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THREE-DIMENSIONAL COMPRESSIBLE LAMINAR BOUNDARY-LAYER FLOW

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SUMMARY

The equations governing the three-dimensional compressible laminar boundary layer with variable viscosity and thermal conductivity are shown to be simplified by:

1. The introduction of a two-component vector potential

2. The use of a transformation to change the equations into nearly incompressible form

3. The use of a further transformation that changes the equations into nearly Cartesian form when a coordinate system appropriate to axial or conical symmetry is used

Flow over flat plates with arbitrary leading-edge contours is discussed and it is deduced that, under certain circumstances, the boundary-layer equations are inapplicable in certain restricted regions of the boundary layer.

Problems involving conical potential flow are discussed, and it is shown that use of the Blasius similarity variable permits reduction of the number of independent variables.

INTRODUCTION

The techniques for predicting the behavior of the two-dimensional laminar flow of a viscous fluid are rather well developed. Because of the interest in aircraft applications, most of the effort has been applied to problems of the boundary layer, which is associated with flight at substantial speeds through a medium of low kinematic viscosity. An investigation of methods for treating the three-dimensional compressible laminar boundary layer was conducted at the NACA Lewis laboratory and is presented herein.

Certain features of the present body of theory for plane cases furnish guidance:

1. Lagrange's stream function may be introduced in order identically to satisfy the equation of continuity, thus permitting two of the dependent variables (velocity components) to be expressed in terms of a single function.

2. In the absence of a pressure gradient, Blasius (reference 1, paragraph 53) shows that, for the incompressible case, the two independent variables (space coordinates) may be combined into a single coordinate of similarity, thus reducing the problem to that of solving an ordinary differential equation for the stream function.

3. The Kármán-Pohlhausen integral method (reference 1, paragraph 60) provides a valuable engineering approach, the simplicity of which is obtained at the expense of restriction to a particular family of velocity profiles.

4. Howarth (reference 2) shows that a certain transformation of coordinates results in momentum equations of nearly incompressible form, provided that a linear dependence of viscosity on temperature is assumed.

5. With regard to the energy equation, if the Prandtl number is 1 and the wall is an insulating surface, the enthalpy remains constant through the boundary layer. Furthermore, if no pressure gradient exists, a term in the solution for temperature depending linearly on the velocity profile may be introduced in order to take into account heat transfer from a wall at constant temperature (Crocco, reference 3, and von Kármán, reference 4).

In all the theories previously mentioned and in the present report, use is made of the usual order-of-magnitude assumptions proceeding from the concept of a thin boundary layer beyond which exists potential flow undisturbed by the presence of the boundary layer. (See, for example, reference 1, paragraphs 44 and 45.)

The three-dimensional problem is, of course, greatly complicated by the number of dependent and independent variables appearing in the equations of motion. When such difficulty is encountered, an integral approach would appear to provide adequate simplification of the problem. Prandtl (reference 5) proposes a method (for incompressible flow) utilizing two parameters; namely, a boundary-layer "thickness" and the local angle of divergence between the outer streamlines and the limiting streamline at the wall. This method is not the only possible way of formulating an integral approach; in fact, the very complexity of the three-dimensional problem tends to make it possible to devise a number of such formulations. For example, the two independent parameters can be defined as the two boundary-layer thicknesses δ_u and δ_w corresponding to the two velocity components u and w in planes parallel to the wall. This formulation would presumably be useful if it were expected that one component would approach its stream value much more rapidly than the other.

The relative merits of Prandtl's formulation and some other, such as the one just described, are difficult to assess; nor is it a simple matter to make a judicious selection of profile functions to be used in a three-dimensional problem. Prandtl (reference 5) points out that the setting-up of an integral method requires a background of experience gained from solutions to the complete equations of motion. For this reason, the present report is concerned with means of attacking the differential boundary-layer equation and will not discuss integral methods further.

The equations of motion have been solved in several three-dimensional cases: Prandtl (reference 5), Sears (reference 6), and R. T. Jones (reference 7) have contributed to the incompressible solution for a yawed infinite cylinder. The most striking result in this problem is that the velocity components in a plane normal to the cylinder may be obtained by two-dimensional theory and may then be used to determine the axial component. Problems involving spherical or axial symmetry have received considerable attention. Mangler (reference 8) provides an analysis of the axially symmetric incompressible flow over bodies of revolution, wherein various changes of variable permit the direct use of plane boundary-layer theory. In unavailable work, Mangler has extended this analysis to permit consideration of compressibility.

The solutions mentioned in the preceding paragraph are obtained by discovering methods by which plane-flow results may be directly adapted. Many problems in the mechanics of the laminar boundary layer remain in which this simplification is impossible and the analysis to follow will investigate methods of solving this type of problem.

SYMBOLS

The following symbols are used in this report:

C constant appearing in viscosity-temperature relation

c_p specific heat at constant pressure

c_v specific heat at constant volume

E	internal energy, $c_v T$
F	component of vector potential in implicit coordinates
f	component of vector potential in conical coordinates
G	component of vector potential in implicit coordinates
g	component of vector potential in conical coordinates
H	total enthalpy in boundary layer, $c_p T + \frac{1}{2}(u^2+w^2)$
k	arbitrary constant
Pr	Prandtl number, $c_p \mu / \kappa$
p	pressure
R	gas constant
r(x)	distance used in definition of implicit coordinates
S=s	dimensionless coordinates
T	absolute temperature
t	time
u,v,w	velocity components
x,y,z	coordinates
X,Y,Z	equal to x, $\left(\frac{p}{p_\infty}\right)^{-\frac{1}{2}} \int_0^y \frac{\rho}{\rho_\infty} dy$, and z, respectively
δ	boundary-layer thickness
θ	enthalpy, $c_p T$
κ	coefficient of thermal conductivity

λ Blasius similarity variable, $\eta\xi^{-\frac{1}{2}}$

μ coefficient of viscosity

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ρ density

σ similarity variable for flat plate, $Y(X-X_0)^{-\frac{1}{2}}$

$\left.\begin{matrix}\varphi,\varphi\\ \psi,\psi\end{matrix}\right\}$ components of vector potential

Subscripts:

1 denotes conditions in flow at outer edge of boundary layer

∞ denotes evaluation at some reference condition

0 denotes conditions at leading edge (See figs. 1 and 2.)

Subscript notation for partial differentiation is used where convenient. Primes denote ordinary differentiation.

THEORY

Equations of Laminar Compressible Boundary Layer

In Cartesian tensor notation, the equations governing the motion of a compressible viscous gas may be written as

$$\left.\begin{aligned}
&\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} - \frac{2}{3}\frac{\partial}{\partial x_i}\left(\mu\frac{\partial u_j}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] \\
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0 \\
&\rho\frac{\partial E}{\partial t} + u_j\frac{\partial E}{\partial x_j} = \left[-\delta_{ij}\left(p + \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]\frac{\partial u_i}{\partial x_j} + \\
&\qquad \frac{\partial}{\partial x_j}\left(\kappa\frac{\partial T}{\partial x_j}\right) \\
&p = \rho R T
\end{aligned}\right\} \quad (1)$$

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Equations of motion in Cartesian coordinates. - On the assumption of "boundary-layer flow" an obvious extension to three dimensions of the argument presented in reference 1 (paragraph 44) permits equations (1) to be reduced to the following form, where a Cartesian system of coordinates (x,y,z) and velocity (u,v,w) is used; y and v are taken normal to a plane surface bounding the flow:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \quad (2a)$$

$$\frac{\partial p}{\partial y} \approx 0 \quad (2b)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) \quad (2c)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad (2d)$$

$$\rho\left(\frac{\partial E}{\partial t} + u\frac{\partial E}{\partial x} + v\frac{\partial E}{\partial y} + w\frac{\partial E}{\partial z}\right) = -\,p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right) \tag{2e}$$

$$p = \rho RT \tag{2f}$$

The argument contained in reference 1 (paragraph 45) indicates that for plane flow, these Cartesian equations apply in general orthogonal coordinates, where y is taken normal to the surface, and thus apply to flow over curved surfaces, provided that the surface curvature is moderate and with the exception that, for curved surfaces, $\partial p/\partial y$ is, in general, of order 1. Of course, even through $\partial p/\partial y$ is of order 1, the change in pressure across a thin boundary layer is small and may be neglected. This argument applies equally well to three-dimensional flows.

Equations of motion in implicit coordinates. - The boundary-layer equations are now written in an orthogonal coordinate system wherein:

1. The body surface is defined by $y = 0$.

2. A point is defined by the distances x, y, and $r(x)\,s$ where the distance $r(x)\,s$ depends implicitly on the distance x, and where $r(x)$ has the dimensions of length.

