https://ntrs.nasa.gov/search.jsp?R=19930083134 2020-06-17T19:44:46+00:00Z

N 62 54280

CASE FILE COPY

NACA TN 2280

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2280

A THEORETICAL ANALYSIS OF THE EFFECTS OF FUEL MOTION ON AIRPLANE DYNAMICS

By Albert A. Schy

Langley Aeronautical Laboratory
Langley Field, Va.

Washington
January 1951

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2280

A THEORETICAL ANALYSIS OF THE EFFECTS OF FUEL MOTION ON AIRPLANE DYNAMICS

By Albert A. Schy

SUMMARY

The general equations of motion for an airplane with a number of spherical fuel tanks are derived. The motion of the fuel is approximated by the motion of solid pendulums. The same type of derivation and equations are shown to apply to any type of fuel tank where the motion of the fuel may be represented in terms of undamped harmonic oscillators.

Motions are calculated for a present-day high-speed airplane and a free-flying airplane model with two spherical tanks in the symmetry plane. These calculations show that the normal airplane motion may be considerably modified and that residual oscillations may result. The ratio of the natural fuel frequency to the natural airplane frequency is shown to be the most important parameter for determining the effect of the fuel motion on the airplane motion. The stabilizing effect of turbulence in the fuel is discussed, and it is suggested that the stabilizing effect of artificially induced turbulence be investigated experimentally.

INTRODUCTION

Small-amplitude lightly damped lateral oscillations are a troublesome characteristic of certain high-speed airplanes. Several possible explanations for these oscillations, which are adequate in specific cases, have been offered. For example, reference 1 shows that nonlinear aerodynamic derivatives could cause such oscillations, and it has been shown that atmospheric turbulence is another possible cause. It has also been suggested that a possible cause of such oscillations is the motion of fuel in the tanks. In some recently designed airplanes the mass of the fuel relative to the airplane mass is much larger than has been common in the past; therefore, the effects of fuel motion can be expected to be relatively more important. In fact, in several cases baffling the fuel tanks was found to have considerable effect on the general handling qualities of the airplane and sometimes actually eliminated the troublesome lightly damped lateral oscillations which had been present.

An experimental investigation of the effects of fuel motion on the lateral motion of a free-flying airplane model is described in reference 2. The results indicated that the effects of fuel motion were noticeable and caused the lateral motion of the model to be very erratic.

The present analysis treats each fuel tank as a pendulum oscillating in two degrees of freedom and applies Lagrange's equations of motion to obtain the interaction between these pendulums and the airplane. Thus, for small motions the fuels are treated as simple harmonic oscillators. The results are applied to obtain the general equations of motion of this system and, in particular, the lateral motion of an airplane with internal fuel tanks in the plane of symmetry of the airplane. Since the general solution of the equations is extremely complicated, an attempt is made to evaluate the results by carrying out numerical calculations for specific cases. This approach is shown to be adequate in yielding the most general effects of fuel motion.

The discussion of the numerical application of the equations of motion to specific cases is given in detail following the derivation of the equations of motion. This discussion of results is understandable quite independently of the derivation of the equations of motion.

SYMBOLS

X, Y, Z	airplane stability axes with origin determined by equations (13); also components of applied forces along these axes
L, M, N	components of applied moments about X-, Y-, and Z-axes, respectively
$\underline{i}$, $\underline{j}$, $\underline{k}$	unit vectors along X-, Y-, and Z-axes, respectively
x, y, z	components of translational displacement of airplane
$\dot{\underline{r}}$	vector translational velocity of airplane $(\underline{i}\dot{x} + \underline{j}\dot{y} + \underline{k}\dot{z} = \underline{i}U_o + \underline{v})$
U_o	magnitude of steady-state velocity
u, v, w	components of disturbance translational velocity of airplane
$\underline{v}$	vector disturbance velocity of airplane $(\underline{i}u + \underline{j}v + \underline{k}w)$

$\underline{R}$	vector position of a point in airplane $(\underline{i}R_X + \underline{j}R_Y + \underline{k}R_Z)$
$\underline{R}_f'$	vector position of center of gravity of fuel in a particular fuel tank
$\underline{V}$	total vector velocity of a point in airplane
β	sideslip angle $\left(\tan^{-1} \frac{v}{U_o}\right)$
ϕ, θ, ψ	infinitesmal rotations of airplane about X-, Y-, and Z-axes, respectively
$\underline{\omega}$	vector rotational velocity of airplane $(\underline{i}\dot{\phi} + \underline{j}\dot{\theta} + \underline{k}\dot{\psi})$
ζ, η	components of angular displacement from vertical of line joining fuel center of gravity to tank center, taken on mutually perpendicular planes; ζ positive in direction of positive roll and η positive in direction of positive pitch
l	distance from tank center to fuel center of gravity
h	vertical displacement of fuel center of gravity from equilibrium position
k	number of fuel tanks
m	mass
m_t	total mass of airplane and fuel
I_X, I_Y, I_Z, I_{XZ}	total moments and product of inertia of airplane about X-, Y-, and Z-axes
$I_X', I_Y', I_Z', I_{XZ}', I_{YZ}', I_{XY}'$	rigid-body moments and products of inertia about axes through center of gravity
I_ζ, I_η	fuel moments of inertia about ζ- and η-axes through tank center

$I_f = I_{\zeta_f}$

I'_ζ , I'_η fuel moments of inertia about ζ- and η-axes through fuel center of gravity

K_X nondimensional radius of gyration in roll $\left(\sqrt{\frac{I_X}{m_t b^2}}\right)$

K_Z nondimensional radius of gyration in yaw $\left(\sqrt{\frac{I_Z}{m_t b^2}}\right)$

K_{XZ} nondimensional product-of-inertia parameter $\left(\frac{-I_{XZ}}{m_t b^2}\right)$

$K_f = \sqrt{\frac{I_f}{m_f l_f b}}$

E_k kinetic energy

E_p potential energy

P period

$T_{1/2}$, T_2 time for exponentially damped or increasing oscillation to halve or double amplitude, respectively

t time

s nondimensional time parameter $\left(\frac{U_o t}{b}\right)$

g acceleration due to gravity

$G = \frac{bg}{U_o^2}$

γ flight-path angle with respect to horizontal

ρ air density

S wing area

b wing span

μ_b, μ_{b_f} lateral nondimensional mass coefficient $\left(\mu_b = \frac{m_t}{\rho S b}, \quad \mu_{b_f} = \frac{m_f}{\rho S b}\right)$

D differentiation operator $\left(\frac{d}{ds}\right)$

C_L trim lift coefficient $\left(\frac{m_t g \cos \gamma_o}{\frac{1}{2} \rho U_o^2 S}\right)$

C_l rolling-moment coefficient $\left(\frac{\text{Rolling moment}}{\frac{1}{2} \rho U_o^2 S b}\right)$

C_n yawing-moment coefficient $\left(\frac{\text{Yawing moment}}{\frac{1}{2} \rho U_o^2 S b}\right)$

C_Y lateral-force coefficient $\left(\frac{\text{Lateral force}}{\frac{1}{2} \rho U_o^2 S}\right)$

$$C_{l_p} = \frac{\partial C_l}{\partial\left(\frac{\dot{\phi} b}{2U_o}\right)}$$

$$C_{n_p} = \frac{\partial C_n}{\partial\left(\frac{\dot{\phi} b}{2U_o}\right)}$$

$$C_{Y_p} = \frac{\partial C_Y}{\partial\left(\frac{\dot{\phi} b}{2U_o}\right)}$$

$$C_{l_r} = \frac{\partial C_l}{\partial\left(\frac{\dot{\psi}b}{2U_o}\right)}$$

$$C_{n_r} = \frac{\partial C_n}{\partial\left(\frac{\dot{\psi}b}{2U_o}\right)}$$

$$C_{Y_r} = \frac{\partial C_Y}{\partial\left(\frac{\dot{\psi}b}{2U_o}\right)}$$

$$C_{l_\beta} = \frac{\partial C_l}{\partial\beta}$$

$$C_{n_\beta} = \frac{\partial C_n}{\partial\beta}$$

$$C_{Y_\beta} = \frac{\partial C_Y}{\partial\beta}$$

Subscripts:

f — particular fuel tank, or summation index over fuel tanks $(f = 1, 2, \ldots k)$

a — airplane without fuel

o — initial conditions at $t = 0$

DERIVATION OF EQUATIONS OF MOTION

Assumptions for Derivation of General Equations of Motion

As a first approximation, only the effect of the motion of the fuel as a whole is considered; that is, only the fundamental mode of the wave motion is considered, and this mode is approximated by rigid-body motion. The main effect of the internal wave motion is to introduce damping into the fuel oscillation. This damping is caused by the conversion of kinetic energy into heat through the turbulence caused by the splashing of the fuel. A strictly analytic consideration of such damping effects is extremely difficult; on the other hand, the damping caused by the viscous tangential forces between the fuel and the tank is completely negligible (see reference 3). The analysis of the problem is therefore confined to the motion with no fuel damping and the effect of the damping is considered in the discussion of the results.

In a spherical tank the fuel can oscillate approximately as a rigid body if no splashing is assumed for small oscillations. The motion may be pictured as the "rocking" of a spherical segment of constant shape. The restraining force of the tank, which always acts in a direction normal to the motion, is exactly analogous to the tension in a pendulum. Thus, the small motions of the fuel in a spherical tank may be represented quite well by the well-known simple properties of small pendulum motions. This approach is used in the mathematical analysis of the problem.

The effect of aspherical tank shape can be approximated by replacing the tank by an equivalent harmonic oscillator with an arbitrary amount of turbulence damping added even for small motions. For example, the representation of rectangular tanks as harmonic oscillators is discussed in reference 4. Thus in this case also the most general effects of the fuel motion on the airplane motion should be qualitatively obtainable by this type of analysis.

The effects of large-amplitude fuel motions will be discussed qualitatively after the discussion of the results of the mathematical analysis. As usual in stability analysis all motions are assumed small and second-order terms are ignored.

Derivation of General Equations of Motion

With the preceding assumptions the physical problem can be considered as the interaction between two or more rigid bodies, namely the airplane and the several fuel pendulums, with each fuel pendulum considered as suspended from the tank center. The only potential energy considered in the system is that of the pendulums. If the inertial characteristics of the airplane and the fuel are known, the kinetic energy of the system can be obtained from the translational and rotational velocities of the airplane and the fuels. With this information the interactions in the system can be obtained by using Lagrange's equations of motion in the form (see reference 5)

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}_i}\right) - \frac{\partial E_k}{\partial q_i} = - \frac{\partial E_p}{\partial q_i} + Q_i \tag{1}$$

$$(i = 1, 2, \ldots n)$$

where q_i is one of the n generalized coordinates of the system corresponding to the n degrees of freedom, $\dot{q}_i$ is the corresponding velocity, and Q_i is the corresponding generalized force. The q_i will be lengths and angles and the corresponding Q_i will be forces and moments, respectively.

The airplane itself introduces the customary six degrees of freedom, which are the three displacements of the airplane system along axes fixed in the airplane (x, y, z) and the corresponding angles of rotation of the airplane about these axes (ϕ, θ, ψ). For small displacements the pendulum motion can be described by two angles, ζ and η, since the vertical motion can be neglected (see fig. 1). The angle ζ is measured from a vertical line through the tank center to the projection of the line joining the tank center to the fuel center of gravity on the vertical plane parallel to the Y-axis and η is the corresponding angle in the vertical plane parallel to the X-axis. For small angles, ζ and η may be represented as in figure 1. In effect this figure makes use of the fact that small angles may be added vectorially. When the two additional coordinates ζ and η are used to describe the pendulum motion, the whole system has two additional degrees of freedom for each fuel tank.

Expressions must be obtained for E_k and E_p in terms of the coordinates of the system and their time derivatives in order to use equation (1). The only potential energy is that of the fuel pendulums, which can be written as follows for k fuel tanks

$$E_p = \sum_{f=1}^{k} m_f h_f g \tag{2}$$

For the height of the center of gravity in each fuel tank, as can be seen from figure 1,

$$h = h_\zeta + h_\eta \approx l - l \cos \zeta \cos \eta \approx l\left[1 - \left(1 - \frac{1}{2}\zeta^2\right)\left(1 - \frac{1}{2}\eta^2\right)\right]$$

or

$$h \approx \frac{1}{2} l\left(\zeta^2 + \eta^2\right) \tag{3}$$

Note that the vertical displacement h is of second order in the small quantities η and ζ. This fact justifies the previous statement that the vertical displacement could be neglected in describing the pendulum motion only by the two coordinates η and ζ. As might be expected, equations (2) and (3) indicate that each fuel pendulum is being considered as an undamped oscillator with two degrees of freedom in a horizontal plane.

The kinetic energy of the total system can be written as the sum of the kinetic energy of the airplane and the kinetic energies of the fuels. Also the kinetic energy of each rigid body can be expressed as the sum of the translational energy of the mass moving with the velocity of its center of gravity and the rotational kinetic energy of the mass about its center of gravity. Thus, when the inertial characteristics of the airplane and the fuels are known, the kinetic energy can be obtained as a function of the generalized coordinates and velocities if the translational velocity of each center of gravity and the angular velocities of the airplane and fuels about their respective centers of gravity can be expressed in terms of these generalized coordinates and velocities.

In order to obtain the required expressions for these velocities a system of axes fixed in the airplane with the X-axis along the steady-state velocity at $t = 0$ is used, as is customary in stability analysis. For the present the origin of the coordinates will not be specified. However, these stability axes are not inertial axes and Newton's second law applies only in an inertial system of axes. The inertial axes may be taken as axes fixed in the earth. Then in the equations of motion the velocities and accelerations must be measured with respect to the earth, and their expressions in terms of components in the moving airplane axes may be obtained as shown in reference 6. These expressions will give the kinetic-reaction forces, which for the case of a rotating system are often referred to as "gyroscopic" forces. For the velocity, referred to the inertial system, of any point defined by the vector $\underline{R}$ in the airplane axes (in particular, for the centers of gravity previously discussed)

$$\underline{V} = \dot{\underline{r}} + \underline{\omega} \times \underline{R} + \dot{\underline{R}} = \underline{i}U_0 + \underline{v} + \underline{\omega} \times \underline{R} + \dot{\underline{R}} \tag{4}$$

where all vectors are given in terms of the airplane axes and

$$\underline{\dot{r}} \equiv \underline{i}\dot{x} + \underline{j}\dot{y} + \underline{k}\dot{z} = \underline{i}U_o + \underline{v}$$

is the velocity of the origin of the airplane system with respect to the earth, while $\underline{v}$ and $\underline{\omega}$ are the translational and rotational disturbance velocities of the airplane axes.

