NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2270

THEORETICAL DAMPING IN ROLL AND ROLLING EFFECTIVENESS OF SLENDER CRUCIFORM WINGS

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Washington
January 1951

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SUMMARY

The theory of slender wings is applied to the determination of the characteristics in roll of slender cruciform wings. The analysis treats the damping in roll and the rolling moment supplied by differential incidence of opposite wing panels. The methods employed in the solution can be applied to slender-cruciform-wing problems having arbitrarily assigned boundary conditions. It is found that the coefficient of damping in roll (based on the horizontal-wing area) for the cruciform wing is 62 percent greater than that of the plane wing having the same aspect ratio, and that the rolling effectiveness (wing-tip helix angle per unit of surface deflection) of the cruciform wing having four equally deflected panels is 6 percent less than that of the plane wing.

INTRODUCTION

Little information is currently available which will permit an evaluation of the stability and control problems associated with the use of cruciform wings. In some instances the characteristics (e.g., the important case of lift) of these wings may be estimated from known solutions for planar-wing systems, but in other cases the effect of interference between components may be so large as to invalidate such estimates. Additional theoretical treatment is therefore required to establish the magnitude of these interference effects.

The present analysis considers the case of a slender cruciform wing. The problem will be treated by the well-known methods of slender-wing theory, as introduced by Jones (reference 1), and extended by Ribner and others to determine the aerodynamic characteristics of plane slender wings. Spreiter (reference 2) has used an extension of this method to treat the case of a slender-cruciform-wing and body combination inclined at small angles of pitch and yaw. In the present report the method is applied to the estimation of the damping in roll and the rolling moment due to differential incidence of opposite wing panels of a slender cruciform wing.

The use of slender-wing theory reduces the problem to that of finding the velocity potential defining the two-dimensional flow of an ideal fluid about a cruciform lamina; solutions satisfying the prescribed boundary conditions may therefore be obtained by the methods of classical hydrodynamics, in particular, the method of conformal transformation. It has been brought to the author's attention, since the completion of the present analysis, that Westwater (reference 3) has previously applied a conformal transformation similar to that given herein to the case of a multibladed, infinite pitch propeller (i.e., a rotating two-dimensional lamina). The surface velocity potential is obtained as a Fourier series which is summable in closed form in the case of a cruciform lamina. Westwater's approach may also be used to find the potential satisfying any assigned boundary conditions on the surfaces of a cruciform wing; the present analysis differs in that the potential is determined in the form of a definite integral.

In a recent paper (reference 4) Bleviss has studied the case of a cruciform triangular wing having supersonic leading edges. The analysis, which was based on the linearized theory, included an approximation of the rolling moment due to a small differential deflection of the horizontal surfaces.

SYMBOLS

A aspect ratio $\left(\frac{b^2}{S}\right)$

b span of wing

c_o root chord of wing

C_l rolling-moment coefficient $\left(\frac{L'}{qSb}\right)$

C_{l_p} coefficient of damping in roll $\left(\frac{C_l}{pb/2U}\right)$

C_{l_δ} coefficient of rolling-moment effectiveness $\left(\frac{\partial C_l}{\partial \delta}\right)$

$\left.\begin{array}{l}\text{cn } u\\ \text{sn } u\end{array}\right\}$ Jacobian elliptic functions, argument u and modulus k

$E(t,k)$ elliptic integral of the second kind, argument t and modulus k

E complete elliptic integral of the second kind, modulus k

$F(t,k)$	elliptic integral of the first kind, argument t and modulus k
i	$\sqrt{-1}$
k	modulus of an elliptic integral or function
K	complete elliptic integral of the first kind, modulus k
L	lift
L'	rolling moment
m	strength of a point source or sink
M	Mach number
p	rate of roll, radians per second (constant)
q	dynamic pressure $\left(\frac{1}{2}\rho U^2\right)$
s	local semispan
s_o	maximum semispan
S	wing area (area of horizontal surface)
v	velocity component in the y direction
U	free-stream velocity
w	velocity component in the z direction
w_o	constant value of w
x,y,z	Cartesian coordinates
X	complex coordinate $(y+iz)$
α	exterior angle of a rectilinear polygon, radians
δ	angle of incidence of wing panel, radians $(\delta \ll 1)$
Δp	local pressure difference
$\frac{\Delta p}{q}$	pressure coefficient

$\Delta\varphi$ $\varphi_l - \varphi_u$

ϵ semivertex angle of a plane triangular wing

ξ complex coordinate $(\eta+i\zeta)$

θ_o angle between a source or sink radius vector and a coordinate axis

η, ζ coordinates in the complex ξ plane

ρ mass density of air

φ velocity potential in the X or ξ planes

Φ complex potential $(\varphi+i\psi)$ in the X or ξ planes

Φ_1 complex potential due to a combination of point sources and sinks

ψ stream function

Subscripts

— value for a plane wing

\+ value for a cruciform wing

H horizontal wing

L.E. value at leading edge

l value on lower surface

m dummy index used in denoting points

T.E. value at trailing edge

u value on upper surface

V vertical wing

ANALYSIS

General

A number of methods, based on the linearized theory of supersonic flow, have been developed for determining the aerodynamic characteristics of planar-wing systems of finite span. However, the application of these methods to the calculation of the characteristics of a cruciform wing (fig. 1) leads to considerable mathematical difficulties, since the effects of interference between components cannot be neglected and it is, in general, not practicable to construct solutions from the solutions for planar systems. (A notable exception is the case of lift.) It is therefore desirable to introduce simplifying assumptions which permit estimation of the characteristics of cruciform wings within reasonable limits of accuracy.

The linearized partial-differential equation for the perturbation velocity potential φ in subsonic and supersonic flow is

$$(1-M^2)\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \tag{1}$$

where the free stream is directed parallel to the positive x axis, and M is the free-stream Mach number. If the longitudinal velocity gradient φ_{xx} is sufficiently small, and the Mach number is not excessively high, then the first term in equation (1) is small compared to the velocity gradients in the y and z directions, and may be neglected. Equation (1) then reduces to

$$\varphi_{yy} + \varphi_{zz} = 0 \tag{2}$$

which is the familiar two-dimensional form of Laplace's equation. For slender wings and bodies the velocity gradient φ_{xx} is small, so that a satisfactory approximation to the aerodynamic characteristics of slender wings and wing-body configurations may be obtained by means of equation (2). The results will be independent of Mach number and will be valid for both subsonic and supersonic Mach numbers, as was pointed out in reference 1.

It was pointed out in reference 4, and discussed in greater detail in reference 5, that equation (1) is still valid if M is replaced by unity, in which case equation (1) again reduces to the two-dimensional form of Laplace's equation.

In the present application of the theory, no point on the trailing edge may lie ahead of the most forward point of maximum span. If the latter condition is not satisfied, lift is obtained off the surface of the wing, which violates the boundary conditions. Furthermore, it should be noted that the slender-wing theory and its extensions cannot be used to solve thickness problems. For a more detailed discussion of slender wing and wing-body theory, the reader is referred to references 1, 2, 4, 5, and 6.

The present problem is solved by finding a solution of equation (2) which satisfies the following boundary conditions:

1. The velocity components $\frac{\partial\varphi}{\partial y}$ and $\frac{\partial\varphi}{\partial z}$ vanish at infinity.

2. At all points on the $y = 0$ or $z = 0$ planes and not on the wing surfaces, $\Delta\varphi = 0$.

3. At all points on the $y = 0$ and $z = 0$ planes $\Delta \frac{\partial\varphi}{\partial y} = 0$ and $\Delta \frac{\partial\varphi}{\partial z} = 0$, respectively.

4. At all points on the $y = 0$ and $z = 0$ planes, within the wing plan-form boundaries, $\left(\frac{\partial\varphi}{\partial y}\right)_{y=0}$ and $\left(\frac{\partial\varphi}{\partial z}\right)_{z=0}$, respectively, are specified.

If the region outside a symmetric cross (chosen symmetric for simplicity) is mapped conformally on the region outside a circle, with points on the circumference of the circle corresponding to points on the arms of the cross, a potential function satisfying the boundary conditions stated above may be found by integrating a suitable combination of infinitesimal sources and sinks over the circumference of the circle.

