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EXTENSION OF THE THEORY OF OSCILLATING AIRFOILS OF FINITE SPAN IN SUBSONIC COMPRESSIBLE FLOW

By Eric Reissner

Massachusetts Institute of Technology

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SUMMARY

As part of an investigation of flows around and forces on an oscillating finite wing in subsonic compressible flow, the exact double integrals occurring in the theoretical formulation of the three-dimensional nonstationary aerodynamic forces have previously been reduced to single integrals over the range of either one of the two independent variables. The present report describes a method for solving the resultant three-dimensional problem based on a generalization of approximate methods for solving two-dimensional problems. It is shown that the calculation of three-dimensional corrections to the two-dimensional theory involves only the solution of a one-dimensional integral equation for the spanwise variation of circulation, provided tabulated values for the kernel of the integral equation are available.

INTRODUCTION

The present report is a sequel to an earlier report (reference 1) on the problem of the oscillating lifting surface of finite span in subsonic compressible flow. In the earlier report the exact double-integral equation of the problem had been reduced to an approximate integral equation in such a way that the double integrals are reduced to single integrals over the range of either one of the two independent variables. The present report describes a method of solution of the resultant three-dimensional problem. The proposed method is a generalization of any one of the known approximate methods for the solution of the two-dimensional problem. It is shown that the calculation of three-dimensional corrections to the two-dimensional theory involves, within the framework of the proposed procedure, nothing more than the solution of a one-dimensional integral equation for the spanwise variation of circulation. The solution of this one-dimensional integral equation will be no more difficult than the corresponding solution for the problem of incompressible flow, once the kernel of the integral equation has been tabulated for an appropriate range of parameters. Determination of the numerical values of this kernel function is not a straightforward matter because of the particular analytical form in which this function appears. Applications of the present theory depend on the execution of this numerical task.

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SYMBOLS

g^*,λ^*	quantities defined as $g^*(x^*,y^*) = g(x,y)$; $\lambda^*(x^*,y^*) = \lambda(x,y)$
x^*	dimensionless coordinate $\left((x - x_m^*)/b^*\right)$
y^*	dimensionless spanwise coordinate $\left(y/s\sqrt{1 - M^2}\right)$
ξ^*	dimensionless coordinate
G,K	kernels of integral equation (1)
$\nu^* = b^*\nu$	
x_m^*	quantity indicating amount of sweep
Ω^*	quantity defined by equation (2)
$\eta^* = \eta/s\sqrt{1 - M^2}$	
b^*	local semichord divided by semichord b at midspan
x,y,z	dimensionless coordinates
k	reduced-frequency parameter
Ω	circulation function
g	function defined by equation (8)
λ	function defined by equation (6)
ξ,η	variables of integration
$\nu = \dfrac{k}{1 - M^2}$	
$\bar{p}_a$	pressure amplitude at lifting surface
ψ	modified potential function

$\psi_a = \psi(x,y,0)$

ρ_o	density of stream flowing with velocity U
U	main-stream velocity
b	semichord at midspan

$\mu = \frac{kM^2}{1 - M^2}$

H	instantaneous shape of lifting surface
M	Mach number of main stream
α	arbitrary constant
ζ	auxiliary variable of integration
P	quantity defined by equation (17)
Q,w_i	functions defined by equations (19) and (20)
S_M	function defined by equation (22)
s	ratio of semispan to semichord at midspan
$H_0^{(2)},H_1^{(2)}$	Hankel functions of second kind and of zeroth and first order, respectively
F_M	function defined by equation (23)
σ,τ	auxiliary variables of integration
$\Delta\overline{p}_a$	component of pressure distribution $\overline{p}_a$ representing solution of special two-dimensional problem
$\overline{p}_a^{(2)}$	component of pressure distribution $\overline{p}_a$ representing solution of problem when three-dimensional effects are neglected
$P^{(2)}$	quantity defined by equation (28)
B	quantity defined by equation (A4)
I,I_1,I_2	functions defined by equations (A5)

a	arbitrary constant
J_0, Y_0	Bessel functions of zero order
F_1, F_2	real and imaginary parts, respectively, of function F_M defined by equations (A12)