Equation (4) may now be used to express the inertial velocities of the airplane center of gravity and the fuel centers of gravity in terms of the generalized coordinates by inserting for $\underline{R}$ the values $\underline{R_a}$ and $\underline{R}_f'$, where $\underline{R_a}$ is the vector position of the airplane center of gravity and $\underline{R}_f'$ indicates the vector position of any particular fuel center of gravity. The vector $\underline{R_a}$ is constant; therefore, $\underline{\dot{R}_a} = 0$. To obtain $\underline{\dot{R}}_f'$, note that to first order

$$\underline{R}_f' \approx \underline{R}_f - \underline{i}l_f \sin\left(\gamma_o + \theta - \eta_f\right) - \underline{j}l_f\left(\zeta_f - \phi \cos \gamma_o - \psi \sin \gamma_o\right) +$$

$$\underline{k}l_f \cos\left(\gamma_o + \theta - \eta_f\right)$$

$$\underline{R}_f' \approx \underline{R}_f - \underline{i}l_f\left[\sin \gamma_o + \left(\theta - \eta_f\right)\cos \gamma_o\right] - \underline{j}l_f\left(\zeta_f - \phi \cos \gamma_o - \psi \sin \gamma_o\right) +$$

$$\underline{k}l_f\left[\cos \gamma_o - \left(\theta - \eta_f\right)\sin \gamma_o\right]$$

where $\underline{R}_f$ is the fixed position of the tank center. Since γ_o is constant, to first order

$$\underline{\dot{R}}_f' \approx \underline{i}l_f\left(\dot{\eta}_f - \dot{\theta}\right)\cos \gamma_o - \underline{j}l_f\left(\dot{\zeta}_f - \dot{\phi} \cos \gamma_o - \dot{\psi} \sin \gamma_o\right) +$$

$$\underline{k}l_f\left(\dot{\eta}_f - \dot{\theta}\right)\sin \gamma_o$$

Again keeping only first-order terms leads to the following equation:

$$\underline{\omega} \times \underline{R}_f' \approx \underline{\omega} \times \underline{R}_f + \underline{i} l_f \dot{\theta} \cos \gamma_o - \underline{j} l_f \left(\dot{\phi} \cos \gamma_o + \dot{\psi} \sin \gamma_o \right) + \underline{k} l_f \dot{\theta} \sin \gamma_o$$

Now combining the last two equations gives

$$\underline{\omega} \times \underline{R}_f' + \dot{\underline{R}}_f' \approx \underline{\omega} \times \underline{R}_f + \underline{i} \dot{\eta}_f l_f \cos \gamma_o - \underline{j} l_f \dot{\zeta}_f + \underline{k} l_f \dot{\eta}_f \sin \gamma_o \qquad (5)$$

This equation shows, as could be predicted physically since no viscous force is assumed between the tank and the fuel, that the airplane rotation affects only the motion of the tank center.

From equations (4) and (5) the necessary translational velocities can be obtained for the translational kinetic energies. The rotational velocity of the airplane is simply $\underline{\omega}$. The spinning motion of the fuel about the vertical axis is ignored; then, the rotational velocity of the fuel may be given by the components $\dot{\zeta}$ and $\dot{\eta}$. The two corresponding horizontal axes of rotation through the fuel center of gravity are principal axes of the spherical segment of fuel; consequently, no product-of-inertia terms occur in the fuel rotational energy. Also, since the airplane center of gravity is in the airplane symmetry plane, $I_{YZ} = I_{XY} = 0$ and only the I_{XZ} product of inertia will appear in the airplane rotational energy.

By use of equation (4), the airplane velocity can be shown to be

$$\underline{V}_a = \underline{i}\left(U_o + u + \dot{\theta} R_{z_a} - \dot{\psi} R_{y_a}\right) + \underline{j}\left(v + \dot{\psi} R_{x_a} - \dot{\phi} R_{z_a}\right) +$$

$$\underline{k}\left(w + \dot{\phi} R_{y_a} - \dot{\theta} R_{x_a}\right) \qquad (6a)$$

When equation (5) is substituted in equation (4), the velocity of any particular fuel center of gravity is

$$\underline{V}_f = \underline{i}\left(U_o + u + \dot{\theta} R_{z_f} - \dot{\psi} R_{y_f} + \dot{\eta}_f l_f \cos \gamma_o\right) + \underline{j}\left(v + \dot{\psi} R_{x_f} - \dot{\phi} R_{z_f} - \dot{\zeta}_f l_f\right) +$$

$$\underline{k}\left(w + \dot{\phi} R_{y_f} - \dot{\theta} R_{x_f} + \dot{\eta}_f l_f \sin \gamma_o\right) \qquad (6b)$$

If V is the magnitude of the translational velocity of the center of gravity of a rigid body, I' its moment-of-inertia tensor about the axes through its center of gravity, and $\underline{\omega}$ the rotational velocity of the rigid body, the kinetic energy is

$$E_k = \frac{1}{2} mV^2 + \frac{1}{2}\left(I_X'\omega_x^2 + I_Y'\omega_y^2 + I_Z'\omega_z^2 - 2I_{YZ}'\omega_y\omega_z - 2I_{XZ}'\omega_x\omega_z - 2I_{XY}'\omega_x\omega_y\right) \tag{7}$$

Thus, for the kinetic energy of the airplane, substitution of equation (6a) in equation (7) gives

$$E_{k_a} = \frac{m_a}{2}\Big[\left(U_o + u + \dot{\theta}R_{z_a} - \dot{\psi}R_{y_a}\right)^2 + \left(v + \dot{\psi}R_{x_a} - \dot{\phi}R_{z_a}\right)^2 +$$

$$\left(w + \dot{\phi}R_{y_a} - \dot{\theta}R_{x_a}\right)^2\Big] + \frac{I_X'}{2}\dot{\phi}^2 + \frac{I_Y'}{2}\dot{\theta}^2 + \frac{I_Z'}{2}\dot{\psi}^2 - I_{XZ}\dot{\phi}\dot{\psi} \tag{8a}$$

and, for the kinetic energy of each fuel, substitution of equation (6b) in equation (7) gives

$$E_{k_f} = \frac{m_f}{2}\Big[\left(U_o + u + \dot{\theta}R_{z_f} - \dot{\psi}R_{y_f} + \dot{\eta}_f l_f \cos\gamma_o\right)^2 +$$

$$\left(v + \dot{\psi}R_{x_f} - \dot{\phi}R_{z_f} - \dot{\zeta}_f l_f\right)^2 + \left(w + \dot{\phi}R_{y_f} - \dot{\theta}R_{x_f} + \dot{\eta}_f l_f \sin\gamma_o\right)^2\Big] +$$

$$\frac{I_{\zeta_f}'}{2}\dot{\zeta}_f^2 + \frac{I_{\eta_f}'}{2}\dot{\eta}_f^2 \tag{8b}$$

For the total kinetic energy, $E_k = E_{k_a} + \sum_{f=1}^{k} E_{k_f}$; therefore, equations (2), (3), and (8) may be used in equation (1) to obtain the $(2k + 6)$ equations of motion. However, it must again be recalled that the coordinate system is rotating. The whole system is therefore subject to an additional gyroscopic acceleration since the time-derivative operator contains an additional gyroscopic term (see reference 6) when the components of the velocity (or any vector) are taken in the rotating system:

$$\frac{\delta}{\delta t}\dot{\underline{r}} = \ddot{\underline{r}} + \underline{\omega} \times \dot{\underline{r}}$$

Thus the gyroscopic acceleration acting on the whole rotating system is

$$\underline{\omega} \times \dot{\underline{r}} = \underline{\omega} \times \left(\underline{i}U_o + \underline{v}\right) \approx (\underline{\omega} \times \underline{i})U_o = \underline{j}\dot{\psi}U_o - \underline{k}\dot{\theta}U_o$$

The effect of this acceleration can be brought into the equations of motion by considering the inertial reaction of the total mass to this acceleration as an additional applied force. If the total mass is $m_t = m_a + \sum_{f=1}^{k} m_f$, this reaction has the following components:

$$\left.\begin{aligned} Y' &= -m_t U_o \dot{\psi} \\ Z' &= m_t U_o \dot{\theta} \end{aligned}\right\} \qquad (9)$$

In addition there is the inertial reaction torque $\underline{M}_f'$ on the fuel; this torque acts about the tank center and is caused by the acceleration of the tank center. For each tank, the vector reaction torque is

$$\underline{M}_f' = \left(\underline{R}_f' - R_f\right) \times m_f U_o\left(-\underline{j}\dot{\psi} + \underline{k}\dot{\theta}\right)$$

$$= m_f U_o\left[\left(-\underline{i}l_f \sin \gamma_o + \underline{k}l_f \cos \gamma_o\right) \times \left(-\underline{j}\dot{\psi} + \underline{k}\dot{\theta}\right) + \text{Second-order terms}\right]$$

$$\approx m_f l_f U_o\left(\underline{i}\dot{\psi} \cos \gamma_o + \underline{j}\dot{\theta} \sin \gamma_o + \underline{k}\dot{\psi} \sin \gamma_o\right)$$

Since $M_\zeta' = \left(\underline{M}_f'\right)_x \cos \gamma_o + \left(\underline{M}_f'\right)_z \sin \gamma_o$ and $M_\eta' = \left(\underline{M}_f'\right)_y$,

$$M'_\zeta = m_f l_f U_o \dot{\psi}\left(\cos^2\gamma_o + \sin^2\gamma_o\right) = m_f l_f U_o \dot{\psi} \tag{10a}$$

$$M'_\eta = m_f l_f U_o \dot{\theta} \sin \gamma_o \tag{10b}$$

The forces and moments in equations (9) and (10) must be added to the weight and aerodynamic forces to obtain the Q_i in equation (1).

For convenience, the results of equations (2), (3), and (8) are as follows:

$$E_p = \frac{1}{2} g \sum_{f=1}^{k} m_f l_f \left(\zeta_f^{\,2} + \eta_f^{\,2}\right) \tag{11a}$$

$$\begin{aligned} E_k = {} & \frac{m_a}{2}\left[\left(U_o + u + \dot{\theta}R_{z_a} - \dot{\psi}R_{y_a}\right)^2 + \left(v + \dot{\psi}R_{x_a} - \dot{\phi}R_{z_a}\right)^2 + \right. \\ & \left.\left(w + \dot{\phi}R_{y_a} - \dot{\theta}R_{x_a}\right)^2\right] + \sum_{f=1}^{k}\left\{\frac{m_f}{2}\left[\left(U_o + u + \dot{\theta}R_{z_f} - \dot{\psi}R_{y_f} + \dot{\eta}_f l_f \cos \gamma_o\right)^2 + \right.\right. \\ & \left.\left.\left(v + \dot{\psi}R_{x_f} - \dot{\phi}R_{z_f} - \dot{\zeta}_f l_f\right)^2 + \left(w + \dot{\phi}R_{y_f} - \dot{\theta}R_{x_f} + \dot{\eta}_f l_f \sin \gamma_o\right)^2\right]\right\} + \\ & \frac{I'_X}{2}\dot{\phi}^2 + \frac{I'_Y}{2}\dot{\theta}^2 + \frac{I'_Z}{2}\dot{\psi}^2 - I'_{XZ}\dot{\phi}\dot{\psi} + \frac{1}{2}\sum_{f=1}^{k}\left(I'_{\zeta_f}\dot{\zeta}_f^{\,2} + I'_{\eta_f}\dot{\eta}_f^{\,2}\right) \end{aligned} \tag{11b}$$

Equations (9), (10), and (11) may now be used in equation (1) to obtain the equations of motion. It should be noted that $\frac{\partial}{\partial \dot{x}} \equiv \frac{\partial}{\partial\left(U_o + u\right)} = \frac{\partial}{\partial u}$, $\frac{\partial}{\partial \dot{y}} = \frac{\partial}{\partial v}$, and $\frac{\partial}{\partial \dot{z}} = \frac{\partial}{\partial w}$. For example, to obtain the equation of motion in the x-direction, note that

$$\frac{\partial E_k}{\partial x} = \frac{\partial E_p}{\partial x} = 0$$

Then, equation (1) may be written as follows:

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{x}}\right) = \frac{d}{dt}\left(\frac{E_k}{\partial u}\right)$$

$$= m_a\left(\dot{u} + \ddot{\theta}R_{z_a} - \ddot{\psi}R_{y_a}\right) + \sum_{f=1}^{k} m_f\left(\dot{u} + \ddot{\theta}R_{z_f} - \ddot{\psi}R_{y_f} + \ddot{\eta}_f l_f \cos \gamma_o\right)$$

$$= m_t\dot{u} + \ddot{\theta}\left(m_aR_{z_a} + \sum_{f=1}^{k} m_fR_{z_f}\right) - \ddot{\psi}\left(m_aR_{y_a} + \sum_{f=1}^{k} m_fR_{y_f}\right) +$$

$$\sum_{f=1}^{k} \ddot{\eta}_f m_f l_f \cos \gamma_o \tag{12}$$

The position of the origin of airplane coordinates has not yet been specified. Equation (12) and the similar equations obtained in the other degrees of freedom suggest that the position of the origin be determined by the following three conditions:

$$m_aR_{x_a} + \sum_{f=1}^{k} m_fR_{x_f} = 0 \tag{13a}$$

$$m_aR_{y_a} + \sum_{f=1}^{k} m_fR_{y_f} = 0 \tag{13b}$$

$$m_aR_{z_a} + \sum_{f=1}^{k} m_fR_{z_f} = 0 \tag{13c}$$

Equations (13) imply that the origin is at the position of the total center of gravity when the fuel mass is treated as being concentrated at the tank center. This choice of the origin greatly simplifies expressions such as equation (12). The physical reason for this choice is again that the fuel does not rotate with the airplane; thus, a force acting on a line through this point, the center of gravity where the fuel reaction is assumed concentrated at the tank center, will produce no rotation of the airplane.

The following substitutions will also greatly simplify the writing of the final equations of motion:

$$\left.\begin{aligned}
I_X &\equiv I_X' + m_a\left[\left(R_{y_a}\right)^2 + \left(R_{z_a}\right)^2\right] + \sum_{f=1}^{k} m_f\left[\left(R_{y_f}\right)^2 + \left(R_{z_f}\right)^2\right] \\
I_Y &\equiv I_Y' + m_a\left[\left(R_{x_a}\right)^2 + \left(R_{z_a}\right)^2\right] + \sum_{f=1}^{k} m_f\left[\left(R_{x_f}\right)^2 + \left(R_{z_f}\right)^2\right] \\
I_Z &\equiv I_Z' + m_a\left[\left(R_{x_a}\right)^2 + \left(R_{y_a}\right)^2\right] + \sum_{f=1}^{k} m_f\left[\left(R_{x_f}\right)^2 + \left(R_{y_f}\right)^2\right] \\
I_{XZ} &\equiv I_{XZ}' + m_a R_{x_a} R_{z_a} + \sum_{f=1}^{k} m_f R_{x_f} R_{z_f}
\end{aligned}\right\} \quad (14a)$$

Note that the quantities defined by equations (14a) are the total moments and product of inertia about the origin of the airplane coordinates when the fuel mass is assumed to be concentrated at the tank center. Finally, the necessary moments of inertia of each fuel pendulum about the tank center are

$$\left.\begin{aligned}
I_{\zeta_f} &= I_{\zeta_f}' + m_f l_f^2 \\
I_{\eta_f} &= I_{\eta_f}' + m_f l_f^2
\end{aligned}\right\} \quad (14b)$$

Without loss of generality then, equations (13) and (14) are used in the equations of motion obtained by substituting equations (9), (10), and (11) in equation (1). The general equations of motion can now be given as follows:

$$\left.\begin{aligned}
&m_t\dot{u} + \sum_{f=1}^{k} m_f l_f \ddot{\eta}_f \cos\gamma_o = X \\
&m_t\left(\dot{w} - U_o\dot{\theta}\right) + \sum_{f=1}^{k} m_f l_f \ddot{\eta}_f \sin\gamma_o = Z \\
&I_Y\ddot{\theta} - \left(m_a R_{x_a} R_{y_a} + \sum_{f=1}^{k} m_f R_{x_f} R_{y_f}\right)\ddot{\phi} - \left(m_a R_{z_a} R_{y_a} + \sum_{f=1}^{k} m_f R_{z_f} R_{y_f}\right)\ddot{\psi} + \\
&\quad \sum_{f=1}^{k}\left(R_{z_f}\cos\gamma_o - R_{x_f}\sin\gamma_o\right) m_f l_f \ddot{\eta}_f = M \\
&I_{\eta_1}\ddot{\eta}_1 + m_1 l_1\Big[g\eta_1 + \dot{u}\cos\gamma_o + \left(\dot{w} - U_o\dot{\theta}\right)\sin\gamma_o + R_{y_1}\ddot{\phi}\sin\gamma_o + \\
&\quad \ddot{\theta}\left(R_{z_1}\cos\gamma_o - R_{x_1}\sin\gamma_o\right) - R_{y_1}\ddot{\psi}\cos\gamma_o\Big] = 0 \\
&\qquad\vdots
\end{aligned}\right\} \quad (15a)$$

$$\left.\begin{aligned}
&m_t\left(\dot{v} + U_o\dot{\psi}\right) - \sum_{f=1}^{k} m_f l_f \ddot{\zeta}_f = Y \\
&I_Z\ddot{\psi} - I_{XZ}\ddot{\phi} - \left(m_a R_{y_a} R_{z_a} + \sum_{f=1}^{k} m_f R_{y_f} R_{z_f}\right)\ddot{\theta} - \\
&\quad \sum_{f=1}^{k} m_f l_f\left(R_{x_f}\ddot{\zeta}_f + R_{y_f}\ddot{\eta}_f\cos\gamma_o\right) = N \\
&I_X\ddot{\phi} - I_{XZ}\ddot{\psi} - \left(m_a R_{x_a} R_{y_a} + \sum_{f=1}^{k} m_f R_{x_f} R_{y_f}\right)\ddot{\theta} + \\
&\quad \sum_{f=1}^{k} m_f l_f\left(R_{z_f}\ddot{\zeta}_f + R_{y_f}\ddot{\eta}_f\sin\gamma_o\right) = L \\
&I_{\zeta_1}\ddot{\zeta}_1 + m_1 l_1\left(g\zeta_1 - \dot{v} - U_o\dot{\psi} + R_{x_1}\ddot{\psi} - R_{z_1}\ddot{\phi}\right) = 0 \\
&\qquad\vdots
\end{aligned}\right\} \quad (15b)$$

In equations (15a) and (15b) only the fuel equations for the first fuel tank have been written. In each set there are k similar fuel equations. As has been previously stated, the forces on the right-hand sides of these equations are the applied forces and the weight and aerodynamic forces.

