no. 1

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2252

FORMULAS FOR SOURCE, DOUBLET, AND VORTEX

DISTRIBUTIONS IN SUPERSONIC WING THEORY

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and Franklyn B. Fuller

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NACA

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IN SUPERSONIC WING THEORY

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SUMMARY

The formulas of supersonic wing theory for source, doublet, and vortex distributions are reviewed and a systematic presentation is provided which relates these distributions to the pressure and to the vertical induced velocity in the plane of the wing. It is shown that care must be used in treating the singularities involved in the analysis and that the order of integration is not always reversible. Further, it is shown that the use of the complex variable can often facilitate the calculation of the integrals involved. Certain special applications are included to illustrate the concepts presented.

INTRODUCTION

One of the most fundamental approaches to the analytical investigation of linearized wing theory, throughout the subsonic and supersonic Mach number range, stems from the use of certain elementary mathematical expressions which are identified physically with sources, doublets, and vortices in the fluid medium. By means of these expressions, boundary-value problems involving wings with thickness, camber, and angle of attack can be solved. These problems are divided into two categories: one, involving symmetrical bodies with thickness and no lift, is analyzed by means of source distributions; and the other, involving lifting plates without thickness, is analyzed by means of doublet and vortex distributions.

All these distributions require the treatment of singularities in the mathematical analysis. Thus, for subsonic Mach numbers, the concept of Cauchy's principal part plays an important role in the calculation of integrals arising in the development of lifting-line and two-dimensional section problems. In supersonic wing theory, the Cauchy principal part is again used in the treatment, for example, of conical-flow problems as in reference 1, but, because of the Mach lines and cones appearing in the physical flow and in the hyperbolic geometry of the differential equation, other techniques in handling improper integrals are needed.

The integrals in supersonic wing theory thus require, in general, more
careful attention to the discontinuities in the integrand and, as an
illustration, indiscriminate use of such standard devices as inversion
of the order of integration may lead to incorrect results.

When problems of the first kind are involved, that is, when pre-
scribed distributions are to be integrated (as for the problem of finding
the pressure on a wing with symmetrical thickness), a guide to the proper
method of calculation is often furnished by physical intuition. However,
when problems of the second kind arise, that is, problems, the solution of
which depends upon the inversion of an integral equation (as the flat
plate of arbitrary plan form), the mathematical methods are more abstract.

The purpose of the present report is: first, to review the formulas
of linearized wing theory in which source, doublet, and elementary-
vortex distributions are introduced and to relate these distributions to
the pressure and to the vertical induced velocity in the plane of the
wing; second, to show that the use of the complex variable can often
facilitate the calculation of the integrals involved; and finally, to
present certain special applications which will illustrate the basic
concepts.

LIST OF IMPORTANT SYMBOLS

C_p pressure coefficient $\left(-2\,\dfrac{u}{V_o}\right)$

M_o free-stream Mach number

$\dfrac{\Delta p}{q}$ loading coefficient (pressure on the lower surface minus pressure
on the upper surface divided by free-stream dynamic pressure)

q free-stream dynamic pressure $\left(\dfrac{1}{2}\,\rho_o\,V_o^{2}\right)$

r_c hyperbolic distance between points x,y,z and $x_1,y_1,0$;
$$r_c = \sqrt{(x-x_1)^2 - \beta^2(y-y_1)^2 - \beta^2 z^2}$$

V_o velocity of the free stream

x,y,z Cartesian coordinates

r,s,z characteristic coordinates

ξ,η,z oblique coordinates

m_1 cotangent of the angle between the η and x axes

m_2 cotangent of the angle between the ξ and x axes

μ_1 $\sqrt{1+m_1{}^2}$

μ_2 $\sqrt{1+m_2{}^2}$

u perturbation velocity in x direction

w perturbation velocity in z direction

β $\sqrt{M_0{}^2-1}$

λ slope of stream surface (w/V_0)

φ perturbation velocity potential

Δ jump in value of the quantity considered across the $z = 0$ plane

Subscript

u value of a quantity on the upper surface of a wing ($z = 0$ plane)

GENERAL THEORY

It is well known that the analysis of thin wings at supersonic speeds and at small angles of attack can be expressed in mathematical terms as a boundary-value problem for the wave equation. If Ω represents a velocity potential or any one of the velocity components themselves, this equation can be written

$$\beta^2 \; \Omega_{xx} - \Omega_{yy} - \Omega_{zz} = 0 \qquad\qquad (1)$$

where the $z = 0$ plane is the plane of the wing, the free-stream velocity V_0 is directed along the x axis and $\beta^2 = M_0{}^2-1$, M_0 being the free-stream Mach number.

Of the many ways of solving the boundary-value problems associated with the wave equation, the most convenient for the present purpose is the Volterra solution. In reference 1, a discussion was given of the application to aerodynamic problems of Volterra's method. Thus, the solution for Ω in terms of $\Delta\Omega$ and $\Delta\frac{\partial\Omega}{\partial z}$, the jump in Ω and its gradient in crossing the $z = 0$ plane can be written, if

$$r_c = \sqrt{(x-x_1)^2 - \beta^2 (y-y_1)^2 - \beta^2 z^2} \qquad (2)$$

in the form

$$\Omega = -\frac{1}{2\pi} \frac{\partial}{\partial x} \int_\tau \int \Delta \frac{\partial \Omega}{\partial z} \text{ arc cosh } \frac{x-x_1}{\beta \sqrt{(y-y_1)^2 + z^2}} \, dx_1 \, dy_1 +$$

$$\frac{1}{2\pi} \frac{\partial}{\partial x} \int_\tau \int \Delta \Omega \frac{z(x-x_1) \, dx_1 \, dy_1}{[(y-y_1)^2 + z^2] \, r_c} \qquad (3)$$

where the region τ is that portion of the $z = 0$ plane lying within the forecone from the point x,y,z (the forebranch of the hyperbola $r_c^2 = 0$).

Equation (3) represents a general solution to the wave equation and has yet to be put in a form which represents directly a solution to a problem arising in the study of wings. When equation (3) is so adapted in the next section, it will represent the velocity potential due to a distribution of sources, doublets, and elementary horseshoe vortices, the strength of which are given in terms of the wing shape and loading.

The following study of the adaptation of equation (3) to the particular boundary-value problems of wing theory requires the introduction of two axial systems other than the x,y,z Cartesian coordinates already defined. First, the ξ,η,z coordinate system is defined so that z is normal to the plane of the wing while ξ, η are both normal to z (i.e., lie in the plane of the wing) and make arbitrary angles with the x,y system (see sketch). If

$$\mu_1 = \sqrt{1+m_1^2}, \quad \mu_2 = \sqrt{1+m_2^2} \qquad (4)$$

The equations which relate the ξ, η, z to the x,y,z system are
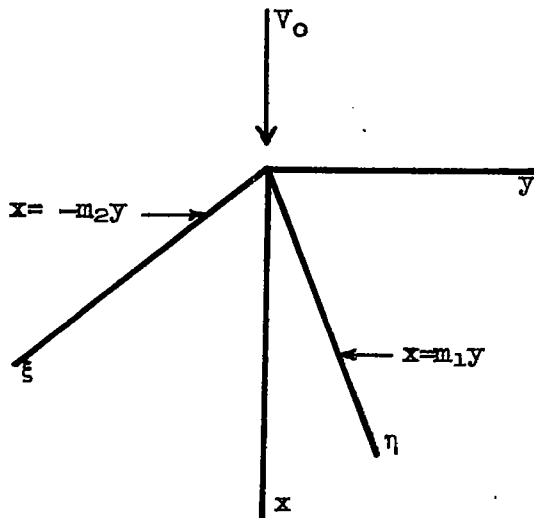
$$x = \frac{\eta m_1}{\mu_1} + \frac{\xi m_2}{\mu_2} \qquad \xi = \frac{\mu_2(x-m_1 y)}{m_1 + m_2}$$

$$y = \frac{\eta}{\mu_1} - \frac{\xi}{\mu_2} \qquad \eta = \frac{\mu_1(x+m_2 y)}{m_1 + m_2} \qquad\qquad (5)$$
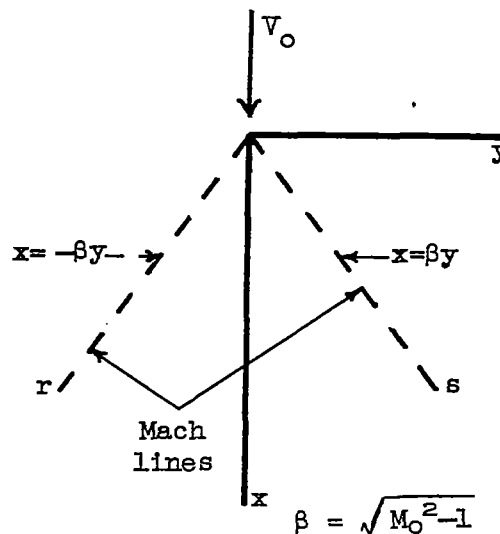
$$z = z \qquad\qquad\qquad z = z$$

and the Jacobian, the relation between the differential areas in the two systems, is

$$dx\ dy = \frac{m_1 + m_2}{\mu_1 \mu_2}\ d\xi\ d\eta \qquad\qquad (6)$$

Finally, the value of r_c is transformed by the equation

$$r_c = \sqrt{\frac{(\eta-\eta_1)^2(m_1{}^2-\beta^2)}{\mu_1{}^2} + \frac{2(\eta-\eta_1)(\xi-\xi_1)(m_1 m_2+\beta^2)}{\mu_1 \mu_2} + \frac{(\xi-\xi_1)^2(m_2{}^2-\beta^2)}{\mu_2{}^2} - \beta^2 z^2}$$