This coordinate system would be useful in the analysis of flow about bodies for which a coordinate x can be defined such that body cross sections are similar for various values of x. The quantity $r(x)$ then gives the variation of scale of these cross sections. For a body of revolution, x may be measured along generators; the cross sections are circular; and $r(x)$ may be taken as the radius of a cross section. For a conical body, x may be measured along rays from the apex, and $r(x)$ is a linear function of x, giving the scale change of the (in general) noncircular cross sections. The velocity components are taken to be u, v, and w in the directions x, y, and s, respectively. The equations of motion are:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{1}{r}\frac{\partial u}{\partial s} - \frac{r'}{r}w^2\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{3a}$$

$$\frac{\partial p}{\partial y} = \text{order of } 1 \tag{3b}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{1}{r}\frac{\partial w}{\partial s} + \frac{r'}{r}uw\right) = -\frac{1}{r}\frac{\partial p}{\partial s} + \frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) \tag{3c}$$

$$\frac{\partial}{\partial t}(\rho r) + \frac{\partial}{\partial x}(\rho ru) + \frac{\partial}{\partial y}(\rho rv) + \frac{1}{r}\frac{\partial}{\partial s}(\rho rw) = 0 \tag{3d}$$

$$\rho\left(\frac{\partial E}{\partial t} + u\frac{\partial E}{\partial x} + v\frac{\partial E}{\partial y} + \frac{1}{r}w\frac{\partial E}{\partial s}\right) = -p\left[\frac{1}{r}\frac{\partial}{\partial x}(ru) + \frac{\partial v}{\partial y} + \frac{1}{r}\frac{\partial w}{\partial s}\right] + \mu\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right) \tag{3e}$$

$$p = \rho RT \tag{3f}$$

Simplification of energy equation. - Equations (2d) and (2f) may be combined with equation (2e) to yield

$$\rho\left(\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z}\right) = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + w\frac{\partial p}{\partial z} + \mu\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{1}{Pr}\frac{\partial}{\partial y}\left(\mu\frac{\partial \theta}{\partial y}\right) \tag{4}$$

where $\theta \equiv c_p T$. The specific heats c_v and c_p, and the Prandtl number $Pr = c_p\mu/\kappa$ are considered constant. Furthermore, if the Prandtl number is assumed equal to 1, the sum of equation (2a) multiplied by u, equation (2c) multiplied by w, and equation (4) yields, for steady flow,

$$\rho\left(u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} + w\frac{\partial H}{\partial z}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial H}{\partial y}\right) \tag{5}$$

where

$$H \equiv c_pT + \frac{1}{2}(u^2+w^2) \tag{6}$$

A special solution of equation (5) is, in accordance with definition (6),

$$H = c_pT + \frac{1}{2}(u^2+w^2) = \text{constant} \tag{7}$$

This solution is presented in references 3 and 4 for the plane case. Evaluation of the derivative of equation (7) at the wall, where $u = w = 0$, shows that equation (7) is the solution of the energy equation (5) for the case of zero heat transfer through the wall $y = 0$. The result of equation (7) is, of course, independent of the coordinate system used and hence is a special solution of equation (3e).

In references 3 and 4, it was observed that for plane flow, when $Pr = 1$ and $\frac{\partial p}{\partial x} = 0$, a solution to the energy equation is

$$H = A + Bu \tag{8}$$

where A and B are constants to be determined from the boundary conditions on the temperature profile. In three dimensions, it is clear that if $\frac{\partial p}{\partial x}$ (but not necessarily $\frac{\partial p}{\partial z}$) vanishes, then the steady-state form of equation (2a) for u is identical to equation (5) for H. Thus, equation (8) is a solution of equation (5). Inasmuch as the velocity components vanish at the wall, equation (8) must correspond to cases involving heat transfer from a wall at constant temperature. Because H must be constant in the outer potential flow, solution (8) is further restricted to cases wherein u_1 (the value of u at outer edge of boundary layer) is constant. An example of a situation wherein solution (8) applies is furnished by the yawed infinite cylinder, where u is in the direction of the cylinder axis. The analogous result for incompressible flow is mentioned in reference 6.

Vector Potential

Definitions. - In order to reduce the number of dependent variables appearing in the boundary-layer equations, it is customary in problems of steady plane flow to introduce the stream function, which identically satisfies the equation of continuity (equation (2d)); that is,

$$\left.\begin{aligned} \rho u &\equiv \frac{\partial \psi}{\partial y} \\ \rho v &\equiv -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \quad (9)$$

It would be desirable to secure a similar advantage in the analysis of three-dimensional steady flows, which may be done by writing

$$\left.\begin{aligned} \rho u &\equiv \frac{\partial \psi}{\partial y} \\ \rho v &\equiv -\frac{\partial \psi}{\partial x} - \frac{\partial \varphi}{\partial z} \\ \rho w &\equiv \frac{\partial \varphi}{\partial y} \end{aligned}\right\} \quad (10)$$

Equations (10) represent one of several ways in which two functions can be defined in order to satisfy equation (2d) identically for steady motion. The particular arrangement of equations (10) is chosen to provide symmetry of ψ and x against φ and z. Of course, for plane flow, equations (10) reduce to equations (9).

For every steady plane flow satisfying the equation of continuity, a stream function exists, according to definition (9). Proof of this theorem is provided by Lamb (reference 9, paragraph 59). Existence of the functions φ and ψ defined by equations (10) for every three-dimensional flow should, however, be proved.

A well-known theorem of vector analysis states that any continuously differentiable vector having a vanishing divergence may be expressed as the curl of a vector potential. By the equation of continuity, the mass-flow vector $\rho \underline{q} \equiv \rho(\underline{i}u+\underline{j}v+\underline{k}w)$ has vanishing divergence. Thus, for any steady flow, a vector potential $\underline{i}A_1+\underline{j}A_2+\underline{k}A_3$ exists that is defined by the equation

$$\rho \underline{q} \equiv \text{curl } \underline{A}$$

or, in Cartesian coordinates,

$$\left.\begin{aligned} \rho u &= \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \\ \rho v &= \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \\ \rho w &= \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{aligned}\right\} \tag{11}$$

Therefore, for every three-dimensional steady flow the following functions exist:

$$\psi \equiv A_3 - \int \frac{\partial A_2}{\partial z} \, dy$$

$$\varphi \equiv - A_1 + \int \frac{\partial A_2}{\partial x} \, dy$$

Differentiation of these expressions yields equations (10), provided that

$$\frac{\partial}{\partial x} \int \frac{\partial A_2}{\partial z} \, dy = \frac{\partial}{\partial z} \int \frac{\partial A_2}{\partial x} \, dy \tag{12}$$

The foregoing argument suffices to show the existence of the functions ψ and φ for three-dimensional steady flows in general. Requirement (12) is certainly met by most flows encountered in practice.

Apparently, the theorem quoted at the beginning of the preceding paragraph may be extended to provide that the three-dimensional vector potential may be written as any one of the three possible two-component vector potentials. The existence of each of the three arrangements may be contingent on a restriction similar to equation (12), which, it might be noted, has been shown here to be sufficient but has not been shown to be a necessary condition.

Equation (3d) is satisfied in two-dimensional (axially symmetric) steady flow by defining ψ such that

$$\left.\begin{aligned} \rho r u &\equiv \frac{\partial\psi}{\partial y} \\ \rho r v &\equiv -\frac{\partial\psi}{\partial x} \end{aligned}\right\} \qquad (13)$$

and in three-dimensional flow by defining ψ and φ such that

$$\left.\begin{aligned} \rho r u &\equiv \frac{\partial\psi}{\partial y} \\ \rho r v &\equiv -\frac{\partial\psi}{\partial x} - \frac{1}{r}\frac{\partial\varphi}{\partial s} \\ \rho r w &\equiv \frac{\partial\varphi}{\partial y} \end{aligned}\right\} \qquad (14)$$

For two-dimensional flow, definitions (14) reduce to definitions (13). The existence of the functions ψ and φ defined in equations (14) may be established in a manner similar to that employed to show the existence of the analogous functions appearing in equations (10).

The quantities ψ and φ are hereinafter referred to as "components of the vector potential" because (in the Cartesian system, for example) equations (10) may formally be obtained from equations (11) by setting $A_3 = \psi$, $A_1 = -\varphi$, $A_2 = 0$.

