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ON THE APPLICATION OF MATHIEU FUNCTIONS IN THE THEORY OF SUBSONIC COMPRESSIBLE FLOW PAST OSCILLATING AIRFOILS

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SUMMARY

An account is given of explicit solutions in terms of Mathieu function functions of the problem of two-dimensional subsonic compressible flow past oscillating airfoils. The results are applied to the calculation of three-dimensional corrections for the two-dimensional theory and the effect of the incorporation of the three-dimensional effects on the Mathieu function solution of the two-dimensional problem is shown. The developments are formal and must be supplemented by an appreciable amount of numerical calculations before the theory can be applied to specific problems.

INTRODUCTION

The present report is concerned with the linearized theory of oscillating airfoils in two-dimensional and three-dimensional subsonic compressible flow.

The problem of two-dimensional flow was first treated in 1938 by Possio (reference 1) who reduced it to an integral equation and obtained approximate solutions of the integral equation by collocation methods. Subsequently, various authors have extended Possio's work and have applied other approximate methods to the solution of Possio's integral equation. Detailed references and a survey of the existing results may be found in a monograph by Karp, Shu, and Weil (references 2 and 3).

A second method of treatment deals directly with the boundary-value problem of the differential equation for the velocity potential or the acceleration potential. If this is done, it is found that an explicit solution of the problem may be obtained by introduction of a suitable curvilinear coordinate system and that this explicit solution is in terms of Mathieu functions. The earliest report containing this Mathieu function approach to the problem under consideration appears to be that of Sherman and the present author (reference 4). Independently, a similar approach, differing in details, was used by Biot (paper presented at the Sixth International Congress for Applied Mechanics, Sept. 1946, but not available in published form), Timman (reference 5), and Haskind (reference 6). It appears that Haskind has gone furthest in presenting the solution in a form suitable for the by no means simple numerical evaluation.

Part of the present report consists in a reproduction, with minor modifications, of the existing results in terms of Mathieu functions for the problem of two-dimensional flow. One of the reasons for this reproduction is the application of these results, in the remainder of the present report, to the problem of calculating three-dimensional corrections for the results of the two-dimensional theory.

Just as the problem of two-dimensional flow can be formulated in terms of a one-dimensional integral equation, so the problem of three-dimensional flow can be formulated as a two-dimensional integral equation. This has been done in an earlier report (reference 7). It has also been shown in this earlier report that an approximate theory for three-dimensional effects can be obtained by making certain approximations in the kernel of the integral equation of the three-dimensional problem. On the basis of these approximations, the three-dimensional problem is reduced to what amounts to a succession of two two-dimensional problems. One of these two problems is of the nature of the problem of the two-dimensional theory proper. The other problem consists in determining the spanwise variation of circulation around the airfoil. In the present report it will be shown in which way the Mathieu function solution of the two-dimensional problem is affected by the incorporation of three-dimensional effects according to reference 7.

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SYMBOLS

X, Y, Z	Cartesian coordinates
H	defined by equation of lifting surface $Z = H(X,t)$
t	time
U	main-stream velocity in X-direction
u, w	components of velocity change caused by presence of lifting surface
ρ_o	density of undisturbed fluid

p, ρ	pressure and density changes, respectively, caused by presence of lifting surface
a	velocity of sound in undisturbed fluid
b	semichord
ϕ	perturbation velocity potential
ω	circular frequency of oscillation
$\overline{\phi}$	potential amplitude
$\overline{p}$	pressure amplitude
x, y, z	dimensionless coordinates defined by equations (17) and (122)
M	Mach number of main stream (U/a)
k	reduced-frequency parameter $(\omega b/U)$
ψ	modified potential amplitude defined by equation (20)

$$\mu = \frac{kM^2}{1 - M^2}$$

$$\kappa = \frac{kM}{1 - M^2}$$

$$\nu = \frac{k}{1 - M^2}$$

$$r = \sqrt{x^2 + z^2}$$

ψ_1	part of ψ representing noncirculatory portion of flow
ψ_2	part of ψ representing circulatory flow around flat plate at rest and at zero angle of attack
ξ, ζ	elliptical coordinates

g	downwash function defined by equation (41)
$\bar{p}_a$	pressure amplitude at airfoil
F, G	functions determined by equations (45)
c	separation constant
ce_m, ce_n	even, periodic Mathieu functions
se_m, se_n	odd, periodic Mathieu functions
A_{mn}, B_{mn}	coefficients of Fourier series expressing ce_m and se_m, respectively
Ce_m, Ce_n	Mathieu-Hankel functions corresponding to ce_m and ce_n, respectively
Se_m, Se_n	Mathieu-Hankel functions corresponding to se_m and se_n, respectively
c_{cm}, c_{sm}	separation constants
a_m, a_n, b_m, b_n	coefficients defined by equations (84) and (56)
$\bar{p}_a^{(1)}$	contribution to pressure at airfoil for noncirculatory-flow component
$\bar{L}^{(1)}$	noncirculatory portion of lift
l_m	coefficient defined by equations (59) and (68)
$\bar{M}^{(1)}$	moment about midchord
m_m	coefficient defined by equations (62) and (69)
$\bar{M}_c^{(1)}$	hinge moment about $x = c$
m_{cm}	aileron hinge-moment coefficient
$J_n(\mu)$, $J_n'(\mu)$	Bessel functions and their derivatives, respectively
$\bar{h}$, $\bar{\alpha}$	functions defined by equations (70)
g_h	function defined by equation (71)

b_{mh}	coefficient defined by equation (72)
I_m	coefficient defined by equation (73)
g_x	function defined by equation (74)
$b_{m\alpha}$	coefficient defined by equation (75)
C	arbitrary constant
$H_0^{(2)}$, $H_1^{(2)}$	Hankel functions of second kind and of zeroth and first order, respectively
$b_n^{(2)}$	coefficients in equation (78)
W	function defined by equation (79)
A, B	arbitrary parameters defined by equations (143)
α_m, α_n, β_m, β_n	coefficients defined by equations (85)
$\bar{p}_a^{(2)}$	pressure distribution at airfoil for circulatory-flow component
$\bar{L}^{(2)}$, $\bar{M}^{(2)}$	circulatory portions of lift and moment, respectively
ϵ	auxiliary variable of integration
ξ_0	arbitrary constant in equation (102)
λ	function defined by equations (107) and (125)
K_0, K_1	modified Bessel functions
Ω	circulation function
s	aspect ratio of surface
y^*	dimensionless spanwise coordinate given by equation (123)
η^*	variable of integration
K	function defined by equation (128)
F_M	function defined by equation (129)
σ, τ	auxiliary variables of integration

Q	function defined by equation (131)
$\overline{G}$	function defined by equation (132)
P_A, P_B	coefficients defined with reference to equation (98)
Q_A, Q_B	coefficients defined in accordance with equation (114)
Q_C	coefficient defined by equation (139)

$$R_A = 2\sum \alpha_n \int_0^\pi e^{i\nu\cos\zeta} ce_n(\zeta)\, d\zeta$$

$$R_B = 2\sum \beta_n \int_0^\pi e^{i\nu\cos\zeta} ce_n(\zeta)\, d\zeta$$

$\Omega^{(2)}$ function defined by equation (146)

FORMULATION OF TWO-DIMENSIONAL PROBLEM

Let $Z = H(X,t)$ be the equation of a nearly plane lifting surface in the path of an inviscid compressible fluid flowing with uniform velocity U in the direction of the positive X-axis. Because of the presence of the lifting surface, the velocity field $(U,0)$ is changed to $(U + u,w)$. The disturbances caused by the presence of the lifting surface are assumed to be small in the sense that the differential equations and boundary conditions of the problem may be linearized.

The differential equations are, in linearized form,

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial X} = -\frac{\partial}{\partial X}\left(\frac{p}{\rho_o}\right) \tag{1}$$

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial X} = -\frac{\partial}{\partial Z}\left(\frac{p}{\rho_o}\right) \tag{2}$$

$$\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial X} = -\rho_o\left(\frac{\partial u}{\partial X} + \frac{\partial w}{\partial Z}\right) \tag{3}$$

$$p = a^2\rho \tag{4}$$

In equations (1) to (4), ρ_0 is the density of the undisturbed fluid, a is the velocity of sound in the undisturbed fluid, and p and ρ are the changes of pressure and density caused by the presence of the lifting surface.

