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DOWNWASH IN THE VERTICAL AND HORIZONTAL PLANES OF SYMMETRY BEHIND A TRIANGULAR WING IN SUPERSONIC FLOW

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SUMMARY

A method developed in a previous report for finding the induced velocity field behind a supersonic wing with known load distribution is used to find the downwash behind a triangular wing with subsonic leading edges. Results are given for the chord plane in the extended vortex wake of the wing and for the vertical plane of symmetry up to about 20 percent of the wing span above the plane of the wing.

INTRODUCTION

A method has been developed in reference 1 for computing the downwash behind wings of known loading flying at supersonic speeds. The solution was based on the distribution of supersonic doublets over the plan form and wake of the wing in a manner determined by the load distribution. The method was applied in reference 1 to the calculation of the downwash behind a triangular wing with leading edges swept behind the Mach cone from the vertex. Near the trailing edge, however, these calculations were limited to a region close to an extension of the center line of the wing, that is, the x axis (fig. 1), and in fact were exact only on this line itself. A simple first approximation was advanced for the downwash variation about the x axis; namely, that the difference in the downwash at the position (x,y,z) from the value at the position $(x,0,0)$ was a linear function of the z distance and independent of the y distance. However, the accuracy of this approximation could be tested only for large values of x where the downwash was known for all values of y and z.

The purpose of this report is threefold: First, to continue the exact calculations so as to include all points on the xy and xz planes behind the trailing edge for distances up to about a semispan from the x axis in the xy plane and up to about 40 percent of a semispan in the xz plane; second, to compare these values of downwash with those obtained from the approximate method given in reference 1; and, third, to serve as a guide through some of the more difficult mathematical manipulations so that the calculations can be extended to other plan forms.

LIST OF IMPORTANT SYMBOLS

a_o velocity of sound in the free stream

c_o root chord of wing

E,E_o complete elliptic integral of the second kind with modulus k,k_o, respectively $\left(E = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt\right)$

$E(t,k)$ incomplete elliptic integral of the second kind with argument t and modulus k $\left[E(t,k) = \int_0^t \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt\right]$

H $\frac{2\alpha V_o}{E_o\beta}$

k_o $\sqrt{1-\theta_o{}^2}$

K complete elliptic integral of the first kind with modulus k $\left[K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-k^2t^2)}}\right]$

$F(t,k)$ incomplete elliptic integral of the first kind with argument t and modulus k $\left[F(t,k) = \int_0^t \frac{dt}{\sqrt{(1-t^2)\,(1-k^2t^2)}}\right]$

M_o free-stream Mach number $\left(\frac{V_o}{a_o}\right)$

p static pressure

Δp $p_l - p_u$

q free-stream dynamic pressure $\left(\frac{1}{2}\rho_o V_o{}^2\right)$

r_c	$\sqrt{(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2(z-z_1)^2}$
u,v,w	perturbation velocity components in the direction of the x, y, and z axes, respectively
Δu_s	u_u-u_l
V_o	free-stream velocity
w_P	z component of velocity induced by doublet distribution over plan form
w_W	z component of velocity induced by doublet distribution over wake
w_o	$-V_o\alpha$
x,y,z	Cartesian coordinates of an arbitrary point
x_1,y_1,z_1	Cartesian coordinates of source or doublet position
x_o	$\frac{x}{c_o}$
y_o	$\frac{\beta y}{c_o}$
z_o	$\frac{\beta z}{c_o}$
α	angle of attack
β	$\sqrt{M_o^2-1}$
θ_o	$\beta \tan \psi$
ρ_o	density in free stream
Φ	perturbation velocity potential
$\Delta\Phi_s$	$\Phi_u-\Phi_l$
ψ	semivertex angle of triangular wing
⌜	sign denoting finite part of integral

Subscripts

u	conditions on upper portion of surface
l	conditions on lower portion of surface
L.E.	conditions at leading edge
T.E.	conditions at trailing edge
W	wake
P	plan form
s	conditions on discontinuity surface (at $z_1=0$)
A,B,C,D,E	conditions in regions A,B,C,D, or E, in wake of triangular wing (figs. 1 and 2)

THEORY

The theoretical development in this report is subdivided into two parts. First, the problem of determining the downwash behind a flat-plate triangular wing swept back of the Mach cone is reduced to the solution of certain types of elliptic integrals. This material has already been presented in reference 1, and the brief description presented here should be a sufficient review. Second, an evaluation of the resulting integrals is made in both the xy and xz planes. Some of the detailed evaluations are given because the methods and substitutions presented may be useful in the solution of downwash problems for other plan forms.