$$(7)$$

Second, the r,s,z coordinate system is also defined so that z is normal to the plane of the wing while r,s are both normal to z and lie along the traces of the Mach cone emanating from the origin (see sketch). It is apparent that the r,s,z coordinates are a special case of the ξ,η,z system formed when $m_1 = m_2 = \beta$. (Notice also that the x,y,z coordinates are obtained from ξ,η,z when $m_1 = 0$ and $m_2 = \infty$.) The equations which relate r,s and z to x,y and z are

$$x = -\beta y \qquad x = \beta y$$

Mach lines

$$\beta = \sqrt{M_0{}^2 - 1}$$

$$x = \frac{\beta}{M_O} (s+r) \qquad r = \frac{M_O}{2\beta} (x-\beta y)$$

$$y = \frac{1}{M_O} (s-r) \qquad s = \frac{M_O}{2\beta} (x+\beta y) \qquad \Bigg\} \qquad (8)$$

$$z = z \qquad z = z$$

and the Jacobian is

$$dx \; dy = \frac{2\beta}{M_O^2} dr \; ds \qquad (9)$$

The value of r_c is transformed by the equation

$$r_c = \sqrt{(4\beta^2/M_O^2)(r-r_1)(s-s_1) - \beta^2 z^2} \qquad (10)$$

THE THREE FUNDAMENTAL FORMULAS

As has been indicated already, the next purpose is to relate equation (3) to the three fundamental formulas arising in wing theory which are those relating the velocity potential to source, doublet, and vortex distributions. These distributions can be expressed in terms of the discontinuity in either φ or its gradients in the plane of the wing. In this way, the expression for a source distribution is obtained when the perturbation velocity potential φ is an even function with respect to the $z = 0$ plane and is expressed as a double integral involving $\Delta(\partial\varphi/\partial z)$ where the Δ notation denotes the jump in the value of $\partial\varphi/\partial z$ in crossing this plane; that is

$$\Delta \frac{\partial\varphi}{\partial z} = \left(\frac{\partial\varphi}{\partial z}\right)_{z=0+} - \left(\frac{\partial\varphi}{\partial z}\right)_{z=0-}$$

Similarly, a distribution of doublets is obtained when φ is an odd function with respect to the $z = 0$ plane and is expressed as a double integral involving $\Delta\varphi$. Finally, a vortex distribution results when φ is an odd function and given as a double integral involving the loading coefficient $\Delta p/q$ which is, in turn, equal to $2\Delta\varphi_x/V_O$. In these formulas $\Delta(\partial\varphi/\partial z)$, $\Delta\varphi$, and $\Delta p/q$ are, respectively, the strengths per unit area of the sources, doublets, and elementary horseshoe vortices.

The Source Distribution

The velocity potential at the point x,y,z of a unit source at the point $x_1,y_1,0$ is given by the equation $\varphi = -1/2\pi r_c$. The manner in which a distribution of these sources affects the potential will now be developed from equation (3). If, in equation (3), Ω is set equal to the velocity potential, and the potential in turn is assumed to be symmetrical above and below the x,y plane (as in the case of a symmetrical airfoil at zero lift), then equation (3) becomes

$$\varphi = -\frac{1}{2\pi}\frac{\partial}{\partial x}\int_\tau\int \Delta\frac{\partial\varphi}{\partial z} \text{ arc cosh } \frac{x-x_1}{\beta\sqrt{(y-y_1)^2+z^2}} \, dx_1 \, dy_1$$

Since the inverse hyperbolic term vanishes on the Mach forecone, the partial derivative can be carried through the double integral sign and there results

$$\varphi(x,y,z) = -\frac{1}{\pi}\int_\tau\int \frac{w_u(x_1,y_1)}{r_c} \, dx_1 \, dy_1 \tag{11}$$

where w_u is the vertical induced velocity on the upper side of the $z = 0$ plane and $\Delta(\partial\varphi/\partial z) = 2w_u$ by reasons of symmetry. Equation (11) is the familiar equation for the velocity potential due to a distribution of sources in the xy plane.

In the ξ,η,z coordinate system equation (11) becomes

$$\varphi(\xi,\eta,z) = -\frac{m_1+m_2}{\pi\mu_1\mu_2}\int_{\tau_1}\int \frac{w_u(\xi_1,\eta_1)}{r_c} \, d\xi_1 \, d\eta_1 \tag{12}$$

where the area τ is transferred to the ξ,η plane, and r_c is given in these coordinates by equation (7).

The Doublet Distribution

The velocity potential at the point x,y,z of a doublet at the point $x_1,y_1,0$ is given by the equation $\varphi = \beta^2 z/2\pi r_c^3$. The effect on the potential of a distribution of these doublets is not so obvious as it was in the case of the sources and considerable care must be shown in the development.

If the vertical induced velocity is considered always to be equal above and below the xy plane, but the potential to be discontinuous across it, then equation (3) for the velocity potential becomes

$$\varphi = \frac{1}{2\pi} \frac{\partial}{\partial x} \int_T \int \frac{z(x-x_1)\Delta\varphi(x_1,y_1)dx_1\ dy_1}{[(y-y_1)^2+z^2]\ r_c}$$

In this case the integrand does not vanish at the Mach cone and the partial derivative cannot be moved directly through the double integral sign. Writing in the limits of integration so that the first integration[1] is made with respect to y_1,

$$\varphi = \frac{1}{2\pi} \frac{\partial}{\partial x} \int_{-\infty}^{x-\beta z} dx_1 \int_{Y_1}^{Y_2} dy_1 \cdot \frac{z(x-x_1)\ \Delta\varphi(x_1,y_1)}{[(y-y_1)^2+z^2]\ \sqrt{(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2}}$$

(13a)

where

$$Y_1 = y - \frac{1}{\beta}\sqrt{(x-x_1)^2-\beta^2z^2}$$

and

$$Y_2 = y + \frac{1}{\beta}\sqrt{(x-x_1)^2-\beta^2z^2}$$

If in the y_1 integral x_1 is replaced by the value $x-\beta z$, the result is indeterminate. Such an indeterminate form can be evaluated, however, by excluding the limit $x-\beta z$ from the area of integration. Hence consider the integral

$$\varphi = \lim_{\epsilon \to 0} \frac{1}{2\pi} \frac{\partial}{\partial x} \int_{-\infty}^{x-\beta\sqrt{z^2+\epsilon^2}} dx_1 \int_{Y_1}^{Y_2} dy_1$$

$$\frac{z(x-x_1)\ \Delta\varphi(x_1,y_1)}{[(y-y_1)^2+z^2]\ \sqrt{(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2}}$$

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[1]Since in what follows the order of integration is important, the notation will be adopted that $\int dy \int dx\ f(x,y) = \int [\int f(x,y)dx\ ]dy$; that is, the integration is made first with respect to x. When the notation $\int\int f(x,y)dx\ dy$ is used, the order of integration is immaterial.

By formal manipulation

$$\varphi = \lim_{\epsilon \to 0} \frac{z}{2\pi} \int_{y-\epsilon}^{y+\epsilon} \frac{\sqrt{z^2+\epsilon^2} \; \Delta \varphi (x-\beta\sqrt{z^2+\epsilon^2}, \; y_1) dy_1}{[(y-y_1)^2+z^2] \; \sqrt{\epsilon^2-(y-y_1)^2}} +$$

$$\lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{x-\beta\sqrt{z^2+\epsilon^2}} dx_1 \; \frac{\partial}{\partial x} \int_{Y_1}^{Y_2} dy_1$$

$$\frac{z(x-x_1) \; \Delta \varphi (x_1,y_1)}{[(y-y_1)^2+z^2] \; \sqrt{(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2 z^2}} \qquad (13b)$$

Application of the mean value theorem to the first integral in equation (13b) yields

$$\lim_{\epsilon \to 0} \frac{z}{2\pi} \sqrt{z^2+\epsilon^2} \; \Delta \varphi (x-\beta\sqrt{z^2+\epsilon^2}, \; y+\theta\epsilon) \int_{y-\epsilon}^{y+\epsilon} \frac{dy_1}{[(y-y_1)^2+z^2]\sqrt{\epsilon^2-(y-y_1)^2}}$$

which becomes

$$\lim_{\epsilon \to 0} \frac{1}{2} \Delta \varphi (x-\beta\sqrt{z^2+\epsilon^2}, \; y+\theta\epsilon) = \frac{1}{2} \Delta \varphi (x-\beta z, y)$$

The second integral in equation (13b) is simplified by introducing the notation of the finite part. Define the symbol $\not\!\!\int$ by the equation

$$\int_0^{a(x)} \frac{\partial}{\partial x} \frac{f(y) dy}{\sqrt{a-y}} = -\frac{1}{2} \frac{\partial a}{\partial x} \int_0^{a(x)} \frac{f(y) dy}{(a-y)^{3/2}} \equiv \frac{\partial}{\partial x} \not\!\!\int_0^{a(x)} \frac{f(y) dy}{\sqrt{a-y}}$$

When applied to a single integral this definition is consistent with the usual ones for the finite part given as

$$\overline{\int_0^a \frac{f(y)dy}{(a-y)^{3/2}}} = \int_0^a \frac{f(y)-f(a)}{(a-y)^{3/2}} dy - 2 \frac{f(a)}{\sqrt{a}} = -2\left[\int_0^a \frac{f'(y)dy}{\sqrt{a-y}} + \frac{f(o)}{\sqrt{a}}\right]$$

However, when applied to double integrals an inconsistency with regard to the order of integration between the two symbols $\sqcap$ and $\not\!\!\int$ arises. Hadamard (reference 2) and Robinson (reference 3) both use the convention that the order of integration in the operation $\overline{\iint f(x,y)dy\ dx}$ is reversible; that is,

$$\overline{\int dy \int dx\ f(x,y)} = \overline{\int dx \int dy\ f(x,y)}$$

Such a convention excludes from the area of integration all singularities over which the order of integration is not reversible. These singular regions are then treated separately. This convention has the disadvantage that, in constructing a series of integrals, the value of a given integral is not independent of succeeding integrals.

