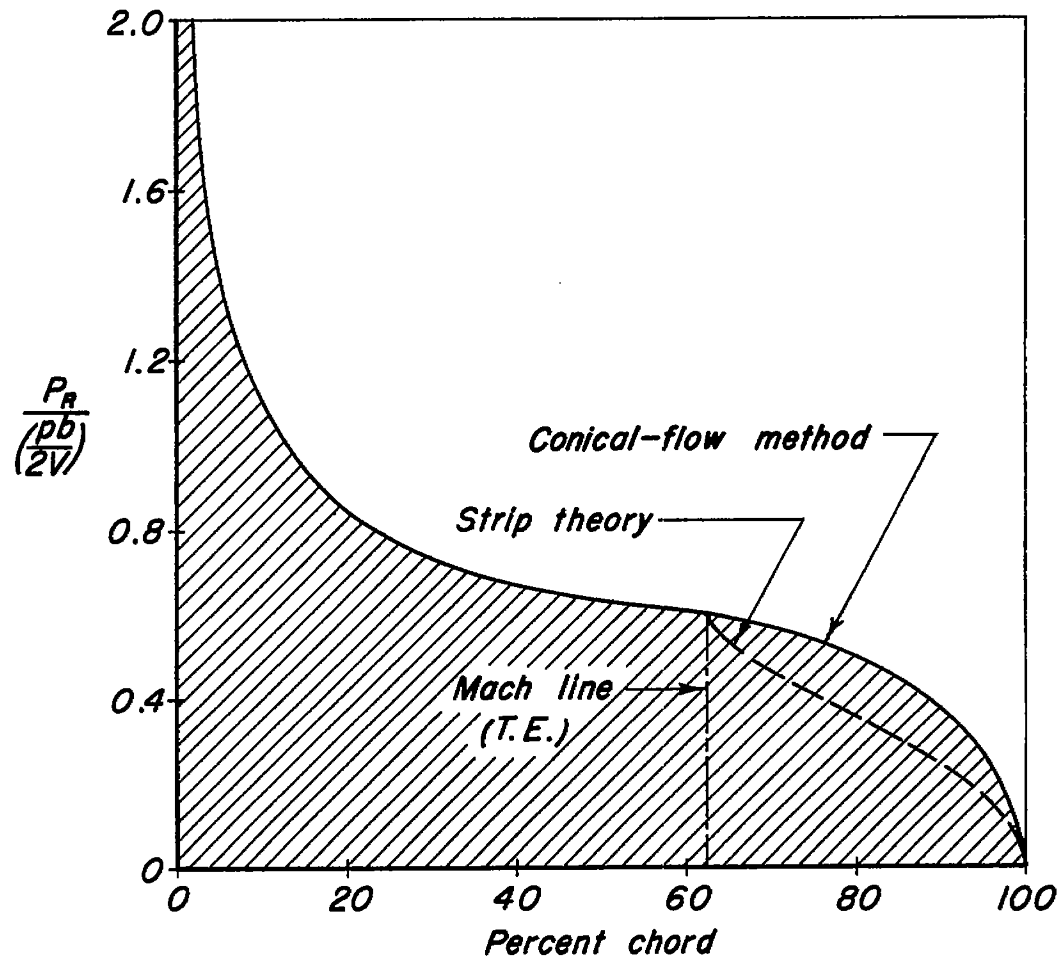
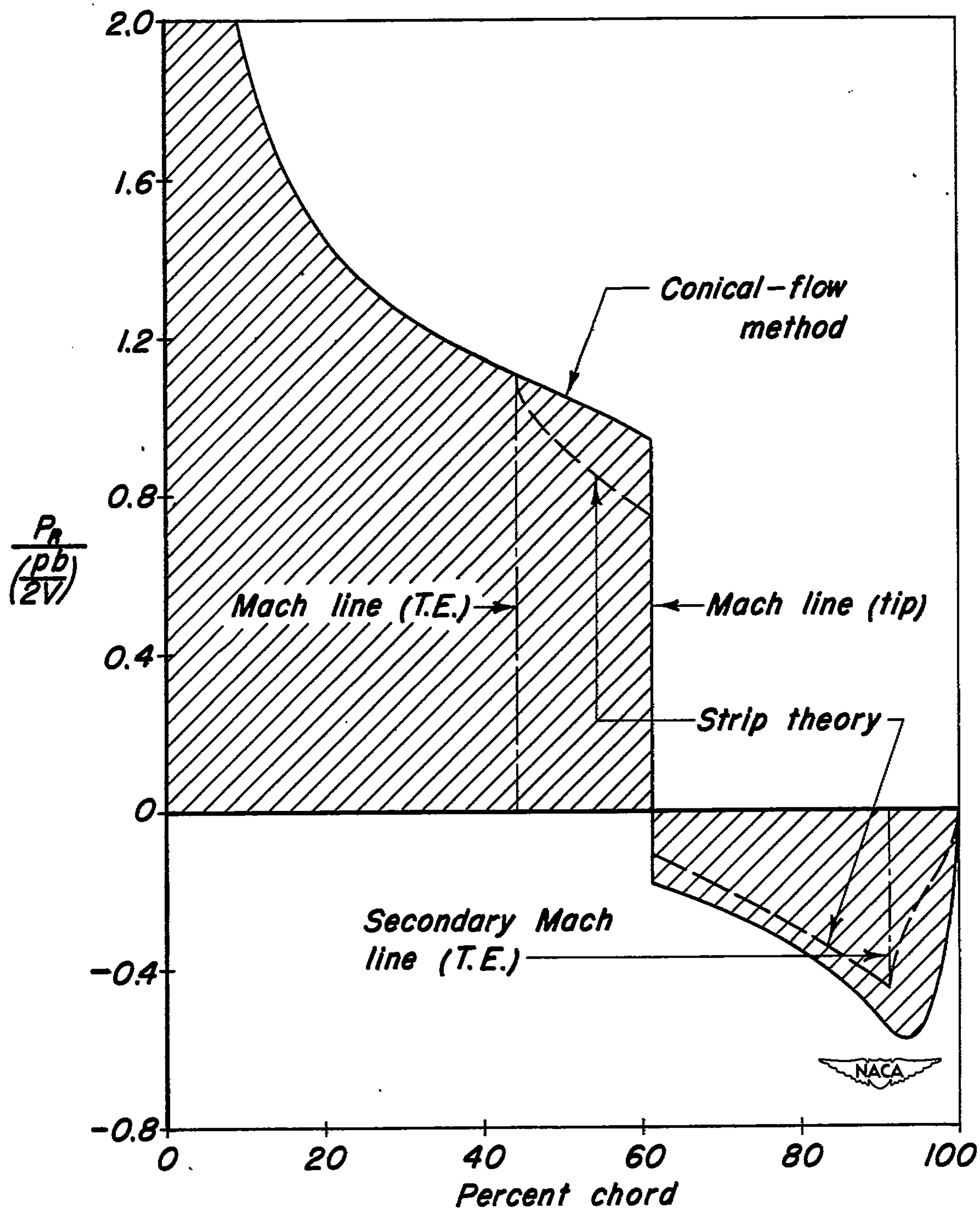


Figure 16.- Variation with Mach number of components of damping-in-roll derivative of illustrative tapered wing (fig. 15).



(a) Section A-A.

Figure 17.- Comparison of chordwise pressure distributions on illustrative untapered wing as calculated by means of strip theory and of conical-flow method.



(b) Section B-B.

Figure 17.— Concluded.