Boundary Conditions and Their Solution

The load distribution for a triangular flat-plate wing (fig. 1) has been determined under the assumptions of thin-airfoil theory and is given as

$$\frac{\Delta p}{q} = \frac{2\Delta u_s}{V_o} = \frac{4\theta_o^2 \alpha x_1}{E_o \beta \sqrt{\theta_o^2 x_1^2 - \beta^2 y_1^2}} \qquad (1)$$

Since the loading coefficient determines the jump in the u induced velocity in the plane of the wing, the definition of the perturbation velocity potential Φ gives

$$\Delta\Phi_s = \int_{\text{L.E.}}^{x_1} \Delta u_s \, dx$$

where $\Delta\Phi_s$ represents the jump in the velocity potential in the xy plane. In any lifting-surface problem there are three regions in the xy plane for which the equations for $\Delta\Phi_s$ are different: the region defined by the boundaries of the lifting surface itself, the region of the vortex wake behind the lifting surface, and the remainder of the plane. For the triangular wing the jump in potential over the lifting surface is given by the equation

$$\Delta\Phi_s = H\sqrt{\theta_o^2 x_1^2 - \beta^2 y_1^2} \tag{2}$$

where

$$x_1 \leq c_o$$

and

$$H = \frac{2\alpha V_o}{E_o \beta}$$

In the vortex wake (the semi-infinite strip behind the wing extending from tip to tip with sides parallel to the free-stream direction), $\Delta\Phi_s$ is independent of x_1 and is given by

$$\Delta\Phi_s = H\sqrt{\theta_o^2 c_o^2 - \beta^2 y_1^2} \tag{3}$$

where

$$x_1 \geq c_o$$

Everywhere else in the xy plane $\Delta\Phi_s$ is zero.

The basic partial differential equation satisfied by the perturbation velocity potential in supersonic flow is well known to be

$$\beta^2 \frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{4}$$

The general solution of equation (4) is given in reference 2 in the form

$$\Phi(x,y,z) = -\frac{1}{2\pi}\,\overline{\left|\iint_{\tau} \left[\left(\frac{1}{r_c}\right)_s \left(\frac{\partial\Phi_u}{\partial z_1} - \frac{\partial\Phi_l}{\partial z_1}\right)_s - \Delta\Phi_s \left(\frac{\partial}{\partial z_1}\frac{1}{r_c}\right)_s\right] dx_1\, dy_1\right.} \qquad (5)$$

where

$$r_c = \sqrt{(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2(z-z_1)^2}$$

and the subscript s indicates that the function is to be evaluated at z_1 equals zero. The region τ is the portion of the combined area of the plan form and wake bounded by the leading edge of the wing, the sides of the vortex wake and the trace in the $z_1=0$ plane of the Mach forecone with vertex at the point (x,y,z). The sign $\overline{|\quad}$ is to be read "finite part of" and is used in both references 1 and 2. It has the property that

$$\overline{\left|\int_a^b \frac{f(x)}{(b-x)^{3/2}}\,dx\right.} = \int_a^b \frac{f(x)-f(b)}{(b-x)^{3/2}}\,dx - \frac{2f(b)}{\sqrt{b-a}} \qquad (6)$$

Since the particular problem of this report is a lifting plate without thickness

$$\frac{\partial\Phi_u}{\partial z_1} = \frac{\partial\Phi_l}{\partial z_1}$$

and equation (5) reduces to the form

$$\Phi(x,y,z) = \Phi(x,y,z)_P + \Phi(x,y,z)_W$$

$$= -\frac{z\beta^2}{2\pi}\,\overline{\left|\iint_{\text{plan form}} \frac{\Delta\Phi_s\, dx_1\, dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}\right.}$$

$$-\frac{z\beta^2}{2\pi}\,\overline{\left|\iint_{\text{wake}} \frac{\Delta\Phi_s\, dx_1\, dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}\right.} \qquad (7)$$

Solution in the xy Plane

It is convenient to divide the study of downwash into two parts: one, the study of the effects of the doublets distributed over the plan form; and, two, the study of the effects of the doublets distributed over the wake. Such a division has already been implied by the arrangement of equation (7). Hence, in the following w_W and w_P will be derived separately and the total downwash will be their sum, $w = w_W + w_P$.