The operator $\not\!\!\int$ avoids the above difficulty. The value of an integral defined by $\not\!\!\int_0^a f(y)dy$ is independent of succeeding operations.[2] At the same time, however, the order of integration of operations involving the sign $\not\!\!\int$ cannot be reversed. Hence

$$\int dy \not\!\!\int dx\ f(x,y) \neq \int dx \not\!\!\int dy\ f(x,y)$$

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[2]For example, according to reference 2

$$\overline{\int_0^\xi \frac{d\eta}{(\xi-\eta)^{3/2}\sqrt{\eta}}} = 0$$

but according to the same reference

$$\overline{\int_0^x d\xi \int_0^\xi \frac{d\eta}{(\xi-\eta)^{3/2}\sqrt{\eta}}} = -2\pi$$

However,

$$\not\!\!\int_0^\xi \frac{d\eta}{(\xi-\eta)^{3/2}\sqrt{\eta}} = \int_0^x d\xi \not\!\!\int_0^\xi \frac{d\eta}{(\xi-\eta)^{3/2}\sqrt{\eta}} = 0$$

Return now to the second integral in equation (13b). Applying the definition already mentioned for f, it is found for this type of integral that

$$\frac{\partial}{\partial x} \int_{A(x)-G(x)}^{A(x)+G(x)} \frac{f(x,y)\ dy}{\sqrt{G^2-(A-y)^2}} = \int_{A-G}^{A+G} \frac{\partial}{\partial x} \left[\frac{f(x,y)}{\sqrt{G^2-(A-y)^2}} \right] dy$$

By means of the last formula, the equation for φ becomes

$$\varphi = \frac{1}{2} \Delta\varphi(x-\beta z,y) - \frac{z\beta^2}{2\pi} \int_\tau dx_1 \int dy_1 \frac{\Delta\varphi(x_1,y_1)}{r_c^3} \tag{14a}$$

where r_c is given by equation (2).

Notice that if the integration had been made first with respect to x_1 the result would be

$$\varphi = \frac{1}{2\pi} \frac{\partial}{\partial x} \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{x-\beta\sqrt{(y-y_1)^2+z^2}} dx_1$$

$$\frac{z(x-x_1)\Delta\varphi(x_1,y_1)}{[(y-y_1)^2+z^2]\sqrt{(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2}}$$
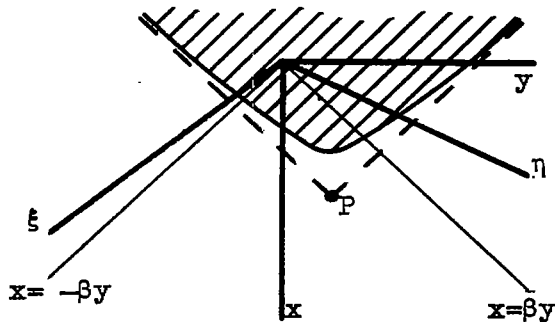
which reduces immediately to

$$\varphi = -\frac{z\beta^2}{2\pi} \int dy_1 \int_\tau dx_1 \frac{\Delta\varphi(x_1,y_1)}{r_c^3} \tag{14b}$$

Equations (14a) and (14b) illustrate the vital importance of the order of integration. In fact by subtraction the result can be derived that

$$\int dx_1 \int_\tau dy_1 \frac{\Delta\varphi}{r_c^{3/2}} - \int dy_1 \int_\tau dx_1 \frac{\Delta\varphi}{r_c^{3/2}} = \frac{\pi}{z\beta^2} \Delta\varphi(x-\beta z,y)$$
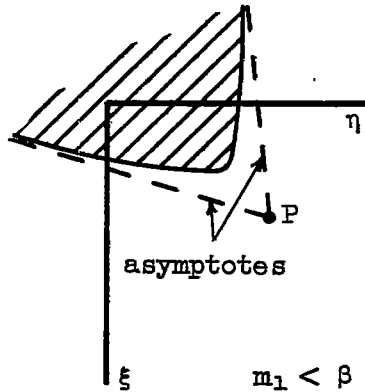
If the ξ,η,z coordinates are used, equation (13) becomes in the notation presented in equation (5)

$$\varphi = \frac{1}{2\pi}\left(\frac{1}{\mu_2}\frac{\partial}{\partial\eta} + \frac{1}{\mu_1}\frac{\partial}{\partial\xi}\right)\int\int_\tau \frac{z\left[(\eta-\eta_1)\frac{m_1}{\mu_1} + (\xi-\xi_1)\frac{m_2}{\mu_2}\right]\Delta\varphi\,d\eta_1 d\xi_1}{\left\{\left[(\eta-\eta_1)\frac{1}{\mu_1} - (\xi-\xi_1)\frac{1}{\mu_2}\right]^2 + z^2\right\} r_c} \tag{15}$$

The limits of integration depend now upon the position of the axes ξ and η with respect to the Mach lines in the x,y plane. The area τ is still bounded by the curve $r_c{}^2 = 0$ and infinity. The asymptotes for the curve $r_c{}^2 = 0$ are given by the two equations

$$\left.\begin{array}{l} \eta-\eta_1 = -\dfrac{\mu_1}{\mu_2}(\xi-\xi_1)\dfrac{m_2+\beta}{m_1-\beta} \\[3mm] \eta-\eta_1 = -\dfrac{\mu_1}{\mu_2}(\xi-\xi_1)\dfrac{m_2-\beta}{m_1-\beta} \end{array}\right\} \tag{16}$$

$m_1 < \beta$
$m_2 < \beta$

The sketch shows how the area τ in the xy plane transfers to the ξ,η plane. In the case in which both m_1 and m_2 are less than β (the case for which the sketch was drawn), the asymptotes are straight lines having positive slopes and the limits of integration are always from one side of the cone to the other and from minus infinity to some maximum value. Consider the case in which the integration is made first with respect to η_1. Then defining

$$L_0 = \eta + \frac{\mu_1}{\mu_2}(\xi-\xi_1)\frac{m_1 m_2 + \beta^2}{m_1{}^2 - \beta^2}$$

$$\tag{17}$$

$$L_1 = \frac{\mu_1\beta}{\beta^2 - m_1{}^2}\sqrt{\left[(\xi-\xi_1)\left(\frac{m_1+m_2}{\mu_2}\right)\right]^2 + z^2(m_1{}^2 - \beta^2)}$$

there results

$$\varphi = \lim_{\epsilon \to 0} \frac{z}{2\pi} \left(\frac{1}{\mu_2} \frac{\partial}{\partial \eta} + \frac{1}{\mu_1} \frac{\partial}{\partial \xi} \right) \int_{-\infty}^{\xi - \frac{\mu_2 \sqrt{(z^2 + \epsilon^2)} \sqrt{(\beta^2 - m_1^2)}}{m_1 + m_2}} d\xi_1 \int_{L_0 - L_1}^{L_0 + L_1} d\eta_1 \frac{\left[(\eta - \eta_1) \frac{m_1}{\mu_1} + (\xi - \xi_1) \frac{m_2}{\mu_2} \right] \Delta \varphi}{\left\{ \left[(\eta - \eta_1) \frac{1}{\mu_1} - (\xi - \xi_1) \frac{1}{\mu_2} \right]^2 + z^2 \right\} r_c}$$

(18)

where the z^2 has been increased by ϵ^2 since again when the ξ derivative is taken through the first integral an indeterminate form results. The evaluation of equation (18) proceeds just as for equation (13). Thus

$$\varphi = \lim_{\epsilon \to 0} \frac{z}{2\pi} \frac{1}{\mu_1} \int_{L_0' - L_1'}^{L_0' + L_1'} d\eta_1 \frac{(\eta - \eta_1) \frac{m_1}{\mu_1} + \frac{z m_2 \sqrt{\beta^2 - m_1^2}}{m_1 + m_2}}{\left\{ \left[(\eta - \eta_1) \frac{1}{\mu_1} - \frac{z \sqrt{\beta^2 - m_1^2}}{m_1 + m_2} \right]^2 + z^2 \right\}} \frac{\Delta \varphi \left(\xi - \frac{z \mu_2 \sqrt{\beta^2 - m_1^2}}{m_1 + m_2}, \eta_1 \right)}{r_c} +$$

$$\frac{z}{2\pi} \int_{-\infty}^{\xi - \frac{\mu_2 z}{m_1 + m_2} \sqrt{\beta^2 - m_1^2}} d\xi_1 \left(\frac{1}{\mu_2} \frac{\partial}{\partial \eta} + \frac{1}{\mu_1} \frac{\partial}{\partial \xi} \right) \int_{L_0 - L_1}^{L_0 + L_1} d\eta_1 \frac{(\eta - \eta_1) \frac{m_1}{\mu_1} + (\xi - \xi_1) \frac{m_2}{\mu_2}}{\left\{ \left[(\eta - \eta_1) \frac{1}{\mu_1} - (\xi - \xi_1) \frac{1}{\mu_2} \right]^2 + z^2 \right\}} \frac{\Delta \varphi (\xi_1, \eta_1)}{r_c}$$