The boundary condition of no relative normal flow at the lifting surface of chord $2b$ is, in linearized form,

$$|X| \leqq b, \quad Z = \pm 0, \quad w = \frac{\partial H}{\partial t} + U \frac{\partial H}{\partial X} \tag{5}$$

The form of equation (5) indicates that w may be taken as an even function of Z. The differential equations (1) and (2) then imply that u and p may be taken as odd functions of Z. As the pressure change p must be continuous except when crossing the lifting surface, it follows that a further boundary condition may be taken in the form

$$b \leqq |X|, \quad Z = 0, \quad p = 0 \tag{6}$$

Further conditions are the conditions of finite trailing-edge velocity,

$$X = b, \quad Z = 0, \quad u \text{ finite} \tag{7}$$

and the condition that the motion of the lifting surface produces energy radiation without reflection at infinity.

The problem as stated may be solved by means of a perturbation velocity potential ϕ, in terms of which

$$\left.\begin{aligned} u &= \frac{\partial \phi}{\partial X} \\ w &= \frac{\partial \phi}{\partial Z} \end{aligned}\right\} \tag{8}$$

Combination of equations (8), (1), and (2) expresses p in terms of ϕ, as follows:

$$p = -\rho_0 \left(\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial X}\right) \tag{9}$$

Combination of equations (9), (8), (4), and (3) leads to a differential equation for ϕ of the form

$$\frac{\partial^2\phi}{\partial X^2} + \frac{\partial^2\phi}{\partial Z^2} - \frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial X}\right)^2\phi = 0 \tag{10}$$

Boundary conditions for ϕ, besides the radiation condition at infinity, are

$$|X| \leqq b, \quad Z = 0, \quad \frac{\partial\phi}{\partial Z} = \frac{\partial H}{\partial t} + U\frac{\partial H}{\partial X} \tag{11}$$

$$X = b, \quad Z = 0, \quad \frac{\partial\phi}{\partial X} \text{ finite} \tag{12}$$

$$b \leqq |X|, \quad Z = 0, \quad \frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial X} = 0 \tag{13}$$

Attention is now restricted to the case of simple harmonic motion, by setting

$$\phi(X,Z,t) = e^{i\omega t}\bar{\phi}(X,Z) \tag{14}$$

Corresponding expressions are written for H and p.

The differential equation for $\bar{\phi}$ is

$$\frac{\partial^2\bar{\phi}}{\partial X^2} + \frac{\partial^2\bar{\phi}}{\partial Z^2} - \frac{1}{a^2}\left(i\omega + U\frac{\partial}{\partial X}\right)^2\bar{\phi} = 0 \tag{15}$$

while the relation between pressure amplitude $\bar{p}$ and potential amplitude $\bar{\phi}$ is of the form

$$\bar{p} = -\rho_o\left(i\omega\bar{\phi} + U\frac{\partial\bar{\phi}}{\partial X}\right) \tag{16}$$

At this stage it is convenient to introduce the following dimensionless variables and parameters:

$$\left.\begin{aligned} x &= \frac{X}{b} \\ z &= \sqrt{1 - M^2}\,\frac{Z}{b} \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} M &= \frac{U}{a} \\ k &= \frac{\omega b}{U} \end{aligned}\right\} \tag{18}$$

The differential equation (15) then becomes

$$\frac{\partial^2\bar{\phi}}{\partial x^2} + \frac{\partial^2\bar{\phi}}{\partial z^2} - i\,\frac{2kM^2}{1 - M^2}\,\frac{\partial\bar{\phi}}{\partial x} + \frac{k^2M^2}{1 - M^2}\,\bar{\phi} = 0 \tag{19}$$

In order to eliminate the first-derivative term in equation (19) a function ψ is introduced, defined by

$$\psi = e^{-i\mu x}\bar{\phi} \tag{20}$$

Setting

$$\mu = \frac{kM^2}{1 - M^2} \tag{21}$$

makes equation (19) read

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} + \kappa^2\psi = 0 \tag{22}$$

where

$$\kappa = \frac{kM}{1 - M^2} \tag{23}$$

Combination of equations (16), (20), and (21) gives for the pressure amplitude $\overline{p}$ the relation

$$\overline{p} = -\frac{\rho_o U}{b} e^{i\mu x}\left(i\nu\psi + \frac{\partial\psi}{\partial x}\right) \tag{24}$$

where

$$\nu = \frac{k}{1 - M^2} \tag{25}$$

In terms of the function ψ boundary conditions (11) to (13) assume the form

$$|x| \leqq 1, \quad z = 0, \quad \frac{\partial\psi}{\partial z} = \frac{Ue^{-i\mu x}}{\sqrt{1 - M^2}}\left(ik\overline{H} + \frac{\partial\overline{H}}{\partial x}\right) \tag{26}$$

$$x = 1, \quad z = 0, \quad \frac{\partial\psi}{\partial x} \text{ finite} \tag{27}$$

$$1 \leqq |x|, \quad z = 0, \quad i\nu\psi + \frac{\partial\psi}{\partial x} = 0 \tag{28}$$

The condition of no radiation-energy reflection at infinity is taken in the form

$$z \rightarrow \pm\infty, \quad \psi \approx f(x,z)e^{-i\kappa r} \tag{29}$$

where $r^2 = x^2 + z^2$ and where f tends to zero as z tends to plus or minus infinity.

Determination of the function ψ is facilitated by introducing two functions ψ_1 and ψ_2 such that

$$\psi = \psi_1 + \psi_2 \tag{30}$$

and where both ψ_1 and ψ_2 satisfy the differential equation (22). The boundary conditions for ψ_1 and ψ_2 are chosen as follows:

$$|x| \leqq 1, \quad z = 0, \quad \frac{\partial \psi_1}{\partial z} = \frac{Ue^{-i\mu x}}{\sqrt{1 - M^2}}\left(ik\overline{H} + \frac{\partial \overline{H}}{\partial x}\right), \quad \frac{\partial \psi_2}{\partial z} = 0 \tag{31}$$

$$x = 1, \quad z = 0, \quad \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial x} \text{ finite} \tag{32}$$

$$1 \leqq |x|, \quad z = 0, \quad \psi_1 = 0, \quad i\nu\psi_2 + \frac{\partial \psi_2}{\partial x} = 0 \tag{33}$$

It may be seen that the function ψ_1 represents the noncirculatory portion of the flow to be determined, while the function ψ_2 represents the circulatory flow around a flat plate at rest and at zero angle of attack. The intensity of the circulatory flow must be chosen such that its infinite trailing-edge velocity cancels the infinite trailing-edge velocity of the noncirculatory flow.

BOUNDARY-VALUE PROBLEM WITH REFERENCE TO ELLIPTICAL COORDINATES

Explicit solutions of the problem as formulated by means of equations (22) and (30) to (33) may be obtained through the introduction of elliptical coordinates ξ and ζ. The Cartesian coordinates (x,z) are related to the coordinates (ξ,ζ) by means of the equations

$$\left.\begin{aligned} x &= \cosh \xi \cos \zeta \\ z &= \sinh \xi \sin \zeta \end{aligned}\right\} \tag{34}$$

The differential equation (22) becomes, in terms of the elliptical coordinates,

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \zeta^2} + \kappa^2(\cosh^2\xi - \cos^2\zeta)\psi = 0 \tag{35}$$

Partial derivatives with respect to the old and new coordinates are related as follows:

$$\left.\begin{aligned}\frac{\partial\psi}{\partial z} &= \frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial z} + \frac{\partial\psi}{\partial\zeta}\frac{\partial\zeta}{\partial z}\\ \frac{\partial\psi}{\partial x} &= \frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\psi}{\partial\zeta}\frac{\partial\zeta}{\partial x}\end{aligned}\right\} \tag{36}$$

where

$$\left.\begin{aligned}\frac{\partial\xi}{\partial z} &= \frac{\cosh\xi\,\sin\zeta}{\Delta}, \quad \frac{\partial\zeta}{\partial z} = \frac{\sinh\xi\,\cos\zeta}{\Delta}\\ \frac{\partial\xi}{\partial x} &= \frac{\sinh\xi\,\cos\zeta}{\Delta}, \quad \frac{\partial\zeta}{\partial x} = \frac{-\cosh\xi\,\sin\zeta}{\Delta}\\ \Delta &= \cosh^2\xi - \cos^2\zeta\end{aligned}\right\} \tag{37}$$

The boundary $z = 0, \ |x| \leqq 1$ becomes the boundary $\xi = 0, \ 0 \leqq \zeta \leqq 2\pi$. The boundary $z = 0, \ x < -1$ becomes the boundary $0 \leqq \xi \leqq \infty, \ \zeta = \pi$. The boundary $z = 0, \ 1 < x$ becomes the boundary $0 \leqq \xi \leqq \infty, \ \zeta = 0$. The boundary $z = \pm\infty$ becomes the boundary $\xi = \infty$. The coordinate curves $\xi =$ Constant are confocal ellipses with $\xi = 0$ as the limiting ellipse $z = 0, \ |x| \leqq 1$. The coordinate curves $\zeta =$ Constant are hyperbolas intersecting the ellipses $\xi =$ Constant at right angles.