Effect of doublets in the plan form.– For the purpose of integration, it is convenient to divide the area behind the wing into three regions as shown in figure 1. The division line separating these regions are formed by the Mach cone traces from the trailing-edge tips. The limits of integration which form the bases of the division differ in each of the three regions and are discussed in more detail in reference 1.

The following symbols will be used in the derivation of the expressions for downwash, in the xy plane, induced by the distribution of doublets over the plan form of a triangular wing swept behind the Mach cone:

E_1, E_2, E_3 complete elliptic integrals of the second kind with moduli k_1, k_2, and k_3, respectively

K_1, K_2, K_3 complete elliptic integrals of the first kind with moduli k_1, k_2, and k_3, respectively

k_1 $\sqrt{\dfrac{\delta_1}{\gamma_1}\left(\dfrac{\mu'-\gamma_1}{\delta_1-\mu}\right)}$

k_2 $\left(\dfrac{\gamma_2-\mu}{\mu-\delta_2}\right)\left(\dfrac{\delta_2-\mu'}{\gamma_2-\mu'}\right)$

k_3 $\sqrt{\dfrac{\gamma_3}{\delta_3}\left(\dfrac{\mu'-\delta_3}{\mu'-\gamma_3}\right)}$

γ_1 $\dfrac{(\mu\mu'+\xi^2)-\sqrt{(\xi^2-\mu^2)\ (\xi^2-\mu'^2)}}{\mu+\mu'}$

γ_2 $\dfrac{\xi(\mu+\mu')-\sqrt{2\xi(\mu'-\mu)\ (\mu+\xi)\ (\mu'-\xi)}}{\mu-\mu'+2\xi}$

γ_3 $\dfrac{(\mu\mu'+\xi^2)+\sqrt{(\xi^2-\mu^2)\ (\xi^2-\mu'^2)}}{\mu+\mu'}$

δ_1	$\dfrac{(\mu\mu'+\xi^2)+\sqrt{(\xi^2-\mu^2)(\xi^2-\mu'^2)}}{\mu+\mu'}$
δ_2	$\dfrac{\xi(\mu+\mu')+\sqrt{2\xi(\mu'-\mu)(\mu+\xi)(\mu'-\xi)}}{\mu-\mu'+2\xi}$
δ_3	$\dfrac{(\mu\mu'+\xi^2)-\sqrt{(\xi^2-\mu^2)(\xi^2-\mu'^2)}}{\mu+\mu'}$
η	$\dfrac{\beta(y-y_1)}{c_o}$
μ	$y_o - \dfrac{\theta_o x_1}{c_o}$
μ'	$y_o + \dfrac{\theta_o x_1}{c_o}$
ξ	$\dfrac{x-x_1}{c_o}$
ξ_o	x_o-1

The downwash $w_p(x,y,0)$ may be obtained by considering $\lim_{z\to 0} \dfrac{\partial\Phi_p}{\partial z}$ in equation (7). It can be shown that, in this case, this limiting process corresponds to taking the partial derivative of equation (7) with respect to z and then simply setting z equal to zero. Thus the expressions for w_p in the regions A, B, and C are, respectively,

$$w_{p_A} = -\frac{H\beta}{2\pi}\left|\int_{\xi_o}^{x_o} d\xi \int_{\mu}^{\mu'} \frac{\sqrt{(\mu'-\eta)(\eta-\mu)}}{(\xi^2-\eta^2)^{3/2}}\, d\eta = -\frac{H\beta}{2\pi}\right| \int_{\xi_o}^{x_o} I_1\, d\xi \qquad (8)$$

$$w_{P_B} = -\frac{H\beta}{2\pi}\left[\int_{\frac{y_0+\theta_0 x_0}{1+\theta_0}}^{x_0} d\xi \int_{\mu}^{\mu'} \frac{\sqrt{(\mu'-\eta)\ (\eta-\mu)}}{(\xi^2-\eta^2)^{3/2}}\, d\eta + \int_{\xi_0}^{\frac{y_0+\theta_0 x_0}{1+\theta_0}} d\xi \int_{\mu}^{\xi} \frac{\sqrt{(\mu'-\eta)\ (\eta-\mu)}}{(\xi^2-\eta^2)^{3/2}}\, d\eta\right]$$