Simplifying Assumptions

The equations of motion have been separated into what would generally be considered the longitudinal motions, equations (15a), and the lateral motions, equations (15b). In the ordinary six-degree-of-freedom case, as can be shown from considerations of symmetry, no cross-coupling terms exist between these motions in the aerodynamic forces (see reference 7). Although such terms are known to exist in practice, they are small and generally neglected. However, many cross-coupling terms occur between equations (15a) and equations (15b) because of the fuel motions, even when the aerodynamic coupling forces are ignored. The magnitudes of these fuel forces can be seen to depend on the masses of the fuels, the vector positions of the fuel tanks, the "pendulum length" (i.e., the radius of the tank and the height of the fuel), and the accelerations involved in the fuel motion. Increasing the magnitude of any of these parameters will increase the effect of fuel motion. For most airplanes these fuel forces will be relatively small, but the present investigation is primarily concerned with all first-order fuel effects. In any particular case, the actual magnitudes of these forces may of course be obtained by inserting the values of the previously mentioned parameters.

Symmetrical fuel distribution.- Some of the terms in equations (15a), for example, the $\ddot{\phi}$ and $\ddot{\psi}$ terms in the θ equation, are essentially product-of-inertia terms arising from unsymmetrical distribution of the fuel about the $y = 0$ plane (i.e., the plane of symmetry). That this is so can be seen if the fuel tanks are assumed to be distributed symmetrically with respect to the symmetry plane; that is

$$\sum_{f=1}^{k} m_f R_{x_f} R_{y_f} = \sum_{f=1}^{k} m_f R_{z_f} R_{y_f} = \sum_{f=1}^{k} m_f R_{y_f} = 0 \qquad (16a)$$

and from equations (13)

$$R_{y_a} = 0 \qquad (16b)$$

therefore, these terms vanish for symmetrical fuel distributions. In most cases the fuel will be symmetrically distributed, and substitution of equations (16) in equations (15) yields the equations of motion for symmetrical fuel distribution:

$$\left.\begin{aligned}
&m_t\dot{u} + \sum_{f=1}^{k} m_f l_f \ddot{\eta}_f \cos\gamma_o = X \\
&m_t\left(\dot{w} - U_o\dot{\theta}\right) + \sum_{f=1}^{k} m_f l_f \ddot{\eta}_f \sin\gamma_o = Z \\
&I_Y\ddot{\theta} + \sum_{f=1}^{k}\left(R_{z_f}\cos\gamma_o - R_{x_f}\sin\gamma_o\right)m_f l_f \ddot{\eta}_f = M \\
&I_{\eta_1}\ddot{\eta}_1 + m_1 l_1\Big[g\eta_1 + \dot{u}\cos\gamma_o + \left(\dot{w} - U_o\dot{\theta}\right)\sin\gamma_o + R_{y_1}\ddot{\phi}\sin\gamma_o + \\
&\qquad \ddot{\theta}\left(R_{z_1}\cos\gamma_o - R_{x_1}\sin\gamma_o\right) - R_{y_1}\ddot{\psi}\cos\gamma_o\Big] = 0 \\
&\qquad\vdots
\end{aligned}\right\} \quad (17a)$$

$$\left.\begin{aligned}
&m_t\left(\dot{v} + U_o\dot{\psi}\right) - \sum_{f=1}^{k} m_f l_f \ddot{\zeta}_f = Y \\
&I_Z\ddot{\psi} - I_{XZ}\ddot{\phi} - \sum_{f=1}^{k} m_f l_f\left(R_{x_f}\ddot{\zeta}_f + R_{y_f}\ddot{\eta}_f\cos\gamma_o\right) = N \\
&I_X\ddot{\phi} - I_{XZ}\ddot{\psi} + \sum_{f=1}^{k} m_f l_f\left(R_{z_f}\ddot{\zeta}_f + R_{y_f}\ddot{\eta}_f\sin\gamma_o\right) = L \\
&I_{\zeta_1}\ddot{\zeta}_1 + m_1 l_1\left(g\zeta_1 - \dot{v} - U_o\dot{\psi} + R_{x_1}\ddot{\psi} - R_{z_1}\ddot{\phi}\right) = 0 \\
&\qquad\vdots
\end{aligned}\right\} \quad (17b)$$

In equations (17) even though the terms arising from unsymmetrical fuel distribution have vanished, some cross-coupling terms still remain between equations (17a) and (17b). These terms occur in the η equations of set (17a) and in the ϕ and ψ equations of set (17b). The significance of these terms is evident since each contains a factor R_{y_f}. Thus, these terms arise when the airplane has fuel tanks with centers not in the plane of symmetry, even though they are symmetrically distributed with respect to this plane. For example, they would arise for wing-tip tanks. Physically, these terms clearly give the interaction between the longitudinal fuel motion η_f in the wing-tip tanks and the airplane rotation about the vertical axis, which consists of the lateral motions ϕ and ψ. For example, assume for simplicity that $\gamma_o = 0$; then, a yawing acceleration of the airplane will cause a longitudinal fuel acceleration $\ddot{\eta}_f$ in the wing-tip tanks, and vice versa.

From this discussion the η motion appears to couple the lateral and longitudinal airplane motions even for the perfectly symmetrical fuel distribution described by equation (16a). However, the fact that this coupling does not occur can be seen by considering any pair of symmetrically placed and loaded tanks. Designate the η motion in this pair of tanks by η_1 and η_2. Then, the η equations in equations (17a) show that a longitudinal horizontal acceleration of the tank gives rise as expected to η accelerations. Since the system is linear this portion of the η motion may be considered independently, and because of the symmetry of the two tanks it is seen that $\ddot{\eta}_1 = \ddot{\eta}_2$ for the portion of the η motion arising from the longitudinal motion. Therefore, in the ψ and ϕ equations of set (17b) the effects of this $\ddot{\eta}$ will vanish since $R_{y_1} = -R_{y_2}$. In a similar manner the laterally caused η motion can be shown to have no effect on the longitudinal motion. Essentially the argument is that the η motion for each pair of tanks can be split up for perfectly symmetrical fuel distribution into symmetrical and antisymmetrical motions. The symmetrical portion of the η motion for each pair of tanks couples only with the longitudinal motion; the antisymmetrical η motion couples only with the lateral motion. Thus, for perfectly symmetrical fuel distributions the η equations of set (17a) could be combined with set (17b), only lateral degrees of freedom in the η equations being used (since the symmetrical portion of the η motion is of no interest); or the η equations may be used as shown with equations (17a), the lateral degrees of freedom in the η equations being ignored.

Fuel tanks centered in symmetry plane.- Many airplanes have large internal fuel tanks which are centered in the airplane symmetry plane. For such airplanes the equations of motion may always be separated into independent lateral and longitudinal modes since $R_{y_f} = 0$. Using this value in equations (17) gives the equations of motion for tanks centered in the symmetry plane:

$$
\left.
\begin{aligned}
& m_t\dot{u} + \sum_{f=1}^{k} m_f l_f \ddot{\eta}_f \cos\gamma_o = X \\
& m_t\left(\dot{w} - U_o\dot{\theta}\right) + \sum_{f=1}^{k} m_f l_f \ddot{\eta}_f \sin\gamma_o = Z \\
& I_Y\ddot{\theta} + \sum_{f=1}^{k} \left(R_{z_f}\cos\gamma_o - R_{x_f}\sin\gamma_o\right) m_f l_f \ddot{\eta}_f = M \\
& I_{\eta_1}\ddot{\eta}_1 + m_1 l_1 \Big[g\eta_1 + \dot{u}\cos\gamma_o + \left(\dot{w} - U_o\dot{\theta}\right)\sin\gamma_o + \\
& \qquad \left(R_{z_1}\cos\gamma_o - R_{x_1}\sin\gamma_o\right)\ddot{\theta}\Big] = 0 \\
& \qquad\qquad \vdots
\end{aligned}
\right\} \qquad (18a)
$$

$$
\left.
\begin{aligned}
& m_t\left(\dot{v} + U_o\dot{\psi}\right) - \sum_{f=1}^{k} m_f l_f \ddot{\zeta}_f = Y \\
& I_Z\ddot{\psi} - I_{XZ}\ddot{\phi} - \sum_{f=1}^{k} m_f l_f R_{x_f}\ddot{\zeta}_f = N \\
& I_X\ddot{\phi} - I_{XZ}\ddot{\psi} + \sum_{f=1}^{k} m_f l_f R_{z_f}\ddot{\zeta}_f = L \\
& I_{\zeta_1}\ddot{\zeta}_1 + m_1 l_1\left(g\zeta_1 - \dot{v} - U_o\dot{\psi} - R_{x_1}\ddot{\psi} - R_{z_1}\ddot{\phi}\right) = 0 \\
& \qquad\qquad \vdots
\end{aligned}
\right\} \qquad (18b)
$$

Each set of equations, (18a) and (18b), contains $(k + 3)$ variables; the two sets can be seen to be independent of each other since the ζ motion of each tank couples with only the lateral motion and the η motion couples with only the longitudinal motion.

For the case of a single fuel tank at the airplane center of gravity the modification of equations (18) is obvious. Then $\underline{R}_a = \underline{R}_f = 0$, and all coupling between the rotational motion and the fuel motion vanishes; that is, all the fuel terms in the rotational equations vanish and all rotational terms in the fuel equations vanish. For an aspherical tank the rotational coupling in this case will be small.

Limitations Inherent in the Approximations

Before proceeding to the application of equations (18b) it is appropriate to consider somewhat more explicitly the assumptions involved in the indiscriminate dropping of all second-order terms which appeared during the derivation of the equations of motion. In this connection the correction, arising from the airplane accelerations, to the constant acceleration field g involved in the pendulum potential energy should be considered. The assumption which is implied in neglecting these accelerations is that the accelerations of the tank centers are small with respect to g.

Considering this and previous approximations, it can be seen that three essential assumptions were made in dropping second-order terms:

(1) The fuel and airplane angular displacement variables are small enough so the angle approximates its sine. However, this approximation sometimes took the form that the angle was much less than 1 radian.

(2) The disturbance velocities are much less than U_o, and products of the linear or angular velocities can be ignored.

(3) The accelerations of the tank centers must be small compared to g.

Strictly speaking then, the statement that the equations of motion (15), and also the simplified equations, are accurate equations of motion to first order is to be taken to mean that the motions to which these equations apply are restricted by the preceding three conditions. Thus, the equations would appear to remain accurate at least at the beginning of a disturbance. Moreover, when the motion becomes large enough that these assumptions break down, the fundamental physical assumption that the fuel may be considered to move as a rigid body also breaks down; therefore, nothing can essentially be gained by keeping higher-order terms in the mathematical expressions.

Since the pendulum motion is little changed even up to angles of 30° to 40°, it could be expected that aside from splashing effects these equations should remain a good to fair approximation even at such angles. On the other hand, even the splashing effects, although they would introduce some damping and change the inertial characteristics of the pendulum somewhat, could certainly not be expected to cause the general assumptions to break down completely for fuel motions up to angles of 30° to 40°. Therefore, the equations of motion derived are assumed to present a fair picture of the disturbance motion even up to fuel displacements of this magnitude.

APPLICATION TO SEVERAL CASES

Nondimensional Equations for Tanks in Symmetry Plane

The equations of motion (18b) have been applied to the lateral motion in several cases with two fuel tanks in the plane of symmetry. In these cases the lateral motion can be considered independently. The applied forces are the weight, the usual aerodynamic forces linear in the disturbance velocities, and any disturbing forces that may be present. In order to put the equations in nondimensional form, the nondimensional lateral airplane equations are used as obtained in reference 8. The fuel equations are made nondimensional by making the standard transformation to nondimensional time derivative, as in the airplane equations, and then dividing through by $\frac{m_f l_f U_o^2}{b}$. The resulting nondimensional expressions in the following equations are defined in equations (20):

$$\left(\mu_b D - \frac{1}{2} C_{Y_\beta}\right)\beta + \left(\mu_b D - \frac{1}{2} C_L \tan \gamma_o\right)\psi - \frac{1}{2} C_L \phi - \mu_{b_1} \frac{l_1}{b} D^2 \zeta_1 -$$

$$\mu_{b_2} \frac{l_2}{b} D^2 \zeta_2 = C_Y \tag{19a}$$

$$-\frac{1}{2} C_{n_\beta}\beta + \left(\mu_b K_Z^2 D^2 - \frac{1}{4} C_{n_r} D\right)\psi + \left(\mu_b K_{XZ} D^2 - \frac{1}{4} C_{n_p} D\right)\phi - \mu_{b_1} \frac{l_1 R_{x_1}}{b^2} D^2 \zeta_1 -$$

$$\mu_{b_2} \frac{l_2 R_{x_2}}{b^2} D^2 \zeta_2 = C_n \tag{19b}$$

$$-\frac{1}{2} C_{l_\beta}\beta + \left(\mu_b K_{XZ} D^2 - \frac{1}{4} C_{l_r} D\right)\psi + \left(\mu_b {K_X}^2 D^2 - \frac{1}{4} C_{l_p} D\right)\phi + \mu_{b_1} \frac{l_1 R_{z_1}}{b^2} D^2\zeta_1 +$$

$$\mu_{b_2} \frac{l_2 R_{z_2}}{b^2} D^2\zeta_2 = C_l \tag{19c}$$

$$-D\beta - \left(\frac{R_{x_1}}{b} D^2 + D\right)\psi + \frac{R_{z_1}}{b} D^2\phi + \left({K_1}^2 D^2 + G\right)\zeta_1 = 0 \tag{19d}$$

$$-D\beta - \left(\frac{R_{x_2}}{b} D^2 + D\right)\psi + \frac{R_{z_2}}{b} D^2\phi + \left({K_2}^2 D^2 + G\right)\zeta_2 = 0 \tag{19e}$$

where

$$G \equiv \frac{bg}{{U_o}^2} \qquad {K_f}^2 \equiv \frac{I_f}{m_f l_f b} \qquad I_f \equiv I_{\zeta_f} \tag{20}$$

The derivatives C_{Y_p} and C_{Y_r} were assumed to be zero.

Methods of Solution

In the present case the two fuel degrees of freedom introduce two additional oscillatory modes into the characteristic solution, in addition to modifying the original airplane mode. The motion will therefore be a combination of three oscillations (aside from the less important exponential modes), but just knowing the three oscillatory roots is insufficient to indicate the type of motion since the relative magnitudes of the oscillatory modes must also be known. For this reason motions must be calculated in order to see the actual effects of the fuel motion. However, in several cases the characteristic roots were also found in order to facilitate the interpretation of the motions. These cases will be discussed subsequently.