Boundary conditions (31) to (33) may now be written in the following form

$$\xi = 0, \quad \frac{1}{\sin\zeta}\frac{\partial\psi_1}{\partial\xi} = g(\zeta), \quad \frac{\partial\psi_2}{\partial\xi} = 0 \tag{38}$$

$$\xi = 0, \quad \lim_{\xi\to 0}\left[\frac{1}{\sin\zeta}\left(\frac{\partial\psi_1}{\partial\zeta} + \frac{\partial\psi_2}{\partial\zeta}\right)\right] \text{finite} \tag{39}$$

$$\zeta = 0,\pi, \quad \psi_1 = 0, \quad i\nu\psi_2 + \frac{1}{\sinh\xi}\frac{\partial\psi_2}{\partial\xi} = 0 \tag{40}$$

The function $g(\zeta)$ follows from a comparison of equation (38) and equation (31) in the form

$$g(\zeta) = \frac{Ue^{-i\mu \cos \zeta}}{\sqrt{1 - M^2}} \left(ik\overline{H} + \frac{\partial \overline{H}}{\partial x}\right) \tag{41}$$

From equation (24) it follows that the pressure amplitude $\overline{p}_a$ at the airfoil may be written in the form

$$\overline{p}_a = - \frac{\rho_o U}{b} e^{i\mu \cos \zeta} \left[i\nu\psi(0,\zeta) - \frac{1}{\sin \zeta} \frac{\partial\psi(0,\zeta)}{\partial\zeta}\right] \tag{42}$$

The problem now is to solve equation (35) in such a way that boundary conditions (38) to (40) are satisfied. Solutions suitable for this purpose will be discussed in the following section.

MATHIEU FUNCTION SOLUTIONS OF TWO-DIMENSIONAL WAVE EQUATION

In the present section are summarized some known results which will be needed in what follows. These, and further developments, not necessarily employing the same notation as that used here, may be found, for instance, in a recent book by McLachlan (reference 8).

Suitable solutions of the differential equation (35) for ψ are obtained by separation of variables, as follows. Setting

$$\psi = F(\xi)G(\zeta) \tag{43}$$

there is obtained

$$\frac{F''(\xi)}{F(\xi)} + \kappa^2 \cosh^2\xi + \frac{G''(\zeta)}{G(\zeta)} - \kappa^2 \cos^2\zeta = 0 \tag{44}$$

Equation (44) implies, with a separation constant c, the following two equations:

$$\left.\begin{aligned} F'' + (\kappa^2 \cosh^2\xi - c)F = 0 \\ G'' - (\kappa^2 \cos^2\zeta - c)G = 0 \end{aligned}\right\} \tag{45}$$

Possible values of the separation constant c are determined by the requirement that the functions $G(\zeta)$ are periodic of period 2π. The periodic solutions of the equation determining G may be either even or odd functions of ζ. They may be written in the following form:

$$ce_m(\zeta) = \sum_{n=0}^{\infty} A_{mn} \cos n\zeta \tag{46}$$

$$se_m(\zeta) = \sum_{n=1}^{\infty} B_{mn} \sin n\zeta \tag{47}$$

The function $ce_m(\zeta)$ has the same significance for elliptical coordinates as has the function $\cos m\zeta$ for polar coordinates. The function $se_m(\zeta)$ is a generalization of $\sin m\zeta$, in the same sense. The coefficients A_{mn} and B_{mn} may be written as power series in κ^2. The same is true for the separation constants c_{cm} and c_{sm}. Evidently these statements imply the following limiting behavior of the results:

$$\kappa = 0 \begin{cases} ce_m(\zeta) = A_{mm} \cos m\zeta \\ se_m(\zeta) = B_{mm} \sin m\zeta \\ c_{cm} = c_{sm} = m^2 \end{cases} \tag{48}$$

The functions ce_m and se_m may be normalized in various ways. The following normalization condition is chosen here:

$$\int_0^{\pi} \left[ce_m(\zeta)\right]^2 d\zeta = \int_0^{\pi} \left[se_m(\zeta)\right]^2 d\zeta = 1 \tag{49}$$

The possibility of obtaining an explicit solution of the problem considered in this report is due to the following orthogonality properties of the Mathieu functions (when $m \neq n$)

$$\left.\begin{aligned} \int_0^{\pi} ce_m(\zeta)ce_n(\zeta)\ d\zeta &= 0 \\ \int_0^{\pi} se_m(\zeta)se_n(\zeta)\ d\zeta &= 0 \end{aligned}\right\} \tag{50}$$

Coming now to the functions $F(\xi)$, as determined by the first of the differential equations (45), there will be two linearly independent solutions $F(\xi)$ for each value of the separation constant c. These two solutions must be combined in such a way that their behavior at infinity is as follows:

$$\xi \longrightarrow \infty, \quad F(\xi) \approx \frac{1}{\sqrt{e^{\xi}}} e^{-i\kappa e^{\xi}} \tag{51}$$

This behavior is required in order that the energy-radiation condition (29) be satisfied by the solution to be obtained. Functions $F(\xi)$ which have this property and which correspond to the functions $ce_m(\zeta)$ and $se_m(\zeta)$ are here designated by $Ce_m(\xi)$ and $Se_m(\xi)$.

In what follows these two systems of functions are considered normalized by the requirement that

$$Ce_m'(0) = Se_m'(0) = 1 \tag{52}$$

Note that the functions Ce_m and Se_m may be obtained from functions Ce_{m*} and Se_{m*} which are normalized in a different manner by the relations $Ce_m = Ce_{m*}/Ce_{m*}'(0)$ and $Se_m = Se_{m*}/Se_{m*}'(0)$. A corresponding statement applies to the functions ce_m and se_m.

There exist various infinite-series representations for Ce_m and Se_m for which reference is made to the literature (reference 8), as no use of them is made in the present report.

As the problem under consideration is linear, all solutions of the type here discussed may be combined and written

$$\psi = \sum_{m=0}^{\infty} a_m ce_m(\zeta) Ce_m(\xi) + \sum b_m se_m(\zeta) Se_m(\xi) \tag{53}$$

The constants a_m and b_m will be determined by the application of appropriate boundary conditions.

DETERMINATION OF NONCIRCULATORY-FLOW COMPONENT

Considered first is the simpler problem of determining the function ψ_1 which, as can readily be seen, corresponds to a flow without circulation. Boundary conditions (38) to (40) indicate that ψ_1 will be an odd function of ζ; thus

$$\psi_1 = \sum_{m=1}^{\infty} b_m se_m(\zeta) Se_m(\xi) \tag{54}$$

Boundary condition (38) for ψ_1 leads to the following relation:

$$\frac{1}{\sin \zeta} \sum b_m se_m(\zeta) Se_m'(0) = g(\zeta) \tag{55}$$

In view of the orthogonality relations (50) and the normalization conditions (49) and (52), the coefficients b_m are obtained from equation (55) in the form

$$b_m = \int_0^{\pi} \sin \zeta \; g(\zeta) \; se_m(\zeta) \; d\zeta \tag{56}$$

The contribution $\bar{p}_a^{(1)}$ to the pressure at the airfoil follows from equations (42) and (54) in the following form:

$$\bar{p}_a^{(1)} = -\frac{\rho_o U}{b} e^{i\mu \cos \zeta} \left[i\nu \sum b_m se_m(\zeta) Se_m(0) - \frac{1}{\sin \zeta} \sum b_m se_m'(\zeta) Se_m(0) \right] \tag{57}$$

From equation (57) it follows that the noncirculatory portion $\bar{L}^{(1)}$ of the lift may be written as

$$\begin{aligned} \bar{L}^{(1)} &= 2b \int_{-1}^{1} \bar{p}_a^{(1)} \; dx \\ &= 2b \int_0^{\pi} \bar{p}_a \sin \zeta \; d\zeta \\ &= -2\rho_o U \sum b_m Se_m(0) \int_0^{\pi} e^{i\mu \cos \zeta} \left[i\nu \sin \zeta \; se_n(\zeta) - se_n'(\zeta) \right] d\zeta \end{aligned} \tag{58}$$