Differential equations for vector potential. -

(a) Cartesian coordinates: Substituting equations (10) into equations (2) and (4) for steady flow and adopting the subscript notation for partial differentiation yield

$$\psi_y\left(\frac{1}{\rho}\psi_y\right)_x - (\psi_x+\varphi_z)\left(\frac{1}{\rho}\psi_y\right)_y + \varphi_y\left(\frac{1}{\rho}\psi_y\right)_z = -p_x(x,z) +$$

$$\left[\mu\left(\frac{1}{\rho}\psi_y\right)_y\right]_y \tag{15a}$$

$$\psi_y\left(\frac{1}{\rho}\varphi_y\right)_x - (\psi_x+\varphi_z)\left(\frac{1}{\rho}\varphi_y\right)_y + \varphi_y\left(\frac{1}{\rho}\varphi_y\right)_z = -p_z(x,z) +$$

$$\left[\mu\left(\frac{1}{\rho}\varphi_y\right)_y\right]_y \tag{15b}$$

$$\psi_y\theta_x - (\psi_x+\varphi_z)\theta_y + \varphi_y\theta_z$$

$$= \frac{1}{\rho}\psi_y p_x + \frac{1}{\rho}\varphi_y p_z + \mu\left\{\left[\left(\frac{1}{\rho}\psi_y\right)_y\right]^2 + \left[\left(\frac{1}{\rho}\varphi_y\right)_y\right]^2\right\} + \frac{1}{Pr}(\mu\theta_y) \tag{15c}$$

or, for $Pr = 1$ and zero heat transfer, equation (15c) is replaced by

$$c_pT + \frac{1}{2}\left[\left(\frac{1}{\rho}\psi_y\right)^2 + \left(\frac{1}{\rho}\varphi_y\right)^2\right] = \text{constant} \tag{15d}$$

(b) Implicit coordinates: Substituting definitions (14) into equations (3) and an equation analogous to equation (4) yields, for steady flow,

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$$\frac{1}{r}\psi_y\left(\frac{1}{\rho r}\psi_y\right)_x - \frac{1}{r}\left(\psi_x+\frac{1}{r}\varphi_s\right)\left(\frac{1}{\rho r}\psi_y\right)_y + \frac{1}{r^2}\varphi_y\left(\frac{1}{\rho r}\psi_y\right)_s - \frac{r'}{r^2}\frac{1}{\rho r}\varphi_y\varphi_y$$

$$= -p_x(x,s) + \left[\mu\left(\frac{1}{\rho r}\psi_y\right)_y\right]_y \tag{16a}$$

$$\frac{1}{r}\psi_y\left(\frac{1}{\rho r}\varphi_y\right)_x - \frac{1}{r}\left(\psi_x+\frac{1}{r}\varphi_s\right)\left(\frac{1}{\rho r}\varphi_y\right)_y + \frac{1}{r^2}\varphi_y\left(\frac{1}{\rho r}\varphi_y\right)_s + \frac{r'}{r^2}\frac{1}{\rho r}\psi_y\varphi_y$$

$$= -\frac{1}{r}p_s(x,s) + \left[\mu\left(\frac{1}{\rho r}\varphi_y\right)_y\right]_y \tag{16b}$$

$$\frac{1}{r}\psi_y\theta_x - \frac{1}{r}\left(\psi_x+\frac{1}{r}\varphi_s\right)\theta_y + \frac{1}{r^2}\varphi_y\theta_s$$

$$= \frac{1}{\rho r}\psi_y p_x + \frac{1}{\rho r^2}\varphi_y p_s + \mu\left\{\left[\left(\frac{1}{\rho r}\psi_y\right)_y\right]^2 + \left[\left(\frac{1}{\rho r}\varphi_y\right)_y\right]^2\right\} + \frac{1}{Pr}(\mu\theta_y)_y \tag{16c}$$

or, for $Pr = 1$ and zero heat transfer, equation (16c) is replaced by

$$c_pT + \frac{1}{2}\left[\left(\frac{1}{\rho r}\psi_y\right)^2 + \left(\frac{1}{\rho r}\varphi_y\right)^2\right] = \text{constant} \tag{16d}$$

Boundary conditions on vector potential. - The usual boundary conditions on velocities applied in problems of the steady laminar boundary layer are, using Cartesian coordinates,

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$$\left.\begin{aligned} u(x,\infty,z) &= u_1(x,z) \quad \text{or} \quad \Psi_y(x,\infty,z) = \rho_1 u_1(x,z) \\ w(x,\infty,z) &= w\ (x,z) \quad \text{or} \quad \varphi_y(x,\infty,z) = \rho_1 w_1(x,z) \\ u(x,0,z) &= 0 \quad \text{or} \quad \Psi_y(x,0,z) = 0 \\ w(x,0,z) &= 0 \quad \text{or} \quad \varphi_y(x,0,z) = 0 \end{aligned}\right\} \tag{17}$$

$$v(x,0,z) = 0 \quad \text{or} \quad \Psi_x(x,0,z) + \varphi_z(x,0,z) = 0 \tag{18}$$

Another boundary condition is usually required; for example, at the leading edge of an airfoil it is necessary to establish the conditions under which the initial growth of the boundary layer takes place.

Boundary condition (18), which involves a combination of Ψ and φ, would probably be rather awkward to apply; the separation of Ψ and φ with respect to their boundary conditions therefore seems desirable. The following argument shows that this separation can be made, so that condition (18) may be replaced by

$$\psi(x,0,z) = \varphi(x,0,z) = 0 \tag{19}$$

Suppose that a solution has been obtained for a given problem, subject to conditions (17) and (18). Equation (18) implies that

$$\psi_x(x,0,z) = h_1(x,z)$$

$$\varphi_z(x,0,z) = h_1(x,z)$$

where h_1 is some arbitrary function. Thus,

$$\psi(x,0,z) = \int h_1 \, dx + h_2(z)$$

$$\varphi(x,0,z) = -\int h_1 \, dz + h_3(x)$$

where h_2 and h_3 are arbitrary functions. New quantities ψ^* and φ^* are now defined:

$$\psi^* \equiv \psi - \int h_1 \, dx - h_2(z)$$

$$\varphi^* \equiv \varphi + \int h_1 \, dz - h_3(x)$$

Thus, in accordance with definitions (10),

$$\psi^*_y = \psi_y = \rho u$$

$$-\psi^*_x - \varphi^*_z = -\psi_x - \varphi_z = \rho v$$

$$\varphi^*_y = \varphi_y = \rho w$$

The functions ψ^* and φ^* yield the correct velocity components, satisfy the same differential equation as ψ and φ, and satisfy conditions (17) and condition (19), which is a special case of condition (18). It is therefore correct to replace condition (18) by condition (19), with the sole effect that the solutions for ψ and φ will be made unique.

The same argument applies in cases wherein definitions (14) are used.

Transformations of Equations of Motion

Viscosity-temperature relation. - It is assumed that the equation

$$\frac{\mu}{\mu_\infty} = \frac{T}{T_\infty} C \tag{20}$$

may be used to represent adequately the variation of viscosity in the boundary layer. The constant C and the reference state T_∞ may be chosen to give the best possible agreement with, for example, Sutherland's formula, over the temperature range contemplated. A complete discussion of equation (20), as applied to flow with vanishing pressure gradient, is provided by Chapman and Rubesin in reference 10.

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<u>Howarth's transformations.</u> - From reference 2, the following transformations are made in order to bring equations (15) and (16) into forms approaching those of the corresponding incompressible equations:

$$\left.\begin{aligned} Y &\equiv \left(\frac{p}{p_\infty}\right)^{-1/2} \int_0^y \frac{\rho}{\rho_\infty}\, dy \\ X &\equiv x \\ Z &\equiv z \quad (\text{or } S \equiv s) \\ \psi &\equiv \left(\frac{p}{p_\infty}\right)^{1/2} \overline{\psi} \\ \varphi &\equiv \left(\frac{p}{p_\infty}\right)^{1/2} \overline{\varphi} \end{aligned}\right\} \tag{21}$$

Before transforming equations (15) and (16) according to definitions (21), it is desirable to make the physical quantities involved dimensionless. Hereinafter, the following quantities on the left will be considered to be measured relative to the quantities on the right, where the subscript ∞ denotes some reference condition:

$$\left.\begin{aligned} &u,v,w \text{ relative to } u_\infty \\ &x,y,z,r,X,Y,Z \text{ relative to } \mu_\infty C/\rho_\infty u_\infty \\ &\rho \text{ relative to } \rho_\infty \\ &T \text{ relative to } u_\infty^2/2c_p \\ &p \text{ relative to } \rho_\infty u_\infty^2 \\ &\psi, \varphi \text{ (as in equations (10) and (15)) relative to } (\mu_\infty C/\rho_\infty u_\infty)\rho_\infty u_\infty \\ &\psi, \varphi \text{ (as in equations (14) and (16)) relative to } (\mu_\infty C/\rho_\infty u_\infty)^2\rho_\infty u_\infty \\ &\theta \text{ relative to } u_\infty^2 \end{aligned}\right\} \tag{22}$$