The most convenient method for the analytical solution of a set of linear ordinary differential equations such as equations (19) is probably the Laplace transform method (see reference 8). However this method is extremely cumbersome and difficult to check since it involves the expansion of fifth-order determinants in which the elements are often quadratic functions of the characteristic root. Therefore, it seemed preferable to use some step-by-step method which would be more amenable to machine computation.

Reference 9 gives a matrix method for getting the step-by-step solution of a set of linear ordinary differential equations. When applied in the present case to equations (19) this method results in a simultaneous solution for the motion in each of the five degrees of freedom and also for the motion in $D\phi$, $D\psi$, $D\zeta_1$, and $D\zeta_2$. The calculations were carried out on the Bell Telephone Laboratories X-66744 relay computer in use at the Langley Laboratory. The essential details of the method are given in appendix A.

Solutions for Several Cases

The two basic cases for which motions were calculated were case A, a present-day high-speed airplane with two fuel tanks satisfying the conditions for equations (19), and case B, which corresponds essentially to case B of the model used in reference 2. The essential parameters for these two cases are given in tables I and II. Table II gives the conditions for case A when both tanks are one-half full (A_1) and when the fuel height equals one-half the radius (A_2) and for case B when the fuel heights in both tanks are 2 inches (B_2), 3 inches (B_3), and 4 inches (B_4).

In case A the tanks are spherical, somewhat over 4 feet in diameter, and centered on the body axis approximately 4 feet in front of and behind the airplane center of gravity. The flight conditions are given in table I. The fuel weight in the half-full condition is approximately 25 percent of the total weight.

In case B the tanks are spherical, centered in the plane of symmetry slightly less than 5 inches below the airplane axis and 4 inches in front of and behind the model center of gravity. The diameters are 8 inches and the total fuel weight in the half-full condition is approximately 46 percent of the total weight. The flight conditions are the same as in reference 2.

Motions were calculated for certain subcases of the basic cases which were obtained by varying certain significant parameters. By comparing the resulting motions an attempt was made to evaluate the effect of varying such factors as amount of fuel, position of tanks, and relative natural fuel and airplane frequencies on the disturbance of the airplane motion caused by fuel motion. Also various initial conditions were considered to show the effect of initial conditions on the resulting motion. In some cases an initial disturbance in sideslip was assumed, and in other cases an initial fuel disturbance was assumed. An initial sideslip of 5° and a fuel displacement of 10° were arbitrarily chosen as standard. Since the equations are linear, multiplying the initial displacements by a common factor simply multiplies the resulting motions by the same factor. In this connection it must be emphasized that, if at any time the calculated motion in any degree of freedom becomes too large to satisfy the equations, the following motion is meaningless For example, if a 5° displacement in β gives rise to a fuel motion much greater than 30° to 40°, the accompanying β motion is meaningless because the assumption of small displacements is violated. However, if multiplying the fuel motion by some arbitrary factor, for example 2/5, will bring its peaks down to less than 30° to 40°, then the β motion resulting from an initial β disturbance of 2° can be obtained by simply multiplying the previous β motion by 2/5 also. The effect of large fuel displacements must be discussed qualitatively.

The motion in sideslip and the motion of the two fuel pendulums in the various subcases are shown in figures 2 to 12. Comments on these motions are presented to facilitate interpretation of the figures. The period of each fuel pendulum is called the natural fuel period. The period and damping of the airplane, the fuel being disregarded, are called the natural airplane period and damping.

Case A_1.- The natural fuel periods for case A_1 (half-full tank) are approximately 1.66 seconds and the natural period of the airplane alone is 1.40 seconds. Damping to half-amplitude occurs in 2 cycles.

The motion in figure 2 is for initial $\beta_o = 2^\circ$. The early β motion seems to have more damping than the natural airplane mode. The disturbance arising from the fuel modes is evident after 2 cycles. That the irregular residual oscillation of amplitude $1/4^\circ$ to $1/2^\circ$ is essentially due to the fuel modes is seen from the fact that the dominant period in the later motion is approximately 1.6 seconds. Notice that in this case a 5° initial β_o would almost immediately cause fuel displacements of over 80°, so that the following motion would be radically changed.

The motions shown in figure 3 for $(\zeta_1)_o = (\zeta_2)_o = 10^\circ$ are quite regular and indicate one dominant mode in each motion. The fuel period is 1.6 seconds. The airplane period starts at 1.4 seconds and builds up to 1.7 seconds and averages 1.6 seconds. The amplitude of the sideslip motion is very small. The largest such motion which could occur for this type of disturbance would be for initial $(\zeta_1)_o = (\zeta_2)_o = 30^\circ$ and would give β amplitude slightly more than 0.1°.

The small amplitude of the sideslip motion in figure 3 was surprising. It was conjectured that for this fuel configuration the fuel displacement $(\zeta_1)_o = 10^\circ$ and $(\zeta_2)_o = -10^\circ$, corresponding essentially to an initial yawing moment, might be more effective in inducing an airplane oscillation. (See fig. 4.) Apparently, this configuration is more effective inasmuch as the sideslip motion now builds up to an amplitude of approximately 0.4°. The energy necessary to induce this considerable "snaking" type of oscillation seems to be obtained initially from the rear-tank motion, which is in the proper phase relation with the sideslip motion to feed energy into it at the start of the motion.

The fuel periods in the regular motion are slightly over 1.6 seconds. The airplane period increases from 1.5 to 1.8 seconds and has an average period of 1.6 seconds.

Case A_2.- The natural fuel periods in case A_2 are approximately 1.52 seconds and the natural airplane period is 1.49 seconds. Damping to half-amplitude occurs in 1.3 cycles.

The motion in figure 5 is for initial $\beta_o = 0.5^\circ$. The early sideslip motion seems to be of greater damping than the natural airplane mode. The residual airplane motion arising from the fuel modes sets in very quickly and is a regular unstable motion of very large relative amplitude, with a period of approximately 1.7 seconds. Both fuels start with a period of approximately 1.5 seconds, which increases to 1.7 seconds.

In figure 6, $(\zeta_1)_o = -10^\circ$ and $(\zeta_2)_o = 10^\circ$. The sideslip builds up to a fairly regular oscillation of 0.4° amplitude, with the period increasing from 1.5 seconds to 1.7 seconds. The fuel motion has a period somewhat under 1.7 seconds, with amplitude quickly building up to the limits where splashing must become important.

Case B_2.- The sideslip motion shown for case B_2 appears to be a normal damped oscillation for the first 4 seconds (fig. 7, $\beta_o = 5^\circ$), but then the peaks show a slight irregularity instead of damping smoothly. The period of the early sideslip motion appears to be somewhat over 0.9 second and the motion damps to half-amplitude in less than 2 cycles. This motion is very close to the undisturbed airplane mode (period of 0.92 sec and damping constant of $1\frac{3}{4}$ cycles). The natural fuel periods are 0.61 second. The fuel motion is very irregular and obviously contains considerable amounts of at least two characteristic modes.

Cases B_3 and B_4.- The motion in cases B_3 and B_4 shown in figures 8 and 9, respectively, for $\beta_o = 5^\circ$ is very much the same as in the previous one, except that the disturbance of the airplane mode in the sideslip motion appears somewhat more pronounced as the amount of fuel increases.

Figure 10 $\left(\text{case } B_4,\ (\zeta_1)_o = (\zeta_2)_o = 10^\circ\right)$ shows that the motion in sideslip resulting from the fuel displacement is much more irregular than in case A. The dominant mode corresponds to a fuel frequency, but apparently the airplane mode is present with considerable amplitude. The maximum oscillations are approximately $\pm 1/4^\circ$. The sideslip motion in this case was much more irregular than for the corresponding initial conditions in case A. It was conjectured that this might be caused by the fact that in this model both tanks are below the X-axis, so that the coupling of the fuel motion with yawing and rolling motions does not have the same phase relationship as in case A where one tank is above and one below the X-axis. Therefore, in case B_{4a} shown in figure 11, the front fuel tank was assumed to be above the X-axis, all other conditions remaining as in case B_4. In this case the general type of motion does seem to resemble that in figure 3. The sideslip, which builds up to $\pm 1/4^\circ$, shows a snaking at the fuel frequency.

In case A the fuel natural periods are very close to the airplane period. In case B, however, the fuel period is approximately two-thirds of the airplane period. In case B_{4b}, shown in figure 12 for $\beta_o = 5^\circ$, the value of C_{n_β} of the model has been arbitrarily changed to give the model a period very close to the fuel period of 0.66 second. Comparison with figures 2 and 5 shows that the motion in this case is very much like the motion in case A.

Transverse accelerations.- In evaluating pilots' reactions to snaking oscillations, the magnitude of the transverse accelerations involved in the oscillations has been found to be an important factor. Acceleration amplitudes above 0.025g are found to be bothersome, and amplitudes above 0.08g are considered very unsatisfactory. Calculations of the transverse accelerations involved in several of the previous motions were carried out. The magnitudes of the acceleration peaks in the residual oscillations were found to be approximately 0.04g to 0.05g. The actual motions are not shown since all the airplane oscillations are essentially of the same type as the β motions.

DISCUSSION OF RESULTS

The motions described in the preceding section are sufficient to give a fair picture of the types of possible fuel effects. Moreover, since each motion is just a superposition of the characteristic modes of the total system, these motions are often easier to understand if the characteristic roots are known. Physically it is clear that the characteristic modes will not differ much from the natural (uncoupled) modes when the interaction between the airplane and fuel is small. Comparison of figures 2; 5, and 12 with figures 7, 8, and 9 indicates that the interaction between airplane and fuel is strongest when the frequency of the airplane is close to that of the fuel, as might be expected from comparison with the resonance phenomena exhibited by an oscillator driving a system at its natural frequency. For this reason the characteristic modes of the total system were calculated in cases A_2 and B_{4b}, where the frequency ratio between airplane and fuel natural frequencies was practically unity. The natural modes are given for purposes of comparison. The results are given as follows in terms of periods and times to halve or double amplitudes, in seconds:

Case A_2, natural modes:

$$P_a = 1.49,\ T_{1/2} = 1.91;\ P_1 = P_2 = 1.52$$

Case A_2, total system:

$$P_a = 1.47,\ T_{1/2} = 61;\ P_1 = 1.29,\ T_{1/2} = 1.34;\ P_2 = 1.67,\ T_2 = 4.56$$

Case B_{4b}, natural modes:

$$P_a = 0.66,\ T_{1/2} = 1.29;\ P_1 = P_2 = 0.66$$

Case B_{4b}, total system:

$$P_a = 0.65,\ T_{1/2} = 30;\ P_1 = 0.54,\ T_2 = 1.5;\ P_2 = 0.71,\ T_{1/2} = 0.48$$

In these cases of large interaction it is difficult to identify one of the characteristic modes as the airplane mode. The characteristic mode in which the period is changed least from the natural airplane period has been called the airplane characteristic mode. However, in figures 5 and 12 this mode is not obviously the dominant one, as the airplane mode is in figures 7, 8, and 9.

The most important effects to be noted in these particular cases, where the fuel and airplane frequencies are equal, are that a characteristic mode which is very lightly damped with a frequency close to the natural airplane frequency exists and that an unstable mode appears. In connection with the first of these effects, it would seem that, theoretically, certain initial conditions might be found that would excite mainly this lightly damped mode in the characteristic solution for β, so that the resulting motion would be a typical snaking. Of course the required initial conditions might or might not be practical ones.

The total characteristic modes were also calculated for cases B_4 and B_{4a} to investigate the changes in motion caused by a hypothetical shift of one of the fuel tanks. The natural modes and the characteristic modes of the total system are given as follows for comparison:

Cases B_4 and B_{4a}, natural modes:

$$P_a = 0.84,\ T_{1/2} = 1.64;\ P_1 = P_2 = 0.66$$

Case B_4, total system:

$$P_a = 0.88,\ T_{1/2} = 1.91;\ P_1 = 0.63,\ T_{1/2} = 5.45;\ P_2 = 0.53,\ T_{1/2} = 1.61$$

Case B_{4a}, total system:

$$P_a = 0.86,\ T_{1/2} = 1.14;\ P_1 = 0.61,\ T_{1/2} = 19.5;\ P_2 = 0.60,\ T_2 = 10.2$$

It is interesting to note that both fuels in figure 10 and also in figure 11 seem to follow the more stable fuel mode in the part of the motion shown. The β motion in figure 11 seems to show the effect of the unstable mode. It appears in this case that, when the fuel tanks are in front of and behind the center of gravity, the configuration with one tank above and one below the X-axis gives rise to an unstable mode, whereas the configuration with both tanks below the X-axis makes both fuel modes stable.

A comparison of figures 2 to 4 or figures 5 and 6 clearly shows that the initial conditions can have a very important effect, since the least stable mode does not necessarily become dominant for a long time. The β disturbance was chosen as a typical airplane disturbance. On the other hand the disturbance of the fuels as an initial condition would seem to be completely artificial. However, these motions are believed to give a rough idea of the residual oscillations caused by fuel motion, at least insofar as magnitude is concerned, since, if the fuels were still displaced after the airplane motion had practically died out, the remaining motion might be considered to be the type caused by a fuel displacement. From this point of view figures 4, 6, 10, and 11 seem to indicate that residual oscillations of the order of magnitude of $1/4^{\circ}$ to $1/2^{\circ}$ might be expected in these cases. Actually, figures 10 and 11 would show oscillations of the order of $3/4^{\circ}$ for 30° fuel displacements.

Because of the lengthiness of the calculations, only cases A_1, A_2, and B_{4b} were carried out far enough to show the residual oscillations following a β disturbance. It is evident that the motion in case B_{4b} (fig. 12) resembles the motion in case A (figs. 2 and 5) much more than it resembles the unmodified case B motion (figs. 7 to 9). The reason for the smaller relative fuel motion in case B_{4b} is probably the fact that the relative fuel mass is considerably larger than in case A. In case A_2 (fig. 5) the residual oscillation dominates the motion almost immediately. These results show that the importance of the residual oscillation depends mainly on the closeness of the natural airplane and fuel frequencies, that is, on the parameter which might be called the frequency ratio. Inasmuch as the previous discussion of the characteristic modes indicated that the frequency ratio was also the most important factor affecting the characteristic modes of the system, the frequency ratio generally can be seen to be the most important factor determining the disturbance of the normal airplane motion caused by the fuel. Moreover, case A indicates that for spherical tanks the fuel frequency may easily be of the same order of magnitude as the airplane frequency. Reference 4 indicates that the same is true for rectangular tanks and for arbitrarily shaped tanks of reasonable dimensions. Thus, even though the residual oscillations might occur at fuel frequencies, these frequencies would not be distinguishable from the normal airplane frequency in the cases where the fuel effect is most pronounced, since in these cases the frequency ratio approaches unity.

The effects of unstable modes cannot be understood without considering the nonlinear effects due to splashing of the fuel. For linear systems the presence of an unstable mode would imply that the total system is unstable. It has been shown, however, that in an actual motion if the coefficient of the unstable mode in the solution for the airplane motion is very small compared with the coefficient of one of the stable modes, then the unstable mode will not appear in the early part of the motion. Now even in cases where the interaction is weak, one of the fuel modes (with no natural damping assumed) may be unstable. In such cases the unstable mode in the airplane motions will be relatively very small, while the unstable mode may be dominant in one of the fuel motions. Then this fuel motion may become very large before the effect on the airplane motion can be observed. At this point splashing will set in and the equations of motion no longer hold. Actually the energy lost in splashing may damp out the fuel motion so that the effect of this fuel motion - and therefore of the unstable mode - may never appreciably appear in the airplane motion.

In case A_2, however, the interaction is large, and figure 5 shows that the unstable mode does soon become very important in the β motion. Here again the effects of large-amplitude fuel motion and, in particular, the nonlinear effects of fuel splashing should be considered qualitatively. In cases A_1 and A_2, because of the strong interaction, a sideslip of 5° would very quickly give rise to fuel displacements of the order of magnitude of 90° or greater. Physically, it is clear that the airplane motion is feeding energy into the fuel motion in this early part of the motion. Because of the large amplitudes of the fuel motion, a large component of this motion is in the vertical direction, so that a considerable part of the energy in the fuel motion will go into creating longitudinal airplane motion. Because of the symmetry conditions previously cited this energy will not be fed back into lateral motion, so that the fuel tends to stabilize the lateral motion by feeding some of the energy from the lateral into the longitudinal motion. Also the turbulence due to splashing will absorb energy which will then be lost altogether from the motion.