$$= -\frac{H\beta}{2\pi}\left(\int_{\frac{y_0\theta_0 x_0}{1+\theta_0}}^{x_0} I_1\, d\xi + \int_{\xi_0}^{\frac{y_0+\theta_0 x_0}{1+\theta_0}} I_2\, d\xi\right) \qquad (9)$$

$$w_{P_C} = -\frac{H\beta}{2\pi}\left[\int_{\frac{y_0+\theta_0 x_0}{1+\theta_0}}^{x_0} d\xi \int_{\mu}^{\mu'} \frac{\sqrt{(\mu'-\eta)\ (\eta-\mu)}}{(\xi^2-\eta^2)^{3/2}}\, d\eta + \int_{\frac{y_0-\theta_0 x_0}{1+\theta_0}}^{\frac{y_0+\theta_0 x_0}{1+\theta_0}} d\xi \int_{\mu}^{\xi} \frac{\sqrt{(\mu'-\eta)\ (\eta-\mu)}}{(\xi^2-\eta^2)^{3/2}}\, d\eta\right.$$

$$\left. + \int_{\xi_0}^{\frac{y_0-\theta_0 x_0}{1+\theta_0}} d\xi \int_{-\xi}^{\xi} \frac{\sqrt{(\mu'-\eta)\ (\eta-\mu)}}{(\xi^2-\eta^2)^{3/2}}\, d\eta\right]$$

$$= -\frac{H\beta}{2\pi}\left(\int_{\frac{y_0+\theta_0 x_0}{1+\theta_0}}^{x_0} I_1\, d\xi + \int_{\frac{y_0-\theta_0 x_0}{1+\theta_0}}^{\frac{y_0+\theta_0 x_0}{1+\theta_0}} I_2\, d\xi + \int_{\xi_0}^{\frac{y_0-\theta_0 x_0}{1+\theta_0}} I_3\, d\xi\right) \qquad (10)$$

The solution of the three integrals I_1, I_2, and I_3, will be discussed in Appendix A.

The expressions for downwash in regions A, B, and C may then be expressed as the following single integrals which can be handled by standard numerical methods:

$$w_{P_A} = -\frac{H\beta}{\pi}\int_{\xi_0}^{x_0} \frac{1}{2\xi^2}\sqrt{\frac{2\xi(\mu'-\mu)}{k_1}}\,(K_1-E_1)\,d\xi \tag{11}$$

$$\begin{aligned} w_{P_B} = -\frac{H\beta}{\pi}\Bigg[& \int_{\frac{y_0+\theta_0 x_0}{1+\theta_0}}^{x_0} \frac{1}{2\xi^2}\sqrt{\frac{2\xi(\mu'-\mu)}{k_1}}\,(K_1-E_1)\,d\xi \\ & + \int_{\xi_0}^{\frac{y_0+\theta_0 x_0}{1+\theta_0}} \left\{K_2\sqrt{\frac{k_2}{(\xi-\mu)(\xi+\mu')}}\left[\frac{\delta_2(\mu-\mu'+2\xi)-2\xi\mu}{\xi^2}\right] - \frac{E_2}{2\xi^2}\sqrt{\frac{(\xi-\mu)(\xi+\mu')}{k_2}}\right\} d\xi\Bigg] \end{aligned} \tag{12}$$

$$\begin{aligned} w_{P_C} = -\frac{H\beta}{\pi}\Bigg[& \int_{\frac{y_0+\theta_0 x_0}{1+\theta_0}}^{x_0} \frac{1}{2\xi^2}\sqrt{\frac{2\xi(\mu'-\mu)}{k_1}}\,(K_1-E_1)\,d\xi \\ & + \int_{\frac{y_0-\theta_0 x_0}{1+\theta_0}}^{\frac{y_0+\theta_0 x_0}{1+\theta_0}} \left\{K_2\sqrt{\frac{k_2}{(\xi-\mu)(\xi+\mu')}}\left[\frac{\delta_2(\mu-\mu'+2\xi)-2\xi\mu}{\xi^2}\right] - \frac{E_2}{2\xi^2}\sqrt{\frac{(\xi-\mu)(\xi+\mu')}{k_2}}\right\} d\xi \\ & + \int_{\xi_0}^{\frac{y_0-\theta_0 x_0}{1+\theta_0}} \frac{1}{2\xi^2}\sqrt{\frac{2\xi(\mu'-\mu)}{k_3}}\,(K_3-E_3)\,d\xi\Bigg] \end{aligned} \tag{13}$$