FORMULATION OF PROBLEM

The problem which is considered in what follows consists in the solution of an integral equation of the following form:

$$g^*(x^*,y^*) = -\oint_{-1}^{1} \lambda^*(\xi^*,y^*)G(x^* - \xi^*)\,d\xi^* +$$

$$i\nu^* e^{-i\nu^* x_m^*}\,\Omega^*(y^*)\int_{1}^{\infty} e^{-i\nu^*\xi^*}\,G(x^* - \xi^*)\,d\xi^* +$$

$$e^{-i\nu^*(x^*+x_m^*)}\oint_{-1}^{1} \frac{d\Omega^*}{d\eta^*}\,K(y^* - \eta^*)\,d\eta^* \qquad (1)$$

The form of the functions G and K follows from a comparison of the foregoing equation with equation (88) of reference 1. Equation (1) is to be solved for λ^*, in terms of g^* and Ω^*, and then use is to be made of the définition of Ω^* in terms of λ^*; namely,

$$\Omega^* = e^{i\nu}\int_{-1}^{1} \lambda^*\,dx^* \qquad (2)$$

Attention shall here be restricted to the problem of the rectangula lifting surface for which

$$\left.\begin{array}{ll} b^* = 1 & \nu^* = \nu \\ x^* = x & \Omega^* = \Omega \\ x_m^* = 0 & g^* = g \end{array}\right\} \qquad (3)$$

Equation (1) may then be rewritten in the form

$$g(x,y^*) = -\oint_{-1}^{1} \lambda(\xi,y^*)G(x - \xi)\ d\xi +$$

$$i\nu\Omega(y^*)\int_{1}^{\infty} e^{-i\nu\xi}\ G(x - \xi)\ d\xi +$$

$$e^{-i\nu x}\oint_{-1}^{1} \frac{d\Omega}{d\eta^*} K(y^* - \eta^*)\ d\eta^* \tag{4}$$

$$\Omega(y^*) = e^{i\nu}\int_{-1}^{1} \lambda(x,y^*)\ dx. \tag{5}$$

It is now recalled that the function λ is related to the pressure amplitude $\overline{p}_a$ at the lifting surface by means of the equations

$$\lambda = 2\,\frac{\partial\psi(x,y,0)}{\partial x} = 2\,\frac{\partial\psi_a}{\partial x} \tag{6}$$

and

$$\overline{p}_a = -\frac{\rho_o U}{b}\, e^{i\mu x}\left(i\nu\psi_a + \frac{\partial\psi_a}{\partial x}\right) \tag{7}$$

and that the function g is determined in terms of the instantaneous shape H of the lifting surface by the relation

$$g = \frac{Ue^{-i\mu x}}{\sqrt{1 - M^2}}\left(ik\overline{H} + \frac{\partial\overline{H}}{\partial x}\right) \tag{8}$$

Note also that

$$\left.\begin{aligned} \nu &= \frac{k}{1 - M^2} \\ \mu &= \frac{kM^2}{1 - M^2} \\ \nu - \mu &= k \end{aligned}\right\} \tag{9}$$

The proposed procedure for the solution of equations (4) and (5) depends on the introduction of the function $\overline{p}_a$ instead of λ into the integral equation, that is, on the reduction of this equation, when $d\Omega/d\eta^* = 0$, to Possio's form. This step will be carried out in the next section.

INTEGRAL EQUATION FOR PRESSURE DISTRIBUTION AT AIRFOIL

Combination of equations (7) and (9) leads to the following alternate relation between ψ_a and $\overline{p}_a$:

$$\psi_a(x,y^*) = -\frac{b}{\rho_o U} e^{-i\nu x}\int_{-1}^{x} \overline{p}_a(x',y^*)e^{ikx'}\, dx' \qquad (10)$$

A corresponding expression for the circulation function Ω is obtained as follows:

$$\Omega = 2e^{i\nu}\int_{-1}^{1} \frac{\partial\psi_a}{\partial x}\, dx = 2e^{i\nu}\, \psi_a(1,y^*)$$

$$= -\frac{2b}{\rho_o U}\int_{-1}^{1} \overline{p}_a(x',y^*)e^{ikx'}\, dx' \qquad (11)$$

From equations (6) and (10) it follows that

$$\lambda = -\frac{2b}{\rho_o U}\left[e^{i(k-\nu)x}\, \overline{p}_a - i\nu e^{-i\nu x}\int_{-1}^{x} \overline{p}_a e^{ikx'}\, dx'\right] \qquad (12)$$