The following abbreviation may be used:

$$l_m = \int_0^{\pi} e^{i\mu \cos \zeta} \left[i\nu \sin \zeta \; se_n(\zeta) - se_n'(\zeta) \right] d\zeta \tag{59}$$

Then,

$$\overline{L}^{(1)} = -2\rho_0 U \sum b_m Se_m(0) l_m \tag{60}$$

A corresponding expression for the moment about midchord becomes

$$\overline{M}^{(1)}(0) = 2b^2 \int_{-1}^{1} x\overline{p}_a^{(1)}\ dx$$

$$= -2\rho_0 Ub \sum b_m Se_m(0) m_m \tag{61}$$

where

$$m_m = \int_0^{\pi} \cos\zeta\ e^{i\mu\cos\zeta}\left[i\nu \sin\zeta\ se_m(\zeta) - se_m'(\zeta)\right] d\zeta$$

$$= -i\frac{\partial l_m}{\partial \mu} \tag{62}$$

Analogously, there is obtained for the hinge moment about $x = c$,

$$\overline{M}_c^{(1)} = 2b^2 \int_c^1 (x - c)\overline{p}_a^{(1)}\ dx = -2\rho_0 Ub \sum b_m Se_m(0) m_{cm} \tag{63}$$

where

$$m_{cm} = \int_0^{\cos^{-1}c} (\cos\zeta - c)e^{i\mu\cos\zeta}\left[i\nu\sin\zeta\ se_m(\zeta) - se_m'(\zeta)\right] d\zeta \tag{64}$$

It may be further indicated in which way the coefficients m_m and l_m may be evaluated. By suitable integration by parts in equation (59) there is first obtained

$$l_m = \left(\frac{\nu}{\mu} - 1\right)\int_0^{\pi} e^{i\mu\cos\zeta} se_m'(\zeta)\ d\zeta \tag{65}$$

There are introduced into this equation the series

$$se_m'(\zeta) = \sum_n B_{mn} n \cos n\zeta \tag{66}$$

and the relation $\nu/\mu = M^{-2}$. This gives

$$l_m = \left(\frac{1}{M^2} - 1\right) \sum_n B_{mn} n \int_0^\pi e^{i\mu \cos \zeta} \cos n\zeta \, d\zeta \tag{67}$$

The integrals in equation (67) are well-known formulas for Bessel functions and equation (67) may therefore be written in the alternate form

$$l_m = \pi \left(\frac{1}{M^2} - 1\right) \sum_n B_{mn} n i^n J_n(\mu) \tag{68}$$

From equation (62) it follows then that

$$m_m = -\pi i \left(\frac{1}{M^2} - 1\right) \sum_n B_{mn} n i^n J_n'(\mu) \tag{69}$$

As tables exist for the coefficients B_{mn} and the Bessel functions $J_n(\mu)$ and their derivatives $J_n'(\mu)$, the coefficients l_m and m_m may readily be evaluated for given values of M and k.

Before a similar statement can be made about the hinge-moment coefficients m_{cm} it will be necessary to tabulate the function

$$\int_0^{\cos^{-1}c} e^{-i\mu \cos \zeta} \cos m\zeta \, d\zeta$$

and its derivative with respect to μ for various values of m and c.

Having evaluated the coefficients l_m, m_m, and m_{cm}, it is next necessary to evaluate the remaining coefficients b_m in equations (60), (61), and (63), which depend on the motion of the wing, through equations (56) and (41). As will next be shown, this evaluation is readily carried out for the two basic degrees of motion

$$\left.\begin{aligned} \overline{H}(x) &= \overline{h} \\ \overline{H}(x) &= \overline{\alpha} b x \end{aligned}\right\} \tag{70}$$

There is obtained

$$g_h(\zeta) = \frac{Ue^{-i\mu x}}{\sqrt{1 - M^2}} ik\bar{h} \tag{71}$$

and

$$b_{mh} = \frac{Uik\bar{h}}{\sqrt{1 - M^2}} I_m \tag{72}$$

where

$$I_m = \int_0^\pi e^{-i\mu \cos \zeta} \sin \zeta \; se_m(\zeta) \; d\zeta$$

$$= \sum_n B_{mn} \int_0^\pi e^{-i\mu \cos \zeta} \sin \zeta \sin n\zeta \; d\zeta$$

$$= \sum_n B_{mn} \frac{1}{2} \int_0^\pi e^{-i\mu \cos \zeta} \left[\cos (n - 1)\zeta - \cos (n + 1)\zeta\right] d\zeta$$

$$= \frac{\pi}{2} \sum_n B_{mn} \left[i^{(n-1)} J_{n-1}(-\mu) - i^{(n+1)} J_{n+1}(-\mu)\right]$$

$$= - \frac{\pi}{2} \frac{1}{\mu} \sum_n B_{mn} (-1)^n i^n J_n(\mu) \tag{73}$$

$$g_\alpha(\zeta) = \frac{Ue^{-i\mu x}}{\sqrt{1 - M^2}} \bar{\alpha} b(ikx + 1) \tag{74}$$

and, with I_m from equation (73),

$$b_{m\alpha} = \frac{U\bar{\alpha}b}{\sqrt{1 - M^2}} \left(I_m - k \frac{\partial I_m}{\partial \mu}\right) \tag{75}$$

In order to evaluate the corresponding coefficients for aileron motion one is again led to nontabulated integrals of the same form as those occurring in the evaluation of the aileron hinge-moment coefficients m_{cm}.

In summary it may be said that existing tables are adequate for direct calculation of the noncirculatory part of lift and moment for bending and torsion motion of the wing section but that additional tables are required for aileron motion and for aileron hinge-moment calculation.

DETERMINATION OF CIRCULATORY-FLOW COMPONENT

The essential difficulty in the present problem is the determination of the function ψ_2. Boundary conditions (38) and (40) for ψ_2 are homogeneous and so the solution ψ_2 will contain an arbitrary multiplicative constant. This may also be seen by rewriting the boundary conditions for ψ_2 in the form

$$\left.\begin{aligned} &z = 0, \quad |x| \leqq 1, \quad \frac{\partial \psi_2}{\partial z} = 0 \\ &z = 0, \quad 1 \leqq |x|, \quad i\nu\psi_2 + \frac{\partial \psi_2}{\partial x} = 0 \end{aligned}\right\} \tag{76}$$

The conditions for $1 \leqq |x|$ may be rewritten in the form

$$\left.\begin{aligned} &z = 0, \quad x \leqq -1, \quad \psi_2 = 0 \\ &z = 0, \quad 1 \leqq x, \quad \psi_2 = Ce^{-i\nu x} \end{aligned}\right\} \tag{77}$$

where C is an arbitrary constant and where account has been taken of the fact that the uniform main flow is undisturbed far in front of the airfoil.

As ψ_2 is an odd function of z, it follows that ψ_2 and $\partial\psi_2/\partial x$ are discontinuous across the line $(z = 0, 1 \leqq x)$ and this is what is responsible for the inconvenience of the problem.

It had previously been proposed (reference 4) to take ψ_2 in the following form:

$$\psi_2 = -\frac{C}{2i}\int_1^\infty e^{-i\nu x'} \frac{\partial H_0^{(2)}(\kappa r)}{\partial z} dx' + \sum b_n^{(2)} se_n(\zeta) Se_n(\xi) \tag{78}$$

where

$$r^2 = (x - x')^2 + z^2$$

The coefficients $b_n^{(2)}$ in equation (78) are then to be determined from the remaining boundary condition, which requires that $\frac{\partial \psi_2}{\partial z} = 0$ when $z = 0$, $|x| \leqq 1$. While this determination is possible in principle, it appears to lead to inconvenient formulas when it comes to the evaluation of the coefficients $b_n^{(2)}$.