If the two-dimensional velocity potential for the flow in transverse planes is given, the wing loading may be written

$$\frac{\Delta p}{q} = \frac{2}{U} \Delta\left(\frac{\partial\varphi}{\partial x}\right) \tag{3}$$

which expresses Bernoulli's equation with the approximation of small disturbances. It follows from equation (3) that the lift of one-half a plane slender wing is

$$\begin{aligned} L &= \rho U \int_0^{s_o} dy \int_{L.E.}^{T.E.} \Delta\left(\frac{\partial\varphi}{\partial x}\right) dx \\ &= \rho U \int_0^{s_o} \left(\Delta\varphi_{T.E.} - \Delta\varphi_{L.E.}\right) dy \end{aligned} \tag{4}$$

Similarly, the rolling moment acting on one-half a plane slender wing is

$$L' = -\rho U \int_0^{s_o} \left(\Delta\varphi_{T.E.} - \Delta\varphi_{L.E.}\right) y\,dy \tag{5}$$

In the following section a conformal transformation is derived which maps the region outside a circle on the region outside a rotationally symmetric cross. It is then shown that, by means of a distribution of infinitesimal sources and sinks on the circumference of the circle, a velocity potential may be found having a normal derivative which satisfies arbitrarily assigned values on the arms of the cross.

In succeeding sections, the velocity potentials are determined for the cases of a slender, rolling, cruciform wing and of a slender cruciform wing for which the horizontal surfaces are differentially deflected through a small angle of incidence. The potential for the latter case, together with the well-known potential for an infinite plate moving normal to itself with constant velocity, may be superposed in various ways to provide solutions to slender, equal-span, cruciform wing problems, the boundary conditions of which involve constant normal velocity components on the surfaces of the wing. The lift and rolling moment, respectively, may then be obtained from the lift formula for a plane slender wing and the moment formula for the slender cruciform wing with differential incidence of the horizontal surfaces.

Conformal Transformation for the Cross Section of an Equal-Span Cruciform Wing

Consider the conformal mapping defined by the equation (see reference 7, p. 395, or reference 8)

$$\frac{dX}{d\xi} = A\left(1 - \frac{\xi_1}{\xi}\right)^{\alpha_1/\pi} \left(1 - \frac{\xi_2}{\xi}\right)^{\alpha_2/\pi} \ldots \left(1 - \frac{\xi_N}{\xi}\right)^{\alpha_N/\pi} \tag{6}$$

where

A = a constant

$\xi_m = se^{i\theta_m}$ for $m = 1, 2, 3, \ldots N$ (s constant)

$$\sum_{m=1}^{N} \alpha_m = 2\pi$$

$$\sum_{m=1}^{N} \alpha_m \xi_m = 0$$

It can be shown that the conformal mapping defined by equation (6) transforms the region outside a closed rectilinear polygon of N sides in the X plane into the region outside a circle of radius s in the ξ plane (fig. 2). The last condition stated in equation (6) is necessary in order that X be a single-valued function of ξ.

By treating the cross of figure 3(a) as a closed polygon having eight sides and exterior angles $-\pi/2$ and π, it is found from equation (6) that the required mapping function for a cruciform wing in which the vertical and horizontal surfaces have equal semispans s is

$$2X^2 = \xi^2 + \frac{s^4}{\xi^2} \tag{7}$$

It can be easily verified that equation (7) maps conformally the region outside the circle $\xi = se^{i\theta}$ in the ξ plane on the region outside the symmetrical cross of width 2s in the X plane. The circumference of the circle is transformed into the cross; corresponding points are shown in figure 3.

An evident generalization of equation (7) is

$$2X^n = \xi^n + \frac{s^{2n}}{\xi^n} \tag{8}$$

where n is a positive integer. Equation (8) maps conformally the region outside the circle $\xi = se^{i\theta}$ in the ξ plane on the region outside a rotationally symmetric figure in the X plane, consisting of n line segments of length 2s, the midpoints of which intersect at the origin. This transformation, together with the method of this report, may be used to study the moment characteristics of a slender, rotationally symmetric wing consisting of n plane wings having a common root chord.

It may be noted that Darwin (reference 9, p. 1) has derived a very general conformal transformation of this type which transforms the region outside a circle into the region outside an arbitrary arrangement of line segments intersecting at the origin. The lengths of the segments and the angles between them are completely arbitrary. Darwin's generalized transformation, in conjunction with the method of this report, may be used to investigate the moment characteristics of a slender multiform wing consisting of any number of plane wing panels having arbitrary semispans and angular spacing, and a common root chord.

Derivation of the Velocity Potential

If a two-dimensional source and sink of equal strengths m are located on the circumference of a circle as shown in figure 4(b), then the circle is a streamline of the resulting flow. If the flow is transformed into the X plane by means of equation (7), the source and sink will be transformed into a special doublet[1] located on the positive part of the horizontal line segment. (See fig. 4(a).) As shown in figure 4(a), the special doublet is characterized by a flow normal to the segment at the point X_o. At all other points on the segments the normal velocity is zero and the segment surfaces are streamlines. In the ξ plane, the complex potential for the special doublet is

$$\Phi_1 = -\frac{m}{2\pi}\log\left(\frac{\xi - se^{i\theta_o}}{\xi - se^{-i\theta_o}}\right) \tag{9}$$

[1]The term is used to avoid confusion with the classical two-dimensional doublet.

The velocity function in the Z plane is

$$\frac{d\Phi_1}{dX} = v - iw = \frac{d\Phi_1}{d\xi}\frac{d\xi}{dX} \tag{10}$$

From equations (7), (9), and (10), it is seen that

$$\frac{d\Phi_1}{dX} = \pm\frac{m}{2\pi}\left(\frac{X}{X^2 - X_o^2}\sqrt{\frac{s^2 - X_o^2}{s^2 - X^2}} + \frac{X}{X^2 - X_o^2}\sqrt{\frac{s^4 - X_o^4}{s^4 - X^4}}\right) \tag{11}$$

where the sign is minus on the upper surface and plus on the lower surface.

This equation gives the velocity function at the variable point X due to the special doublet located at the fixed point X_o. If X is taken very near to X_o, equation (9) is approximately

$$\frac{d\Phi_1}{dX} = \pm\frac{m}{2\pi}\left(\frac{1}{X - X_o}\right) \tag{12}$$

If equation (12) is integrated around two small semicircular regions having a common center at X_o, it is seen that there is an inflow of $m/2$ units per second above the real axis and an outflow of $m/2$ units per second below the real axis. The flow from an infinitesimal source of strength dm (located on the arc element $sd\theta_o$ in the ξ plane) is of course dm units per second. In the X plane the flow across the corresponding element dX_o is $|wdX_o|$ units per second, where w is the vertical velocity component at the point X_o. By the principle of continuity of flow, it is seen that

$$wdX_o = \frac{dm}{2} \tag{13}$$

where w may be any function of X_o and dX_o is obtained from equation (7). The velocity function corresponding to any assigned distribution of w_o may then be found by integrating equation (11) over the proper range of values of X_o. Since the complex potential is desired, however, it is simpler to construct a source-sink distribution which gives the complex potential by a single integration. This procedure will be followed in the succeeding sections.

Rolling Moment Due to Rolling

The case of a slender equal-span, cruciform wing rolling about its longitudinal axis with constant angular velocity p is considered. The complex potential in the ξ plane for the source-sink combination shown in figure 5(b) is

$$\Phi_1 = -\frac{m}{2\pi}\log\left(\frac{\xi^4 - s^4e^{4i\theta_0}}{\xi^4 - s^4e^{-4i\theta_0}}\right)$$

Replacing m by $dm = -2pydy = 2ps^2 \sin 2\theta_0 d\theta_0$, and integrating from $\theta_0=0$ to $\theta_0=\pi/4$, the complex potential in the ξ plane for the rolling cruciform triangular wing is given by

$$\Phi = -\frac{ips^2}{\pi}\int_0^{\pi/4} \sin 2\theta_0 \log\left(\frac{\xi^4 - s^4e^{4i\theta_0}}{\xi^4 - s^4e^{-4i\theta_0}}\right)d\theta_0$$

Substituting $t=2\theta_0$, and integrating once by parts,

$$\Phi = -\frac{ips^2}{\pi}\left(s^8 - \xi^8\right)\int_0^{\pi/2} \frac{\cos t}{\xi^8 + s^8 - 2s^4\xi^4 \cos 2t}\,dt \tag{14}$$

where an imaginary constant has been omitted. Substituting equation (7) into equation (14), the complex potential in the X plane is seen to be

$$\Phi = \frac{ips^2}{\pi}X^2\sqrt{X^4 - s^4}\int_0^{\pi/2} \frac{\cos t}{X^4 - s^4 \cos^2 t}\,dt$$

$$= \frac{p}{\pi}X^2\left[\operatorname{sech}^{-1}\left(\frac{X}{s}\right)^2 \pm \frac{i\pi}{2}\right]$$

On the part of the real axis corresponding to the wing the surface velocity potential is therefore

$$\varphi = \pm\frac{p}{\pi}y^2 \operatorname{sech}^{-1}\left(\frac{y}{s}\right)^2 \tag{15}$$

From equations (15) and (3) it follows that the spanwise load distribution for a slender, rolling, cruciform wing is

$$\left(\frac{\Delta p}{q}\right)_+ = \pm\frac{2psA}{\pi U}\left[\frac{(y/s)^2}{\sqrt{1-(y/s)^4}}\right] \tag{16}$$

This load distribution is shown in figure 6.