Effect of Doublets in the Wake.- The study of the downwash induced on the xy plane by the doublets distributed over the wake will also be divided into the three regions indicated in figure 1, and the symbols listed as follows will be used in the derivations:

a_A $\qquad \frac{\delta_A}{\gamma_A}\left(\frac{\gamma_A-\nu'}{\delta_A-\nu'}\right)$

a_B $\qquad \frac{\delta_B}{\gamma_B}\left(\frac{\gamma_B-\nu'}{\delta_B-\nu'}\right)$

a_C $\qquad \frac{\gamma_C}{\delta_C}\left(\frac{\nu'-\delta_C}{\gamma_C-\nu'}\right)$

E_A, E_B, E_C complete elliptic integrals of the second kind with moduli k_A, k_B, and k_C, respectively

$E\left(\frac{1}{a_A},k_A\right)$, $E\left(\frac{1}{a_B},k_B\right)$, $E\left(\frac{a_C}{k_C},k_C\right)$ incomplete elliptic integrals of the second kind with arguments $1/a_A$, $1/a_B$, and a_C/k_C, and with moduli k_A, k_B, and k_C, respectively

K_A, K_B, K_C complete elliptic integrals of the first kind with moduli k_A, k_B, and k_C, respectively

$F\left(\frac{1}{a_A},k_A\right)$, $F\left(\frac{1}{a_B},k_B\right)$, $F\left(\frac{a_C}{k_C},k_C\right)$ incomplete elliptic integrals of the first kind with arguments $1/a_A$, $1/a_B$, and a_C/k_C, and moduli k_A, k_B, and k_C, respectively

k_A $\sqrt{\frac{\delta_A}{\gamma_A}}\left(\frac{\gamma_A-\nu'}{\delta_A-\nu'}\right)$

k_B $\left(\frac{\delta_B-\nu}{\nu-\gamma_B}\right)\left(\frac{\gamma_B-\nu'}{\delta_B-\nu'}\right)$

k_C' $\sqrt{\frac{\gamma_C}{\delta_C}}\left(\frac{\nu'-\delta_C}{\gamma_C-\nu'}\right)$

L_1L_2 undefined limits of integration.

x_o $\frac{x}{c_o}$

y_o $\frac{\beta y}{c_o}$

z_o $\frac{\beta z}{c_o}$

γ_A $\frac{(\nu\nu'+\xi_o{}^2)-\sqrt{(\xi_o{}^2-\nu^2)\;(\xi_o{}^2-\nu'^2)}}{\nu+\nu'}$

γ_B $\frac{\xi_o(\nu+\nu')-\sqrt{2\xi_o(\nu-\nu')\;(\nu-\xi_o)\;(\xi_o+\nu')}}{\nu'-\nu+2\xi_o}$

γ_C $\frac{(\nu\nu'+\xi_o{}^2)+\sqrt{(\xi_o{}^2-\nu^2)\;(\xi_o{}^2-\nu'^2)}}{\nu+\nu'}$

δ_A $\frac{(\nu\nu'+\xi_o{}^2)+\sqrt{(\xi_o{}^2-\nu^2)\;(\xi_o{}^2-\nu'^2)}}{\nu+\nu'}$

δ_B $\frac{\xi_o(\nu+\nu')+\sqrt{2\xi_o(\nu-\nu')\;(\nu-\xi_o)\;(\xi_o+\nu')}}{\nu'-\nu+2\xi_o}$

δ_C $\frac{(\nu\nu'+\xi_o{}^2)-\sqrt{(\xi_o{}^2-\nu^2)\;(\xi_o{}^2-\nu'^2)}}{\nu+\nu'}$

η $\quad \frac{\beta(y-y_1)}{c_o}$

ν $\quad y_o+\theta_o$

ν' $\quad y_o-\theta_o$

ξ_o $\quad x_o-1$

On integrating the second term in equation (7) with respect to x_1 and using the notation just presented

$$\Phi_W = \frac{H\beta}{2\pi}\frac{\xi_o z_o c_o}{\beta}\int_{L_2}^{L_1}\frac{\sqrt{(\nu-\eta)(\eta-\nu')}}{(\eta^2+z_o^2)\sqrt{\xi_o^2-\eta^2-z_o^2}}\,d\eta \qquad (14)$$

The limits on the integral as previously noted differ in the regions A, B, and C shown in figure 1; however, in each case the limits are roots of one of the two radicals in the integrand.