Equations (11) and (12) are introduced into integral equation (4). There occurs one term which must be transformed by integration by parts, namely, the term

$$\oint_{-1}^{1}\left[e^{-i\nu\xi}\,G(x-\xi)\int_{-1}^{\xi}\overline{p}_a e^{ikx'}\,dx'\right]d\xi =$$

$$\left[\int_{\alpha}^{\xi}e^{-i\nu\xi'}\,G(x-\xi')\,d\xi'\int_{-1}^{\xi}\overline{p}_a e^{ikx'}\,dx'\right]_{-1}^{1} -$$

$$\oint_{-1}^{1}\left\{\left[\int_{\alpha}^{\xi}e^{-i\nu\xi'}\,G(x-\xi')\,d\xi'\right]\overline{p}_a(\xi,y^*)e^{ik\xi}\right\}d\xi \tag{13}$$

The arbitrary constant α may be set equal to infinity. If this is done then the integrated portion of equation (13) can be used to cancel the second term in integral equation (4). Equation (4) becomes

$$g = \frac{2b}{\rho_o U}\oint_{-1}^{1}\overline{p}_a\,(\xi,y^*)\left[e^{i(k-\nu)\xi}\,G(x-\xi) + \right.$$

$$\left. i\nu\int_{\infty}^{\xi}e^{-i\nu\xi'}\,G(x-\xi')\,d\xi' e^{ik\xi}\right]d\xi +$$

$$e^{-i\nu x}\oint_{-1}^{1}\frac{d\Omega}{d\eta^*}\,K(y^*-\eta^*)\,d\eta^* \tag{14}$$

On the left-hand side of equation (14) the value of g from equation (8) is introduced. Then both sides of the resultant equation are multiplied by $(\rho_o U/2b)e^{i\mu x}$. This gives

$$\frac{\rho_o U^2}{2b\sqrt{1-M^2}}\left(ik\overline{H} + \frac{\partial\overline{H}}{\partial x}\right) = \oint_{-1}^{1}\overline{p}_a(\xi,y^*)\left[e^{i\mu(x-\xi)}\,G(x-\xi) - \right.$$

$$\left. i\nu e^{i\mu x+ik\xi}\int_{\xi}^{\infty}e^{-i\nu\xi'}\,G(x-\xi')\,d\xi'\right]d\xi +$$

$$\frac{\rho_o U}{2b}\,e^{-ikx}\oint_{-1}^{1}\frac{d\Omega}{d\eta^*}\,K(y^*-\eta^*)\,d\eta^* \tag{15}$$

Now set

$$\tilde{G}(x - \xi) = e^{i\mu(x-\xi)} G(x - \xi) - i\nu e^{ik(x-\xi)} \int_{-\infty}^{x-\xi} e^{i\nu\zeta} G(\zeta) \, d\zeta \tag{16}$$

and

$$\frac{\rho_o U}{2b} \Omega = - \int_{-1}^{1} \overline{p}_a(x,y^*) e^{ikx} \, dx = -P \tag{17}$$

This leads to the following final form of the integral equation of the problem

$$\frac{\rho_o U^2}{2b\sqrt{1 - M^2}} \left(ik\overline{H} + \frac{\partial \overline{H}}{\partial x} \right) = \oint_{-1}^{1} \overline{p}_a(\xi,y^*) \tilde{G}(x - \xi) \, d\xi - e^{-ikx} \oint_{-1}^{1} \frac{dP}{d\eta^*} K(y^* - \eta^*) \, d\eta^* \tag{18}$$

When $dP/d\eta^* \equiv 0$ this integral equation must reduce to the corresponding equation of the two-dimensional theory; that is, the kernel $\tilde{G}$ must be reducible to the corresponding kernel in Possio's formulation of the problem. It is not necessary for the present purposes to verify explicitly that this requirement is satisfied.

It is useful to introduce as further abbreviations

$$Q = \oint_{-1}^{1} \frac{dP}{d\eta^*} K(y^* - \eta^*) \, d\eta^* \tag{19}$$

$$w_i = \frac{\rho_o U^2}{2b\sqrt{1 - M^2}} \left(ik\overline{H} + \frac{\partial \overline{H}}{\partial x} \right) \tag{20}$$

Finally, before proceeding, the analytical expression for the kernel function K is listed explicitly.