Substituting equation (15) into equation (5), the total rolling moment due to roll is

$$(L')_+ = \frac{-8\rho Up}{\pi}\int_0^{s_0} y^3 \operatorname{sech}^{-1}\left(\frac{y}{s_0}\right)^2 dy$$

$$= \frac{-2}{\pi}\rho Ups_0^{\,4}$$

The coefficient of damping in roll for the slender cruciform wing is therefore simply

$$\left(C_{l_p}\right)_+ = \frac{-2 \tan \epsilon}{\pi}$$

$$= -\frac{A}{2\pi}$$

where the coefficient is based on the area of the horizontal wing only.

For a slender planar wing it is known (reference 10) that the rolling moment due to roll is

$$(L')_- = \frac{-\pi}{8} \rho U p s_o{}^4$$

and the coefficient of damping in roll is

$$\left(C_{l_p}\right)_- = \frac{-\pi}{8} \tan \epsilon$$

$$= -\frac{\pi A}{32}$$

The ratio of the damping moments for the rolling cruciform wing and the rolling planar wing is therefore

$$\frac{(L')_+}{(L')_-} = \frac{16}{\pi^2}$$

and the ratio of the damping-in-roll coefficients for the rolling cruciform wing and the rolling planar wing is

$$\frac{\left(C_{l_p}\right)_+}{\left(C_{l_p}\right)_-} = \frac{16}{\pi^2} = 1.62$$

if the aspect ratios are the same.

The damping-in-roll coefficient for the slender cruciform wing is therefore seen to be only 62 percent greater than that for a plane wing having the same aspect ratio.

If the velocity potential for a slender, plane, rolling wing (reference 10) is substituted into equation (3), it is found that the spanwise load distribution for this case is

$$\left(\frac{\Delta p}{q}\right)_- = \pm \frac{psA}{2U} \left[\frac{y/s}{\sqrt{1 - (y/s)^2}}\right] \tag{17}$$

Figure 6, which presents the load distribution over a spanwise section of a plane rolling wing and that over the horizontal and vertical wings of the cruciform arrangement, shows the effect of the wing interference in reducing the load distribution which opposes the rolling motion.

Rolling Moment Due to Differential Wing Incidence

The case considered here consists of a slender, equal-span, cruciform wing in which each half of the horizontal wing is differentially deflected through a small angle δ. (See fig. 7(a).) The vertical velocity component on the surface of each half of the horizontal wing is constant, and is $w_o = \pm U\delta$; on the surface of the vertical wing in the lateral velocity component must be zero.

The complex potential for the source-sink combination shown in figure 7(c) is

$$\Phi_1 = -\frac{m}{2\pi} \log \left(\frac{\xi^2 - s^2 e^{2i\theta_o}}{\xi^2 - s^2 e^{-2i\theta_o}}\right)$$

A distribution of such sources and sinks over the circular arcs corresponding to $-(\pi/4) \leqq \theta_o \leqq (\pi/4)$ and $(3\pi/4) \leqq \theta_o \leqq (5\pi/4)$, with m replaced by

$$\begin{aligned} dm &= -2w_o\, dy \\ &= \frac{2w_o s \sin 2\theta_o\, d\theta_o}{\sqrt{\cos 2\theta_o}} \end{aligned}$$

satisfies the boundary conditions shown in figure 7(b). The complex potential satisfying these boundary conditions is therefore

$$\Phi = \frac{w_o s}{\pi} \int_{\theta_o=0}^{\theta_o=\pi/4} \left[\log \left(\frac{\xi^2 - s^2 e^{2i\theta_o}}{\xi^2 - s^2 e^{-2i\theta_o}}\right)\right] d \sqrt{\cos 2\theta_o}$$

Substituting $t=2\theta_o$, and integrating once by parts,

$$\Phi = -\frac{i w_o s}{\pi} (s^4 - \xi^4) \int_0^{\pi/2} \frac{\sqrt{\cos t}}{\xi^4 + s^4 - 2s^2\xi^2 \cos t}\, dt \tag{18}$$

If equation (7) is substituted into equation (18), it is found that the complex potential in the X plane is given by

$$\Phi = \frac{i w_o s}{\pi} \sqrt{X^4 - s^4} \int_0^{\pi/2} \frac{\sqrt{\cos t}}{X^2 - s^2 \cos t}\, dt \tag{19}$$

The integral in equation (19) is a complete elliptic integral of the third kind with modulus $k = 1/\sqrt{2}$. Substituting

$$\mathrm{cn}^2\, u = \cos t$$

in equation (19), the complex potential may be written

$$\Phi = \frac{iw_o s\sqrt{2}}{\pi} \sqrt{X^4 - s^4} \int_0^K \frac{\mathrm{cn}^2 u}{X^2 - s^2\mathrm{cn}^2 u}\, du \tag{20}$$

For the evaluation of this and the succeeding moment integrals, and the values of the surface velocity potential on the horizontal and vertical surfaces, see the appendix.

If the surface velocity potentials as given in the appendix for the horizontal and vertical surfaces are substituted in equation (3), it is found that the spanwise load distribution on the horizontal surface is

$$\left(\frac{\Delta p}{q}\right)_{H+} = \pm \frac{\delta A\sqrt{2}}{\pi} \left[\frac{K+(2E-K)(y/s)^2}{\sqrt{1-(y/s)^4}}\right] \tag{21}$$

and that the load distribution on the vertical surface is

$$\left(\frac{\Delta p}{q}\right)_{V+} = \mp \frac{\delta A\sqrt{2}}{\pi} \left[\frac{K-(2E-K)(z/s)^2}{\sqrt{1-(z/s)^4}}\right] \tag{22}$$

where $k = 1/\sqrt{2}$ in both equations. These load distributions are shown in figure 8.

If the surface velocity potential for the horizontal surface (see appendix) is substituted into equation (5) and integrated, the rolling moment due to the horizontal surfaces is seen to be

$$\left.\begin{aligned} L'_H &= -\frac{4\sqrt{2}}{3\pi}\left[\frac{K}{2}\left(\frac{\pi}{2}-1\right)+E\right]\rho U^2\delta s_o{}^3;\ k = 1/\sqrt{2} \\ &= -1.128\ \rho U^2\delta s_o{}^3 \end{aligned}\right\} \tag{23}$$

Similarly, the rolling moment due to the vertical surface is

$$\left.\begin{aligned} L'_V &= +\frac{4\sqrt{2}}{3\pi}\left[\frac{K}{2}\left(\frac{\pi}{2}+1\right)-E\right]\rho U^2\delta s_o{}^3;\ k = 1/\sqrt{2} \\ &= +0.620\ \rho U^2\delta s_o{}^3 \end{aligned}\right\} \tag{24}$$

The total rolling moment is therefore

$$(L')_+ = \frac{-4\sqrt{2}}{3\pi}(2E-K)\ \rho U^2 \delta s_o^3;\ k = 1/\sqrt{2}$$

$$= -0.508\ \rho U^2 \delta s_o^3$$

and the coefficient of rolling-moment effectiveness is

$$(C_{l_\delta})_+ = -\frac{A\sqrt{2}}{3\pi}(2E-K);\ k = 1/\sqrt{2}$$

$$= -0.127A$$

based on the horizontal wing area. If the vertical surfaces are also deflected differentially through a small angle δ, the preceding value would be doubled, or

$$(C_{l_\delta})_+ = -0.254A$$

From reference 11, the rolling moment for a slender plane wing having the panels differentially deflected is

$$(L')_- = \frac{-2}{3}\rho U^2 \delta s_o^3$$

and the coefficient of rolling-moment effectiveness is

$$\left(C_{l_\delta}\right)_- = -\frac{A}{6}$$

The ratio of the rolling moments produced by the horizontal panels of the slender cruciform wing and the slender plane wing is

$$\frac{(L')_+}{(L')_-} = \frac{2\sqrt{2}}{\pi}(2E-K) = 0.762$$

It is seen that, although the rolling moment supplied by the horizontal surfaces of the cruciform wing is larger than for the plane wing, the counter rolling moment induced on the vertical surface is so large that the total rolling moment is 24 percent less than for the plane wing.