(19)

where the prime on the L indicates its value when ξ_1 is given by the upper limit of equation (18). The first term in equation (19) becomes

$$\lim_{\epsilon \to 0} \frac{z}{2\pi\mu_1} \frac{\left[\frac{m_1 z (m_1 m_2 + \beta^2)}{(m_1 + m_2) \sqrt{\beta^2 - m_1^2}} + \frac{z m_2 \sqrt{\beta^2 - m_1^2}}{m_1 + m_2} \right]}{\left\{ \left[\frac{z (m_1 m_2 + \beta^2)}{(m_1 + m_2) \sqrt{\beta^2 - m_1^2}} - z \frac{\sqrt{\beta^2 - m_1^2}}{m_1 + m_2} \right]^2 + z^2 \right\}} \Delta \varphi (\xi_a, \eta_a) \int_{\lambda_1}^{\lambda_2} \frac{d\eta_1}{\sqrt{\frac{\beta^2 - m_1^2}{\mu_2^2}} \sqrt{(\lambda_2 - \eta_1)(\eta_1 - \lambda_1)}}$$

which reduces to

$$\frac{1}{2} \triangle \varphi (\xi_a, \eta_a)$$

where

$$\left. \begin{array}{l} \xi_a = \xi - \dfrac{z\mu_2 \sqrt{\beta^2 - m_1^2}}{m_1 + m_2} \\[4mm] \eta_a = \eta - \dfrac{z\mu_1 (m_1 m_2 + \beta^2)}{(m_1 + m_2) \sqrt{\beta^2 - m_1^2}} \end{array} \right\} \qquad (20)$$

The second term in equation (19) simplifies when the finite-part notation is introduced so that finally

$$\varphi = \frac{1}{2} \triangle \varphi (\xi_a, \eta_a) - \frac{z\beta^2 (m_1 + m_2)}{2\pi\mu_1\mu_2} \int d\xi_1 \int_\tau d\eta_1 \frac{\triangle \varphi(\xi_1, \eta_1)}{r_c^3} \qquad (21)$$

If the integration had been made first with respect to ξ_1, the results would have been

$$\varphi = \frac{1}{2} \triangle \varphi (\xi_b, \eta_b) - \frac{z\beta^2 (m_1 + m_2)}{2\pi\mu_1\mu_2} \int d\eta_1 \int_\tau d\xi_1 \frac{\triangle \varphi(\xi_1, \eta_1)}{r_c^3} \qquad (22)$$
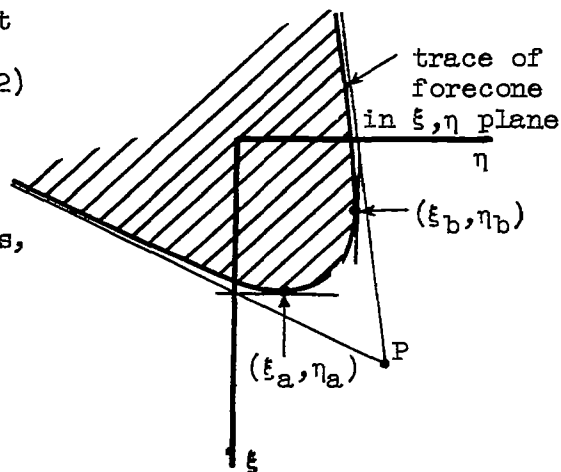
where

$$\left. \begin{array}{l} \xi_b = \xi - \dfrac{z\mu_1 \sqrt{\beta^2 - m_2^2}}{m_1 + m_2} \\[4mm] \eta_b = \eta - \dfrac{z\mu_2 (m_1 m_2 + \beta^2)}{(m_1 + m_2) \sqrt{\beta^2 - m_2^2}} \end{array} \right\} \qquad (23)$$

The geometric interpretation of the points ξ_b, η_b and ξ_a, η_a is as follows: They are the points at which the forecone in the ξ, η

plane (given by one branch of the curve $r_c{}^2=0$) attains maximum values
for η and ξ, respectively, (see sketch).

As the ξ and η axes approach
the Mach lines in the x,y plane, that
is, as m_2 and m_1 approach β, the
residue terms in equations (21) and (22)
approach $(1/2)\Delta\,\varphi(\xi_a,-\infty)$ and
$(1/2)\Delta\varphi(-\infty,\eta_b)$, respectively, which
represent the jump in potential infi-
nitely far distant from the point P
(and hence may be taken as zero). Thus,
when the ξ,η axes lie along the Mach
lines, thereby becoming the r,s axes
of equations (3), the equations for φ
are without the residue terms and the
order of integration is immaterial.
When m_1 and m_2 are greater than β
the same is true (i.e., the terms
$(1/2)\Delta\varphi(\xi_a,\eta_a)$ and $(1/2)\Delta\varphi(\xi_b,\eta_b)$
are missing from equations (21) and
(22), respectively), so that the effect
of a distribution of doublets on the
velocity potential can be summarized as
being

for $0 \leq m_1 < \beta,\ 0 \leq m_2 \leq \infty$

$$\varphi = \frac{1}{2}\,\Delta\,\varphi(\xi_a,\eta_a) - \frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\int d\xi_1 \fint_\tau d\eta_1\,\frac{\Delta\,\varphi(\xi_1,\eta_1)}{r_c{}^3} \tag{24a}$$

for $0 \leq m_2 < \beta,\ 0 \leq m_1 \leq \infty$

$$\varphi = \frac{1}{2}\,\Delta\,\varphi(\xi_b,\eta_b) - \frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\int d\eta_1 \fint_\tau d\xi_1\,\frac{\Delta\,\varphi(\xi_1,\eta_1)}{r_c{}^3} \tag{24b}$$

for $\beta \leq m_1 \leq \infty,\ \beta \leq m_2 \leq \infty$

$$\varphi = -\frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\int\fint_\tau\,\frac{\Delta\,\varphi(\xi_1,\eta_1)}{r_c{}^3}\,d\eta_1 d\xi_1 \tag{24c}$$

where $\xi_a, \eta_a, \xi_b,$ and η_b are given by equations (20) and (23).

There exists the interesting corollary obtained by subtracting equation (24a) from (24b); namely, that the difference between an integration of supersonic doublets made first in one order and then in the reverse is equal to the difference in the magnitude of the distribution at two points in the plane.

The Vortex Distribution

The velocity potential at the point x_1, y_1, z of an elementary horseshoe vortex at the point $x_1, y_1, 0$ is given by the equation $\varphi = -z(x-x_1)/2\pi \ [(y-y_1)^2+z^2]r_c$. If Ω in equation (3) is taken to be the induced velocity u in the direction of the free stream, and, as in the case involving doublets, the flow field is considered to contain no bodies with thickness (so $\Delta(\partial u/\partial z)$ is everywhere zero), then equation (3) reads

$$u = \frac{1}{2\pi} \frac{\partial}{\partial x} \int_T \int \frac{z(x-x_1)\Delta u \ dx_1 dy_1}{[(y-y_1)^2+z^2]r_c} \tag{25}$$

Now, since by definition

$$\varphi = \int_{-\infty}^{x} u \ dx$$

the potential for a distribution of vortices can be written

$$\varphi = \frac{1}{2\pi} \int_T \int \frac{z(x-x_1)\Delta u(x_1,y_1)dx_1 dy_1}{[(y-y_1)^2+z^2]\ r_c} \tag{26}$$

In terms of the ξ, η, z coordinates, equation (26) transforms by simple substitution to the equality

$$\varphi = \frac{(m_1+m_2)z}{2\pi\mu_1\mu_2} \int_T \int \int \frac{(\eta-\eta_1)\frac{m_1}{\mu_1} + (\xi-\xi_1)\frac{m_2}{\mu_2}}{\left\{ \left[(\eta-\eta_1)\frac{1}{\mu_1} - (\xi-\xi_1)\frac{1}{\mu_2}\right]^2 + z^2 \right\}} \frac{\Delta u(\xi_1,\eta_1)}{r_c} d\xi_1 \ d\eta_1 \tag{27}$$

APPLICATION OF FUNDAMENTAL FORMULAS TO THE PLANE OF THE WING

Pressure in Terms of Slope for Symmetrical Bodies

Since in linearized theory the equations for pressure coefficient and surface slope are

$$
\left.
\begin{array}{c}
C_p = -\dfrac{2u}{V_o} \\[2em]
\lambda = \dfrac{w}{V_o}
\end{array}
\right\}
\tag{28}
$$

the equation for the source distribution, equation (11), can be rewritten as

$$
C_p = \frac{2}{\pi} \frac{\partial}{\partial x} \int_T \int \frac{\lambda_u(x_1,y_1)\,dx_1\,dy_1}{r_c}
\tag{29}
$$

where λ_u is the value of the slope on the upper surface of a symmetrical body. In terms of the ξ, η, z system this becomes (compare the transition from equation (13) to equation (15))

$$
C_p = \frac{2}{\pi} \left(\frac{1}{\mu_2} \frac{\partial}{\partial \eta} + \frac{1}{\mu_1} \frac{\partial}{\partial \xi} \right) \int_T \int \frac{\lambda_u(\xi_1,\eta_1)\,d\xi_1\,d\eta_1}{r_c}
\tag{30}
$$

In carrying the partial derivative through the integrals there results (as in equation (19)) for $0 \le m_1 < \beta$ the equation

$$
C_p = \lim_{\epsilon \to 0} \frac{2}{\pi\mu_1} \int_{L_0{}'-L_1{}'}^{L_0{}'+L_1{}'} d\eta_1 \frac{\lambda_u\left(\xi - \dfrac{z\mu_2\sqrt{\beta^2-m_1{}^2}}{m_1+m_2}, \eta_1 \right)}{r_c} -
$$

$$
\frac{2(m_1+m_2)}{\pi\mu_1\mu_2} \int d\xi_1 \int_T d\eta_1 \frac{(\eta-\eta_1)\dfrac{m_1}{\mu_1} + (\xi-\xi_1)\dfrac{m_2}{\mu_2}}{r_c{}^3} \lambda(\xi_1,\eta_1)
$$

and this reduces to the expression

for $0 \leq m_1 < \beta$, $0 \leq m_2 \leq \infty$.