One conclusion which can be drawn from this discussion is that the motion following large disturbances may be more stable than that following small disturbances. For example, the initial disturbances in figures 2 and 5 have been adjusted so that during the motion shown there is little splashing, and considerable residual oscillations are shown. But if the initial disturbances in these cases had been 5° or more in β, the energy lost in splashing in the early motion would possibly be so great that the residual motion in β would be smaller than that shown here. This might explain why some airplanes which definitely showed troublesome fuel oscillations were reported to be more stable in conditions of large atmospheric turbulence than in slightly turbulent atmospheric conditions.

A more important conclusion is that the fuel can be used to stabilize the airplane motion by introducing turbulence, by use of appropriate baffles for instance. This can be seen by noticing that the early part of the β motion in figures 2 and 5 is very stable. But if most of the energy fed into the fuel in this part of the motion were converted into heat through turbulence, then as has been pointed out this energy could not be fed back into the airplane motion and the residual oscillations would not appear. Since the amount of energy lost in turbulence cannot be calculated analytically, it would seem that an experimental investigation of the effects of honeycomb or other turbulence-inducing baffles on the airplane stability would be desirable, especially in cases where the airplane and fuel natural frequencies are approximately equal.

Finally it is possible in a strongly unstable case of residual oscillations, such as shown in figure 5, that the fuel may lose just enough of its energy in splashing to reduce its amplitude to where the motion is again smooth. Then, because of the instability of the system for small motions, the amplitude might again begin to build up. In this way continued oscillations of a more or less regular nature would occur when the calculations neglecting splashing show unstable motion. This result is important because it shows that somewhat irregular small-amplitude oscillations can be expected when the ratio of the airplane natural frequency to the fuel natural frequency approaches unity, even for moderate fuel masses of the order of one-tenth the total mass or less.

EFFECTS OF ASPHERICAL TANKS

The calculations have been carried out for rigid-body motion in spherical tanks only. Actually this assumes that for small oscillations the fundamental wave motion in spherical tanks approximates rigid-body motion. This approximation only applies when the tanks are one-half full or less. This restriction is not too serious, however, since the fuel motion will generally have its greatest effect in this range.

It is important to note that the potential energy of the fuels is simply the potential energy of a set of harmonic oscillators located at the positions $\underline{R}_f$. Thus, the same general analysis will apply whenever the fuel motion in the tank can be represented in terms of harmonic oscillators with given effective mass and spring constant. Reference 4 has already been mentioned as obtaining such a representation for the fundamental mode of a rectangular tank. Usually the fundamental mode will be the most important and involve the greater effective mass. It is conceivable that for long tanks the second mode might be of a frequency closer to that of the airplane, and in that case might be more important. In such a case each mode might be represented by a separate oscillator. As has been pointed out, the damping is mainly due to turbulence and will be more important for aspherical tanks. For small motions, however, the damping may still be neglected.

From the general derivation of the equations of motion, the most important result was the effect of fuel distribution on the coupling of lateral and longitudinal motions. It is plain that these results apply strictly only to spherical tanks. Consider for example a tank of triangular plan form located in the symmetry plane and oriented symmetrically with respect to this plane. Because of the symmetrical orientation it can be seen that, although sideslip motion will give rise to forward and rearward forces (because the pressure forces are normal to the diagonal surfaces), the forward and rearward motion will give antisymmetrical lateral forces which will cancel. In this case the result would be to feed energy from the lateral into the longitudinal motion; this condition would be favorable since the longitudinal motion is generally well-damped. For an unsymmetrically oriented tank of this type, energy could be fed back again from the longitudinal to the lateral motion and the problem would be quite complicated.

In general, the results on coupling for spherical tanks would be valid to first order for such symmetrical plan forms as the rectangular or the diamond-shape ones. For any simple symmetrically oriented shape in the plane of symmetry, a loss of energy from the lateral to the longitudinal motion might occur. This condition would be favorable. Finally, for tanks outside the plane of symmetry the same considerations would be valid if the tanks were symmetrically placed with respect to the symmetry plane and symmetrically shaped with respect to the plane through the tank center parallel to the symmetry plane.

CONCLUSIONS

The following conclusions may be drawn from the theoretical analysis presented:

1. Considerable disturbances of the normal airplane motion can be caused by fuel motion.

(a) The most important factor determining the effect is the ratio between fuel and airplane frequencies. When these are equal even moderate amounts of fuel (one-tenth the total mass or less) may cause considerable disturbances.

(b) The most usual type of disturbed motion is a somewhat irregular small-amplitude oscillation and the type of motion is strongly dependent on the initial conditions.

(c) The effects of splashing will be to make the motion more stable, and the loss of energy in fuel turbulence may make it possible to increase the stability by artificially introducing turbulence in the fuel.

2. The fuel motion may cause coupling between lateral and longitudinal motions.

3. The derivation of the equations of motion for spherical tanks may be applied to any tanks where the fuel motion may be represented in terms of harmonic oscillators.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., November 21, 1950

APPENDIX A

A STEP-BY-STEP SOLUTION OF THE EQUATIONS OF MOTION IN MATRIX NOTATION

A matrix method for solving the equations of motion is given in reference 9. The first step in this method is to reduce equations (19) to a set of first-order equations by introducing as new variables $D\phi$, $D\psi$, $D\zeta_1$, and $D\zeta_2$ as described in reference 9 for $D\phi$ and $D\psi$. This transforms the equations of motion into a set of nine linear first-order equations in the nine variables. In matrix notation the equations may then be written as follows when there are no applied forces C_Y, C_n, or C_l:

$$A(Dq) + Bq = 0 \qquad \text{(A1)}$$

where A and B are ninth-order square matrices and q is the column matrix (or vector), the elements (components) of which are the nine variables. In partitioned form,

$$A \equiv \left[\begin{array}{c|c} I_4 & O_{45} \\ \hline O_{54} & A_{55} \end{array}\right] \qquad B \equiv \left[\begin{array}{c|c} O_{45} & -I_4 \\ \hline C_{55} & E_{54} \end{array}\right] \qquad q \equiv \left[\zeta_2, \zeta_1, \phi, \psi, \beta, D\zeta_2, D\zeta_1, D\phi, D\psi\right]$$

where I_4 is the identity matrix of fourth order; O_{45} and O_{54} are zero matrices of order (4×5) and (5×4), respectively; and

$$A_{55} \equiv \begin{bmatrix} \mu_b & -\mu_{b_2} l_2/b & -\mu_{b_1} l_1/b & 0 & 0 \\ 0 & -\mu_{b_2} l_2 R_{x_2}/b^2 & -\mu_{b_1} l_1 R_{x_1}/b^2 & \mu_b K_{XZ} & \mu_b {K_Z}^2 \\ 0 & \mu_{b_2} l_2 R_{z_2}/b^2 & \mu_{b_1} l_1 R_{z_1}/b^2 & \mu_b {K_X}^2 & \mu_b K_{XZ} \\ -1 & 0 & {K_1}^2 & R_{z_1}/b & -R_{x_1}/b \\ -1 & {K_2}^2 & 0 & R_{z_2}/b & -R_{x_2}/b \end{bmatrix}$$

$$C_{55} \equiv \begin{bmatrix} 0 & 0 & -\frac{1}{2} C_L & -\frac{1}{2} C_L \tan \gamma_o & -\frac{1}{2} C_{Y_\beta} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} C_{n_\beta} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} C_{l_\beta} \\ 0 & G & 0 & 0 & 0 \\ G & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_{54} \equiv \begin{bmatrix} 0 & 0 & 0 & \mu_b \\ 0 & 0 & -\frac{1}{4} C_{n_p} & -\frac{1}{4} C_{n_r} \\ 0 & 0 & -\frac{1}{4} C_{l_p} & -\frac{1}{4} C_{l_r} \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Equation (A1) may be solved for Dq by multiplying through by the inverse of A:

$$Dq = -A^{-1}Bq \equiv Pq \tag{A2}$$

Equation (A2) shows that, when there are no applied forces, the differentiation operator with respect to nondimensional time may be replaced by premultiplication with the matrix $P \equiv -A^{-1}B$. For example, note that $D^2q = D(Dq) = P(Dq) = P^2q$ and that similar relations would result for higher powers.

Now, by Taylor's expansion, the value of $q(s + \Delta s)$ may be obtained from the value of $q(s)$ by the series

$$q(s + \Delta s) = q(s) + \frac{\Delta s}{1!} Dq(s) + \frac{(\Delta s)^2}{2!} D^2q(s) + \ldots$$

By use of equation (A2), this equation can be written

$$q(s + \Delta s) = \left[I_9 + \frac{\Delta s}{1!} P + \frac{(\Delta s)^2}{2!} P^2 + \ldots\right] q(s) \tag{A3}$$

which is the fundamental recurrence relation used in the step-by-step calculation. In equation (A3), I_9 is the ninth-order identity matrix. The set of initial disturbances q_o being given, the magnitude of the step Δs will determine the number of powers necessary to obtain a given accuracy in the solution. Because of the relatively high fuel frequencies and because it was desired to obtain the motion to a rather large number of periods with reasonable accuracy, the series in equation (A2) was used to the sixth power with Δs approximately 1/20 of the airplane period. Thus, the matrix relation (A3) was

$$q(s + \Delta s) = Qq(s) \tag{A4}$$

where

$$Q \equiv I_9 + \frac{\Delta s}{1!} P + \frac{(\Delta s)^2}{2!} P^2 + \ldots + \frac{(\Delta s)^6}{6!} P^6$$

REFERENCES

1. Sternfield, Leonard: Some Effects of Nonlinear Variation in the Directional-Stability and Damping-in-Yawing Derivatives on the Lateral Stability of an Airplane. NACA TN 2233, 1950.

2. Smith, Charles C., Jr.: The Effects of Fuel Sloshing on the Lateral Stability of a Free-Flying Airplane Model. NACA RM L8C16, 1948.

3. Lamb, Horace: Hydrodynamics. Reprint of Sixth ed. (first American ed.), Dover Publications (New York), 1945, pp. 637-638.

4. Graham, E. W.: The Forces Produced by Fuel Oscillation in a Rectangular Tank. Rep. No. SM-13748, Douglas Aircraft Co., Inc., April 13, 1950.

5. Whittaker, E. T.: A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. Fourth ed., Dover Publications (New York), 1944, p. 37.

6. Page, Leigh: Introduction to Theoretical Physics. Second ed., D. Van Nostrand Co., Inc., 1935, pp. 97-105.

7. Von Mises, Richard: Theory of Flight. McGraw-Hill Book Co., Inc., 1945, pp. 572-574.

8. Mokrzycki, G. A.: Application of the Laplace Transformation to the Solution of the Lateral and Longitudinal Stability Equations. NACA TN 2002, 1950.

9. Fossier, M. W., Bratt, R. W., and Dill, D. G.: Comparison of Analytical and High-Speed Mechanical-Calculator Solutions of the Lateral Equations of Motion of an Airplane, Including Product-of-Inertia Terms. Jour. Aero. Sci., vol. 17, no. 5, May 1950, pp. 271-276, 296.

TABLE I

STABILITY DERIVATIVES AND MASS AND GEOMETRICAL CHARACTERISTICS FOR TWO CASES CONSIDERED

Parameters	Case A	Case B
C_{Y_β}	-1.042	-0.80
C_{n_β}	0.17	[a]0.17
C_{l_β}	-0.126	-0.14
C_{n_p}	-0.01552	[b]-0.040
C_{l_p}	-0.342	-0.30
C_{n_r}	-0.28	-0.16
C_{l_r}	0.0796	0.30
Weight of airplane alone, lb	6970	11.25
S, sq ft	130	2.67
b, ft	28	4.0
γ_o, deg	0	-11
ρ, slugs/cu ft	0.00136	0.002378
U_o, ft/sec	704	61.5 to 74.25

[a]In case B_{4b}, $C_{n_\beta} = 0.29$.

[b]Actually, slightly different values of C_{n_p} were used for each of the subcases of case B.

NACA

TABLE II

FUEL DEPENDENT PARAMETERS

Parameters	A_1	A_2	B_2	B_3	B_4
Radius of forward tank, ft . .	2.15	2.15	0.333	0.333	0.333
Radius of rear tank, ft . . .	2.12	2.12	0.333	0.333	0.333
Fuel height in forward tank, ft	2.15	1.075	0.1667	0.250	0.333
Fuel height in rear tank, ft .	2.12	1.06	0.1667	0.250	0.333
R_{x_1}, ft	3.5	3.5	0.333	0.333	0.333
R_{x_2}, ft	-4.1	-4.1	-0.333	-0.333	-0.333
R_{z_1}, ft	0.0123	0.0123	0.407	0.407	0.407
R_{z_2}, ft	-0.0144	-0.0144	0.407	0.407	0.407
Weight of forward fuel, lb . .	1480	462	1.51	3.07	4.85
Weight of rear fuel, lb . . .	1068	334	1.51	3.07	4.85
Total weight of airplane and fuel, lb	9518	7766	14.27	17.39	20.95
C_L	0.19	0.19	1.062	1.178	1.174
I_X, slug-ft^2	1360	1360	0.1081	0.1243	0.1428
I_Z, slug-ft^2	7340	7708	0.2098	0.2206	0.233
I_{XZ}, slug-ft^2	262	277	0	0	0
l_1, ft	0.806	1.45	0.225	0.174	0.125
l_2, ft	0.795	1.43	0.225	0.174	0.125
I_1, slug-ft^2	85.2	40.0	0.00315	0.00507	0.00668
I_2, slug-ft^2	59.7	28.1	0.00315	0.00507	0.00668

NACA

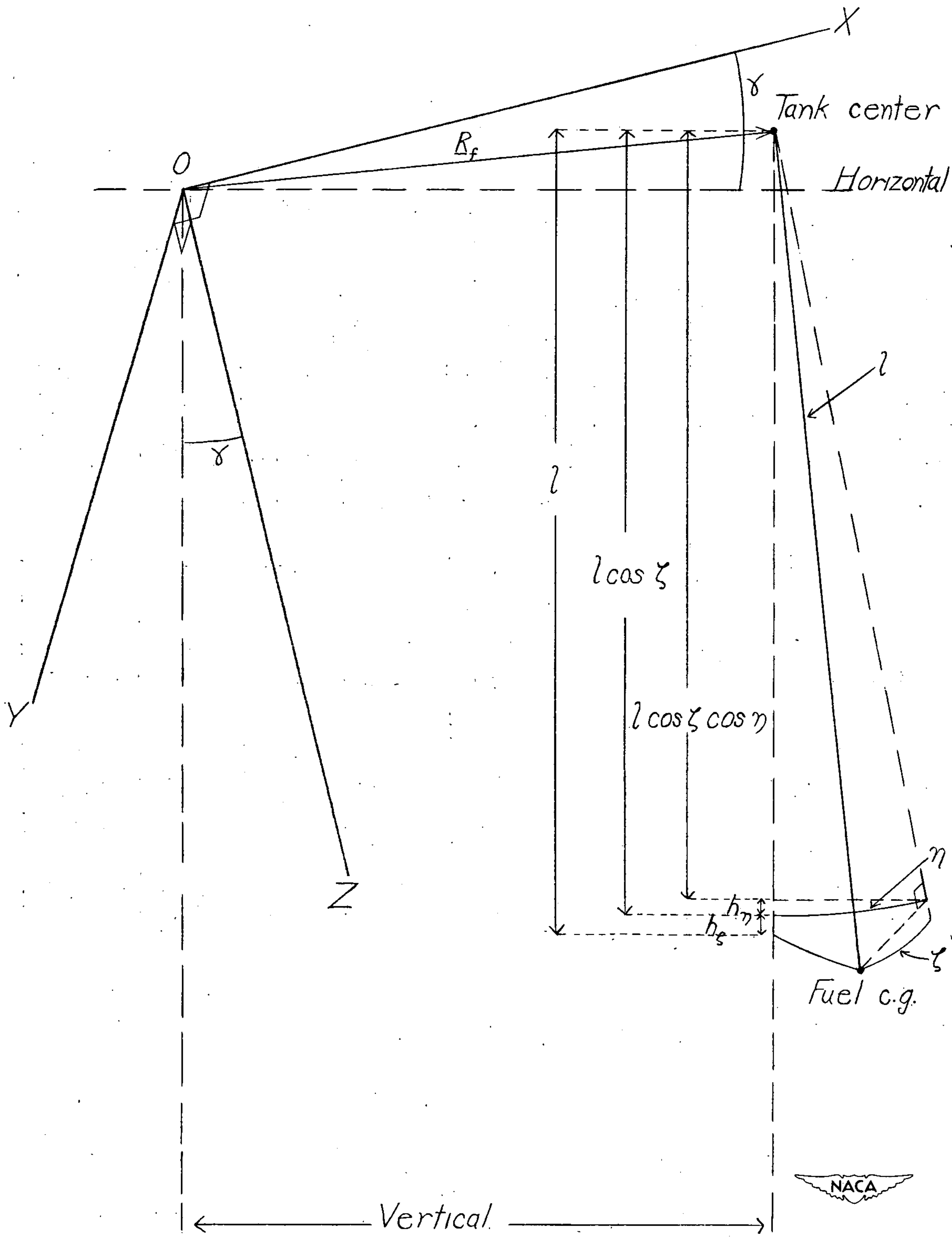

Figure 1.- Decomposition of horizontal fuel motion in terms of angles ζ and η in vertical planes.