If the velocity potential[2] for a slender plane wing with differential incidence of the horizontal surfaces is substituted into equation (3), it is found that the spanwise loading is

$$\left(\frac{\Delta p}{q}\right)_- = \pm\frac{2\delta A}{\pi}\left[\frac{y/s}{\sqrt{1-(y/s)^2}}\right] \qquad (25)$$

[2]The velocity potential for this case may be easily derived by applying the Joukowsky transformation with the method of this report.

Figure 8 shows the pressure distribution over the vertical and horizontal surfaces of the cruciform wing and, for comparison, the pressure distribution over a plane wing.

Rolling Effectiveness of a Slender Cruciform Wing With Differential Incidence of the Horizontal and Vertical Surfaces

A parameter often used in evaluating the rolling effectiveness of a lateral-control system is the rate of change of the wing-tip helix angle $pb/2U$ with differential control surface deflection. This parameter is obtained from the relationship

$$\frac{d}{d\delta}\left(\frac{pb}{2U}\right) = \frac{C_{l_\delta}}{C_{l_p}}$$

From the results of the previous sections, the rolling effectiveness of the cruciform wing having four panels equally deflected is

$$\left(\frac{C_{l_\delta}}{C_{l_p}}\right)_+ = \frac{4\sqrt{2}}{3}(2E-K);\ k = 1/\sqrt{2}$$

$$= 1.594$$

Similarly, the rolling effectiveness of a plane wing (or of a cruciform wing with zero interference) is

$$\left(\frac{C_{l_\delta}}{C_{l_p}}\right)_- = \frac{16}{3\pi} = 1.696$$

The ratio of the rolling effectiveness of a cruciform wing to that for a plane wing is therefore

$$\frac{\left(C_{l_\delta}/C_{l_p}\right)_+}{\left(C_{l_\delta}/C_{l_p}\right)_-} = 0.94$$

It is seen that the rolling effectiveness of a plane wing is reduced 6 percent by the insertion of a wing with similar plan form and surface incidence in the vertical plane of symmetry. If no interference effects existed between the horizontal and vertical surfaces of the cruciform wing, this reduction would be zero. Although the coefficient of damping in roll is 81 percent, and the coefficient of rolling-moment effectiveness is 76 percent of their respective values with zero interference, the combined effect of these reductions is to decrease the rolling effectiveness only 6 percent.

CONCLUDING REMARKS

The rolling-moment characteristics of slender cruciform wings have been investigated by a method based on low-aspect-ratio wing theory. It was found that the coefficient of damping in roll (based on the area of the horizontal wing) of a slender cruciform wing is 62 percent greater than that for a slender plane wing having the same aspect ratio, and that the rolling moment supplied by differential deflection of the opposite panels of the horizontal and vertical surfaces of a slender cruciform wing is only 52 percent greater than that for a slender plane wing in which the panels are similarly deflected. The rolling effectiveness $d(pb/2U)/d\delta$ of the cruciform wing having four panels deflected was found to be 6 percent less than that of the plane wing.

The method may be applied to the investigation of the characteristics of slender, equal-span, cruciform wings which have any specified distribution of normal velocities in the planes of the horizontal and vertical surfaces.

A conformal transformation is given which may be used to investigate the moment characteristics of a rotationally symmetric slender-wing configuration consisting of any number of plane wings having a common root chord.

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National Advisory Committee for Aeronautics,
Moffett Field, Calif., Sept. 29, 1950.

APPENDIX

INTEGRATION FOR ROLLING MOMENT DUE TO DIFFERENTIAL INCIDENCE

OF THE HORIZONTAL SURFACE OF A SLENDER,

CRUCIFORM, TRIANGULAR WING

The complex potential for the cruciform triangular wing is

$$\left.\begin{aligned}\Phi &= \frac{iw_o s\sqrt{2}}{\pi}\sqrt{X^4-s^4}\int_0^K \frac{cn^2u}{X^2-s^2cn^2u}\,du \\ &= -\frac{iw_o K\sqrt{2}}{\pi s}\sqrt{X^4-s^4}+\frac{iw_o\sqrt{2}}{\pi s}X^2\sqrt{X^4-s^4}\int_0^K \frac{du}{X^2-s^2cn^2u}\end{aligned}\right\} \quad (A1)$$

Let

$$I_1 = \int_0^K \frac{du}{X^2-s^2cn^2u} = \frac{1}{X^2-s^2}\int_0^K \frac{du}{1-[s^2/(s^2-X^2)]sn^2u} \quad (A2)$$

On the real axis

$$I_1 = \frac{1}{y^2-s}\int_0^K \frac{du}{1-[s^2/(s^2-y^2)]sn^2u} \quad (A3)$$

The integral is of the form

$$\int_0^K \frac{du}{1-a_1{}^2sn^2u}$$

where

$$a_1{}^2 = \frac{s^2}{s^2-y^2} \quad (1 \leqq a_1{}^2 \leqq \infty) \quad (A4)$$

This is the standard form for the complete elliptic integral of the third kind with parameter a_1. It can be expressed in terms of incomplete elliptic integrals of the first and second kinds, by use of Jacobi's Zeta and Theta functions (see reference 12, ch. 22). The result for the stated range of values of a_1 is

$$\int_0^K \frac{du}{1-a_1^2 \mathrm{sn}^2 u} = \frac{a_1}{\sqrt{(a_1^2-1)(a_1^2-k^2)}} \left[EF\left(\frac{1}{a_1},k\right) - KE\left(\frac{1}{a_1},k\right) \right] \tag{A5}$$

From equations (A1), (A4), and (A5) it is found that the surface velocity potential for the horizontal wing $(z=0, -s \leqq y \leqq s)$ is

$$\varphi_H = \pm \frac{w_o}{\pi} \left\{ \frac{K\sqrt{2}}{s} \sqrt{s^4-y^4} + 2y \left[EF\left(\sqrt{1-\frac{y^2}{s^2}}, \frac{1}{\sqrt{2}}\right) - KE\left(\sqrt{1-\frac{y^2}{s^2}}, \frac{1}{\sqrt{2}}\right) \right] \right\} \tag{A6}$$

Substituting equation (A6) into equation (5), the rolling moment for the horizontal surfaces is then

$$L'_H = \frac{4\rho U w_o}{\pi} \int_0^{s_o} \left[\frac{K\sqrt{2}}{s_o} y \sqrt{s_o^4-y^4} + 2Ey^2 F\left(\sqrt{1-\frac{y^2}{s_o^2}}, \frac{1}{\sqrt{2}}\right) - 2Ky^2 E\left(\sqrt{1-\frac{y^2}{s_o^2}}, \frac{1}{\sqrt{2}}\right) \right] dy \tag{A7}$$

The integrals in equation (A7) may be easily evaluated by integration by parts. The result is

$$L'_H = -\frac{4\sqrt{2}}{3\pi} \left[\frac{K}{2}\left(\frac{\pi}{2} - 1\right) + E \right] \rho U^2 \delta s_o^3 \tag{A8}$$

with

$$k = \frac{1}{\sqrt{2}}$$

On the imaginary axis $(-s \leqq z \leqq s)$ the integral in equation (A1) is

$$
\left.\begin{aligned}
I_2 &= -\int_0^K \frac{du}{z^2+s^2\mathrm{cn}^2u} \\
&= -\frac{1}{s^2+z}\int_0^K \frac{du}{1-[s^2/(s^2+z^2)]\mathrm{sn}^2u}
\end{aligned}\right\} \tag{A9}
$$

Let

$$
a_2{}^2 = \frac{s^2}{s^2+z^2} \qquad \left(\frac{1}{2} \leqq a_2{}^2 \leqq 1\right) \tag{A10}
$$

By applying the Jacobian Zeta and Theta functions as previously noted, it can be shown that[3]

$$
\left.\begin{aligned}
&\int_0^K \frac{du}{1-a_2{}^2\mathrm{sn}^2u} = \frac{a_2}{\sqrt{(1-a_2{}^2)(a_2{}^2-k^2)}}\left[(E-K)F(a_3,k') + KE(a_3,k')\right] \\
&\text{where} \\
&\qquad a_3 = \frac{1}{a_2}\sqrt{\frac{a_2{}^2-k^2}{1-k^2}} \\
&\qquad k^2 \leqq a_2{}^2 \leqq 1 \qquad\qquad k^2 + k'^2 = 1
\end{aligned}\right\} \tag{A11}
$$

From equations (A1), (A9), (A10), and (A11), the surface velocity potential in the plane of the vertical wing is

$$
\varphi_V = \pm \frac{w_o}{\pi}\left\{\frac{K\sqrt{2}}{\varepsilon}\sqrt{s^4-z^4} - 2z\left[(E-K)F\left(\sqrt{1-\frac{z^2}{s^2}},\ \frac{1}{\sqrt{2}}\right) + KE\left(\sqrt{1-\frac{z^2}{s^2}},\ \frac{1}{\sqrt{2}}\right)\right]\right\} \tag{A12}
$$

---

[3]This integral was obtained from a comprehensive table of complete elliptic integrals of the third kind calculated by Paul Byrd, Ames Laboratory, NACA.