$$C_p = \frac{2}{\sqrt{\beta^2 - m_1{}^2}} \lambda_u(\xi_a,\eta_a) - \frac{2(m_1+m_2)}{\pi\mu_1\mu_2} \int d\xi_1 \oint_\tau d\eta_1 \frac{(\eta-\eta_1)\frac{m_1}{\mu_1} + (\xi-\xi_1)\frac{m_2}{\mu_2}}{r_c{}^3} \lambda_u(\xi_1,\eta_1)$$

(31a)

Similarly, for $0 \leq m_2 < \beta$, $0 \leq m_1 \leq \infty$

$$C_p = \frac{2}{\sqrt{\beta^2 - m_2{}^2}} \lambda_u(\xi_b,\eta_b) - \frac{2(m_1+m_2)}{\pi\mu_1\mu_2} \int d\eta_1 \oint_\tau d\xi_1 \frac{(\eta-\eta_1)\frac{m_1}{\mu_1} + (\xi-\xi_1)\frac{m_2}{\mu_2}}{r_c{}^3} \lambda_u(\xi_1,\eta_1)$$

(31b)

and for $\beta \leq m_1 \leq \infty$, $\beta \leq m_2 \leq \infty$

$$C_p = - \frac{2(m_1+m_2)}{\pi\mu_1\mu_2} \int \oint_\tau \frac{(\eta-\eta_1)\frac{m_1}{\mu_1} + (\xi-\xi_1)\frac{m_2}{\mu_2}}{r_c{}^3} \lambda_u(\xi_1,\eta_1) \, d\xi_1 \, d\eta_1 \quad (31c)$$

where $\xi_a, \eta_a, \xi_b,$ and η_b are given by equations (20) and (23)

As special cases of the results given by equations (31), consider ξ and η to represent the x,y and the r,s axes, respectively. In the former case, x replaces ξ as $m_2 \rightarrow \infty$ and y replaces η as $m_1 \rightarrow 0$; hence

$$C_p = \frac{2}{\beta} \lambda_u(x-\beta z,y) - \frac{2}{\pi} \int dx_1 \oint_\tau dy_1 \frac{(x-x_1)\lambda_u(x_1,y_1)}{r_c{}^3} \quad (32a)$$

and

$$C_p = - \frac{2}{\pi} \int dy_1 \oint_\tau dx_1 \frac{(x-x_1)\lambda_u(x_1,y_1)}{r_c{}^3} \quad (32b)$$

In the latter case r replaces ξ as $m_2 \rightarrow \beta$ and s replaces η as $m_1 \rightarrow \beta$; hence

$$C_p = \frac{-2\beta^2}{\pi M_0{}^3} \int\int_T \frac{(r-r_1)+(s-s_1)}{r_c{}^3} \lambda_u(r_1,s_1) \, dr_1 ds_1 \qquad (33)$$

In each of the equations (31) through (33), z can be set equal to zero without affecting the validity of the equalities. When z vanishes, the following identities hold

$$\left.\begin{array}{c} (\xi_b)_{z=0} = (\xi_a)_{z=0} = \xi \\[3mm] (\eta_b)_{z=0} = (\eta_a)_{z=0} = \eta \\[3mm] (r_c)_{z=0} \equiv r_0 \end{array}\right\} \qquad (34)$$

where

$$\left.\begin{array}{l} r_0 = \sqrt{(x-x_1)^2 - \beta^2(y-y_1)^2} \\[3mm] r_0 = \dfrac{2\beta}{M_0}\sqrt{(r-r_1)(s-s_1)} \\[2mm] \text{or} \\[2mm] r_0 = \sqrt{(\eta-\eta_1)^2\left(\dfrac{m_1{}^2-\beta^2}{\mu_1{}^2}\right) + 2(\eta-\eta_1)(\xi-\xi_1)\left(\dfrac{m_1 m_2+\beta^2}{\mu_1\mu_2}\right) + (\xi-\xi_1)^2\left(\dfrac{m_2{}^2-\beta^2}{\mu_2{}^2}\right)} \end{array}\right\} \qquad (35)$$

With these equations the pressure coefficient on the surface of a symmetrical nonlifting wing can be determined if the surface slope is given. The special cases of equations (31) in the plane $z=0$ and in the x,y and r,s coordinate systems are given in the summary of this section (see equations (47) and (50)).

Vertical Induced Velocity in Terms of the Jump in Potential

It is proposed next to find the vertical induced velocity in the $z=0$ plane as a function of the jump in potential across that plane.

Consider equations (24) for the doublet distribution and take the partial derivative with respect to z of both sides; then find the limit

of the resulting expression as z goes to zero. If equation (24a) is used, for example, there results for the first term

$$z \overset{\lim}{\to} 0 \quad \frac{\partial}{\partial z} \frac{1}{2} \Delta\varphi(\xi_a, \eta_a) = \frac{1}{2}\left(\frac{\partial\Delta\varphi}{\partial\xi_a}\right)_{z\to 0} \frac{\partial\xi_a}{\partial z} + \frac{1}{2}\left(\frac{\partial\Delta\varphi}{\partial\eta_a}\right)_{z\to 0} \frac{\partial\eta_a}{\partial z}$$

which becomes

$$-\frac{1}{2(m_1+m_2)\sqrt{\beta^2-m_1^2}}\left[\mu_2(\beta^2-m_1^2)\frac{\partial}{\partial\xi}\Delta\varphi(\xi,\eta)+\mu_1(m_1 m_2+\beta^2)\frac{\partial}{\partial\eta}\Delta\varphi(\xi,\eta)\right]$$

$$(36)$$

and for the second term

$$-z\overset{\lim}{\to}0 \quad \frac{\partial}{\partial z} \frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\int d\xi_1 \oint_\tau d\eta_1 \frac{\Delta\varphi(\xi_1,\eta_1)}{r_c^3}$$

But this reduces to

$$-\frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\int d\xi_1 \oint_\tau d\eta_1 \frac{\Delta\varphi(\xi_1,\eta_1)}{r_o^3} -$$

$$\frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} z\overset{\lim}{\to}0 \quad z\frac{\partial}{\partial z}\int d\xi_1 \oint_\tau d\eta_1 \frac{\Delta\varphi(\xi_1,\eta_1)}{r_c^3} \qquad (37)$$

and since

$$\oint_{\lambda_1}^{\lambda_2} \frac{d\eta_1}{[(\lambda_2-\eta_1)(\eta_1-\lambda_1)]^{3/2}} = 0$$

the second term in expression (37) vanishes. Finally, therefore, the vertical induced velocity in the plane of the wing w_u becomes

for $0 \le m_1 < \beta$, $0 \le m_2 \le \infty$

$$w_u = -\frac{1}{2(m_1+m_2)\sqrt{\beta^2-m_1^2}}\left[\mu_2(\beta^2-m_1^2)\frac{\partial}{\partial\xi}\Delta\varphi(\xi,\eta)+\mu_1(m_1m_2+\beta^2)\frac{\partial}{\partial\eta}\Delta\varphi(\xi,\eta)\right]-$$

$$\frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\int d\xi_1 \oint_\tau d\eta_1 \frac{\Delta\varphi(\xi_1,\eta_1)}{r_0^3} \qquad (38a)$$

and similarly

for $0 \leq m_2 < \beta$, $0 \leq m_1 \leq \infty$

$$w_u = -\frac{1}{2(m_1+m_2)\sqrt{\beta^2-m_2^2}}\left[\mu_1(\beta^2-m_2^2)\frac{\partial}{\partial\xi}\Delta\varphi(\xi,\eta)+\mu_2(m_1m_2+\beta^2)\frac{\partial}{\partial\eta}\Delta\varphi(\xi,\eta)\right]-$$

$$\frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\int d\eta_1 \oint_\tau d\xi_1 \frac{\Delta\varphi(\xi_1,\eta_1)}{r_0^3} \qquad (38b)$$

and for $\beta \leq m_2 \leq \infty$, $\beta \leq m_1 \leq \infty$

$$w_u = -\frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\iint_\tau \frac{\Delta\varphi(\xi_1,\eta_1)}{r_0^3} d\xi_1\,d\eta_1 \qquad (38c)$$

The special cases of equations (38) obtained when the ξ,η coordinates represent the x,y or the r,s systems are given in the summary of this section (see equations (48) and (51)).