An alternate procedure due to Haskind (reference 6) is at present believed to be the most convenient approach. Haskind's procedure makes use of a function W defined by

$$\frac{\partial W}{\partial z} = i\nu\psi_2 + \frac{\partial \psi_2}{\partial x} \tag{79}$$

and required also to satisfy the differential equation $\nabla^2 W + \kappa^2 W = 0$. As $\partial W/\partial z$ is an odd function of z, it is indicated that W itself is an even function of z. Furthermore, the function W is continuous outside the slit $z = 0$, $|x| \leqq 1$. Consequently, W may be taken in the following form:

$$W = \sum_m a_m ce_m(\zeta) Ce_m(\xi) \tag{80}$$

In order to obtain the coefficients a_m it is observed that the first of equations (76) is equivalent to the following boundary condition for W:

$$z = 0, \quad |x| \leqq 1, \quad \frac{\partial^2 W}{\partial z^2} = 0 \tag{81}$$

In view of the form of the differential equation for W, equation (81) is equivalent to the following equation:

$$z = 0, \quad |x| \leqq 1, \quad W = A \cos \kappa x + B \frac{\sin \kappa x}{\kappa} \tag{82}$$

Combination of equations (80) and (82) shows that

$$\sum a_m ce_m(\zeta) Ce_m(0) = A \cos (\kappa \cos \zeta) + B \frac{\sin (\kappa \cos \zeta)}{\kappa} \tag{83}$$

The orthogonality properties of the functions ce_m lead then to the following expressions for the coefficients a_m:

$$a_m = A\alpha_m + B\beta_m \tag{84}$$

where

$$\left.\begin{aligned} \alpha_m &= \frac{\int_0^{\pi} \cos (\kappa \cos \zeta) ce_m(\zeta)\, d\zeta}{Ce_m(0)} \\ \beta_m &= \frac{\int_0^{\pi} \kappa^{-1} \sin (\kappa \cos \zeta) ce_m(\zeta)\, d\zeta}{Ce_m(0)} \end{aligned}\right\} \tag{85}$$

Now

$$W' = A \sum \alpha_m ce_m(\zeta) Ce_m(\xi) + B \sum \beta_m ce_m(\zeta) Ce_m(\xi) \tag{86}$$

and

$$\bar{p}_a^{(2)} = -\frac{\rho_o U}{b} e^{i\mu x}\left(i\nu\psi_2 + \frac{\partial\psi_2}{\partial x}\right)_{z=0,\ |x|\leqq 1}$$

$$= -\frac{\rho_o U}{b} e^{i\mu x}\left(\frac{\partial W}{\partial z}\right)_{z=0,\ |x|\leqq 1}$$

$$= -\frac{\rho_o U}{b} e^{i\mu\cos\zeta}\left(\frac{1}{\sin\zeta}\frac{\partial W}{\partial\xi}\right)_{\xi=0}$$

$$= -\frac{\rho_o U}{b}\frac{e^{i\mu\cos\zeta}}{\sin\zeta}\left[A\sum\alpha_m ce_m(\zeta) + B\sum\beta_m ce_m(\zeta)\right] \qquad (87)$$

Calculation of lift $\bar{L}^{(2)}$ and moments $\bar{M}^{(2)}$ proceeds in a manner which is entirely analogous to the procedure used in equations (58) to (75) in order to obtain expressions for $\bar{L}^{(1)}$ and $\bar{M}^{(1)}$. These expressions may be readily listed as soon as numerical calculations are intended.

The foregoing solution ψ_2 still contains two arbitrary parameters, A and B, instead of one as it should. The reason for this is that, instead of the condition $\partial\psi_2/\partial z = 0$ at the airfoil, only the less restrictive condition $(i\nu + \partial/\partial x)\ \partial\psi_2/\partial z = 0$ at the airfoil has been satisfied. A relation between A and B which takes account of this fact is obtained as follows. From

$$\left(i\nu + \frac{\partial}{\partial x}\right)\frac{\partial\psi_2}{\partial z} = e^{-i\nu x}\frac{\partial}{\partial x}\left(e^{i\nu x}\frac{\partial\psi_2}{\partial z}\right) = \frac{\partial^2 W}{\partial z^2} \qquad (88)$$

there is obtained by integration

$$\left[e^{i\nu x}\frac{\partial\psi_2}{\partial z}\right]_{-\infty}^{x} = \int_{-\infty}^{x} e^{i\nu x'}\frac{\partial^2 W}{\partial z^2}\,dx' \qquad (89)$$

In view of the form of the differential equation for W, and since $\partial\psi_2(-\infty, z)/\partial z = 0$, equation (89) is equivalent to the following equation:

$$e^{i\nu x} \frac{\partial \psi_2}{\partial z} = -\int_{-\infty}^{x} e^{i\nu x'} \left(\frac{\partial^2 W}{\partial x'^2} + \kappa^2 W\right) dx' \tag{90}$$

Equation (90), when suitably applied, leads to the required relation between the constants A and B. Since $\partial\psi_2/\partial z = 0$ at the airfoil, equation (90) implies that

$$\lim_{z \to 0} \int_{-\infty}^{x} e^{i\nu x'} \left(\frac{\partial^2 W}{\partial x'^2} + \kappa^2 W\right) dx' = 0, \quad |x| \leqq 1 \tag{91}$$

Since $\dfrac{\partial^2 W}{\partial x^2} + \kappa^2 W = 0$ when $z = 0$ and $|x| < 1$, there follows from equation (91)

$$\lim_{\substack{z \to 0 \\ \epsilon \to 0}} \int_{-\infty}^{-1+\epsilon} e^{i\nu x} \left(\frac{\partial^2 W}{\partial x^2} + \kappa^2 W\right) dx = 0 \tag{92}$$

Equation (92) may be transformed by integration by parts in various ways. One such form is obtained by writing

$$\left.\begin{aligned} \int_{-\infty}^{-1+\epsilon} e^{i\nu x} \frac{\partial^2 W}{\partial x^2} &= \left[e^{i\nu x} \frac{\partial W}{\partial x}\right]_{-\infty}^{-1+\epsilon} - i\nu \int_{-\infty}^{-1+\epsilon} e^{i\nu x} \frac{\partial W}{\partial x} dx \\ \int_{-\infty}^{-1+\epsilon} e^{i\nu x} W &= \left[\frac{e^{i\nu x}}{i\nu} W\right]_{-\infty}^{-1+\epsilon} - \frac{1}{i\nu} \int_{-\infty}^{-1+\epsilon} e^{i\nu x} \frac{\partial W}{\partial x} dx \end{aligned}\right\} \tag{93}$$

This makes equation (92) appear in the following form:

$$\begin{gathered} (\kappa^2 - \nu^2) \int_{-\infty}^{-1} e^{i\nu x} \frac{\partial W(x, 0)}{\partial x} dx = \\ e^{-i\nu}\left[A(i\nu\kappa \sin \kappa + \kappa^2 \cos \kappa) + B(i\nu \cos \kappa - \kappa \sin \kappa)\right] \end{gathered} \tag{94}$$

Another equation which follows from equation (92) and which is equivalent to Haskind's equation is obtained by writing

$$\int_{-\infty}^{-1+\epsilon} e^{i\nu x} \frac{\partial^2 W}{\partial x^2} dx = \left[e^{i\nu x} \frac{\partial W}{\partial x} - i\nu e^{i\nu x} W\right]_{-\infty}^{-1+\epsilon} - \nu^2 \int_{-\infty}^{-1+\epsilon} e^{i\nu x} W \, dx \qquad (95)$$

Combination of equations (95) and (92) gives

$$(\kappa^2 - \nu^2) \int_{-\infty}^{-1} e^{i\nu x} W(x,0) \, dx =$$

$$e^{-i\nu}\left[A(i\nu \cos \kappa - \kappa \sin \kappa) - B\left(\cos \kappa + \frac{i\nu}{\kappa} \sin \kappa\right)\right] \qquad (96)$$

It may be remarked that equation (96) cannot be used to obtain the correct result for the limiting case of incompressible flow for which $\kappa = 0$. The reason for this is, as will be shown, that when $\kappa = 0$ the integral on the left of equation (96) is not convergent. On the other hand, equation (94) does lead to the correct result in the limiting case of incompressibility.