Substituting equation (A12) into equation (5), with y replaced by z, and evaluating the integrals by parts, the rolling moment for the vertical wing is

$$L'_V = \frac{4\sqrt{2}}{3\pi}\left[\frac{K}{2}\left(\frac{\pi}{2} + 1\right) - E\right]\rho U^2 \delta s_o^3 \tag{A13}$$

with

$$k' = \frac{1}{\sqrt{2}}$$

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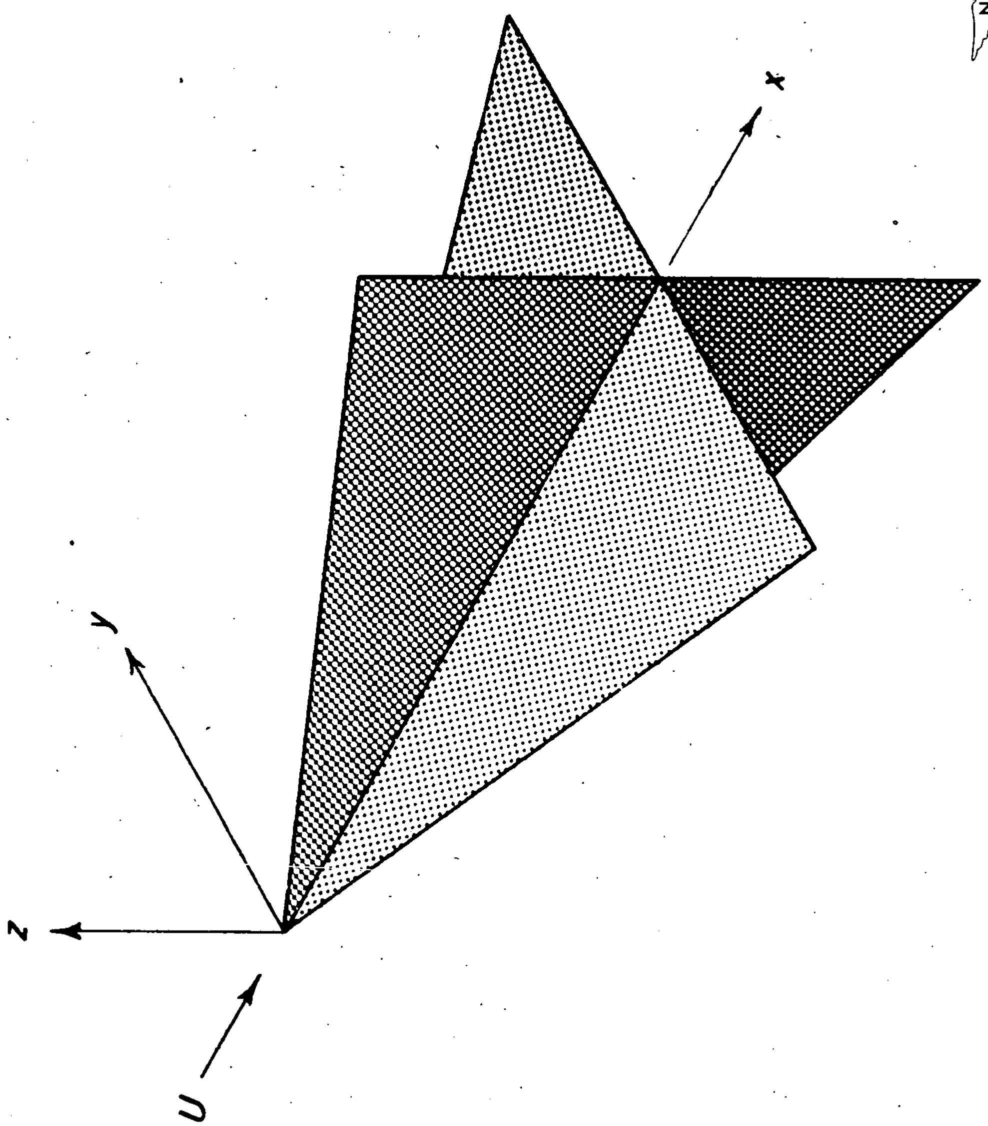
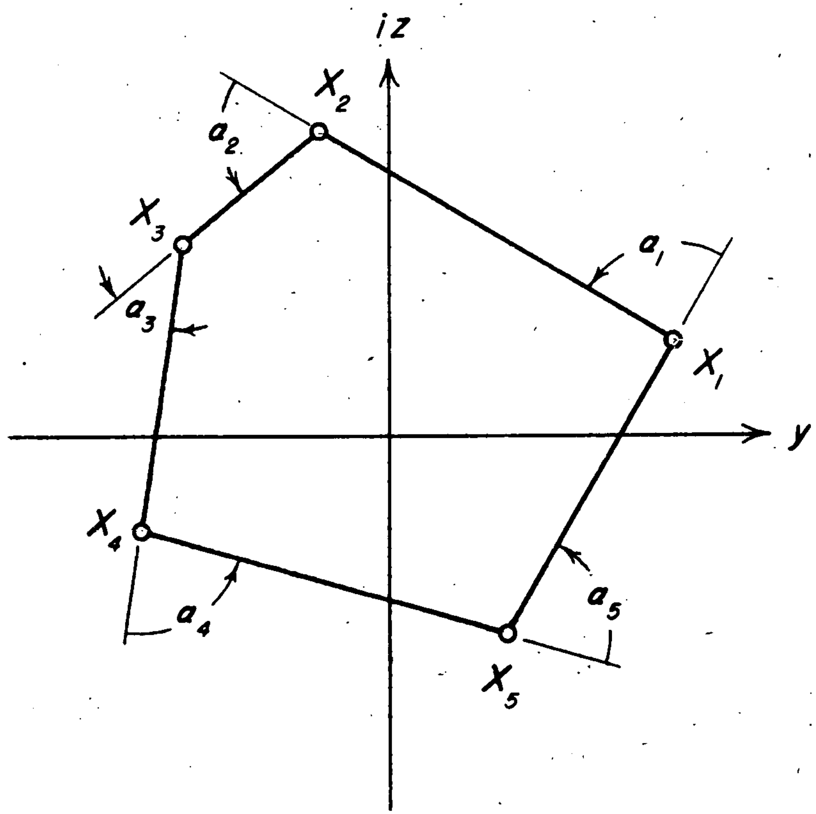
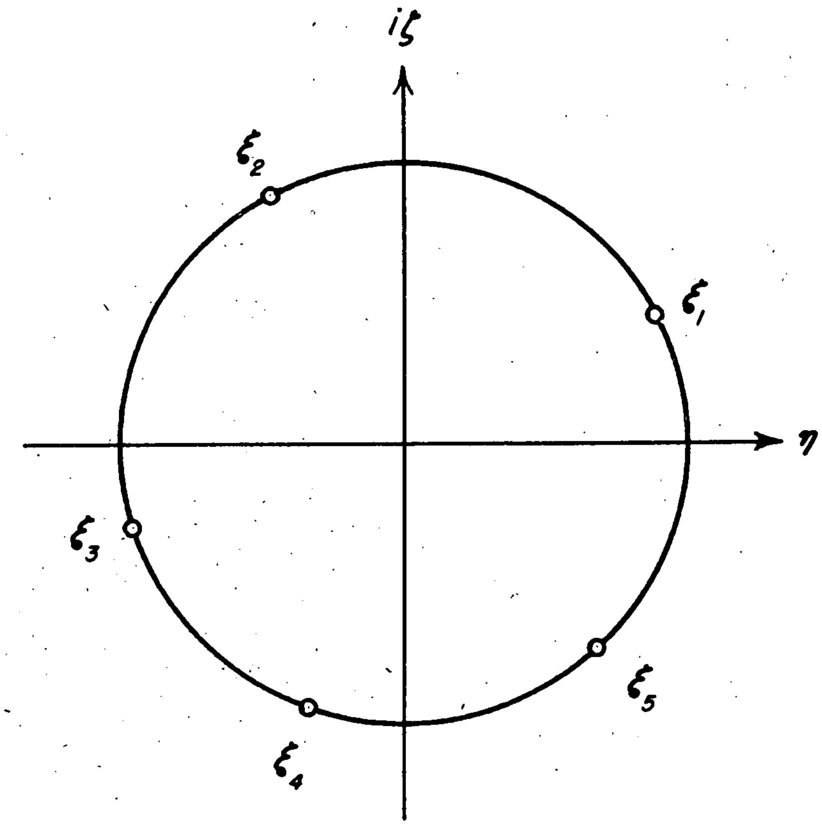