With the conventions (22) and equation (20), equation (15a), for example, becomes

$$\psi_y\left(\frac{1}{\rho}\psi_y\right)_x - (\psi_x+\varphi_z)\left(\frac{1}{\rho}\psi_y\right)_y + \varphi_y\left(\frac{1}{\rho}\psi_y\right)_z = -p_x + \left[\frac{T}{T_\infty}\left(\frac{1}{\rho}\psi_y\right)_y\right]_y \tag{23}$$

From equations (21), inasmuch as p is not to be considered a function of y,

$$\left.\begin{aligned} \frac{\partial}{\partial y} &= \rho\left(\frac{p}{p_\infty}\right)^{-1/2}\frac{\partial}{\partial Y} \\ \frac{\partial}{\partial x} &= \frac{\partial}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial Z} + \frac{\partial Y}{\partial z}\frac{\partial}{\partial Y} \end{aligned}\right\} \tag{24}$$

and from equations (10),

$$\left.\begin{aligned} u &= \frac{1}{\rho}\psi_y = \overline{\psi}_Y \\ w &= \frac{1}{\rho}\varphi_y = \overline{\varphi}_Y \end{aligned}\right\} \tag{25}$$

After relations (24) are introduced, equation (23) becomes

$$\rho\overline{\psi}_Y(\overline{\psi}_{YX}+Y_x\overline{\psi}_{YY}) - \rho\left(\overline{\psi}_X+\overline{\varphi}_Z+Y_x\overline{\psi}_Y+Y_z\overline{\varphi}_Y+\frac{1}{2}\frac{p_X}{p}\overline{\psi}+\frac{1}{2}\frac{p_Z}{p}\overline{\varphi}\right)\overline{\psi}_{YY} +$$

$$\rho\overline{\varphi}_Y\left(\overline{\psi}_{YZ}+Y_z\overline{\psi}_{YY}\right) = -p_X + \rho\frac{p_\infty}{p}\left(\rho\frac{T}{T_\infty}\overline{\psi}_{YY}\right)_Y$$

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or, using the equation of state (2f), equation (26a) is obtained. The rest of equations (26), corresponding to equations (15) in Cartesian coordinates, and equations (28), corresponding to equations (16) in implicit coordinates, are obtained in a similar manner.

In Cartesian coordinates,

$$\bar{\Psi}_Y \bar{\Psi}_{XY} - (\bar{\Psi}_X + \bar{\varphi}_Z)\bar{\Psi}_{YY} + \bar{\varphi}_Y \bar{\Psi}_{ZY} = -\frac{1}{\rho} p_X + \frac{1}{2}\left(\frac{p_X}{p}\bar{\Psi} + \frac{p_Z}{p}\bar{\varphi}\right)\bar{\Psi}_{YY} + \bar{\Psi}_{YYY} \qquad (26a)$$

$$\bar{\Psi}_Y \bar{\varphi}_{XY} - (\bar{\Psi}_X + \bar{\varphi}_Z)\bar{\varphi}_{YY} + \bar{\varphi}_Y \bar{\varphi}_{ZY} = -\frac{1}{\rho} p_Z + \frac{1}{2}\left(\frac{p_X}{p}\bar{\Psi} + \frac{p_Z}{p}\bar{\varphi}\right)\bar{\varphi}_{YY} + \bar{\varphi}_{YYY} \qquad (26b)$$

$$\bar{\Psi}_Y \theta_X - (\bar{\Psi}_X + \bar{\varphi}_Z)\theta_Y + \bar{\varphi}_Y \theta_Z$$

$$= \frac{1}{\rho}(\bar{\Psi}_Y p_X + \bar{\varphi}_Y p_Z) + \frac{1}{2}\left(\frac{p_X}{p}\bar{\Psi} + \frac{p_Z}{p}\bar{\varphi}\right)\theta_Y + \left[(\bar{\Psi}_{YY})^2 + (\bar{\varphi}_{YY})^2\right] + \frac{1}{Pr}\theta_{YY} \qquad (26c)$$

$$\frac{p}{p_\infty} = \rho \frac{T}{T_\infty} \qquad (26d)$$

or, for $Pr = 1$ and zero heat transfer, equation (26c) is replaced by

$$T + (\bar{\Psi}_Y)^2 + (\bar{\varphi}_Y)^2 = \text{constant} \qquad (26e)$$

Combining equations (17), (19), and (25) leads to the following boundary conditions on $\bar{\Psi}$ and $\bar{\varphi}$:

$$\left.\begin{aligned} \bar{\Psi}_Y(X,\infty,Z) &= u_1(X,Z) \\ \bar{\varphi}_Y(X,\infty,Z) &= w_1(X,Z) \\ \bar{\Psi}_Y(X,0,Z) = \bar{\varphi}_Y(X,0,Z) = \bar{\Psi}(X,0,Z) &= \bar{\varphi}(X,0,Z) = 0 \end{aligned}\right\} \qquad (26f)$$

Boundary conditions on temperature are required (only at $Y = \infty$ if equation (26e) is used) and a condition must be imposed to describe the initial growth of the boundary layers. (See <u>Boundary conditions on vector potential</u>.)

In implicit coordinates, from equations (14),

$$\left.\begin{aligned} ru &= \frac{1}{\rho}\psi_y = \overline{\Psi}_Y \\ rw &= \frac{1}{\rho}\varphi_y = \overline{\varphi}_Y \end{aligned}\right\} \tag{27}$$

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Equations (16) become

$$\overline{\Psi}_Y\left(\frac{1}{r}\overline{\Psi}_Y\right)_X - \left(\overline{\Psi}_X + \frac{1}{r}\overline{\varphi}_S\right)\left(\frac{1}{r}\overline{\Psi}_Y\right)_Y + \frac{1}{r}\overline{\varphi}_Y\left(\frac{1}{r}\overline{\Psi}_Y\right)_S - \frac{r'}{r^2}\left(\overline{\varphi}_Y\right)^2$$

$$= -\frac{r}{\rho}p_X + \frac{1}{2}\left(\frac{p_X}{p}\overline{\Psi} + \frac{1}{r}\frac{p_S}{p}\overline{\varphi}\right)\left(\frac{1}{r}\overline{\Psi}_Y\right)_Y + \overline{\Psi}_{YYY} \tag{28a}$$

$$\overline{\Psi}_Y\left(\frac{1}{r}\overline{\varphi}_Y\right)_X - \left(\overline{\Psi}_X + \frac{1}{r}\overline{\varphi}_S\right)\left(\frac{1}{r}\overline{\varphi}_Y\right)_Y + \frac{1}{r}\overline{\varphi}_Y\left(\frac{1}{r}\overline{\varphi}_Y\right)_S + \frac{r'}{r^2}\overline{\Psi}_Y\overline{\varphi}_Y$$

$$= -\frac{1}{\rho}p_S + \frac{1}{2}\left(\frac{p_X}{p}\overline{\Psi} + \frac{1}{r}\frac{p_S}{p}\overline{\varphi}\right)\left(\frac{1}{r}\overline{\varphi}_Y\right)_Y + \overline{\varphi}_{YYY} \tag{28b}$$

$$\frac{1}{r}\left[\overline{\Psi}_Y\theta_X - \left(\overline{\Psi}_X + \frac{1}{r}\overline{\varphi}_S\right)\theta_Y + \frac{1}{r}\overline{\varphi}_Y\theta_S\right]$$

$$= \frac{1}{\rho r}\left(\overline{\Psi}_Y p_X + \frac{1}{r}\overline{\varphi}_Y p_S\right) + \frac{1}{2r}\left(\frac{p_X}{p}\overline{\Psi} + \frac{1}{r}\frac{p_S}{p}\overline{\varphi}\right)\theta_Y + \frac{1}{r^2}\left[\left(\overline{\Psi}_{YY}\right)^2 + \left(\overline{\varphi}_{YY}\right)^2\right] + \frac{1}{Pr}\theta_{YY} \tag{28c}$$

$$\frac{p}{p_\infty} = \rho\frac{T}{T_\infty} \tag{28d}$$

or, for $Pr = 1$ and zero heat transfer, equation (28c) is replaced by

$$T + \frac{1}{r^2}\left[\left(\overline{\Psi}_Y\right)^2 + \left(\overline{\varphi}_Y\right)^2\right] = \text{constant} \tag{28e}$$

Two advantages are provided by use of transformation (21):

1. The velocities are directly related to the vector potential; thus the satisfaction of boundary conditions is easier than under the original formulation, which included a variation with density.