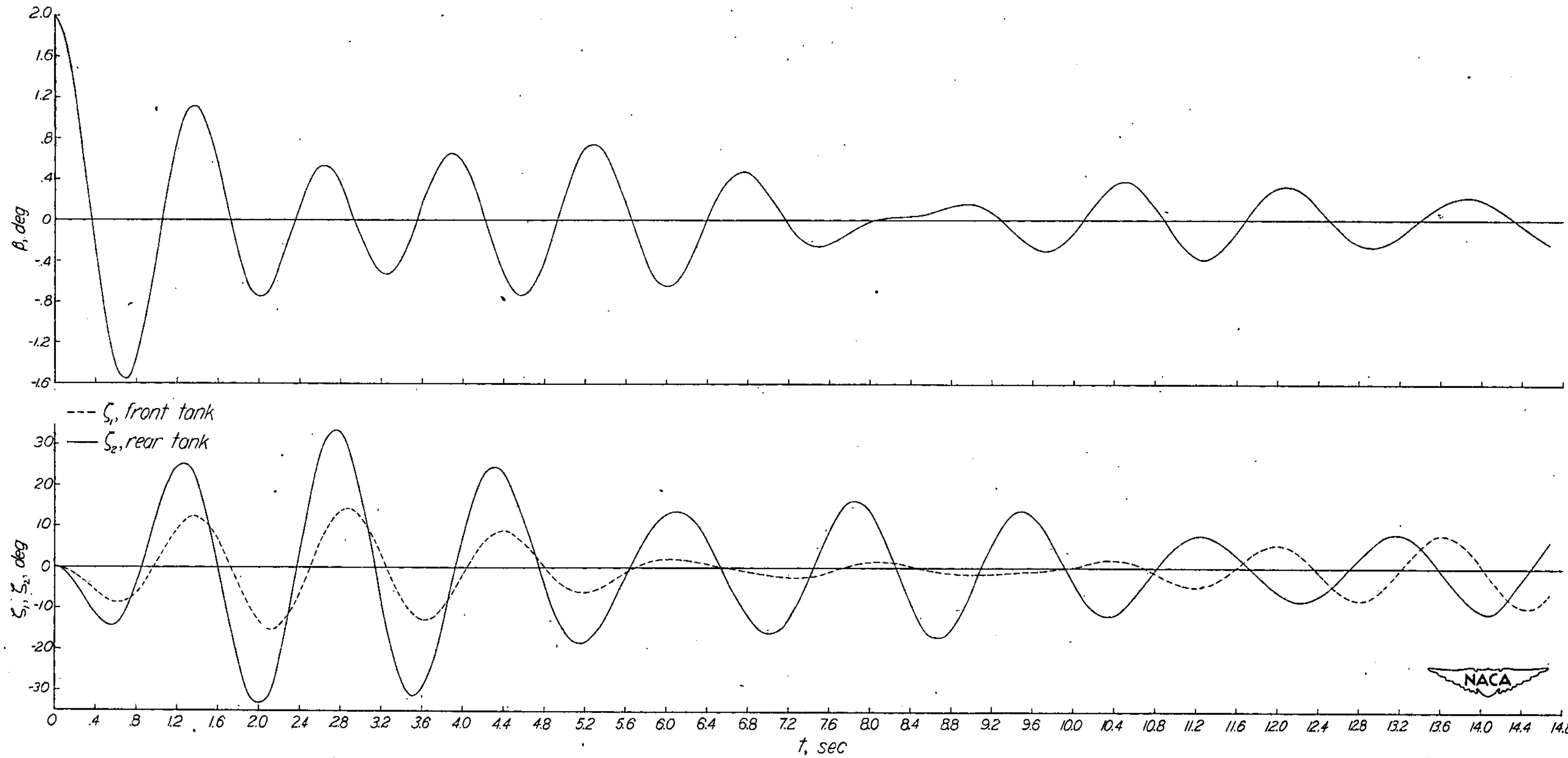

Figure 2.- Present-day high-speed airplane with tanks one-half full (case A_1). Sideslip and fuel motions following initial sideslip, $\beta_o = 2^o$.

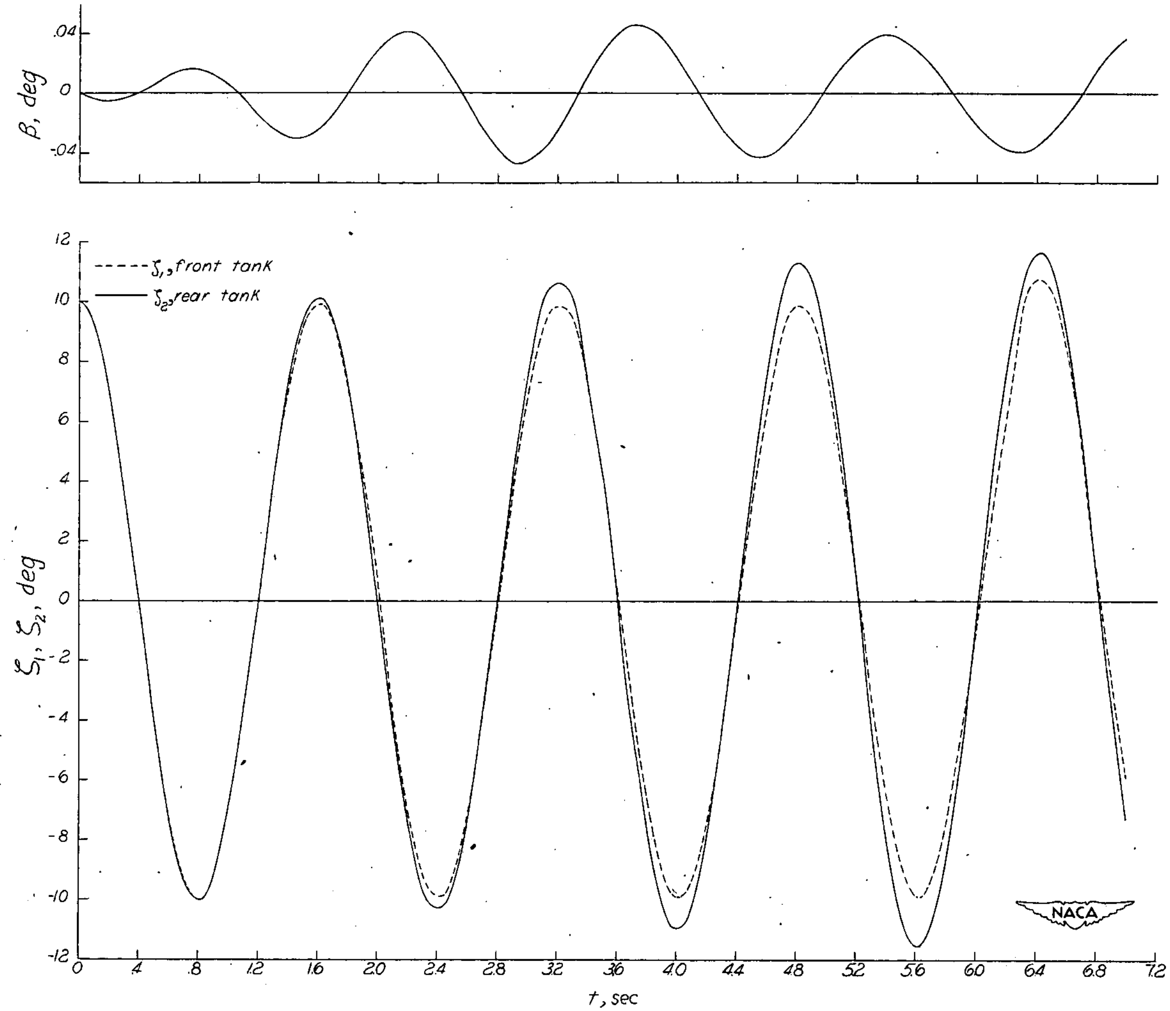

Figure 3.- Present-day high-speed airplane with tanks one-half full (case A_1). Sideslip and fuel motions following initial fuel disturbance, $(\zeta_1)_o = (\zeta_2)_o = 10^\circ$.

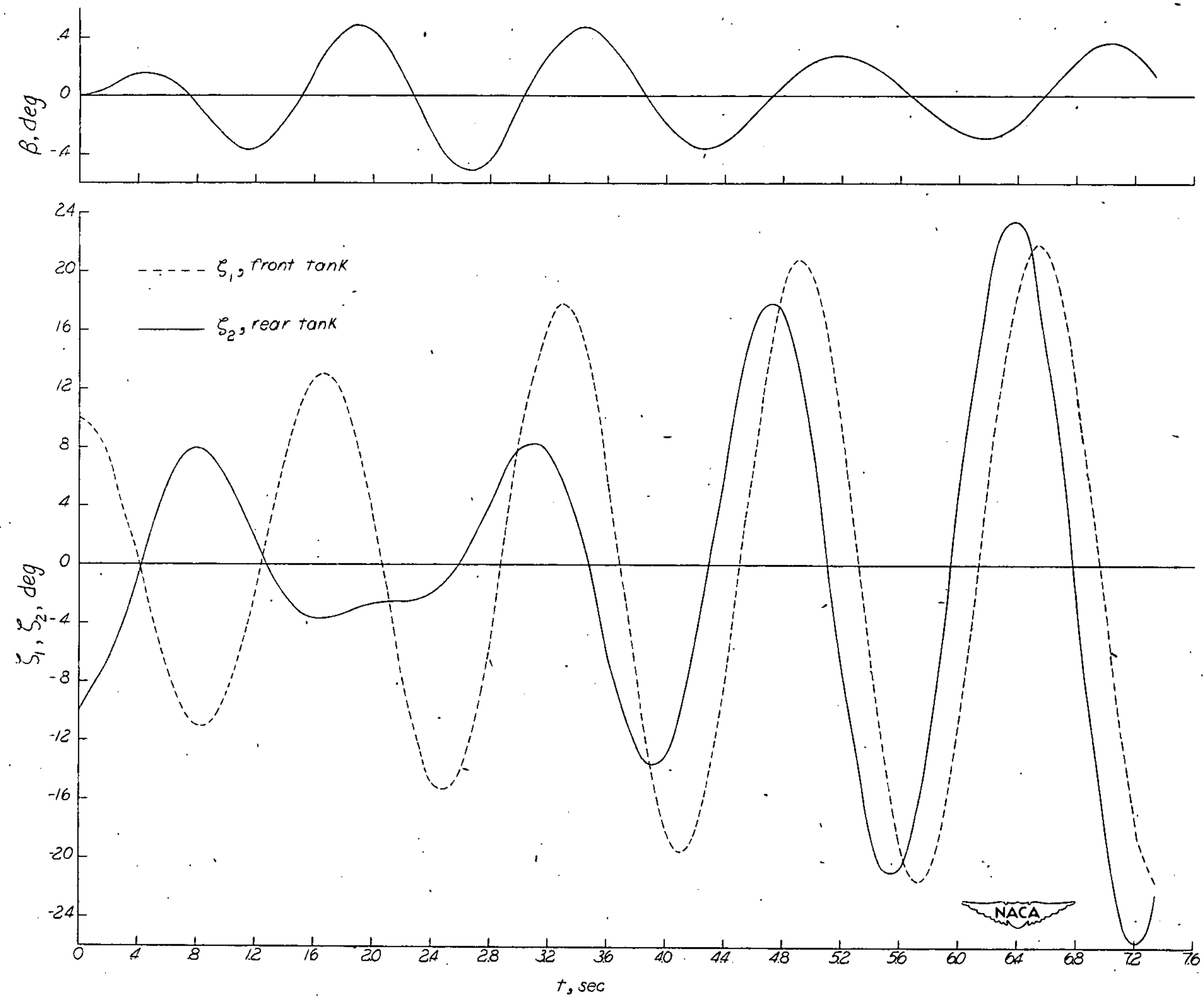

Figure 4.- Present-day high-speed airplane with tanks one-half full (case A_1). Sideslip and fuel motions following initial fuel disturbance, $(\zeta_1)_o = 10^\circ$, $(\zeta_2)_o = -10^\circ$.

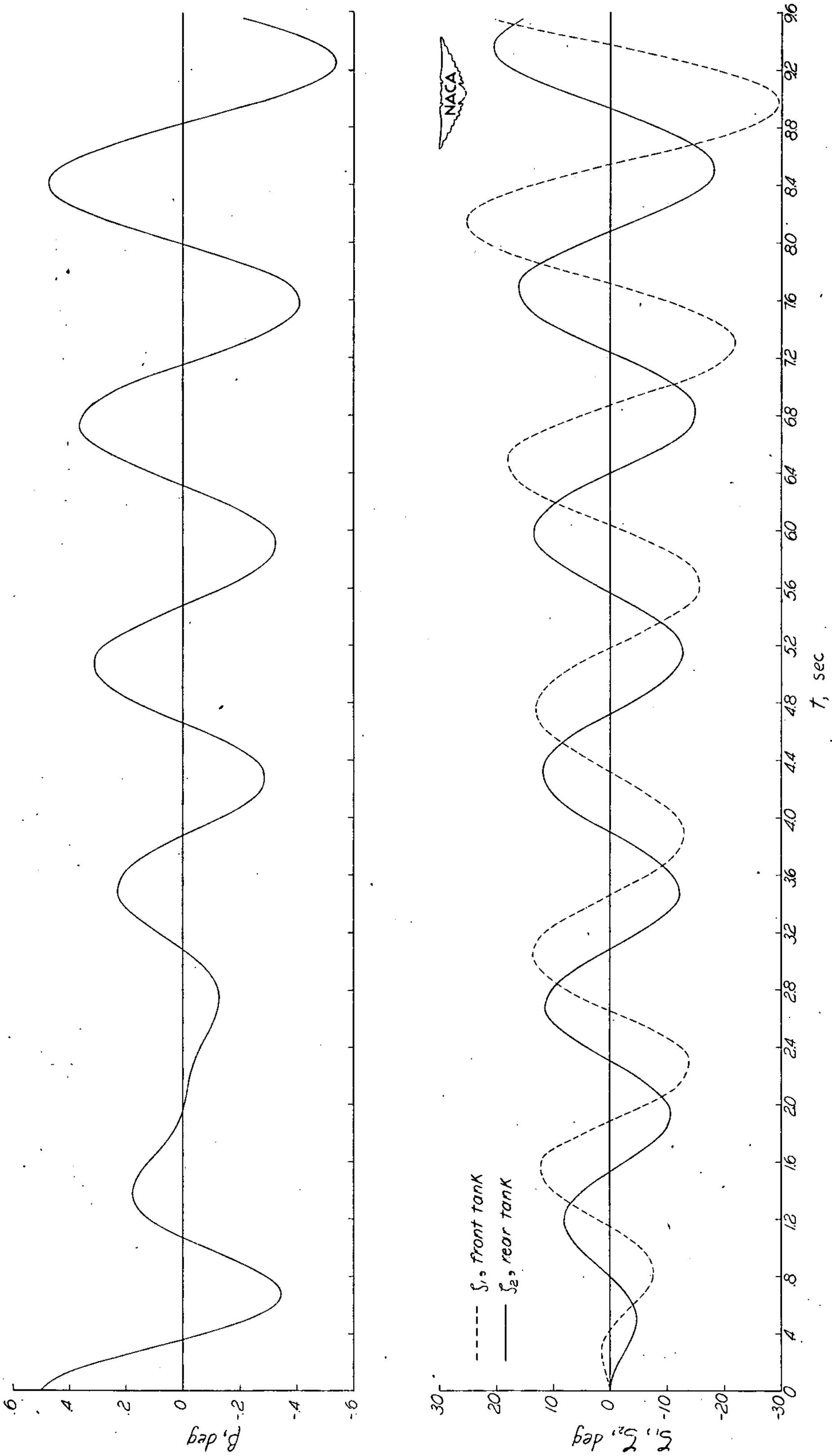

Figure 5.- Present-day high-speed airplane with fuel heights one-half the tank radius (case A_2). Sideslip and fuel motions following initial sideslip, $\beta_o = 0.5^o$.