Figure 1.- Cruciform triangular wing.

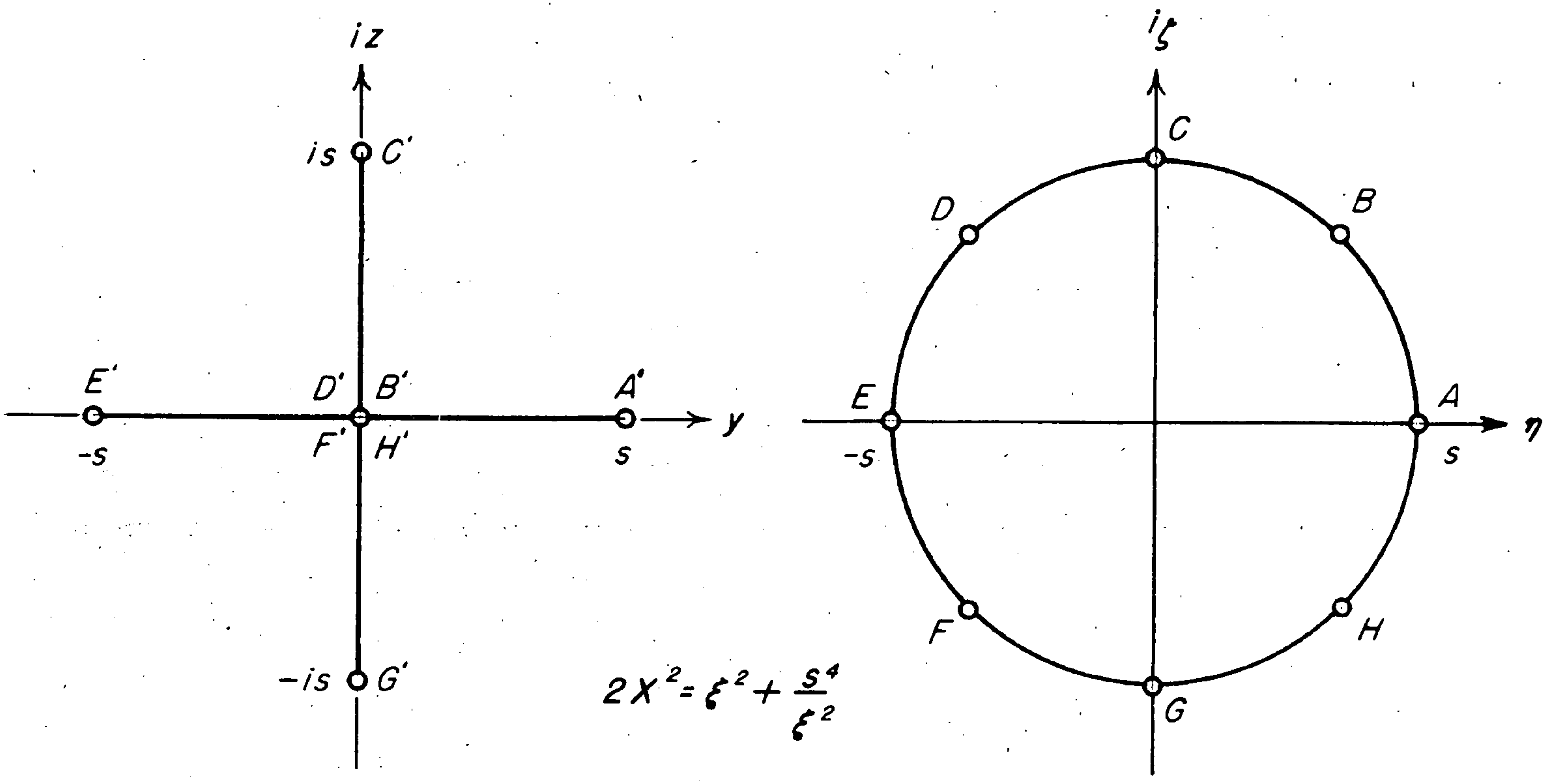


(a) X plane.



(b) ξ plane.

Figure 2.- Conformal mapping of the exterior of a polygon on the exterior of a circle.



$$2X^2 = \xi^2 + \frac{s^4}{\xi^2}$$

(a) X plane.

(b) ξ plane.

Figure 3.- Conformal mapping for a cruciform wing.