2. The transformed equations themselves closely resemble the corresponding equations of incompressible motion.

Mangler's transformations. - Equations (27) and (28) may be brought into forms approaching those of equations (25) and (26) by means of a transformation introduced by Mangler (reference 8) for the purpose of relating axially symmetric flow to plane flow. The following transformation is to be applied:

$$\left.\begin{aligned} \xi &\equiv k^2 \int_0^X r^2\, dX \\ \eta &\equiv krY \\ \zeta &\equiv S \\ F &\equiv k\overline{\Psi} \\ G &\equiv k\overline{\varphi} \end{aligned}\right\} \qquad (29)$$

where k is an arbitrary constant. The functions p, ρ, and T are not transformed. Thus,

$$\frac{\partial}{\partial X} = k^2 r^2 \frac{\partial}{\partial \xi} + \eta_X \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial Y} = kr \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial S} = \frac{\partial}{\partial \zeta}$$

whence,

$$\left.\begin{aligned} u &= \frac{1}{r}\,\overline{\Psi}_Y = F_\eta \\ w &= \frac{1}{r}\,\overline{\varphi}_Y = G_\eta \end{aligned}\right\} \tag{30}$$

Under definitions (29), equation (28a) transforms as follows:

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$$rF_\eta\left(k^2r^2F_{\eta\xi}+\eta_X F_{\eta\eta}\right) - \left(kr^2F_\xi+\frac{1}{k}\eta_X F_\eta+\frac{1}{kr}G_\zeta\right)rkF_{\eta\eta} + G_\eta F_{\eta\zeta} - r'\left(G_\eta\right)^2$$

$$= -\frac{r^3}{\rho}k^2p_\xi + \frac{1}{2}\left(k^2r^3\frac{p_\xi}{p}F+\frac{p_\zeta}{p}G\right)F_{\eta\eta} + k^2r^3F_{\eta\eta\eta}$$

which may be divided throughout by k^2r^3, yielding equation (31a). The rest of equations (31) are obtained in a similar manner:

$$F_\eta F_{\eta\xi} - \left(F_\xi+\frac{1}{k^2r^3}G_\zeta\right)F_{\eta\eta} + \frac{1}{k^2r^3}G_\eta F_{\eta\zeta} - \frac{r'}{k^2r^3}\left(G_\eta\right)^2$$

$$= -\frac{1}{\rho}p_\xi + \frac{1}{2}\left(\frac{p_\xi}{p}F+\frac{1}{k^2r^3}\frac{p_\zeta}{p}G\right)F_{\eta\eta} + F_{\eta\eta\eta} \tag{31a}$$

$$F_\eta G_{\eta\xi} - \left(F_\xi+\frac{1}{k^2r^3}G_\zeta\right)G_{\eta\eta} + \frac{1}{k^2r^3}G_\eta G_{\eta\zeta} + \frac{r'}{k^2r^3}G_\eta F_\eta$$

$$= -\frac{1}{k^2r^3}\frac{1}{\rho}p_\zeta + \frac{1}{2}\left(\frac{p_\xi}{p}F+\frac{1}{k^2r^3}\frac{p_\zeta}{p}G\right)G_{\eta\eta} + G_{\eta\eta\eta} \tag{31b}$$

$$F_\eta\theta_\xi - \left(F_\xi+\frac{1}{k^2r^3}G_\zeta\right)\theta_\eta + \frac{1}{k^2r^3}G_\eta\theta_\zeta$$

$$= \frac{1}{2}\left(\frac{p_\xi}{p}F+\frac{1}{k^2r^3}\frac{p_\zeta}{p}G\right)\theta_\eta + \frac{1}{\rho}\left(F_\eta p_\xi+\frac{1}{k^2r^3}G_\eta p_\zeta\right) + \left[\left(F_{\eta\eta}\right)^2+\left(G_{\eta\eta}\right)^2\right] + \frac{1}{Pr}\theta_{\eta\eta} \tag{31c}$$

$$\frac{p}{p_\infty} = \rho \frac{T}{T_\infty} \tag{31d}$$

or, for $Pr = 1$ and zero heat transfer, equation (31c) is replaced by

$$T+(F_\eta)^2+(G_\eta)^2 = \text{constant} \tag{31e}$$

Equations (27) and the analog of condition (19) lead to the following boundary conditions on F and H:

$$\left.\begin{aligned} F_\eta(\xi,\infty,\zeta) &= u_1(\xi,\zeta) \\ G_\eta(\xi,\infty,\zeta) &= w_1(\xi,\zeta) \\ F_\eta(\xi,0,\zeta) = G_\eta(\xi,0,\zeta) = F(\xi,0,\zeta) &= G(\xi,0,\zeta) = 0 \end{aligned}\right\} \tag{31f}$$

Boundary conditions on temperature are needed (only at $\eta = \infty$ if equation (31e) is used) and the initial growth of the boundary layer must be described. (See Boundary conditions on vector potential.)

Transformation (29) thus confers the following advantages:

1. The velocities are related to the potential in the same manner as in Cartesian coordinates, thus simplifying the application of boundary conditions.

2. The transformed equations approach Cartesian form.

Further direct simplification of the boundary-layer equations does not seem feasible. Further simplifications, expecially with regard to the reduction of the number of independent variables, must be sought in the consideration of special cases or classes of problems.

APPLICATIONS

Reduction to Problems of Known Solution

The following examples are chosen to show how the foregoing theory specializes to some of the known boundary-layer flows.

Two-dimensional flow. - If it is supposed that two-dimensional flow obtains in the x,y-plane, then

$$w = 0$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial Z} = 0$$

whence, from equations (25),

$$u = \overline{\Psi}_Y$$

$$\overline{\varphi} = 0$$

and, for example, equations (26) become

$$\left.\begin{aligned} \overline{\Psi}_Y\overline{\Psi}_{XY} - \overline{\Psi}_X\overline{\Psi}_{YY} &= -\frac{1}{\rho}\, p_X + \frac{1}{2}\,\frac{p_X}{p}\,\overline{\Psi}\overline{\Psi}_{YY} + \overline{\Psi}_{YYY} \\ \overline{\Psi}_Y\theta_X - \overline{\Psi}_X\theta_Y &= \frac{1}{\rho}\,\overline{\Psi}_Y p_X + \frac{1}{2}\,\frac{p_X}{p}\,\overline{\Psi}\theta_Y + \left(\overline{\Psi}_{YY}\right)^2 + \frac{1}{Pr}\,\theta_{YY} \end{aligned}\right\} \quad (32)$$

and so forth. These are the equations of motion for plane flow, in terms of the stream function $\overline{\Psi}$.

Axially symmetric flow. - In the case of axially symmetric flow, the implicit coordinate system is used and $r(x)$ is identified with the radius of a body of revolution. Thus, by the axial symmetry,

$$w = 0$$

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial \zeta} = 0$$

whence, from equations (30),

$$u = F_\eta$$

$$G = 0$$

and equations (31) reduce to equations of the same form as equations (32).

<u>Flow about yawed infinite cylinder.</u> - The Cartesian coordinate system is used for flow about a yawed infinite cylinder; the x,y-plane is taken as the cross-sectional plane of the cylinder and the coordinate z is measured parallel to the axis of the cylinder. Thus, the potential flow has components in both the x- and z-directions. Therefore, in this case,

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$$\frac{\partial}{\partial z} = \frac{\partial}{\partial Z} = 0$$

$$w \neq 0$$

$$\varphi \neq 0$$

and equations (26) become

$$\left.\begin{aligned} \bar{\Psi}_Y\bar{\Psi}_{XY} - \bar{\Psi}_X\bar{\Psi}_{YY} &= -\frac{1}{\rho}p_X + \frac{1}{2}\frac{p_X}{p}\bar{\Psi}\bar{\Psi}_{YY} + \bar{\Psi}_{YYY} \\ \bar{\Psi}_Y\bar{\varphi}_{XY} - \bar{\Psi}_X\bar{\varphi}_{YY} &= \frac{1}{2}\frac{p_X}{p}\bar{\Psi}\bar{\varphi}_{YY} + \bar{\varphi}_{YYY} \\ \bar{\Psi}_Y\theta_X - \bar{\Psi}_X\theta_Y &= \frac{1}{\rho}\bar{\Psi}_Y p_X + \frac{1}{2}\frac{p_X}{p}\bar{\Psi}\theta_Y + \left[(\bar{\Psi}_{YY})^2 + (\bar{\varphi}_{YY})^2\right] + \frac{1}{Pr}\theta_{YY} \end{aligned}\right\} \quad (33)$$

If $Pr = 1$ and the wall temperature is constant, then, from equations (8) and (26e),

$$T + (\bar{\Psi}_Y)^2 + (\bar{\varphi}_Y)^2 = a + b\bar{\Psi}_Y$$

where a and b are constants to be determined from boundary conditions on the temperature profile.