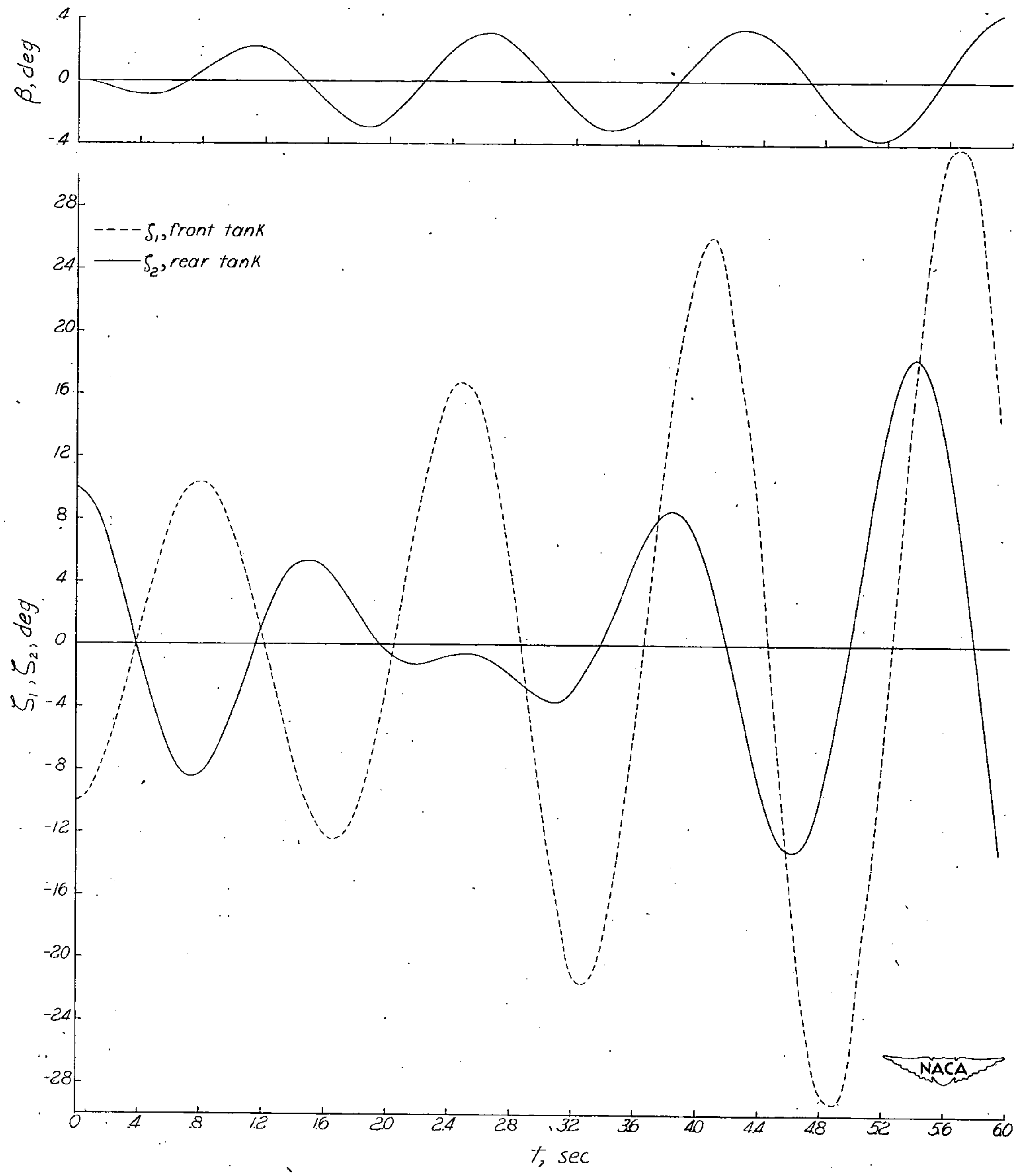

Figure 6.- Present-day high-speed airplane with fuel heights one-half the tank radius (case A_2). Sideslip and fuel motions following initial fuel disturbance, $(\zeta_1)_o = -10^\circ$, $(\zeta_2)_o = 10^\circ$.

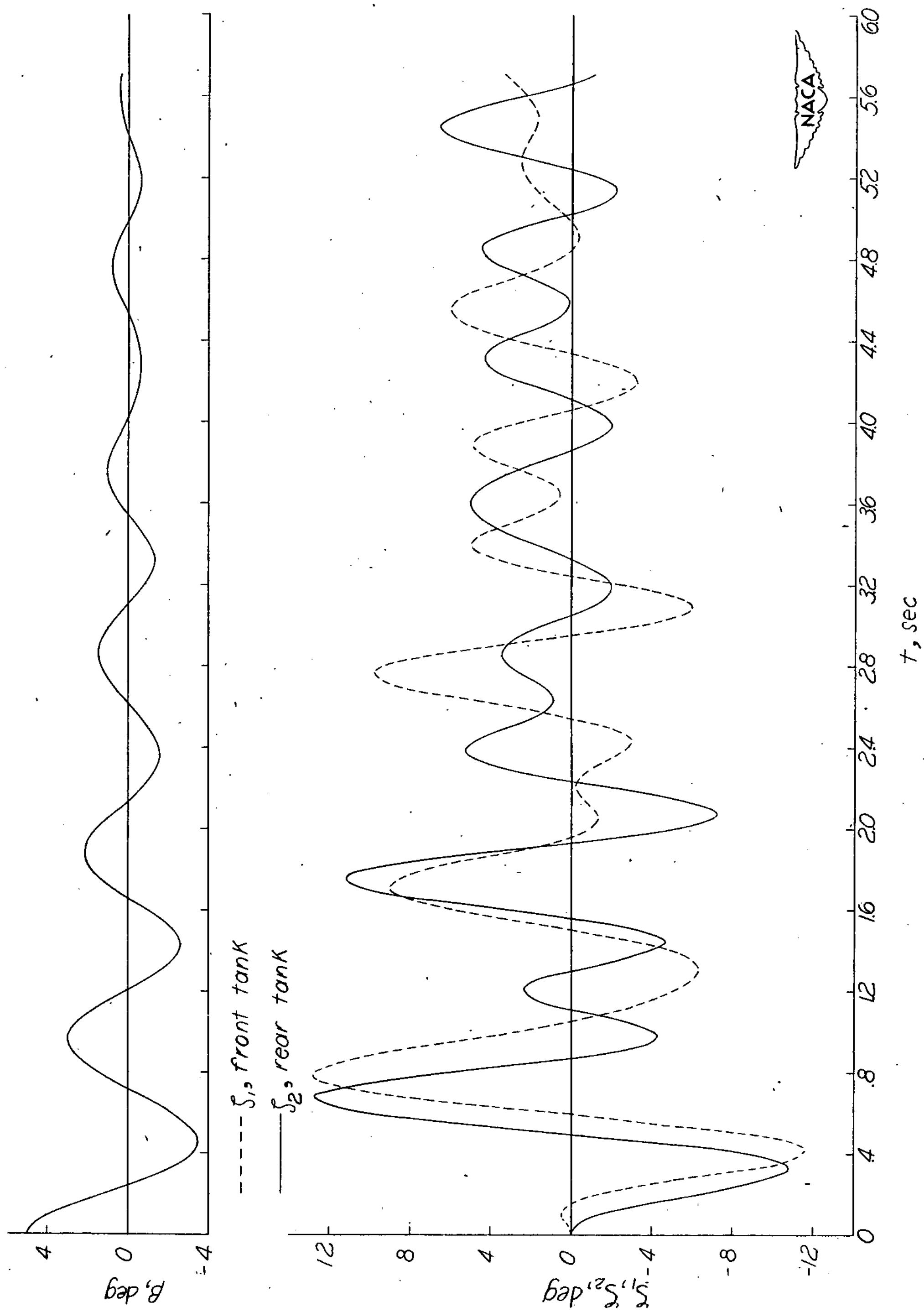

Figure 7.- Free-flying airplane model with fuel heights one-half the tank radius (case B_2). Sideslip and fuel motions following initial sideslip, $\beta_o = 5^o$.

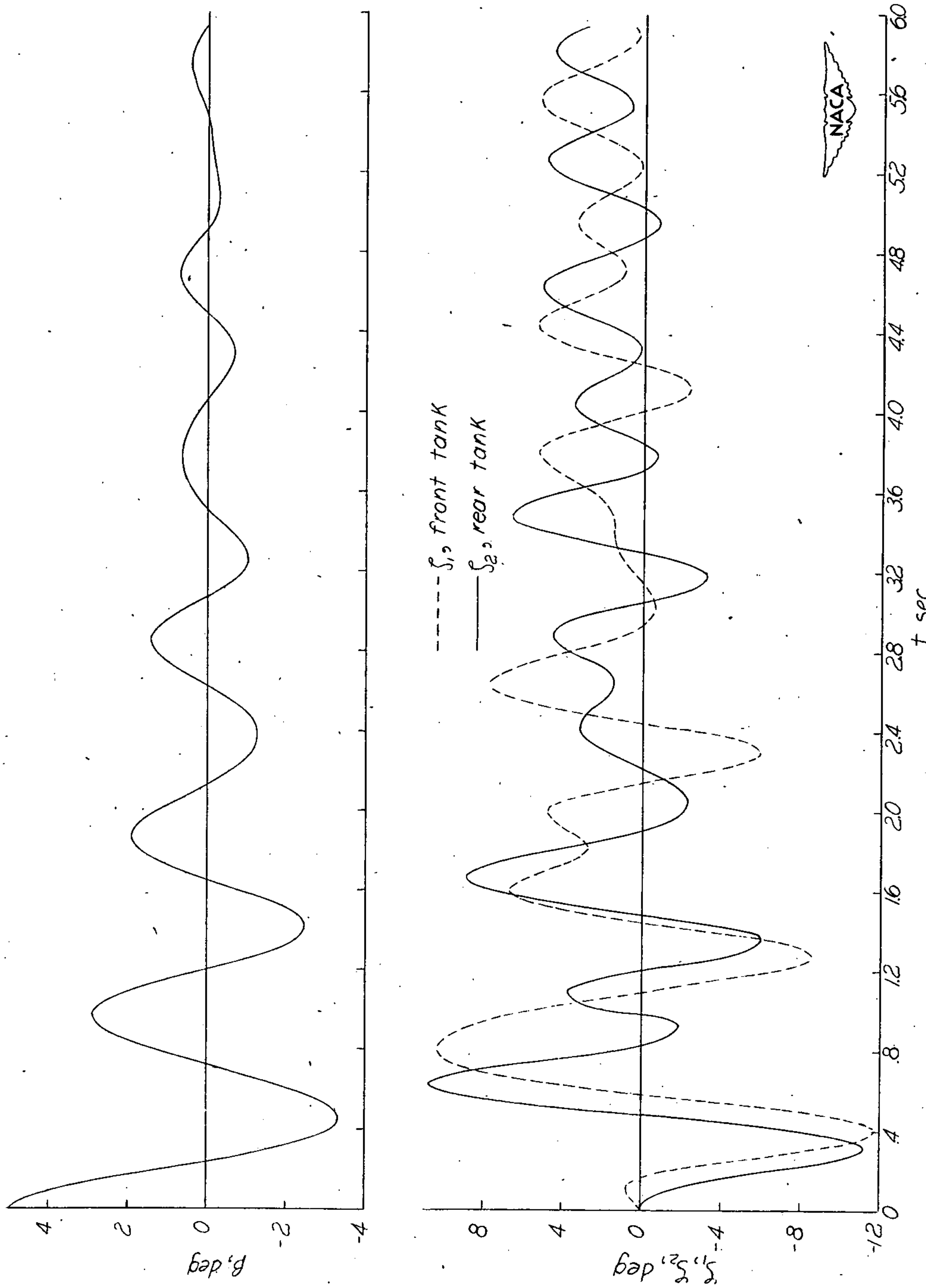

Figure 8.- Free-flying airplane model with fuel heights three-fourths the tank radius (case B_3). Sideslip and fuel motions following initial sideslip, $\beta_o = 5^o$.

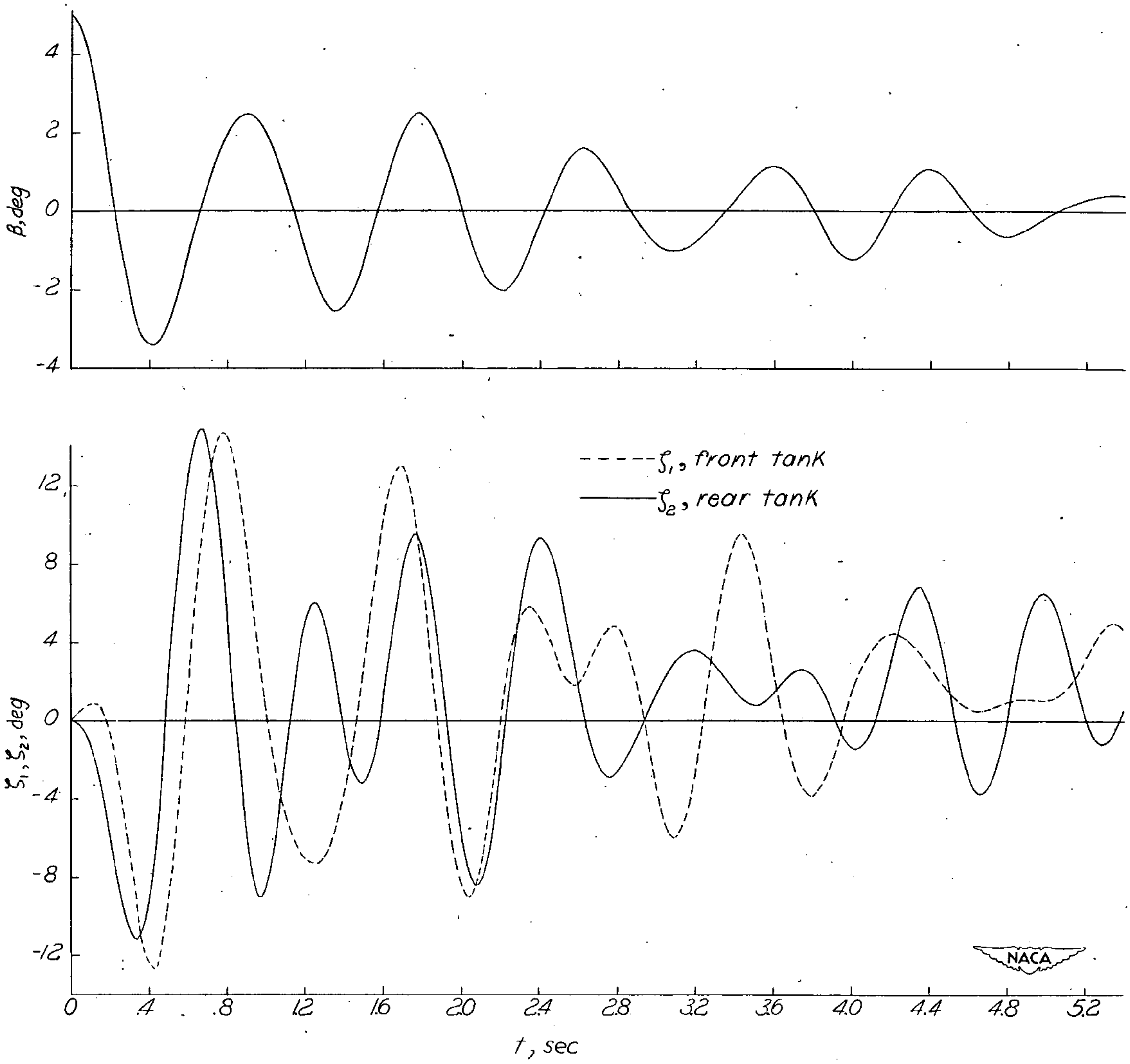

Figure 9.- Free-flying airplane model with tanks one-half full (case B_4). Sideslip and fuel motions following initial sideslip, $\beta_o = 5^\circ$.