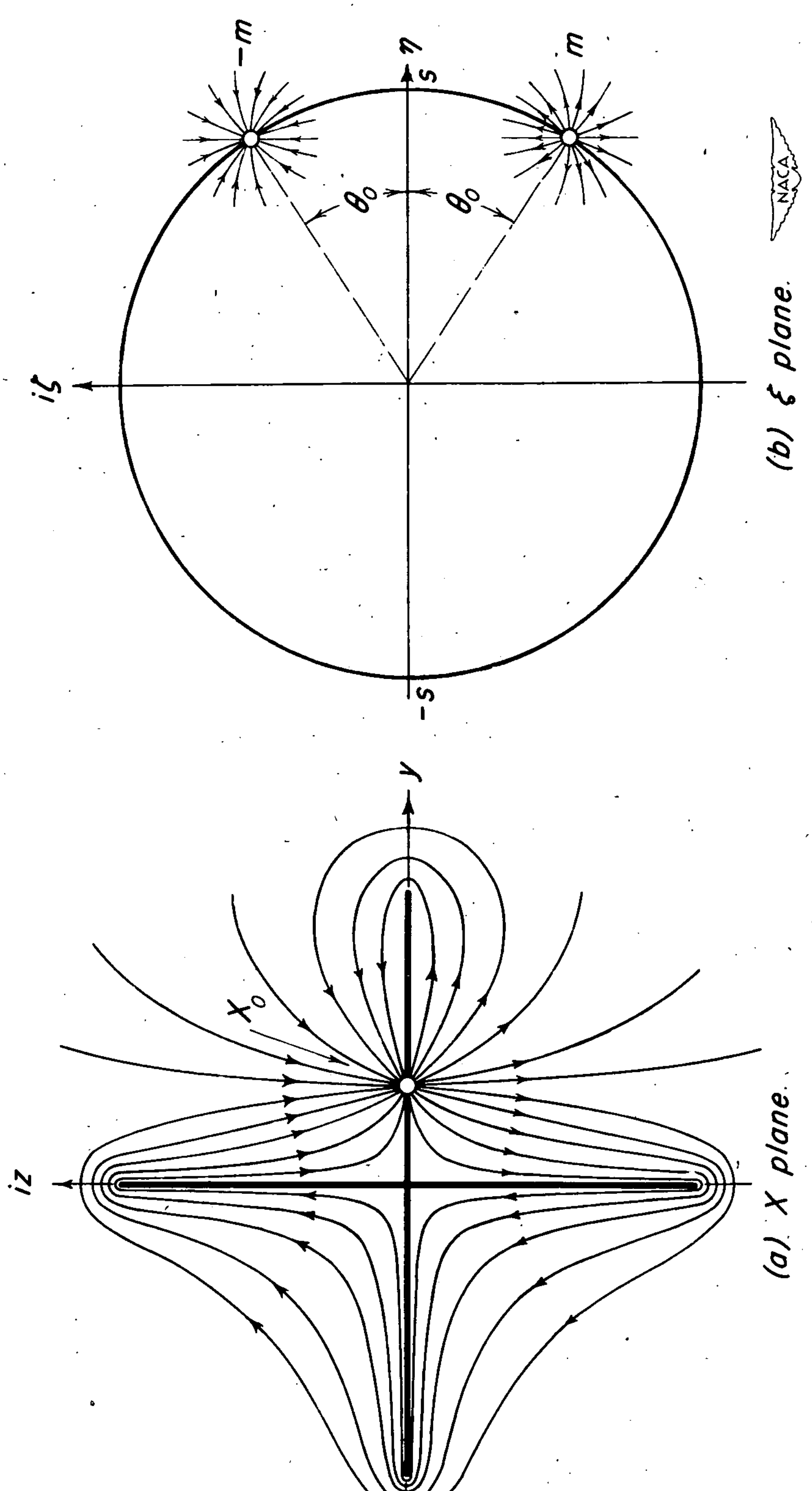
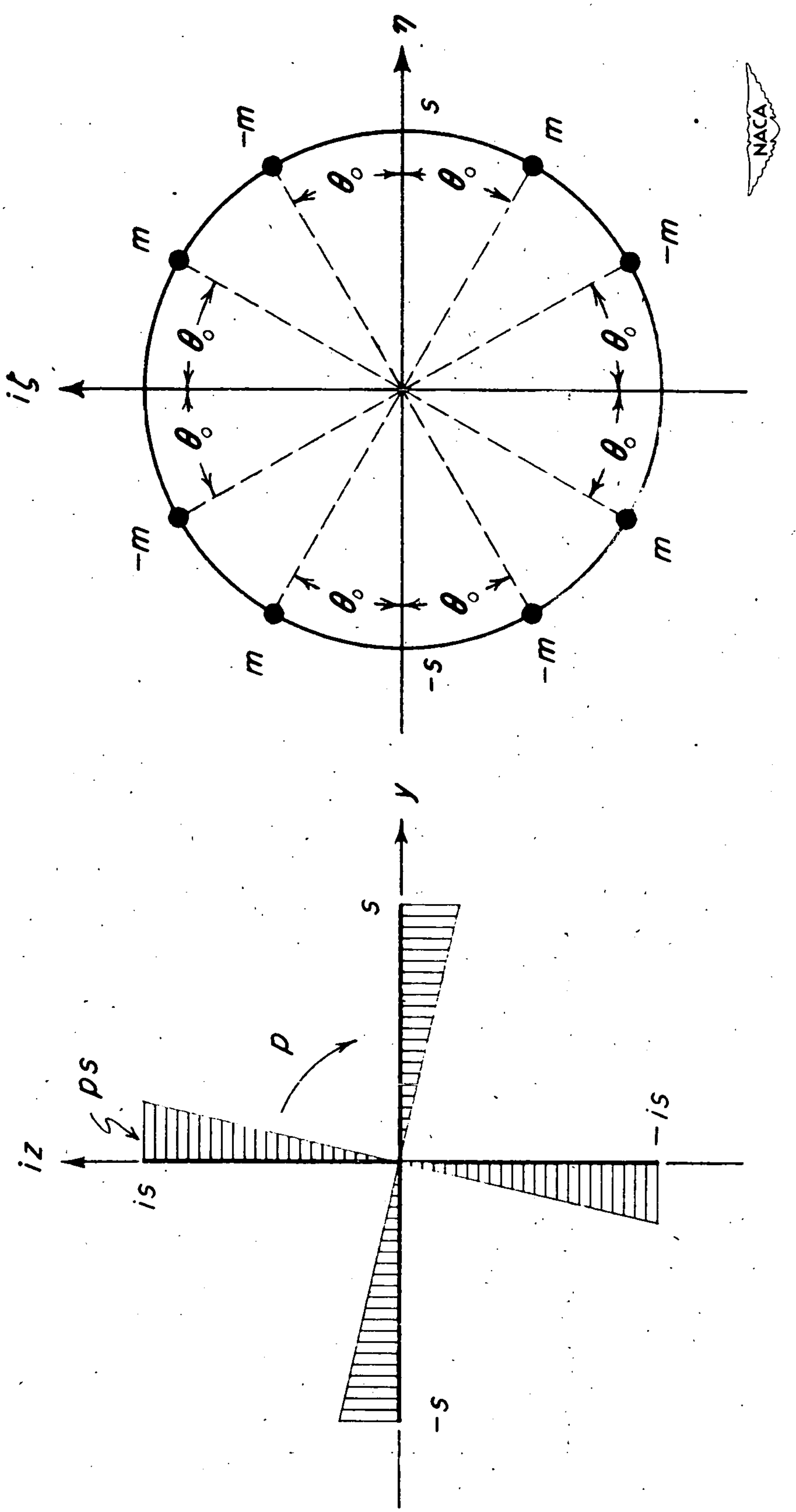


Figure 4.- Special doublet.



(a) X plane.

(b) ξ plane.

Figure 5.- Rolling cruciform wing.

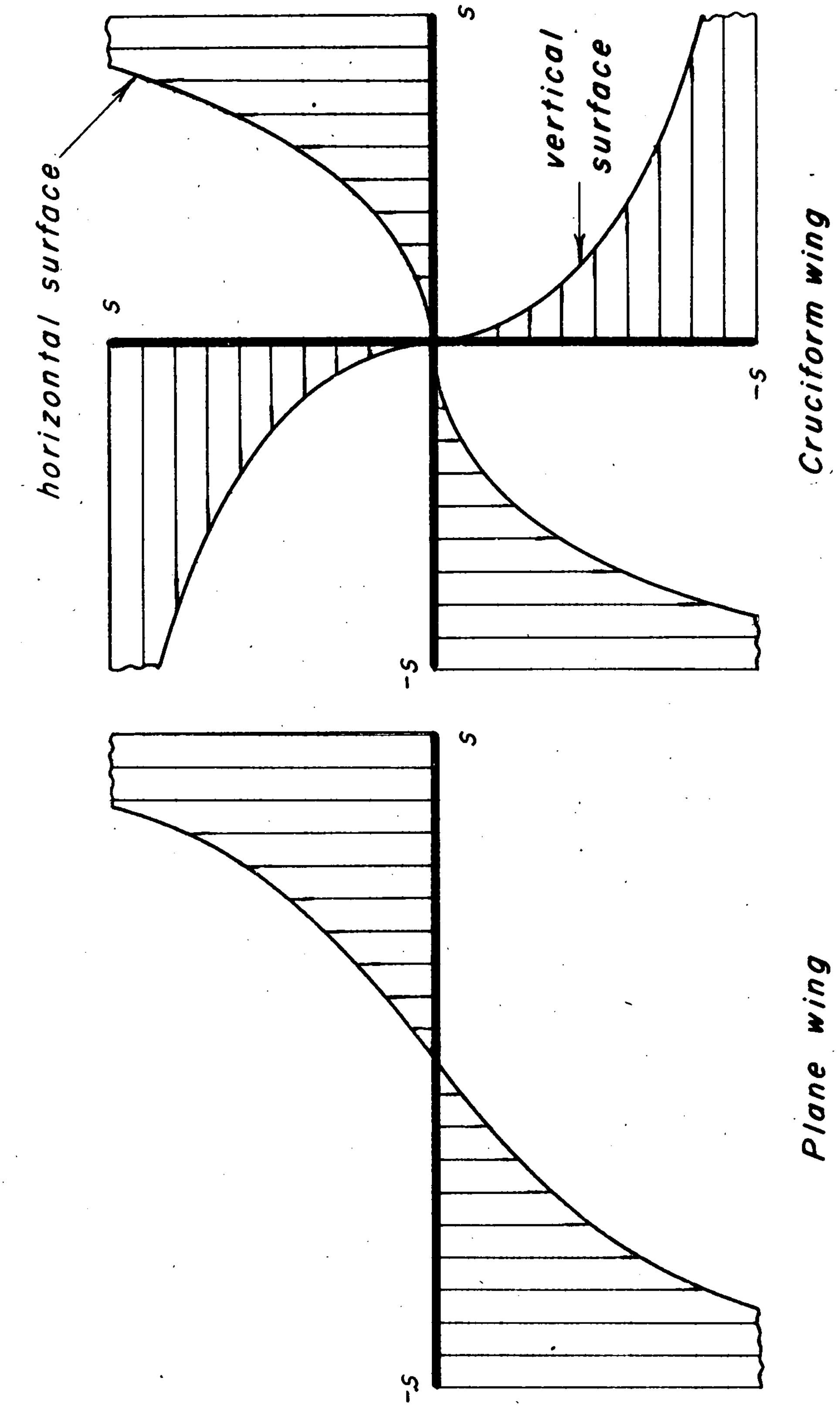


Figure 6.- Load distributions-rolling wings.