It is desirable to express equation (14) in a different form in order to obtain an expression for downwash in the plane of the airfoil. Integrating by parts

$$\Phi_W = \frac{H\beta}{2\pi}\frac{c_o}{\beta}\left\{\left[\sqrt{(\nu-\eta)(\eta-\nu')}\tan^{-1}\frac{\xi_o\eta}{z_o\sqrt{\xi_o^2-\eta^2-z_o^2}}\right]_{L_2}^{L_1}\right.$$

$$\left.-\int_{L_2}^{L_1}\frac{\nu+\nu'-2\eta}{2\sqrt{(\nu-\eta)(\eta-\nu')}}\tan^{-1}\frac{\xi_o\eta}{z_o\sqrt{\xi_o^2-\eta^2-z_o^2}}\,d\eta\right\}$$

When $\lim\limits_{z_o \to 0} \frac{\partial \Phi_W}{\partial z}$ is considered, it can be shown that in all three regions the contribution to the downwash made by the doublets in the wake is given by the expression

$$w_W = \frac{H\beta}{2\pi\xi_o} \int_{L_2}^{L_1} \frac{\nu+\nu'-2\eta}{2\eta\sqrt{(\nu-\eta)(\eta-\nu')}} \sqrt{\xi_o^2-\eta^2}\; d\eta \tag{15}$$

The solution of equation (15) in regions A, B, and C will be considered separately.

Region A

In region A, $L_1=\nu$ and $L_2=\nu'$. The substitution $\eta = \frac{\gamma_A+\delta_A t}{1+t}$ eliminates the linear term in the radical of the integrand, and equation (15) becomes

$$w_{W_A} = \frac{H\beta}{2\pi\xi_o} \sqrt{\frac{\xi_o^2-\gamma_A^2}{(\nu-\gamma_A)(\gamma_A-\nu')}}\,(\delta_A-\gamma_A) \int_{\frac{\nu'-\gamma_A}{\delta_A-\nu'}}^{\frac{\nu-\gamma_A}{\delta_A-\nu}} \left[\frac{\nu+\nu'}{2(\gamma_A+\delta_A t)(1+t)}\right]\left[-\frac{1}{(1+t)^2}\right] \sqrt{\frac{1-\left(\frac{\xi_o^2-\delta_A^2}{\xi_o^2-\gamma_A^2}\right)t^2}{1-\left(\frac{\delta_A-\nu}{\nu-\gamma_A}\right)^2 t^2}}\; dt \tag{16}$$

The identities

$$\xi_o^2 = \gamma_A\delta_A$$

and

$$(\delta_A-\nu)(\nu'-\gamma_A) + (\delta_A-\nu')(\nu-\gamma_A) = 0$$

are useful in the integration.

The transformation $\omega = \dfrac{\delta_A-\nu}{\nu-\gamma_A}\,t$ is next introduced, and the expression for downwash becomes

$$w_{W_A} = \frac{H\beta}{2\pi}\sqrt{\frac{\xi_0{}^2-\gamma_A{}^2}{(\nu-\gamma_A)(\gamma_A-\nu')}}\,(\delta_A-\gamma_A)\left(\frac{\nu-\gamma_A}{\delta_A-\nu}\right)\left[\int_{-1}^{1}\frac{\nu+\nu'}{2\gamma_A(1+a_A\omega)\left(1+\dfrac{\gamma_A}{\delta_A}a_A\omega\right)}\sqrt{\frac{1-k_A{}^2\omega^2}{1-\omega^2}}\,d\omega\right.$$

$$\left.-\int_{-1}^{1}\frac{1}{\left(1+\dfrac{\gamma_A}{\delta_A}a_A\omega\right)^2}\sqrt{\frac{1-k_A{}^2\omega^2}{1-\omega^2}}\,d\omega\right]$$