Vertical Induced Velocity in Terms of the Loading

The equation for the loading coefficient in linearized theory can be written

$$\frac{\Delta p}{q} = \frac{2\Delta u}{V_0} \qquad (39)$$

so that equation (27) for the vortices yields for w_u the expression

$$w_u = \lim_{z \to 0} \frac{\partial}{\partial z} \frac{(m_1+m_2)zV_0}{4\pi\mu_1\mu_2} \times$$

$$\int_\tau \int \frac{(\eta-\eta_1)\frac{m_1}{\mu_1} + (\xi-\xi_1)\frac{m_2}{\mu_2}}{\left\{\left[(\eta-\eta_1)\frac{1}{\mu_1} - (\xi-\xi_1)\frac{1}{\mu_2}\right]^2 + z^2\right\}} \frac{\frac{\Delta p}{q}(\xi_1,\eta_1)}{r_c} d\xi_1\,d\eta_1 \quad (40)$$

The evaluation of w_u can be divided into two steps; first, the pro-
cedure necessary for carrying the derivative through the first integral,
and second, the calculation of I where

$$I = \lim_{z \to 0} \frac{(m_1+m_2)V_0}{4\pi\mu_1\mu_2} \times$$

$$\int_\tau \frac{\partial}{\partial z} \int \frac{\left[(\eta-\eta_1)\frac{m_1}{\mu_1} + (\xi-\xi_1)\frac{m_2}{\mu_2}\right]z}{\left\{\left[(\eta-\eta_1)\frac{1}{\mu_1} - (\xi-\xi_1)\frac{1}{\mu_2}\right]^2 + z^2\right\}} \frac{\frac{\Delta p}{q}(\xi_1,\eta_1)}{r_c} d\xi_1\,d\eta_1 \quad (41)$$

Again the order of integration is important. To begin with, the
first integration will be taken with respect to η_1. Further the case
$0 < m_1 < \beta$ will be considered. Hence, the equation for w_u becomes
(just as in the derivation of equation (18))

$$w_u = \lim_{\substack{z \to 0 \\ \epsilon \to 0}} \frac{(m_1+m_2)V_0}{4\pi\mu_1\mu_2} \frac{\partial}{\partial z} \int_{-\infty}^{\xi - \frac{\mu_2\sqrt{z^2+\epsilon^2}\sqrt{\beta^2-m_1^2}}{m_1+m_2}} d\xi_1 \int_{L_0-L_1}^{L_0+L_1} d\eta_1$$

$$\frac{\left[(\eta-\eta_1)\frac{m_1}{\mu_1} + (\xi-\xi_1)\frac{m_2}{\mu_2}\right]z}{\left\{\left[(\eta-\eta_1)\frac{1}{\mu_1} - (\xi-\xi_1)\frac{1}{\mu_2}\right]^2 + z^2\right\}} \frac{\frac{\Delta p}{q}(\xi_1,\eta_1)}{r_c} \quad (42)$$

where the values of L_0 and L_1 are given by equation (17) and the limit as ϵ goes to zero is to be taken first. Equation (36) reduces to

$$w_u = -\lim_{\substack{z \to 0 \\ \epsilon \to 0}} \frac{V_0 z^2 \sqrt{\beta^2 - m_1^2}}{4\pi\mu_1 \sqrt{z^2 + \epsilon^2}} \int_{L_0' - L_1'}^{L_0' + L_1'} d\eta_1$$

$$\frac{(\eta - \eta_1) \frac{m_1}{\mu_1} + \frac{z m_2 \sqrt{\beta^2 - m_1^2}}{m_1 + m_2}}{\left\{ \left[(\eta - \eta_1) \frac{1}{\mu_1} - \frac{z \sqrt{\beta^2 - m_1^2}}{m_1 + m_2} \right]^2 + z^2 \right\}} \quad \frac{\frac{\Delta p}{q}(\xi_a, \eta_1)}{r_c} + I \qquad (43)$$

The quantity within the integral of the first term of equation (43) is the same as the similar term in equation (19); hence by analogy

$$w_u = -\frac{V_0 \sqrt{\beta^2 - m_1^2}}{4} \frac{\Delta p}{q}(\xi, \eta) + I \qquad (44)$$

The evaluation of I requires some care. Consider the following integral which contains all the difficulties involved in I.

$$I_0 = \lim_{z \to 0} \int_a^b \frac{\partial}{\partial z} \left[\frac{z f(y_1, z)}{z^2 + (y - y_1)^2} \right] dy_1 \qquad b > y > a$$

where $f(y, z)$ and its derivative is bounded and continuous in the interval $a \le y \le b$.

Integrating by parts

$$I_0 = \lim_{z \to 0} \frac{\partial}{\partial z} \left[f(a, z) \text{ arc tan } \frac{y-a}{z} - f(b, z) \text{ arc tan } \frac{y-b}{z} + \right.$$

$$\left. \int_a^b \frac{\partial f(y_1, z)}{\partial y_1} \text{ arc tan } \frac{y-y_1}{z} dy_1 \right]$$

and since

$$\lim_{z \to 0} \int_a^b \frac{\partial f_z(y_1, z)}{\partial y_1} \arctan \frac{y-y_1}{z} \, dy_1 = \pi f_z(y, 0) - \frac{\pi}{2} \left[f_z(a, 0) + f_z(b, 0) \right]$$

this becomes

$$I_o = \frac{f(b)}{y-b} - \frac{f(a)}{y-a} - \int_a^b \frac{f'(y_1) dy_1}{y-y_1} + \pi f_z(y, 0)$$

It is, however, more convenient to write I_o in terms of $f(y_1)$ and not its derivative. In order to do this, it is necessary to introduce another notation involving singular integrals. The concept of Cauchy's principal part is adopted and defined in the following way:

$$\int_a^b \frac{f(y_1) dy_1}{y-y_1} \equiv \frac{\partial}{\partial y} \int_a^b f(y_1) \ln |y-y_1| \, dy_1$$

This procedure can be generalized (see also appendix D in reference 4) so that for

$$\oint_a^b \frac{f(y_1) dy_1}{(y_1-y)^2} = \oint_a^b \frac{\partial}{\partial y} \frac{f(y_1) dy_1}{y_1-y} \equiv \frac{\partial}{\partial y} \int_a^b \frac{f(y_1) dy_1}{y_1-y} \qquad (45a)$$

This definition of the symbol $\oint$ can be made in another way. If the indefinite integral is expressible in the form

$$\int \frac{f(y_1) dy_1}{(y_1-y)^2} = G(y_1, y) + \text{constant}$$

then

$$\oint_a^b \frac{f(y_1) dy_1}{(y_1-y)^2} \equiv G(b, y) - G(a, y) \qquad (45b)$$

Thus the conventional rules for evaluating a definite integral from the indefinite integral can be used.

Using equation (45b), one can derive by integration by parts the relation

$$\fint_a^b \frac{f(y_1,0)\,dy_1}{(y_1-y)^2} = \frac{f(b)}{y-b} - \frac{f(a)}{y-a} - \int_a^b \frac{f'(y_1,0)\,dy_1}{y-y_1}$$

Hence the equation for I_0 becomes

$$I_0 = \fint_a^b \frac{f(y_1,0)\,dy_1}{(y_1-y)^2} + \pi\, f_z\,(y,0) \tag{45c}$$

By means of the concepts introduced by equations (45), I can be evaluated[3] and finally w_u can be written

for $0 \le m_1 \le \beta,\ 0 \le m_2 \le \infty$

$$\frac{w_u}{V_0} = -\frac{\sqrt{\beta^2-m_1{}^2}}{4}\frac{\Delta p}{q}\,(\xi,\eta) +$$

$$\frac{m_1+m_2}{4\mu_1\mu_2\pi}\int_\tau d\xi_1 \fint d\eta_1 \frac{(\eta-\eta_1)\frac{m_1}{\mu_1} + (\xi-\xi_1)\frac{m_2}{\mu_2}}{\left[(\eta-\eta_1)\frac{1}{\mu_1} - (\xi-\xi_1)\frac{1}{\mu_2}\right]^2}\frac{\Delta p/q}{r_0} \tag{46a}$$

Similarly, for $0 \le m_2 \le \beta,\ 0 \le m_1 \le \infty$

$$\frac{w_u}{V_0} = -\frac{\sqrt{\beta^2-m_2{}^2}}{4}\frac{\Delta p}{q}\,(\xi,\eta) +$$

$$\frac{m_1+m_2}{4\pi\mu_1\mu_2}\int_\tau d\eta_1 \fint d\xi_1 \frac{(\eta-\eta_1)\frac{m_1}{\mu_1} + (\xi-\xi_1)\frac{m_2}{\mu_2}}{\left[(\eta-\eta_1)\frac{1}{\mu_1} - (\xi-\xi_1)\frac{1}{\mu_2}\right]^2}\frac{\Delta p/q}{r_0} \tag{46b}$$

and, finally, for $\beta \le m_1 \le \infty,\ \beta \le m_2 \le \infty$

$$\frac{w_u}{V_0} = \frac{m_1+m_2}{4\pi\mu_1\mu_2}\int_\tau\!\!\int \frac{(\eta-\eta_1)\frac{m_1}{\mu_1} + (\xi-\xi_1)\frac{m_2}{\mu_2}}{\left[(\eta-\eta_1)\frac{1}{\mu_1} - (\xi-\xi_1)\frac{1}{\mu_2}\right]^2}\frac{\Delta p/q}{r_0}\,d\xi_1\,d\eta_1 \tag{46c}$$

---

[3] In the evaluation of I the term representing $f(y_1,z)$ can be written as a function of z^2. Hence, the term $\pi f_z(y,0)$ in equation (45c) vanishes.

Again the special cases of equations.(46) when ξ,η become x,y or r,s are given in the following summary (see equations (49) and (52)).