$$K(y^* - \eta^*) = \frac{1}{4\pi s\sqrt{1 - M^2}} S_M\left[\frac{ks}{\sqrt{1 - M^2}}(y^* - \eta^*)\right] \tag{21}$$

where

$$S_M(z) = \frac{i\pi}{2}\frac{ksM}{\sqrt{1 - M^2}}\frac{|z|}{z}\left[H_1^{(2)}(M|z|) + \int_{-\infty}^{-M|z|} H_0^{(2)}(|\zeta|)\,d\zeta\right] - \frac{iks}{\sqrt{1 - M^2}} F_M(z) \tag{22}$$

$$F_M(z) = \frac{|z|}{z}\int_0^{\infty} e^{-i\sigma}\left[\int_{-\infty}^{-|z|} \frac{\sigma e^{-iM\sqrt{\sigma^2+\zeta^2}}}{\sigma^2 + \zeta^2}\left(\frac{1}{\sqrt{\sigma^2 + \zeta^2}} + iM\right)d\zeta + M^2\int_{-\infty}^{-\sigma}\left(\int_{-\infty}^{-|z|}\frac{e^{-iM\sqrt{\tau^2+\zeta^2}}}{\sqrt{\tau^2 + \zeta^2}}\,d\zeta\right)d\tau + \int_{-\infty}^{-\sigma}\frac{|z|e^{-iM\sqrt{\tau^2+z^2}}}{\tau^2 + z^2}\left(\frac{1}{\sqrt{\tau^2 + z^2}} + iM\right)d\tau\right]d\sigma \tag{23}$$

The difficulty of the problem of evaluating the function K lies in the form of the function F_M. In an appendix to this report there are incorporated some preliminary considerations by Z. Kopal concerning evaluation of this function F_M.

SOLUTION OF INTEGRAL EQUATION FOR PRESSURE DISTRIBUTION $\overline{p}_a$

Equation (18) is written in the form

$$w_1(x,y^*) + e^{-ikx}Q(y^*) = \oint_{-1}^{1} \overline{p}_a(\xi,y^*)\widetilde{G}\,d\xi \tag{24}$$

and it is seen that the expression for $\bar{p}_a$ may be considered to be composed of two parts, as follows:

$$\bar{p}_a(x,y^*) = \bar{p}_a^{(2)}(x,y^*) - Q\,\Delta\bar{p}_a(x) \tag{25}$$

The term $\bar{p}_a^{(2)}$ represents the solution of the problem when three-dimensional effects are neglected. The term $\Delta\bar{p}_a$ represents the solution of a special two-dimensional problem, namely, that problem for which $w_i(x,y^*) = -e^{-ikx}$. This result holds regardless of the manner in which the solution of the two-dimensional problem is approached, and for the present purposes $\bar{p}_a^{(2)}$ and $\Delta\bar{p}_a$ may be considered as known.

It remains to determine the values of Q. In order to do this equation (25) may be combined with equation (17). This gives

$$P(y^*) = \int_{-1}^{1} \bar{p}_a^{(2)} e^{ikx}\,dx - Q\int_{-1}^{1} \Delta\bar{p}_a e^{ikx}\,dx \tag{26}$$

Use is now made of the definition of Q by means of equation (26), in order to write

$$P(y^*) = \int_{-1}^{1} \bar{p}_a^{(2)} e^{ikx}\,dx - \left(\int_{-1}^{1} \Delta\bar{p}_a e^{ikx}\,dx\right) \oint_{-1}^{1} \frac{dP}{d\eta^*} K\,d\eta^* \tag{27}$$

In equation (27) the abbreviations

$$P^{(2)}(y^*) = \int_{-1}^{1} \bar{p}_a^{(2)}(x,y^*) e^{ikx}\,dx \tag{28}$$

$$\mu(k,M) = \int_{-1}^{1} \Delta\bar{p}_a e^{ikx}\,dx \tag{29}$$

are used in order to obtain the basic integral equation of the problem in the form

$$P(y^*) + \mu(k,M) \oint_{-1}^{1} \frac{dP}{d\eta^*} K(y^* - \eta^*)\,d\eta^* = P^{(2)}(y^*) \tag{30}$$

Equation (30) is the appropriate generalization of the previously obtained integral equation for the circulation function Ω in incompressible flow (reference 2). It is seen that determination of the terms $P^{(2)}$ and μ, as defined by equations (28) and (29), requires the solution of a two-dimensional problem in accordance with equations (24) and (25). In view of equations (19) and (30) the factor Q in the three-dimensional correction term in equation (25) may also be written in the form

$$Q = \frac{P^{(2)} - P}{\mu} \tag{31}$$

What is left to be done, in order to obtain three-dimensional corrections of the kind here contemplated, is to solve integral equation (30), and there the main difficulty is the tabulation of the function K. Once K has been tabulated the problem of subsonic compressible flow is solved to the same extent as the problem of incompressible flow has been solved in reference 2. The solution of specific problems may then be obtained in the same manner as was done in reference 3 for problems of incompressible flow.