In order to evaluate equations (96) or (94) it is observed that, for $\zeta = \pi$, $x = -\cosh \xi$ and $dx = -\sinh \xi \, d\xi$. Hence,

$$\left.\begin{aligned} \int_{-\infty}^{-1} e^{i\nu x} (W)_{z=0} \, dx &= \int_0^{\infty} e^{-i\nu \cosh \xi} (W)_{\zeta=\pi} \sinh \xi \, d\xi \\ \int_{-\infty}^{-1} e^{i\nu x} \left(\frac{\partial W}{\partial x}\right)_{z=0} dx &= -\int_0^{\infty} e^{-i\nu \cosh \xi} \left(\frac{\partial W}{\partial \xi}\right)_{\zeta=\pi} d\xi \end{aligned}\right\} \qquad (97)$$

Now take W from equation (86) and obtain, first from equation (96),

$$A\left[e^{-i\nu}(\kappa \sin \kappa - i\nu \cos \kappa) + (\kappa^2 - \nu^2) \sum \alpha_n ce_n(\pi) \int_0^\infty e^{-i\nu \cosh \xi} Ce_n(\xi) \sinh \xi \, d\xi\right] + B\left[e^{-i\nu}\left(\cos \kappa + \frac{i\nu}{\kappa} \sin \kappa\right) + (\kappa^2 - \nu^2) \sum \beta_n ce_n(\pi) \int_0^\infty e^{-i\nu \cosh \xi} Ce_n(\xi) \sinh \xi \, d\xi\right] = 0 \quad (98)$$

The corresponding result based on equation (94) is

$$A\left[e^{-i\nu}\kappa(i\nu \sin \kappa + \kappa \cos \kappa) + (\kappa^2 - \nu^2) \sum \alpha_n ce_n(\pi) \int_0^\infty e^{-i\nu \cosh \xi} Ce_n'(\xi) \, d\xi\right] + B\left[e^{-i\nu}(i\nu \cos \kappa - \kappa \sin \kappa) + (\kappa^2 - \nu^2) \sum \beta_n ce_n(\pi) \int_0^\infty e^{-i\nu \cosh \xi} Ce_n'(\xi) \, d\xi\right] = 0 \quad (99)$$

It thus appears that determination of the appropriate values of the ratio A/B involves the calculation of the set of infinite integrals

$$\int_0^\infty e^{-i\nu \cosh \xi} Ce_n'(\xi) \sinh \xi \, d\xi$$

or

$$\int_0^\infty e^{-i\nu \cosh \xi} Ce_n'(\xi) \, d\xi$$

for all integer values of n and for whatever values of κ and ν are of interest.

It may finally be remarked that in the work of Haskind (reference 6) it is shown that equation (98) can be replaced by an alternative equation in which, instead of the above infinite integrals, there occur integrals between the limits $0, \pi$. The integrand in these alternate integrals consists of products of a Mathieu function and a function which itself is defined as an infinite integral involving a Hankel function in the integrand.

DETERMINATION OF CIRCULATORY-FLOW COMPONENT FOR INCOMPRESSIBLE FLOW

The following discussion is intended to show the usefulness of Haskind's function W in the derivation of the known solution of the problem of incompressible flow.

Setting $\kappa = 0$, boundary condition (82) becomes

$$z = 0, \quad |x| \leqq 1, \quad W = A + Bx \tag{100}$$

or

$$\xi = 0, \quad 0 \leqq \zeta \leqq 2\pi, \quad W = A + B \cos \zeta \tag{101}$$

Solution (80) becomes, when $\kappa = 0$,

$$W = \frac{a_0}{\pi} (\xi + \xi_0) + \frac{2}{\pi} \sum_{n=1}^{\infty} a_m \cos m\zeta \ e^{-m\xi} \tag{102}$$

The constant ξ_0 is arbitrary and may be set equal to 1. Comparison of equations (102) and (101) shows that equation (101) is satisfied by setting

$$\left.\begin{aligned} a_0 &= \pi A \\ a_1 &= (\pi/2)B \\ a_m &= 0 \qquad\qquad (m = 2, 3, \ldots) \end{aligned}\right\} \tag{103}$$

and, consequently, the function W assumes the form

$$W = A(1 + \xi) + Be^{-\xi} \cos \zeta \tag{104}$$

The pressure distribution $\overline{p}_a^{(2)}$ at the airfoil becomes, according to equation (87),

$$\overline{p}_a^{(2)} = -\frac{\rho_o U}{b}\left(\frac{1}{\sin\zeta}\frac{\partial W}{\partial\xi}\right)_{\xi=0}$$

$$= -\frac{\rho_o U}{b}\left(\frac{A}{\sin\zeta} - \frac{B}{\sin\zeta}\cos\zeta\right)$$

$$= -\frac{\rho_o U}{b}\left(\frac{A}{\sqrt{1 - x^2}} - \frac{Bx}{\sqrt{1 - x^2}}\right) \tag{105}$$

Equation (94), which gives a relation between A and B, becomes, with $\kappa = 0$,

$$Bi\nu e^{-i\nu} = \nu^2 \int_0^\infty e^{-i\nu\cosh\xi}\left(\frac{\partial W}{\partial\xi}\right)_{\xi=\pi} d\xi$$

$$= \nu^2 \int_0^\infty e^{-i\nu\cosh\xi}\left(A + Be^{-\xi}\right) d\xi \tag{106}$$

The integrals in equation (106) become, with

$$\left.\begin{aligned}\cosh\xi &= \lambda \\ d\xi &= d\lambda\Big/\sqrt{\lambda^2 - 1}\end{aligned}\right\} \tag{107}$$

expressible in terms of modified Bessel functions, as follows:

$$\int_0^\infty e^{-i\nu\cosh\xi}\, d\xi = \int_1^\infty \frac{e^{-i\nu\lambda}}{\sqrt{\lambda^2 - 1}}\, d\lambda = K_0(i\nu) \tag{108}$$

$$\int_0^\infty e^{-i\nu\cosh\xi-\xi}\,d\xi = \int_1^\infty e^{-i\nu\lambda}\,\frac{\lambda - \sqrt{\lambda^2 - 1}}{\sqrt{\lambda^2 - 1}}\,d\lambda$$

$$= \int_1^\infty e^{-i\nu\lambda}\left(\sqrt{\frac{\lambda + 1}{\lambda - 1}} - 1 - \frac{1}{\sqrt{\lambda^2 - 1}}\right)d\lambda$$

$$= K_0(i\nu) + K_1(i\nu) - \frac{e^{-i\nu}}{i\nu} - K_0(i\nu)$$

$$= K_1(i\nu) - \frac{e^{-i\nu}}{i\nu} \tag{109}$$

Combination of equations (106), (108), and (109) relates the constants A and B in the following way:

$$BK_1(i\nu) + AK_0(i\nu) = 0 \tag{110}$$

Combination of equations (105) and (110) gives the required explicit expression for the pressure distribution at the airfoil for the purely circulatory flow component.

It may finally be remarked that use of equation (96) instead of equation (94) is not permitted, when $\kappa = 0$, as the integral on the left of equation (96) does not converge when the function W of equation (104) is substituted in it. The reason for this is that W tends to infinity for large values of ξ when $\kappa = 0$ and does not do so when $\kappa \neq 0$.

SATISFACTION OF TRAILING-EDGE CONDITION

It remains to obtain one further condition, besides equation (98) or equation (99), for the two constants A and B in equation (87) for the pressure component $\overline{p}_a^{(2)}$. This remaining condition is the condition of finite trailing-edge velocity as expressed by equation (32). As ψ_1 and ψ_2 themselves remain finite at the trailing edge, equation (32) may be replaced by the following conditions:

$$z = 0, \quad x = 1, \quad \frac{\partial\psi_1}{\partial x} + i\nu\psi_2 + \frac{\partial\psi_2}{\partial x} \quad \text{finite} \tag{111}$$

From equation (54) it follows that

$$z = 0, \quad \frac{\partial\psi_1}{\partial x} = -\frac{1}{\sin\zeta}\left(\frac{\partial\psi_1}{\partial\zeta}\right)_{\xi=0}$$

$$= -\frac{1}{\sin\zeta}\sum b_n se_n'(\zeta)Se_n(0) \tag{112}$$

From equations (79) and (80) it follows that

$$z = 0, \quad i\nu\psi_2 + \frac{\partial\psi_2}{\partial x} = \frac{1}{\sin\zeta}\left(\frac{\partial W}{\partial\xi}\right)_{\xi=0}$$

$$= \frac{1}{\sin\zeta}\sum\left(A\alpha_n + B\beta_n\right)ce_n(\zeta) \tag{113}$$

Combination of equations (111), (112), and (113) leads to the following further relation between A and B:

$$A\sum\alpha_n ce_n(0) + B\sum\beta_n ce_n(0) = \sum b_n se_n'(0)Se_n(0) \tag{114}$$

It may be recalled that the coefficients α_n and β_n are given by equations (85) and that the coefficients b_n are given by equation (56), where the downwash function g is defined by equation (38).