Summary of Formula for w and C_p in the Plane of the Wing

For the x,y coordinates.— In these cases $\xi \to x$, $y \to \eta$, $m_1 \to 0$, and $m_2 \to \infty$.

$$C_p = \frac{2}{\beta} \lambda_u(x,y) - \frac{2}{\pi} \int dx_1 \fint_\tau dy_1 \frac{(x-x_1)\lambda_u(x_1,y_1)}{[(x-x_1)^2-\beta^2(y-y_1)^2]^{3/2}} \tag{47a}$$

$$C_p = -\frac{2}{\pi} \int dy_1 \fint_\tau dx_1 \frac{(x-x_1)\lambda_u(x_1,y_1)}{[(x-x_1)^2-\beta^2(y-y_1)^2]^{3/2}} \tag{47b}$$

$$w_u = -\frac{\beta}{2} \Delta u(x,y) - \frac{\beta^2}{2\pi} \int dx_1 \fint_\tau dy_1 \frac{\Delta\varphi(x_1,y_1)}{[(x-x_1)^2-\beta^2(y-y_1)^2]^{3/2}} \tag{47c}$$

$$w_u = -\frac{\beta^2}{2\pi} \int dy_1 \fint_\tau dx_1 \frac{\Delta\varphi(x_1,y_1)}{[(x-x_1)^2-\beta^2(y-y_1)^2]^{3/2}} \tag{48}$$

$$\frac{w_u}{V_o} = -\frac{\beta}{4} \frac{\Delta p}{q}(x,y) + \frac{1}{4\pi} \fint dx_1 \fint_\tau dy_1 \frac{(x-x_1)(\Delta p/q)}{(y-y_1)^2\sqrt{(x-x_1)^2-\beta^2(y-y_1)^2}} \tag{49a}$$

$$\frac{w_u}{V_o} = \frac{1}{4\pi} \fint dy_1 \fint_\tau dx_1 \frac{(x-x_1)(\Delta p/q)}{(y-y_1)^2\sqrt{(x-x_1)^2-\beta^2(y-y_1)^2}} \tag{49b}$$

For the r,s coordinates.—. In these cases $\xi \to r$, $\eta \to s$, $m_1 \to \beta$, and $m_2 \to \beta$.

$$C_p = -\frac{1}{2\pi\beta} \iint_\tau \frac{(r-r_1)+(s-s_1)}{[(r-r_1)(s-s_1)]^{3/2}} \lambda_u(r_1,s_1)dr_1ds_1 \tag{50}$$

$$w_u = -\frac{M_o}{8\pi} \iint_\tau \frac{\Delta\varphi(r_1,s_1)dr_1ds_1}{[(r-r_1)(s-s_1)]^{3/2}} \tag{51}$$

$$\frac{w_u}{V_o} = \frac{\beta}{4\pi} \iint_\tau \frac{[(r-r_1)+(s-s_1)]\,(\Delta p/q)dr_1ds_1}{[(s-s_1)-(r-r_1)]^2\sqrt{(r-r_1)(s-s_1)}} \tag{52}$$

It should be emphasized that in the r,s coordinate system the order of integration is immaterial.

SOME ALTERNATIVE EXPRESSIONS

It is sometimes very convenient to be able to express the equations given in the previous sections in a slightly different form. Consider, for example, equation (11) which gives the velocity potential due to a distribution of sources, thus

$$\varphi = -\frac{1}{\pi} \iint_\tau \frac{w_u(x_1,y_1)dx_1dy_1}{r_c}$$

where the area τ can be defined (see sketch) as the area bounded by the wing plan form[4] and the forecone from the point $P(x,y,z)$. Define now the area τ_1 as being the area bounded by the plane $x=x_1$ and the wing plan form ahead of the plane $x=x_1$ (see sketch). It is apparent that r_c is a pure real quantity everywhere inside the area τ and is a pure imaginary quantity everywhere outside τ and inside τ_1. The same is true, of course, of r_c^3. It is clear that all other terms in the integrands of the integrals which have been considered in the preceding section, in particular $w_u(x_1,y_1)$ in equation (11), are

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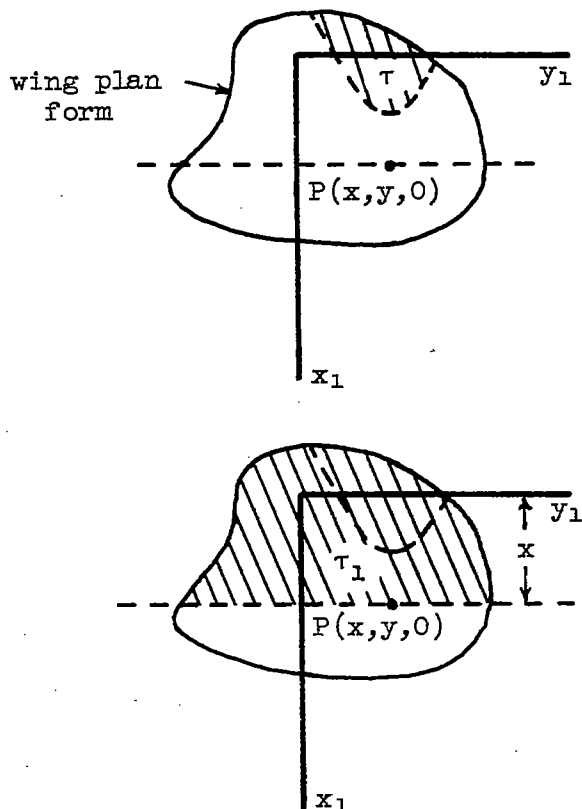
[4]The actual definition of τ, that it is the area within the Mach forecone, is often replaced by the one used here since the strengths of the sources, etc., are zero ahead of the wing.

always real over the entire wing plan forms. Hence the integral

$$-\frac{1}{\pi}\int_{\tau_1}\int \frac{w_u(x_1,y_1)dx_1dy_1}{r_c}$$

is a complex quantity, the real part of which is the velocity potential; thus

$$\varphi = -\frac{1}{\pi}\,r.p.\int_{\tau_1}\int \frac{w_u(x_1,y_1)dx_1dy_1}{r_c} \qquad (53)$$

Similarly, each of the integrals in equations (47) and (48) may be replaced by the real parts of their values taken over the area τ_1.

The evaluation of the terms involving the finite part are particularly simple when the τ_1 area is used since, if

$$\int \frac{f(y)dy}{(a-y)^{3/2}} = F(y)+C$$

then for positive a and $b \neq a$

$$r.p.\int_0^b \frac{f(y)dy}{(a-y)^{3/2}} = r.p.\ [F(b) - F(o)] \qquad (54)$$

For example, consider

$$I = \int_0^a \frac{y^2dy}{(a^2-y^2)^{3/2}} = r.p.\int_0^b \frac{y^2dy}{(a^2-y^2)^{3/2}}$$

where $a < b$. From the relation

$$\int \frac{y^2dy}{(a^2-y^2)^{3/2}} = \frac{y}{\sqrt{a^2-y^2}} - \arcsin\frac{y}{a}$$

together with equation (54)

$$I = r.p.\left(\frac{b}{\sqrt{a^2-b^2}} - \arcsin\frac{b}{a}\right) = -\frac{\pi}{2}$$

Notice the simple extension that

$$\fint_{-a}^{a} \frac{y^2 dy}{(a^2-y^2)^{3/2}} = r.p. \fint_{-b}^{b} \frac{y^2 dy}{(a^2-y^2)^{3/2}} = -\pi$$

Further examples using the real part of the integration taken over the area τ_1 will be given in the next section on special applications.

## SPECIAL APPLICATIONS

An Integral Equation

Applications of the results given by equations (48) through (52) are apparent. One of the more important uses, however, comes in the development of integral equations necessary for the solution of many supersonic wing theory problems.

An example of such an application arises in the analysis of the slender rectangular wing at an angle of attack α. Since the wing chord is long compared to its span, and since along the side edges the loading falls to zero, an approximation to $\Delta p/q$ is given by the equation

$$\frac{\Delta p}{q} = 4\alpha f\left(\frac{x}{s}\right) \sqrt{1-(y/s)^2} \qquad (55)$$

where $f(x/s)$ is an unknown function. The assumption made when using equation (55) is, of course, to fix the spanwise variation of loading but leave the chordwise variation arbitrary.

The function f can be determined by the condition that the value of w_u in equation (49a) is a constant all along the center line of the wing. The area τ is indicated by the shaded region in the last sketch so that equation (49a) for the case $y=0$ (and for added simplicity $\beta=1$) becomes for $x > s$
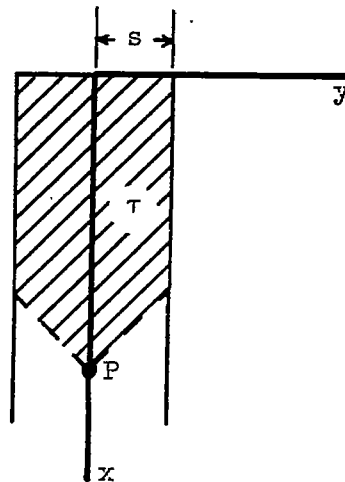
$$\frac{w_u}{V_0} = -\alpha f\left(\frac{x}{s}\right) + \frac{\alpha}{\pi}\int_0^{x-s}(x-x_1)f\left(\frac{x_1}{s}\right)dx_1 \int_{-s}^{s}\frac{\sqrt{1-(y_1/s)^2}\,dy_1}{y_1^2\sqrt{(x-x_1)^2-y_1^2}} +$$

$$\frac{\alpha}{\pi}\int_{x-s}^{x}(x-x_1)f\left(\frac{x_1}{s}\right)dx_1 \int_{-(x-x_1)}^{x-x_1}\frac{\sqrt{1-(y_1/s)^2}\,dy_1}{y_1^2\sqrt{(x-x_1)^2-y_1^2}}$$

and for $0 < x < s$

$$\frac{w_u}{V_0} = -\alpha f\left(\frac{x}{s}\right) + \frac{\alpha}{\pi}\int_0^{x}(x-x_1)f\left(\frac{x_1}{s}\right)dx_1 \int_{-(x-x_1)}^{x-x_1}\frac{\sqrt{1-(y_1/s)^2}\,dy_1}{y_1^2\sqrt{(x-x_1)^2-y_1^2}}$$

Introduce the notation

$$\theta_1 = \frac{x_1}{s}, \quad \theta = \frac{x}{s}, \quad k_1 = \frac{1}{\theta-\theta_1}, \quad k_2 = \theta - \theta_1$$

and these equations become,[5] since $\alpha = -\dfrac{w_u}{V_0}$

for $0 < \theta < 1$

$$1 = f(\theta) + \frac{2}{\pi}\int_0^{\theta}k_2B_2f(\theta_1)d\theta_1$$

for $1 < \theta$

$$1 = f(\theta) + \frac{2}{\pi}\int_{\theta-1}^{\theta}k_2B_2f(\theta_1)d\theta_1 + \frac{2}{\pi}\int_0^{\theta-1}E_1f(\theta_1)d\theta_1$$

$$\left.\right\}(56)$$

---

[5]The symbols B and E indicate elliptic integrals. Thus

$$E_n = \int_0^1\sqrt{\frac{1-k_n^2t^2}{1-t^2}}\,dt; \quad K_n = \int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k_n^2t^2)}}; \text{ and}$$

$$B_n = \frac{E_n-(1-k_n^2)K_n}{k_n^2}$$

The solution to these equations has been obtained numerically and has the form shown in the sketch.