Massachusetts Institute of Technology
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APPENDIX

SOME CONSIDERATIONS ON EVALUATION OF FUNCTION F_M

By Z. Kopal

Given

$$F_M(x) = \frac{|x|}{x}\int_0^{\infty} e^{-i\sigma}\left[\int_{-\infty}^{-|x|} \frac{\sigma e^{-iM\sqrt{\sigma^2+\xi^2}}}{\sigma^2+\xi^2}\left(\frac{1}{\sqrt{\sigma^2+\xi^2}}+iM\right)d\xi + \right.$$

$$\int_{-\infty}^{-\sigma} \frac{|x| e^{-iM\sqrt{\tau^2+x^2}}}{\tau^2+x^2}\left(\frac{1}{\sqrt{\tau^2+x^2}}+iM\right)d\tau +$$

$$\left. M^2\int_{-\infty}^{-\sigma}\int_{-\infty}^{-|x|} \frac{e^{-iM\sqrt{\tau^2+\xi^2}}}{\sqrt{\tau^2+\xi^2}}\,d\tau\,d\xi\right]d\sigma \tag{A1}$$

Since

$$\left.\begin{aligned}\int_{-\infty}^{-|x|} \frac{\sigma e^{-iM\sqrt{\sigma^2+\xi^2}}}{\sigma^2+\xi^2}\left(\frac{1}{\sqrt{\sigma^2+\xi^2}}+iM\right)d\xi &= -\frac{\partial}{\partial\sigma}\int_{|x|}^{\infty} \frac{e^{-iM\sqrt{\sigma^2+\xi^2}}}{\sqrt{\sigma^2+\xi^2}}\,d\xi \\ \int_{-\infty}^{-\sigma} \frac{|x| e^{-iM\sqrt{\tau^2+x^2}}}{\tau^2+x^2}\left(\frac{1}{\sqrt{\tau^2+x^2}}+iM\right)d\tau &= -\frac{|x|}{x}\frac{\partial}{\partial x}\int_{\sigma}^{\infty} \frac{e^{-iM\sqrt{\tau^2+x^2}}}{\sqrt{\tau^2+x^2}}\,d\tau\end{aligned}\right\} \tag{A2}$$

and since, moreover, the order of integration in the last integral (factored by M^2) is interchangeable, the foregoing function $F_M(x)$ can also be written as

$$F_M(x) = -\int_0^{\infty} e^{-i\sigma}\,B(\sigma,x;M)\,d\sigma \tag{A3}$$

where

$$B = \frac{\partial}{\partial \sigma} I(x,\sigma^2) + \frac{\partial}{\partial x} I(\sigma,x^2) - M^2 \int_x^{\infty} I(\sigma,\xi^2)\, d\sigma \tag{A4}$$

In equation (A4) the abbreviations

$$\left.\begin{aligned} I(a,\alpha^2) &\equiv \int_a^{\infty} \frac{e^{-iM\sqrt{\alpha^2+y^2}}}{\sqrt{y^2+\alpha^2}}\, dy \equiv I_1(a,\alpha^2) - iI_2(a,\alpha^2) \\ I_1(a,\alpha^2) &= \int_a^{\infty} \frac{\cos M\sqrt{\alpha^2+y^2}}{\sqrt{\alpha^2+y^2}}\, dy \\ I_2(a,\alpha^2) &= \int_a^{\infty} \frac{\sin M\sqrt{\alpha^2+y^2}}{\sqrt{\alpha^2+y^2}}\, dy \end{aligned}\right\} \tag{A5}$$

are used and (since only positive values of x are of interest here) $|x|/x$ is set equal to 1.