VALUE OF CIRCULATION FUNCTION

For the purpose of the subsequent calculation of three-dimensional corrections to the two-dimensional theory, there is needed the value of the circulation function Ω, defined by

$$\Omega = 2e^{i\nu}\int_{-1}^{1}\left(\frac{\partial\psi}{\partial x}\right)_{z=0} dx = 2e^{i\nu}\psi(1,0) \tag{115}$$

Since $\psi_1(1,0) = 0$ it follows that the circulatory component of flow is the only one which contributes to the value of Ω and

$$\Omega = 2e^{i\nu}\psi_2(1,0) \tag{116}$$

From equation (79) it follows that

$$e^{-i\nu x}\frac{\partial}{\partial x}\left(e^{i\nu x}\psi_2\right) = \frac{\partial W}{\partial z} \tag{117}$$

and, consequently,

$$\left[e^{i\nu x}\psi_2\right]_{-1}^{1} = \int_{-1}^{1} e^{i\nu x}\frac{\partial W}{\partial z}\,dx \tag{118}$$

Since $\psi_2(-1,0) = 0$ there follows from equation (118) the relation

$$e^{i\nu}\psi_2(1,0) = \int_0^{\pi} e^{i\nu\cos\zeta}\left(\frac{\partial W}{\partial \xi}\right)_{\xi=0} d\zeta \tag{119}$$

and, with W from equation (80), there is obtained for Ω

$$\frac{1}{2}\,\Omega = A\sum \alpha_n \int_0^{\pi} e^{i\nu\cos\zeta}ce_n(\zeta)\,d\zeta + B\sum \beta_n \int_0^{\pi} e^{i\nu\cos\zeta}ce_n(\zeta)\,d\zeta \tag{120}$$

For numerical applications it will be necessary to evaluate the integrals in equation (120) for the chosen values of the parameters ν and κ. This can be done in a manner which is analogous to the method of evaluation of the coefficients l_m in equations (65) to (68).

CALCULATION OF THREE-DIMENSIONAL CORRECTIONS TO TWO-DIMENSIONAL THEORY

In what follows there is indicated a procedure of incorporating in the results of the two-dimensional theory corrections taking account approximately of the three-dimensionality of the flow over wings of finite span. This procedure is based on an approximate integral equation of the three-dimensional problem as given in an earlier report (reference 7). For the sake of clarity, attention is restricted in what follows to the case of a lifting surface of rectangular plan form, although the results of reference 7 were obtained for lifting surfaces with taper and moderate sweep as well.

In considering the three-dimensional problem, besides the coordinates X and Z for the two-dimensional problem, a spanwise coordinate Y is introduced. The plan form of the lifting surface is given by

$$\left.\begin{aligned} |X| &\leqq b \\ |Y| &\leqq sb \end{aligned}\right\} \tag{121}$$

so that s is the value of the aspect ratio of the surface. The following two dimensionless coordinates are introduced further:

$$y = \sqrt{1 - M^2}\,\frac{Y}{b} \tag{122}$$

and

$$y^* = \frac{y}{s\sqrt{1 - M^2}} = \frac{Y}{sb} \tag{123}$$

The final result of reference 7 was an integral equation which, for the rectangular-plan-form wing, is of the following form:

$$g(x,y) = -\frac{1}{2\pi}\oint_{-1}^{1} \lambda(\xi,y)G(x - \xi;\ \kappa)\,d\xi + \Omega(y)F(x) +$$

$$e^{-i\nu x}\oint_{-1}^{1} \frac{d\Omega}{d\eta^*}\,K(y^* - \eta^*;\ k,\ s,\ M)\,d\eta^* \tag{124}$$

The quantities g, κ, and ν have been defined earlier. The functions λ, G, F, and K are defined as follows:

$$\lambda = 2\,\frac{\partial\Psi(x,y,0)}{\partial x} \tag{125}$$

$$G(x-\xi;\,\kappa) = -\frac{i\pi\kappa}{2}\left[\frac{|x-\xi|}{x-\xi}H_1^{(2)}\left(\kappa|x-\xi|\right) - \int_{-\infty}^{\kappa(x-\xi)} H_0^{(2)}\left(|\zeta|\right)\,d\zeta\right] \tag{126}$$

$$F(x) = \frac{i\nu}{2\pi}\int_1^{\infty} e^{-i\nu\xi}G(x-\xi;\,\kappa)\,d\xi \tag{127}$$

$$K(y^*-\eta^*;\,k,\,s,\,M) = \frac{ikM}{8(1-M^2)}\,\frac{|y^*-\eta^*|}{y^*-\eta^*}\left\{H_1^{(2)}\left(\frac{ksM}{\sqrt{1-M^2}}|y^*-\eta^*|\right) + \int_{-\infty}^{-\frac{ksM}{\sqrt{1-M^2}}|y^*-\eta^*|} H_0^{(2)}\left(|\zeta|\right)\,d\zeta - \frac{iks}{\sqrt{1-M^2}}F_M\left[\frac{ks(y^*-\eta^*)}{\sqrt{1-M^2}}\right]\right\} \tag{128}$$

$$F_M(z) = \frac{|z|}{z}\int_0^{\infty} e^{-i\sigma}\left[\int_{-\infty}^{-|z|}\frac{\sigma e^{-iM\sqrt{\sigma^2+\zeta^2}}}{\sigma^2+\zeta^2}\left(\frac{1}{\sqrt{\sigma^2+\zeta^2}} + iM\right)d\zeta + M^2\int_{-\infty}^{-\sigma}\left(\int_{-\infty}^{-|z|}\frac{e^{-iM\sqrt{\tau^2+\zeta^2}}}{\sqrt{\tau^2+\zeta^2}}\,d\zeta\right)d\tau + \int_{-\infty}^{-\sigma}\frac{|z|\,e^{-iM\sqrt{\tau^2+z^2}}}{\tau^2+z^2}\left(\frac{1}{\sqrt{\tau^2+z^2}} + iM\right)d\tau\right]d\sigma \tag{129}$$

To effect the solution of the integral equation (124) proceed as follows. Write equation (124) in the form

$$g(x,y) - e^{-i\nu x}Q = -\frac{1}{2\pi}\oint_{-1}^{1} \lambda(\xi,y)\overline{G}(x - \xi)\, d\xi \tag{130}$$

where

$$Q = \oint_{-1}^{1} \frac{d\Omega}{d\eta^*} K\, d\eta^* \tag{131}$$

$$\overline{G} = G - 4\pi e^{i\nu}F(x) \tag{132}$$

Equation (130) indicates that the solution of the three-dimensional problem is equivalent to the solution of a two-dimensional problem with a modified downwash function $g + \Delta g$, where

$$\Delta g = -e^{-i\nu x}Q \tag{133}$$

It must next be shown how this modification of the two-dimensional downwash affects the previously discussed solution of the two-dimensional problem.

Returning first to the noncirculatory portion of the two-dimensional solution, equation (54), it is evident that the Δg term is responsible for a change of the coefficients b_n into $b_n + \Delta b_n$. The coefficients b_n are given as before by equation (56), except that now they are functions of y.

$$b_n(y) = \int_0^{\pi} \sin \zeta\; g(x,y) se_n(\zeta)\, d\zeta \qquad (x = \cos \zeta) \tag{134}$$

The correction coefficients Δb_n are correspondingly given by

$$\Delta b_n = -Q \int_0^{\pi} \sin \zeta\; e^{-i\nu \cos \zeta}\, se_n(\zeta)\, d\zeta \tag{135}$$

The integrals on the right of equation (135) may be evaluated once for all. The integrals on the right of equation (134) are the same as those which must be evaluated for the solution of the two-dimensional problem and for which series representations for bending-torsion motions are listed in equations (72) and (75).

Coming next to the circulatory portion of the flow, it is evident that the only modification here consists in assuming that the arbitrary coefficients A and B in solution (86) are functions of the spanwise coordinate y. The two functions A and B are related to each other, as before, in accordance with equation (98). This relation may be given here in the form

$$A(y)P_A + B(y)P_B = 0 \tag{136}$$

where P_A and P_B are defined with reference to equation (98).

The next step consists in the satisfaction of the trailing-edge condition. This condition is analogous to equation (114) except that now it is necessary to write $b_n + \Delta b_n$ instead of b_n. Thus,

$$A(y)Q_A + B(y)Q_B = \sum (b_n + \Delta b_n)se_n'(0)Se_n(0) \tag{137}$$

where Q_A and Q_B are defined in accordance with equation (114).