Because of compressibility, the analytical separation of $\bar{\Psi}_Y$ and $\bar{\varphi}_Y$ discussed in references 5 to 7 does not strictly apply. If incompressible flow is considered, then equations (33) become

$$\overline{\Psi}_Y \overline{\Psi}_{XY} - \overline{\Psi}_X \overline{\Psi}_{YY} = -\frac{1}{\rho} p_X + \overline{\Psi}_{YYY}$$

$$\overline{\Psi}_Y \overline{\varphi}_{XY} - \overline{\Psi}_X \overline{\varphi}_{YY} = \overline{\varphi}_{YYY}$$

where ρ is constant. It is then possible to solve first for $\overline{\Psi}$, and subsequently to solve for $\overline{\varphi}$, knowing $\overline{\Psi}$.

Boundary Layer on Flat Plates of Arbitrary Leading-Edge Contour

The three-dimensional body considered here and the coordinates used in the analysis of its effect on a uniform stream are shown in figure 1(a). The surface of the body is a flat plate at zero angle of attack with respect to the stream.

The differential equations of motion are, from equations (26a) and (26b), written for zero pressure gradient as

$$\overline{\Psi}_Y \overline{\Psi}_{XY} - (\overline{\Psi}_X + \overline{\varphi}_Z) \overline{\Psi}_{YY} + \overline{\varphi}_Y \overline{\Psi}_{ZY} = \overline{\Psi}_{YYY} \tag{34a}$$

$$\overline{\Psi}_Y \overline{\varphi}_{XY} - (\overline{\Psi}_X + \overline{\varphi}_Z) \overline{\varphi}_{YY} + \overline{\varphi}_Y \overline{\varphi}_{ZY} = \overline{\varphi}_{YYY} \tag{34b}$$

where

$$\left.\begin{aligned} u &= \overline{\Psi}_Y \\ w &= \overline{\varphi}_Y \end{aligned}\right\} \tag{34c}$$

When $u_\infty = u_1 =$ constant, the boundary conditions are

$$\overline{\Psi}_Y(X,\infty,Z) = \overline{\Psi}_Y(X_0,Y,Z) = 1 \tag{35a}$$

$$\overline{\varphi}_Y(X,\infty,Z) = \overline{\varphi}_Y(X_0,Y,Z) = 0 \tag{35b}$$

$$\overline{\Psi}_Y(X,0,Z) = \overline{\Psi}(X,0,Z) = 0 \tag{35c}$$

$$\overline{\varphi}_Y(X,0,Z) = \overline{\varphi}(X,0,Z) = 0 \tag{35d}$$

The result $\varphi \equiv 0$ obviously satisfies equation (34b) and boundary conditions (35b) and (35d). This solution is presumably unique and may be interpreted to mean that in this case there is no secondary flow in the absence of a pressure gradient transverse to the stream. Equation (34a) then becomes

$$\bar{\Psi}_Y \bar{\Psi}_{XY} - \bar{\Psi}_X \bar{\Psi}_{YY} = \bar{\Psi}_{YYY} \tag{36}$$

The solution of equation (36), subject to boundary conditions (35a) and (35c), may be obtained by defining, by analogy with plane flow over a flat plate (reference 1, paragraph 53),

$$\sigma \equiv \frac{Y}{\sqrt{X-X_0(Z)}} \tag{37}$$

and

$$\bar{\Psi} \equiv \sqrt{X-X_0}\ \beta(\sigma) \tag{38a}$$

whence,

$$u = \bar{\Psi}_Y = \beta'(\sigma) \tag{38b}$$

The application of equations (37) and (38) transforms equation (36) as follows:

$$\beta\beta'' + 2\beta''' = 0 \tag{39}$$

The appropriate boundary conditions are

$$\left.\begin{aligned} \beta'(\infty) &= 1 \\ \beta'(0) = \beta(0) &= 0 \end{aligned}\right\} \tag{40}$$

The solution of equation (39) subject to conditions (40) was given by Blasius and is presented in reference 1 (paragraph 53).

Clearly, this solution to the problem of figure 1(a) is valid for any form $X_0(Z)$ of the leading-edge contour. In particular, it is valid for the special configuration shown in figure 1(b). In this case, however, the Z-derivatives of flow properties in the boundary layer (for example u_Z, which is related to the shear stress μu_Z) suffer discontinuities at the plane $Z = 0$. This discontinuity may be shown by writing, using equation (37),

$$\frac{\partial}{\partial Z} = \left[\frac{\sigma}{2} \frac{X_0'(Z)}{X - X_0(Z)}\right] \frac{\partial}{\partial \sigma}$$

and noting that $X_0'(Z)$ discontinuously changes sign at the plane $Z = 0$. Not only is this circumstance physically inadmissable, but it violates the boundary-layer assumption that $\frac{\partial^2}{\partial Y^2} >> \frac{\partial^2}{\partial Z^2}$. Apparently, there exists a narrow region extending in the stream direction from the apex of the body in figure 1(b) wherein the usual boundary-layer assumptions, and hence equations (33), do not apply. Outside of this region, hereinafter referred to as "wake", the usual quasi-two-dimensional solution of the boundary-layer equations for an infinite yawed flat plate is correct.

A qualitative indication of the existence of such a wake in the flow shown in figure 1(b) is obtained by imagining a viscous wake in the form of a parabolic cylinder extending downstream from each point of the leading edge. On either side of the X,Y-plane, these wakes then provide parabolic envelopes that, although not tangent to each other, are each tangent to the cylindrical wake proceeding from the origin. Thus, in the neighborhood of the X,Y-plane, where the parabolic envelopes join to the cylindrical wake, it is clearly incorrect to make the usual boundary-layer assumption $\frac{\partial^2}{\partial Z^2} << \frac{\partial^2}{\partial Y^2}$.

It is natural to suppose that the "wake" shaded region in figure 1(b) has an effective width of the order of the boundary-layer thickness and thus that the gross effects of viscosity on the body are adequately given by the solution of equations (39) and (40). There is the possibility, of course, that the wake herein discussed has an over-all effect on the flow due to a difference in stability characteristics, as compared with the surrounding boundary layer.

A wake such as is discussed in the preceding paragraph extends downstream from any corner in a leading-edge contour and presents a situation similar to that existing close to the leading edge of the flat plate, irrespective of the shape of the leading edge; as is well known, the boundary-layer equations do not apply in the immediate vicinity of the leading edge.

The inapplicability of the boundary-layer assumptions to a wake region extending downstream from a leading-edge corner is also associated with body configurations neighboring that of the flat plate with a leading-edge corner; that is, if the plate of figure 1(b) is given a small thickness, the solution of the boundary-layer equations then shows a variation of u_Z at the plane $Z = 0$ which, though not discontinuous, is too rapid to be consistent with the boundary-layer assumption $\frac{\partial^2}{\partial Y^2} \gg \frac{\partial^2}{\partial Z^2}$. As the thickness is further increased, this variation becomes less and less rapid, and, for some order of thickness, becomes consistent with the boundary-layer assumption. A similar argument applies if the curvature of the leading edge is imagined to be changed from infinity at the corner to some large finite value.

Boundary Layer Associated with Supersonic Conical Potential Flows

If the inviscid equations of motion are subject to boundary conditions on fluid properties that are constant along rays having a common focal point, it has been shown (reference 11) that the solution yields fluid properties that are constant along any such ray in the flow field; that is, that fluid properties (velocity, pressure, and so forth) are constant along each of a family of rays proceeding from a common focal point. This property is referred to as "conical symmetry." Solutions of this type exist, in general, only for supersonic flows.