The whole difficulty of the problem centers around the evaluation of the I integrals when $M \neq 0$. If $a = 0$,

$$\left.\begin{aligned} \int_0^{\infty} \frac{\cos M\sqrt{\alpha^2+y^2}}{\sqrt{\alpha^2+y^2}}\, dy &= \frac{\pi}{2} Y_0(M\alpha) \\ \int_0^{\infty} \frac{\sin M\sqrt{\alpha^2+y^2}}{\sqrt{\alpha^2+y^2}}\, dy &= \frac{\pi}{2} J_0(M\alpha) \end{aligned}\right\} \tag{A6}$$

It can, furthermore, be proved (although the proof is somewhat involved) that the I functions, regarded as functions of M, satisfy the following differential equations:

$$\left.\begin{aligned} M^2 \frac{\partial^2 I_1}{\partial M^2} + M \frac{\partial I_1}{\partial M} + \alpha^2 M^2 I_1 &= Ma \sin M\sqrt{a^2 + \alpha^2} \\ M^2 \frac{\partial^2 I_2}{\partial M^2} + M \frac{\partial I_2}{\partial M} + \alpha^2 M^2 I_2 &= Ma \cos M\sqrt{a^2 + \alpha^2} \end{aligned}\right\} \tag{A7}$$

which, incidentally, can be written as a single equation of the form

$$M^2 \frac{\partial^2 I}{\partial M^2} + M \frac{\partial I}{\partial M} + \alpha^2 M^2 I = -iMae^{iM\sqrt{a^2+\alpha^2}} \tag{A8}$$

If $a = 0$, the homogenous part of the foregoing equations is easily recognized as Bessel's equation of zero order and their solutions are

$$\left.\begin{aligned} I_1 &= \frac{\pi}{2} Y_0(M\alpha) \\ I_2 &= \frac{\pi}{2} J_0(M\alpha) \\ I &= I_1 - iI_2 = -\frac{\pi i}{2} H_0^{(1)}(M\alpha) \end{aligned}\right\} \tag{A9}$$

where $H_0^{(1)}$ denotes the respective Hankel function.

When the two particular solutions of the homogeneous equation (for $a = 0$) are known, the inhomogeneous equations admit of a solution

$$I_1 = \frac{\pi}{2} Y_0(M\alpha) + aJ_0(M\alpha) \int^M \xi^2 Y_0(\xi\alpha) \sin \xi\sqrt{a^2 + \alpha^2}\, d\xi -$$

$$aY_0(M\alpha) \int^M \xi^2 J_0(\xi\alpha) \sin \xi\sqrt{a^2 + \alpha^2}\, d\xi \tag{A10}$$

and

$$I_2 = \frac{\pi}{2} J_0(M\alpha) + aY_0(M\alpha)\int^{M} \xi^2 J_0(\xi\alpha) \cos \xi \sqrt{a^2 + \alpha^2}\, d\xi -$$

$$aJ_0(M\alpha)\int^{M} \xi^2 Y_0(\xi\alpha) \cos \xi \sqrt{a^2 + \alpha^2}\, d\xi \qquad \text{(A11)}$$

Since both transcendent functions occurring behind the integral sign of these integrals can be expanded in rapidly converging power series of ξ and subsequently evaluated, these integrals can henceforward be regarded as known. This leaves one more quadrature to be performed in order to evaluate the function $F_M(x)$ - the real and imaginary parts of which can easily be written as

$$F_M(x) \equiv F_1 - iF_2$$

where

$$\left.\begin{aligned}
F_1 = &\int_0^\infty \left[\frac{\partial}{\partial\sigma} I_1(x,\sigma^2) + \frac{\partial}{\partial x} I_1(\sigma,x^2) - M^2\int_x^\infty I_1(\sigma,\xi^2)\, d\xi\right]\cos\sigma\, d\sigma - \\
&\int_0^\infty \left[\frac{\partial}{\partial\sigma} I_2(x,\sigma^2) + \frac{\partial}{\partial x} I_2(\sigma,x^2) - M^2\int_x^\infty I_2(\sigma,\xi^2)\, d\xi\right]\sin\sigma\, d\sigma \\
F_2 = &\int_0^\infty \left[\frac{\partial}{\partial\sigma} I_1(x,\sigma^2) + \frac{\partial}{\partial x} I_1(\sigma,x^2) - M^2\int_x^\infty I_1(\sigma,\xi^2)\, d\xi\right]\sin\sigma\, d\sigma + \\
&\int_0^\infty \left[\frac{\partial}{\partial\sigma} I_2(x,\sigma^2) + \frac{\partial}{\partial x} I_2(\sigma,x^2) - M^2\int_x^\infty I_2(\sigma,\xi^2)\, d\xi\right]\cos\sigma\, d\sigma
\end{aligned}\right\}\text{(A12)}$$

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