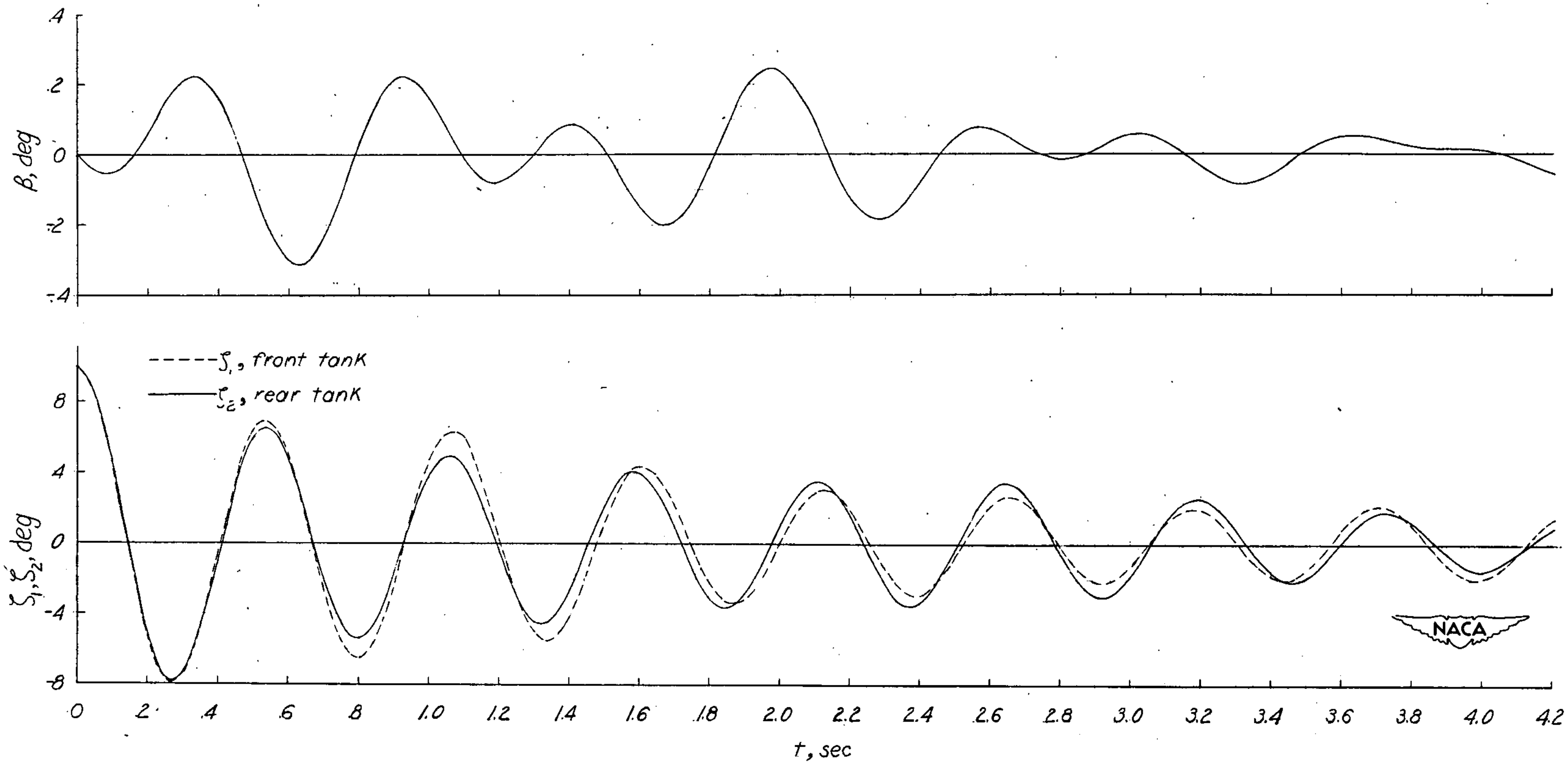

Figure 10.- Free-flying airplane model with tanks one-half full (case B_4). Sideslip and fuel motions following initial fuel disturbance, $(\zeta_1)_o = (\zeta_2)_o = 10^o$.

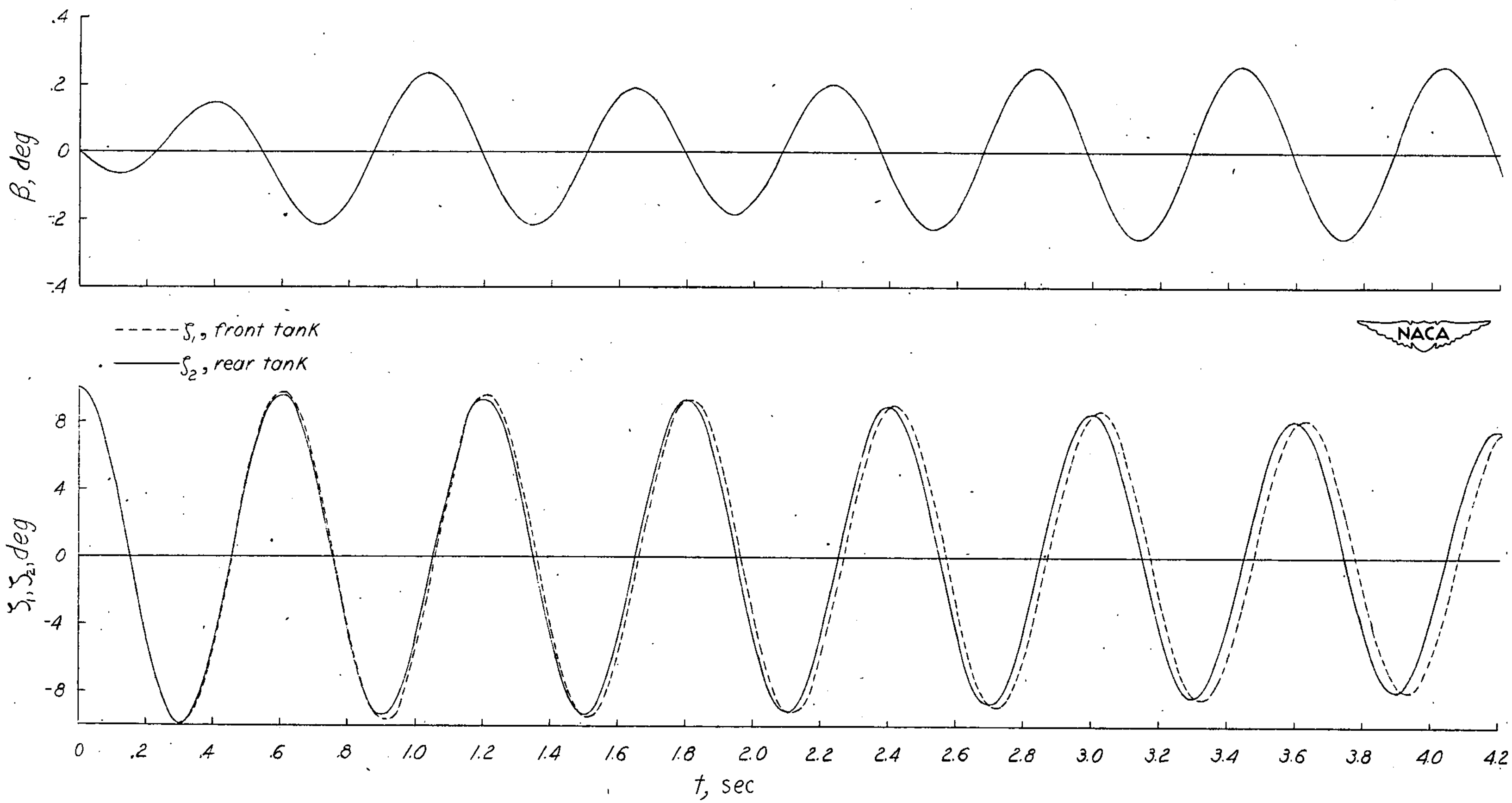

Figure 11.- Free-flying airplane model with tanks one-half full and front tank assumed to be shifted above the X-axis (case B_{4a}). Sideslip and fuel motions following initial fuel disturbance, $(\zeta_1)_o = (\zeta_2)_o = 10^o$.

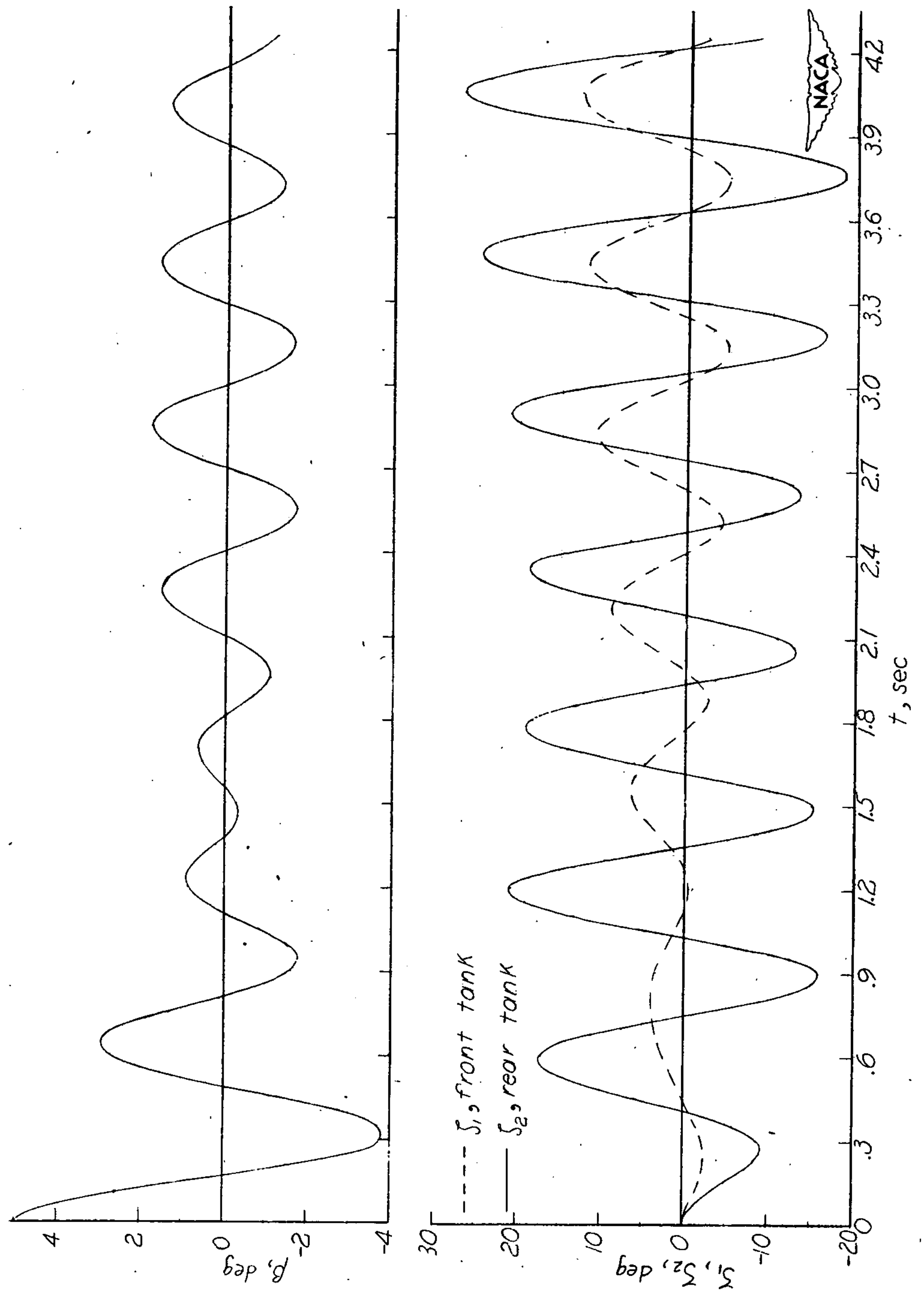

Figure 12.- Free-flying airplane model with tanks one-half full and C_{n_β} adjusted to make natural fuel and airplane frequencies equal (case B4b). Sideslip and fuel motions following initial sideslip, $\beta_o = 5^o$.

Stability, Dynamic 1.8.1.2

A Theoretical Analysis of the Effects of Fuel Motion on Airplane Dynamics.

By Albert A. Schy

NACA TN 2280
January 1951

(Abstract on Reverse Side)

Research Technique, Mathematics 9.2.7

A Theoretical Analysis of the Effects of Fuel Motion on Airplane Dynamics.

By Albert A. Schy

NACA TN 2280
January 1951

(Abstract on Reverse Side)

Schy, Albert A.

A Theoretical Analysis of the Effects of Fuel Motion on Airplane Dynamics.

By Albert A. Schy

NACA TN 2280
January 1951

(Abstract on Reverse Side)

Abstract

The general equations of motion for an airplane with a number of spherical fuel tanks are derived. These equations are applied to two cases with two fuel tanks located in the plane of symmetry. The calculated motions show that the airplane motion may be greatly changed by considering the motion of the fuel and, in particular, that small-amplitude residual oscillations may result. The same type of analysis may be applied to arbitrarily shaped tanks; therefore, the most general conclusions as to the effects of the fuel motion on airplane dynamics also apply for arbitrarily shaped tanks.

Abstract

The general equations of motion for an airplane with a number of spherical fuel tanks are derived. These equations are applied to two cases with two fuel tanks located in the plane of symmetry. The calculated motions show that the airplane motion may be greatly changed by considering the motion of the fuel and, in particular, that small-amplitude residual oscillations may result. The same type of analysis may be applied to arbitrarily shaped tanks; therefore, the most general conclusions as to the effects of the fuel motion on airplane dynamics also apply for arbitrarily shaped tanks.

Abstract

The general equations of motion for an airplane with a number of spherical fuel tanks are derived. These equations are applied to two cases with two fuel tanks located in the plane of symmetry. The calculated motions show that the airplane motion may be greatly changed by considering the motion of the fuel and, in particular, that small-amplitude residual oscillations may result. The same type of analysis may be applied to arbitrarily shaped tanks; therefore, the most general conclusions as to the effects of the fuel motion on airplane dynamics also apply for arbitrarily shaped tanks.