In view of equation (135) there may be written, instead of equation (137),

$$A(y)Q_A + B(y)Q_B + Q(y)Q_c = \sum b_n se_n'(0)Se_n(0) \tag{138}$$

where the coefficient Q_c is defined by

$$Q_c = \sum se_n'(0)Se_n(0) \int_0^\pi \sin \zeta \; e^{-i\nu \cos \zeta} \; se_n(\zeta) \; d\zeta \tag{139}$$

The function $Q(y)$ is defined by equation (131). In equation (131) it is necessary to express Ω in terms of A and B, in accordance with equation (120). To this end, equation (120) may be written in the form

$$\Omega = AR_A + BR_B \tag{140}$$

Then,

$$Q = R_A \oint_{-1}^{1} \frac{dA}{d\eta^*} K \, d\eta^* + R_B \oint_{-1}^{1} \frac{dB}{d\eta^*} K \, d\eta^* \tag{141}$$

Combination of equations (141) and (138) gives

$$Q_A A(y) + Q_c R_A \oint_{-1}^{1} \frac{dA}{d\eta^*} K \, d\eta^* + Q_B B(y) + Q_c R_B \oint_{-1}^{1} \frac{dB}{d\eta^*} K \, d\eta^* =$$

$$\sum b_n se_n'(0) Se_n(0) \tag{142}$$

Equations (142) and (136) represent the main result of the present developments, so far as the calculation of three-dimensional corrections to the two-dimensional theory is concerned.

Having calculated A and B in accordance with equations (142) and (136), the circulatory component of the pressure distribution follows from equation (87), while the noncirculatory pressure distribution follows from equation (57), where b_n is replaced by $b_n + \Delta b_n$. In equation (135) for Δb_n the quantity Q is to be taken from equation (141).

The foregoing explanation indicates that the major difficulty in calculating three-dimensional corrections, once the results of the two-dimensional theory are known, consists in evaluating the integrals

$$\oint_{-1}^{1} \frac{dA}{d\eta^*} K \, d\eta^*$$

and

$$\oint_{-1}^{1} \frac{dB}{d\eta^*} K \, d\eta^*$$

In order to do this, it is necessary to have the numerical values for the function K, as defined by equations (128) and (129). Once the function K is calculated, the determination of three-dimensional corrections to the two-dimensional theory can be effected in a manner analogous to what has previously been done for the corresponding problem of incompressible flow, either in accordance with reference 9 or in accordance with any modifications of the scheme used in reference 9.

The following further development of the preceding scheme establishes a somewhat closer connection with the corresponding results for the problem of incompressible flow. Solve equations (136) and (140) for A and B, as follows:

$$\left.\begin{aligned} A &= \frac{P_B}{R_A P_B - R_B P_A}\,\Omega \\[2ex] B &= \frac{-P_A}{R_A P_B - R_B P_A}\,\Omega \end{aligned}\right\} \tag{143}$$

Combination of equations (143) and (87) gives the circulatory pressure distribution at the airfoil in terms of the circulation function Ω. An integral equation for the function Ω is obtained by combining equations (138) and (143), in the following form:

$$\frac{Q_A P_B - Q_B P_A}{R_A P_B - R_B P_A}\,\Omega(y) + Q_c \oint_{-1}^{1} \frac{d\Omega}{d\eta^*}\,K\,d\eta^* = \sum b_n se_n'(0) Se_n(0) \tag{144}$$

In analogy with the corresponding results for the problem of incompressible flow, a function $\mu(k,M)$ may be defined by

$$\mu(k,M) = Q_c \frac{R_A P_B - R_B P_A}{Q_A P_B - Q_B P_A} \tag{145}$$

It may further be written

$$\Omega^{(2)}(y) = \frac{\mu}{Q_c} \sum b_n se_n'(0) Se_n(0) \tag{146}$$

Equation (144) then assumes the form

$$\Omega(y) + \mu(k,M) \oint_{-1}^{1} \frac{d\Omega}{d\eta^*}\,K\,d\eta^* = \Omega^{(2)}(y) \tag{147}$$

In this form equation (147) is of the same appearance as a corresponding equation for incompressible flow (reference 9). Equation (147) is valid for lifting surfaces with rectangular plan form. The corresponding equation for arbitrary plan form, provided sweep is moderate, may also quite readily be established on the basis of the results in reference 7.

SUMMARY OF RESULTS

It has been shown that the amplitude of the pressure distribution on a rectangular-plan-form airfoil oscillating with frequency ω is given by the following formulas:

$$\bar{p}_a = \bar{p}_a^{(1)} + \bar{p}_a^{(2)}$$

$$\bar{p}_a^{(1)} = -\frac{\rho_o U}{b} e^{i\mu \cos\zeta}\left[i\nu \sum (b_n + \Delta b_n)se_n(\zeta)Se_n(0) - \frac{1}{\sin\zeta} \sum (b_n + \Delta b_n)se_n'(\zeta)Se_n(0)\right]$$

$$\bar{p}_a^{(2)} = -\frac{\rho_o U}{b} \frac{e^{i\mu \cos\zeta}}{\sin\zeta}\left[P_B \sum \alpha_n ce_n(\zeta) - P_A \sum \beta_n ce_n(\zeta)\right]\frac{\Omega(y)}{R_A P_B - R_B P_A}$$

In the equations for $\bar{p}_a^{(1)}$ and $\bar{p}_a^{(2)}$

$$\mu = \frac{kM^2}{1 - M^2} \qquad M = \frac{U}{a}$$

$$\nu = \frac{k}{1 - M^2} \qquad \cos\zeta = x = \frac{X}{b}$$

$$k = \frac{\omega b}{U} \qquad \kappa = \frac{kM}{1 - M^2}$$

The Mathieu functions se_n, ce_n, Se_n, and Ce_n are defined by equations (45) to (49), (53), and (54).

The coefficients b_n are defined by

$$b_n = \int_0^{\pi} g \sin \zeta \; se_n(\zeta) \; d\zeta$$

$$\Delta b_n = -Q \int_0^{\pi} e^{-i\nu \cos \zeta} \sin \zeta \; se_n(\zeta) \; d\zeta$$

where

$$g = \frac{Ue^{-i\mu \cos \zeta}}{\sqrt{1 - M^2}} \left(ik\overline{\overline{H}} + \frac{\partial \overline{\overline{H}}}{\partial x} \right)$$

and

$$Q = \oint_{-1}^{1} \frac{d\Omega}{d\eta^*} K \; d\eta^*$$

$$\eta^* = \frac{H}{sb}$$

The coefficients α_n and β_n are defined by

$$\alpha_n = \frac{\int_0^{\pi} \cos (\kappa \cos \zeta) \; ce_n(\zeta) \; d\zeta}{Ce_n(0)}$$

$$\beta_n = \frac{\int_0^{\pi} \sin (\kappa \cos \zeta) \; ce_n(\zeta) \; d\zeta}{\kappa Ce_n(0)}$$

The coefficients P and R in the equation for $\overline{p}_a^{(2)}$ are defined by

$$P_A = e^{-i\nu}(\kappa \sin \kappa - i\nu \cos \kappa) +$$

$$(\kappa^2 - \nu^2) \sum \alpha_n ce_n(\pi) \int_0^\infty e^{-i\nu \cosh \xi} Ce_n(\xi) \sinh \xi \, d\xi$$

$$P_B = e^{-i\nu}\left(\cos \kappa + \frac{i\nu}{\kappa} \sin \kappa\right) +$$

$$(\kappa^2 - \nu^2) \sum \beta_n ce_n(\pi) \int_0^\infty e^{-i\nu \cosh \xi} Ce_n(\xi) \sinh \xi \, d\xi$$

and

$$R_A = 2 \sum \alpha_n \int_0^\pi e^{i\nu \cos \zeta} ce_n(\zeta) \, d\zeta$$

$$R_B = 2 \sum \beta_n \int_0^\pi e^{i\nu \cos \zeta} ce_n(\zeta) \, d\zeta$$

The function Ω is to be determined from the integral equation

$$\Omega + \mu(k, M) \oint_{-1}^{1} \frac{d\Omega}{d\eta^*} K \, d\eta^* = \Omega^{(2)}$$

where

$$\mu(k,M) = Q_c \frac{R_A P_B - R_B P_A}{Q_A P_B - Q_B P_A}$$

and

$$\Omega^{(2)} = \frac{\mu(k,M)}{Q_c} \sum b_n se_n'(0) Se_n(0)$$

and where the coefficients Q_A, Q_B, and Q_c are given by

$$Q_A = \sum \alpha_n ce_n(0)$$

$$Q_B = \sum \beta_n ce_n(0)$$

$$Q_c = \sum se_n'(0) Se_n(0) \int_0^{\pi} e^{-i\nu \cos \zeta} se_n(\zeta) \sin \zeta \, d\zeta$$

The kernel K of the integral equation is defined by equations (128) and (129) and will not be repeated here.

The results of the two-dimensional theory are contained in the foregoing equations. They appear when $d\Omega/d\eta^* = 0$ which makes $\Delta b_n = 0$ in the equation for $\overline{p}_a^{(1)}$ and $\Omega = \Omega^{(2)}$ in the equation for the circulation function.

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