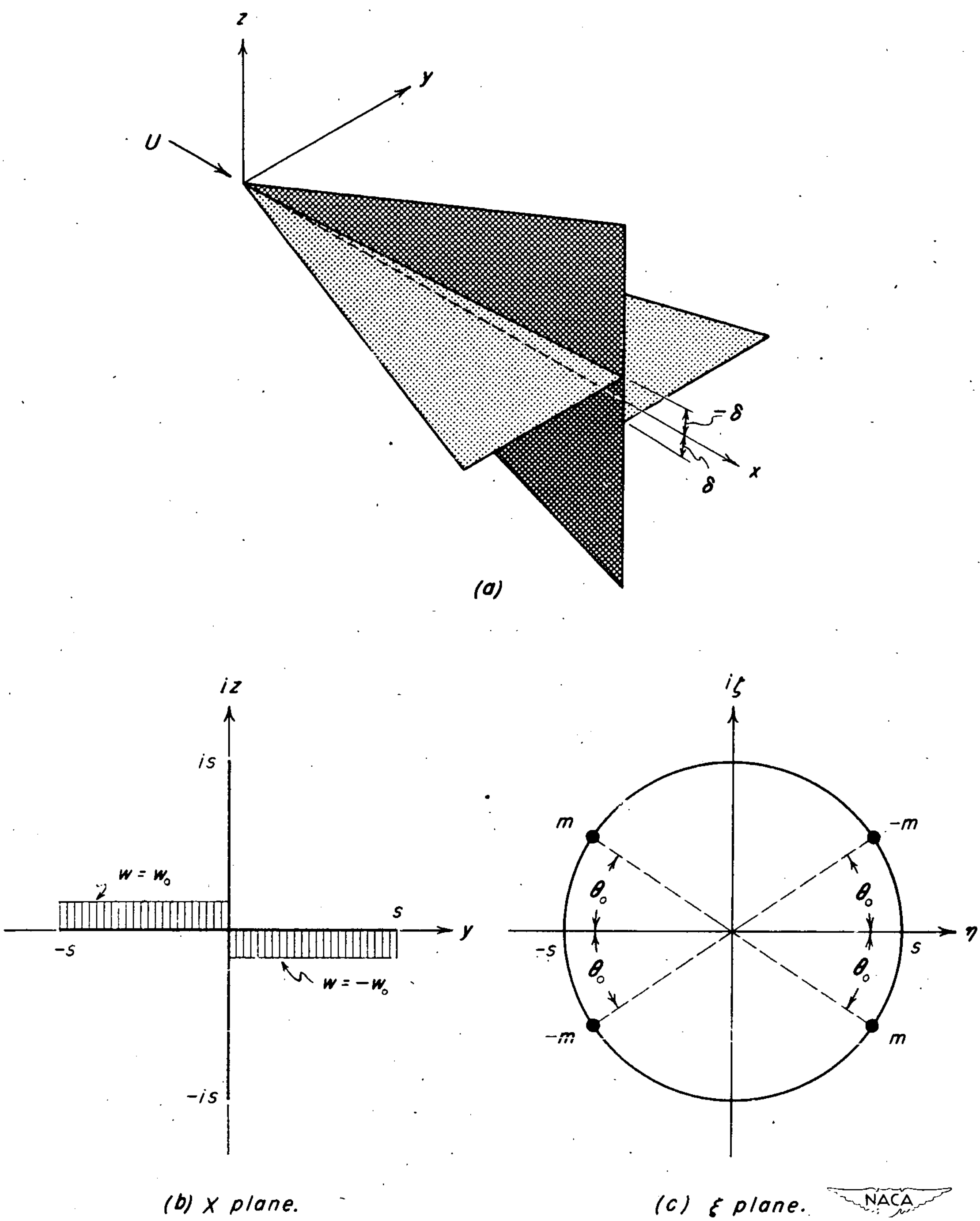


Figure 7.- Cruciform wing with differential incidence of the horizontal surfaces.

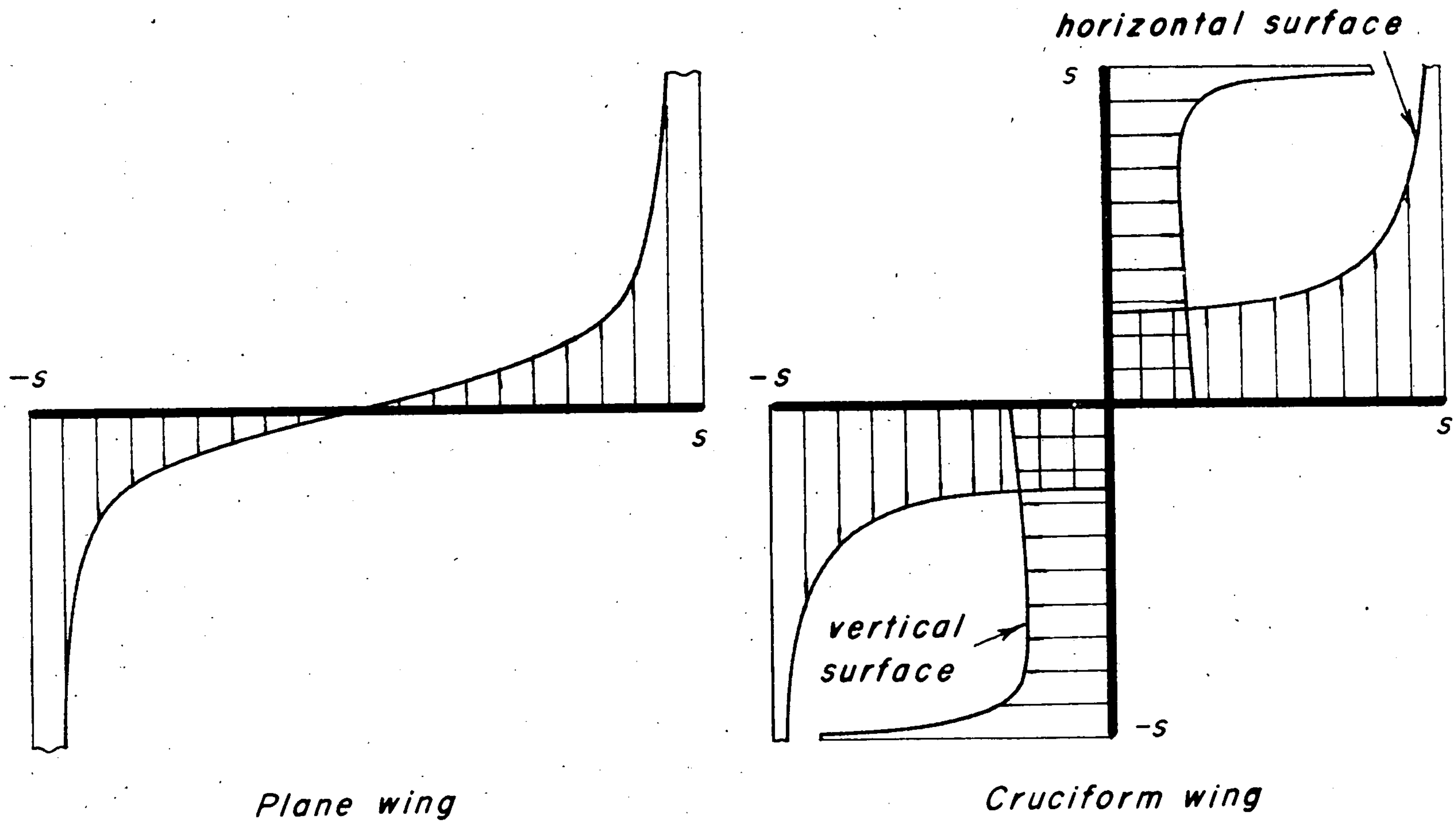


Figure 8 - Load distributions for wings with differential incidence of the horizontal surfaces.

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By Gaynor J. Adams

NACA TN 2270

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Control, Lateral 1.8.2.2

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NACA TN 2270

January 1951

(Abstract on reverse side)

Abstract

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