For boundary-layer calculations, the significant feature of flows wherein the outer (potential) solution is conical is that pressure and velocity gradients at the outer edge of the boundary layer vanish in the direction of rays from a focal point. In this circumstance, the equations of motion (31) appropriate to the implicit coordinate system are used. Distance along a ray is denoted by x, distance normal to the ray in the surface of the body by xs (that is, $r(x) \equiv x$), and distance normal to the surface by y. (See fig. 2(a).) Equations (29) thus become

$$\left.\begin{aligned} \xi &= k^2 \int_0^X x^2\, dX = \tfrac{1}{3}k^2X^3 \\ \eta &= kxY = kXY \\ \zeta &= S \end{aligned}\right\} \qquad (41)$$

Equations (31) may therefore be written, inasmuch as, by conical symmetry of the potential flow, $p_\xi = 0$,

$$F_\eta F_{\eta_\xi} - \left(F_\xi + \frac{1}{3\xi} G_\zeta\right) F_{\eta\eta} + \frac{1}{3\xi} G_\eta F_{\eta\zeta} - \frac{1}{3\xi} (G_\eta)^2 = \frac{1}{6\xi} \frac{p'(\zeta)}{p} GF_{\eta\eta} + F_{\eta\eta\eta} \qquad (42a)$$

$$F_\eta G_{\eta_\xi} - \left(F_\xi + \frac{1}{3\xi} G_\zeta\right) G_{\eta\eta} + \frac{1}{3\xi} G_\eta G_{\eta\zeta} + \frac{1}{3\xi} G_\eta F_\eta$$

$$= -\frac{1}{3\xi\rho} p'(\zeta) + \frac{1}{6\xi} \frac{p'(\zeta)}{p} G\, G_{\eta\eta} + G_{\eta\eta\eta} \qquad (42b)$$

$$F_\eta \theta_\xi - \left(F_\xi + \frac{1}{3\xi} G_\zeta\right) \theta_\eta + \frac{1}{3\xi} G_\eta \theta_\zeta$$

$$= \frac{1}{3\xi\rho} G_\eta p'(\zeta) + \frac{1}{6\xi} \frac{p'(\zeta)}{p} G\theta_\eta + \left[(F_{\eta\eta})^2 + (G_{\eta\eta})^2\right] + \frac{1}{Pr} \theta_{\eta\eta} \qquad (42c)$$

The circumstance that $p_\xi = 0$ suggests that the Blasius similarity variable $\eta\xi^{-1/2}$ (reference 1, paragraph 53) can be employed; that is, the boundary-layer development along rays from the origin may be expected to be parabolic. Thus, the following definitions are made, by analogy with the Blasius analysis:

$$\left.\begin{aligned}
\lambda &\equiv \eta\xi^{-1/2} = \sqrt{3}\,YX^{-1/2} \\
F &\equiv \xi^{1/2} f(\lambda,\zeta) \\
G &\equiv \xi^{1/2} g(\lambda,\zeta) \\
\theta &\equiv \theta(\lambda,\zeta) \\
\rho &= \rho(\lambda,\zeta) \\
T &= T(\lambda,\zeta)
\end{aligned}\right\} \quad (43)$$

whence,

$$u = F_\eta = f_\lambda$$

$$w = G_\eta = g_\lambda$$

Thus, equations (42) become

$$\left(f+\frac{1}{3}\,\frac{p'(\zeta)}{p}\,g+\frac{2}{3}\,g_\zeta\right)f_{\lambda\lambda} - \frac{2}{3}\,g_\lambda f_{\lambda\zeta} + \frac{2}{3}\,(g_\lambda)^2 + 2f_{\lambda\lambda\lambda} = 0 \quad (44a)$$

$$\left(f+\frac{1}{3}\,\frac{p'(\zeta)}{p}\,g+\frac{2}{3}\,g_\zeta\right)g_{\lambda\lambda} - \frac{2}{3}\,g_\lambda g_{\lambda\zeta} - \frac{2}{3}\,g_\lambda f_\lambda - \frac{2}{3\rho}\,p'(\zeta) + 2g_{\lambda\lambda\lambda} = 0 \quad (44b)$$

$$\left(f+\frac{1}{3}\,\frac{p'(\zeta)}{p}\,g+\frac{2}{3}\,g_\zeta\right)\theta_\lambda - \frac{2}{3}\,g_\lambda\theta_\zeta + \frac{2}{3\rho}\,g_\lambda p'(\zeta) +$$

$$2\left[(f_{\lambda\lambda})^2+(g_{\lambda\lambda})^2\right] + 2\,\frac{1}{Pr}\,\theta_{\lambda\lambda} = 0 \quad (44c)$$

or, for $Pr = 1$ and zero heat transfer, equation (44c) is replaced by

$$T + (f_\lambda)^2 + (g_\lambda)^2 = \text{constant} \quad (44d)$$

or, using the equation of state,

$$\frac{p}{p_\infty}\frac{T_\infty}{\rho} + (f_\lambda)^2 + (h_\lambda)^2 = \text{constant} \tag{44e}$$

The value of ρ given by equation (44e) may be substituted directly into equations (44a) and (44b). The following boundary conditions apply:

$$\left.\begin{aligned} f_\lambda(\infty,\zeta) &= u_1(\zeta) \\ g_\lambda(\infty,\zeta) &= w_1(\zeta) \\ f_\lambda(0,\zeta) = g_\lambda(0,\zeta) = f(0,\zeta) &= g(0,\zeta) = 0 \\ T(\infty,\zeta) = T_1(\zeta) \quad &\text{(if equation (44d) is used)} \end{aligned}\right\} \tag{44f}$$

The equations (44) involve (if equation (44e) is used) two dependent variables f and g and two independent variables λ and ζ. It seems reasonable to conclude that the solution of these equations for many cases is feasible provided modern high-speed computing techniques are used.

In the light of the discussion of flat-plate flow contained in the preceding section, certain observations may be made concerning applications of equations (44):

The flow about the body shown in figure 1(b) may be considered a special case of boundary-layer flow about a body with conical symmetry. In figure 2(b) this flat delta wing is drawn with reference to the coordinate system used herein for conical bodies. It may be shown that the solution given in the preceding section in Cartesian coordinates is consistent with the form required by equations (43) in conical coordinates: From equation (38b), the velocity components in conical coordinates (fig. 2(b)) are seen to be

$$\left.\begin{aligned} u &= \beta'(\sigma)\cos\zeta \\ w &= -\beta'(\sigma)\sin\zeta \end{aligned}\right\} \tag{45}$$

The distance $X-X_0$ in Cartesian coordinates (fig. 1(b)) is, in conical coordinates,

$$X \cos \zeta - X \sin \zeta \cot \zeta_0$$

Thus, the σ of equation (37) is, in conical coordinates,

$$\begin{aligned}\sigma &= Y\left(X \cos \zeta - X \sin \zeta \cot \zeta_0\right)^{-1/2} \\ &= YX^{-1/2}\left[\cos \zeta \left(1 - \cot \zeta \cot \zeta_0\right)\right]^{-1/2}\end{aligned}$$

or, from equations (43),

$$\sigma = \frac{1}{\sqrt{3}}\lambda\left[\cos \zeta \left(1 - \cot \zeta \cot \zeta_0\right)\right]^{-1/2} \tag{46}$$

Therefore, when equations (45) and (46) are combined, it is seen that

$$u = u(\lambda, \zeta)$$

$$w = w(\lambda, \zeta)$$

as supposed in equations (43).

The example just discussed concerns one of a class of conical bodies for which the boundary layer contains a wake of the type discussed in the preceding section; that is, the boundary layer contains a narrow region to which the boundary-layer assumptions, and hence equations (43), do not apply. This class of bodies includes flat (or nearly flat) delta wings or rectangular wing tips.

CONCLUSIONS

Certain of the difficulties encountered in dealing with the equations of three-dimensional motion for a laminar compressible boundary layer may be overcome by:

1. The introduction of the vector potential in a manner that permits the expression of the three velocity components in terms of two scalar functions.

2. The use of a transformation of coordinates that has the simplifying effect of relating the vector potential to the velocity components in the same manner as for incompressible flow. Further, the transformed equations are of nearly incompressible form. This transformation requires the use of a linear type of viscosity-temperature relation.

3. The use of a further transformation of coordinates in cases where an implicit coordinate system is employed (for example, when axial or conical symmetry is involved), such that the velocity components and vector potential are related in the same way as in Cartesian coordinates. The transformed equations are of nearly Cartesian form.

The condition of constant total enthalpy is a solution to the energy equation in three- as well as in two-dimensional cases, when the Prandtl number is 1 and for zero heat transfer. When the restriction of zero heat transfer is removed, an additional solution exists for the total enthalpy depending linearly on the velocity component in the direction of isobars of the flow, provided this component is constant in the outer flow.

The equations herein developed for three-dimensional motion reduce directly to two-dimensional form for plane or axially symmetric boundary conditions.