Since from equation (55) the average spanwise loading P can be readily calculated as

$$P = \frac{1}{2s}\int_{-s}^{s}\frac{\Delta p}{q}\,dy = \alpha \pi f\left(\frac{x}{s}\right)$$

the curve for average span loading can easily be constructed. This curve is also shown in a sketch together with a portion of the variation of P obtained from an exact linearized analysis.

In the interval where the comparison can be made (i.e., near the leading edge) the agreement will be the poorest because in this region the spanwise variation deviates most radically from the value assumed in the construction of the integral equation.

$f(\theta)$

θ

exact linearized

from eq. (47)

$P\left(\frac{x}{s}\right)$

0   2   4   6 $\frac{x}{s}$
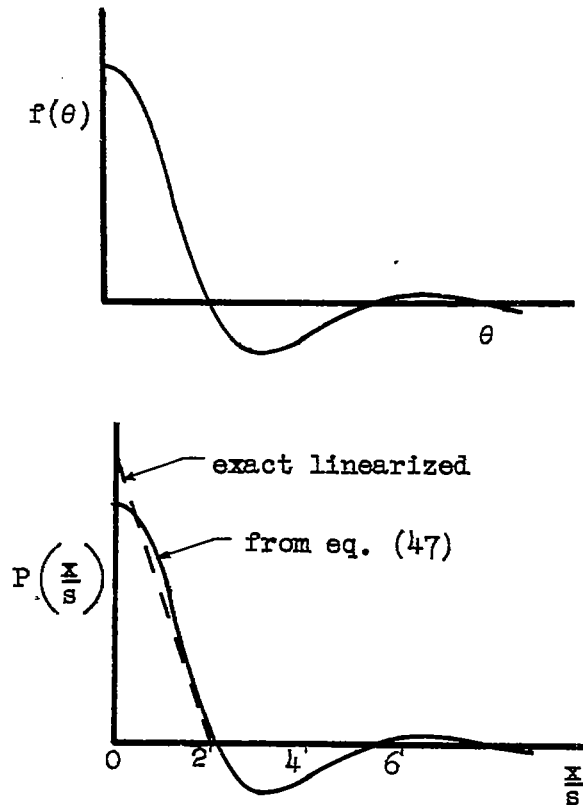
Drag Reversibility Theorem

The well-known theorem that the drag of a symmetrical nonlifting body is the same in forward and reversed flight at the same speed (see reference 5 or 6) can be derived in another way using the results of the preceding sections.

By definition

$$C_D = \frac{1}{S}\int_S \int 2\lambda_u(x,y)C_p(x,y)\,dy\,dx \qquad (57)$$

where

S   area of the wing

λ_u slope of the upper surface

C_p pressure coefficient

Using equation (47b),

$$C_D = -\frac{4}{S\pi} \int_S \int \lambda_u(x,y)\,dy\,dx \left[\text{r.p.} \int dy_1 \oint_{\tau_1} dx_1\, \frac{(x-x_1)\lambda_u(x_1,y_1)}{r_c{}^3} \right] \quad (58)$$

Now the equation for the drag coefficient in reversed flow can be obtained by:

1. Replacing the area τ_1 by τ_2 such that $\tau_1 + \tau_2 = S$

2. Rotating the axial system in the xy plane through 180°

3. Reversing the signs of $\lambda_u(x,y)$ and $\lambda_u(x_1,y_1)$

There results

$$C_{D_r} = -\frac{4}{S\pi} \int_S \int \lambda_u(x,y)\,dy\,dx \left[\text{r.p.} \int dy_1 \oint_{\tau_2} dx_1\, \frac{(x_1-x)\lambda_u(x_1,y_1)}{r_c{}^3} \right] (59)$$

and subtracting equation (59) from (58) gives

$$C_D - C_{D_r} = -\frac{4}{S\pi} \text{ r.p.} \int_S dy \int dx \int_S dy_1 \oint dx_1\, \frac{x\lambda_u(x_1,y_1)\lambda_u(x,y)}{r_c{}^3} +$$

$$\frac{4}{S\pi} \text{ r.p.} \int_S dy \int dx \int_S dy_1 \oint dx_1\, \frac{x_1\lambda_u(x_1,y_1)\lambda_u(x,y)}{r_c{}^3} \quad (60)$$

Since the symbols x_1, y_1, x, y are dummy variables of integration, the last term in equation (60) can be written

$$\frac{4}{S\pi} \text{ r.p.} \int_S dy_1 \int dx_1 \int_S dy \oint dx\, \frac{x\lambda_u(x,y)\lambda_u(x_1,y_1)}{r_c{}^3}$$

and reversing the operators $\int dy_1 \int dx_1$ and $\int dy \int dx$ (but always preserving the same order within the operation) yields for the second term in equation (60) the same expression as the first term except for sign. Hence

$$C_D - C_{D_r} = 0$$

or

$$C_D = C_{D_r}$$

as was to be shown.

Lift on Wings With Supersonic Edges

The lift on any wing can be written

$$\frac{L}{q} = \int\int_S \int \frac{\Delta p}{q} \, dy \, dx$$

Moreover,

$$\int_{L.E.}^{T.E.} \frac{\Delta p}{q} \, dx = \frac{4\phi_{T.E.}}{V_0}$$

where T.E. and L.E. denote the trailing edge and leading edge, respectively, and $\phi_{T.E.}$ is the value of the velocity potential on the upper surface of the wing at the trailing edge.

Consider now a wing with all edges supersonic and a straight trailing edge not necessarily at right angles to the free-stream direction. Let the wing be a plate having arbitrary twist and camber. Then for a point on the wing, the velocity potential from equation (53) can be written
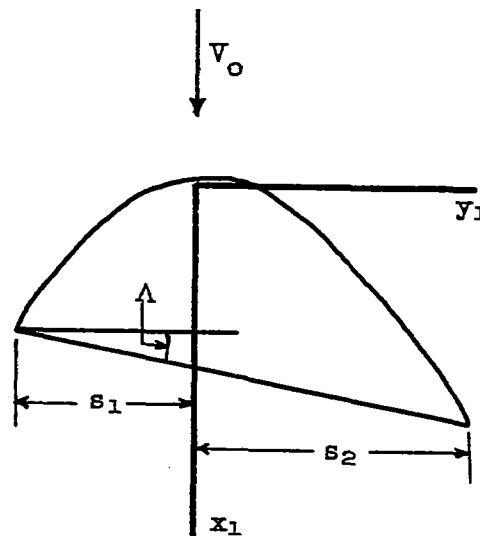
$$\phi = -\frac{1}{\pi} \, r.p. \int_{\tau_1} \int \frac{w_u(x_1,y_1)dx_1dy_1}{\sqrt{(x-x_1)^2-\beta^2(y-y_1)^2}}$$

and if the equation of the trailing edge is

$$x = a + y \tan \Lambda$$

where a is some constant, then

$$\phi_{T.E.} = -\frac{1}{\pi} \, r.p. \int_S \int \frac{w_u(x_1,y_1) \, dx_1 \, dy_1}{\sqrt{(a+y \tan\Lambda-x_1)^2-\beta^2(y-y_1)^2}}$$

so that the total lift on the wing can be written

$$\frac{L}{q} = -\frac{4}{\pi V_o} \; r.p. \int_{s_1}^{s_2} dy \int_S \int \frac{w_u(x_1,y_1)dx_1dy_1}{\sqrt{(a+y\tan\Lambda-x_1)^2-\beta^2(y-y_1)^2}}$$

The area S, being that of the wing plan form, does not depend on y so the y integration can be made first and, since the edges of the wing are supersonic, the interval $s_1 \leq y \leq s_2$ must always contain the roots λ_1 and λ_2 of the expression under the radical. Hence

$$r.p. \int_{-s}^{s} \frac{dy}{\sqrt{(\beta^2-\tan^2\Lambda)(\lambda_1-y)(y-\lambda_2)}} = \int_{\lambda_1}^{\lambda_2} \frac{dy}{\sqrt{(\beta^2-\tan^2\Lambda)(\lambda_1-y)(y-\lambda_2)}}$$

and since

$$\int_{\lambda_1}^{\lambda_2} \frac{dy}{\sqrt{(\lambda_1-y)(y-\lambda_2)}} = \pi$$

then

$$\frac{L}{q} = \frac{-4}{\sqrt{\beta^2-\tan^2\Lambda}} \int_S \int \frac{w_u(x_1,y_1)}{V_o} dx_1dy_1 \qquad (61)$$

An alternative expression for equation (61) is

$$C_L = \frac{4\bar{\alpha}}{\sqrt{\beta^2-\tan^2\Lambda}} \qquad (62)$$

where $\bar{\alpha}$ is the average angle of attack of the surface and by definition

$$\bar{\alpha} = \frac{-1}{S} \int_S \int \frac{w_u(x_1,y_1)}{V_o} dx_1dy_1 \qquad (63)$$

It is interesting to notice that the lift coefficient for such a wing is the same as that for a two-dimensional flat plate flying at an angle of attack $\bar{\alpha}$ into a free stream, the speed of which is given by the component of velocity normal to the trailing edge of the three-dimensional wing just studied. This result has been derived previously in reference 7.

Ames Aeronautical Laboratory,
    National Advisory Committee for Aeronautics,
        Moffett Field, Calif., Oct. 16, 1950.

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