In cases of flow over flat plates at zero angle of attack, when the leading edge is some arbitrary curve, the flow viewed in planes parallel to the stream and perpendicular to the plate is given by the plane Blasius solution. When the leading edge has a corner, this solution contains discontinuous derivatives at the normal plane passing through the corner in the streamwise direction. It is inferred that, in a narrow region extending downstream from the corner, a "smoothing" process occurs to which the boundary-layer equations are not applicable. This effect is not expected to be physically important, although the possibility exists that it affects stability and transition.

When the boundary layer is associated with supersonic potential flow having conical symmetry, the Blasius similarity variable may be applied with respect to variations in normal planes containing the apex. Thus, the laminar boundary-layer development is parabolic along rays from the apex. The use of this information reduces the number of independent variables to two and is thus considered to bring such problems within the range of effectiveness of modern computing techniques, especially if the assumptions of Prandtl number equal to 1 and zero heat transfer are made.

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, September 5, 1950.

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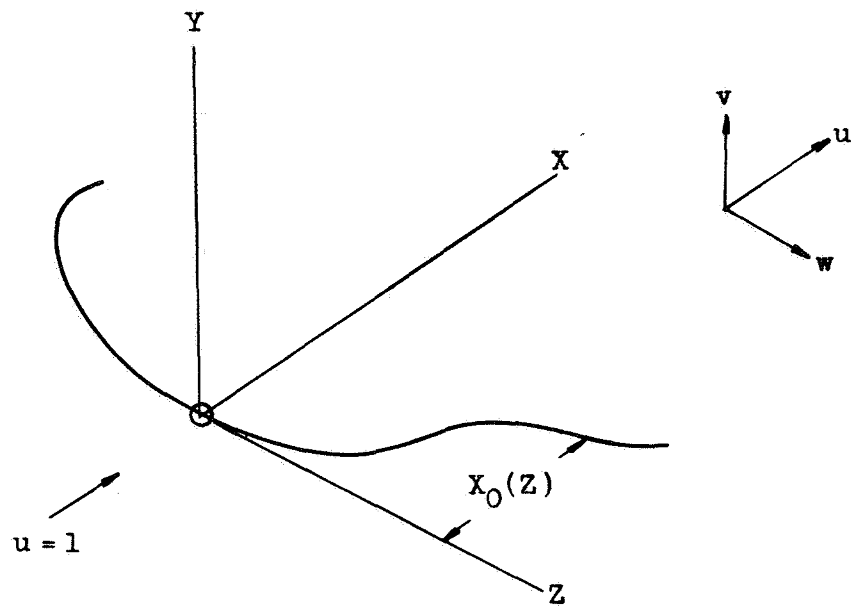
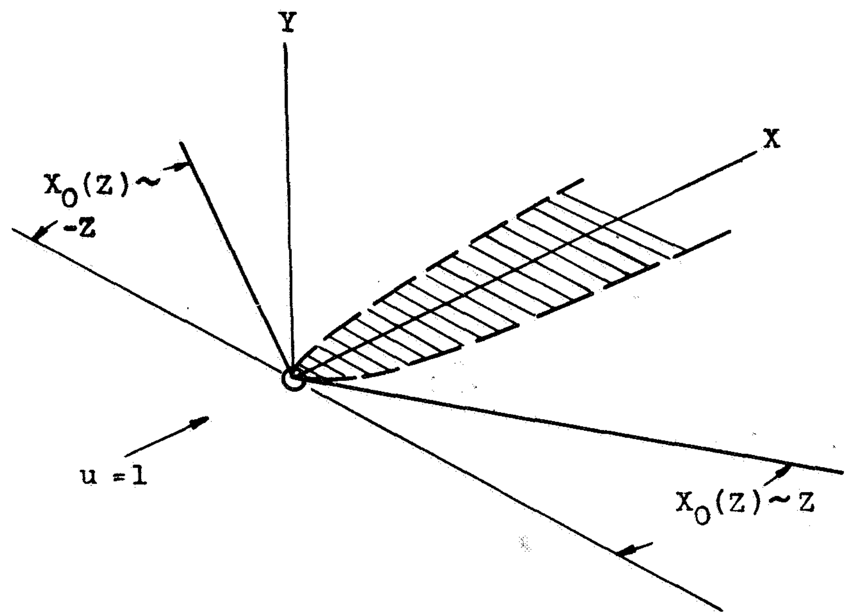
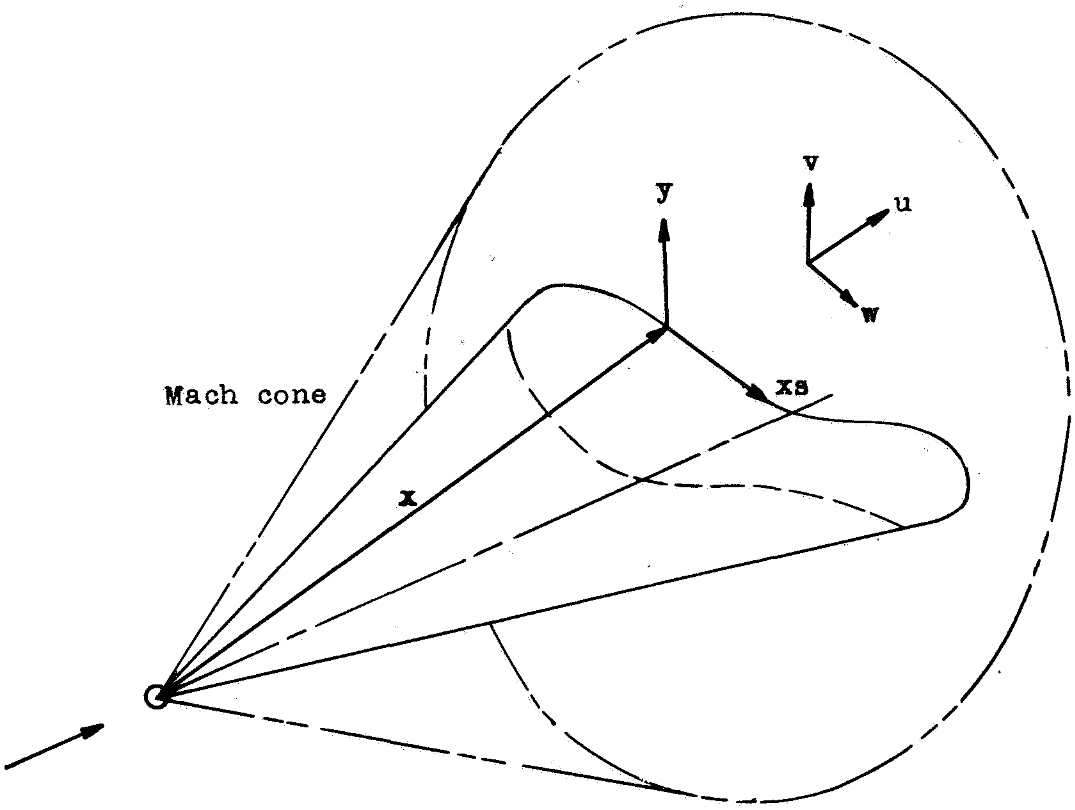


(a) Smooth leading edge.

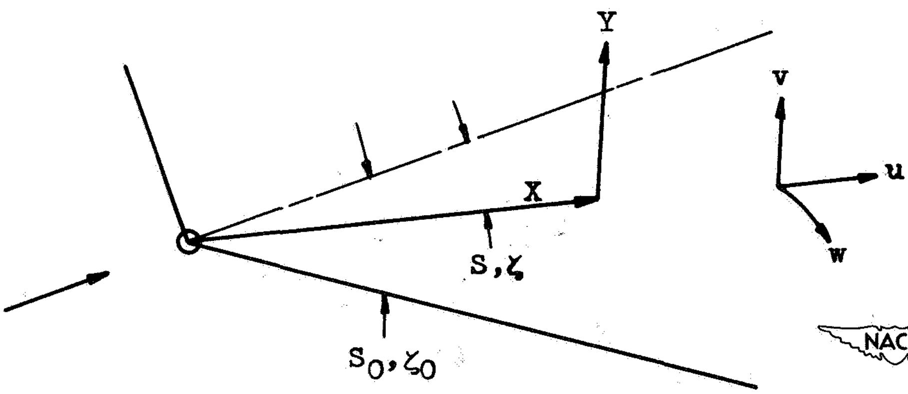


(b) Leading edge with corner.

Figure 1. - Three-dimensional flow over flat plate.



(a) General conical body in supersonic flow.



(b) Flat plate with leading-edge corner.

Figure 2. - Conical bodies described in implicit coordinate system.