Integrating the second term by parts and applying the fundamental properties of even and odd functions yield

$$w_{W_A} = \frac{H\beta}{2\pi\xi_0}\sqrt{\frac{\xi_0^2-\gamma_A^2}{(\nu-\gamma_A)(\gamma_A-\nu')}}(\delta_A-\gamma_A)\left(\frac{\nu-\gamma_A}{\delta_A-\nu}\right)\left\{\int_{-1}^{1} -\frac{\nu+\nu'}{2\gamma_A}\left[1-\frac{1}{1-a_A^2\omega^2}-\frac{1}{1-\left(\frac{\gamma_A}{\delta_A}\right)^2 a_A^2\omega^2}\right]\frac{d\omega}{\sqrt{(1-k_A^2\omega^2)(1-\omega^2)}}\right.$$

$$+\left[\frac{\delta_A}{\gamma_A a_A\left(1+\frac{\gamma_A}{\delta_A}a_A\omega\right)}\sqrt{\frac{1-k_A^2\omega^2}{1-\omega^2}}\right]_{-1}^{1}$$

$$\left.+\int_{-1}^{1}\frac{\delta_A^2(1-k_A^2)}{\delta_A^2-\gamma_A^2a_A^2}\left[\frac{\omega^2}{1-\omega^2}-\frac{1}{1-\left(\frac{\gamma_A a_A}{\delta_A}\right)^2\omega^2}+1\right]\frac{d\omega}{\sqrt{(1-k_A^2\omega^2)(1-\omega^2)}}\right\}$$

$$= \frac{H\beta}{2\pi\xi_0}\sqrt{\frac{\xi_0^2-\gamma_A^2}{(\nu-\gamma_A)(\gamma_A-\nu')}}(\delta_A-\gamma_A)\left(\frac{\nu-\gamma_A}{\delta_A-\nu}\right)\left\{\left[\frac{\delta_A}{\gamma_A a_A\left(1+\frac{\gamma_A}{\delta_A}a_A\omega\right)}\sqrt{\frac{1-k_A^2\omega^2}{1-\omega^2}}\right]_{-1}^{1}\right.$$

$$\left.+\,2\int_{0}^{1}\frac{\delta_A^2(1-k_A^2)}{(\delta_A^2-\gamma_A^2a_A^2)}\frac{\omega^2 d\omega}{(1-\omega^2)^{3/2}\sqrt{1-k_A^2\omega^2}}+2\int_{0}^{1}\frac{\nu+\nu'}{2\gamma_A}\frac{d\omega}{(1-a_A^2\omega^2)\sqrt{(1-k_A^2\omega^2)(1-\omega^2)}}\right\} \qquad (17)$$

The Jacobian transformation $\omega = \operatorname{sn} u$ reduces the integrals in equation (17) to standard elliptic forms (reference 3), and

$$w_{W_A} = -\frac{H\beta}{\pi\xi_o}\left[\left(\xi_o^2-\nu^2\right)\left(\xi_o^2-\nu'^2\right)\right]^{1/4}\sqrt{1-k_A^2}\left\{\frac{a_A}{\sqrt{\left(k_A^2-a_A^2\right)\left(1-a_A^2\right)}}\left[K_A E\left(\frac{1}{a_A},k_A\right)-E_A F\left(\frac{1}{a_A},k_A\right)\right]+\frac{E_A}{1-k_A^2}\right\} \tag{18}$$

Region B

In region B $L_1 = \xi_o$ and $L_2 = \nu'$ and equation (15) may be written

$$w_{W_B} = \frac{H\beta}{2\pi\xi_o}\int_{\nu'}^{\xi_o}\left[\left(\frac{\nu+\nu'}{2\eta}\right)\xi_o^2-\left(\frac{\nu+\nu'}{2}\right)\eta-\xi_o^2+\eta^2\right]\frac{d\eta}{\sqrt{\left[(\nu-\eta)(\xi_o+\eta)\right]\left[(\xi_o-\eta)(\eta-\nu')\right]}} \tag{19}$$

The transformations $\eta = \frac{\gamma_B+\delta_B t}{1+t}$ and $\omega = \frac{\delta_B-\nu'}{\gamma_B-\nu'}t$ where

$$(\gamma_B-\nu')(\xi_o-\delta_B)+(\xi_o-\gamma_B)(\delta_B-\nu')=0$$

and

$$(\xi_o+\gamma_B)(\nu-\delta_B)+(\xi_o+\delta_B)(\nu-\gamma_B)=0$$

reduce equation (19) to

$$w_{W_B} = \frac{H\beta}{2\pi\xi_o}\int_{-1}^{1}\left[\frac{(\nu+\nu')\xi_o^{2}}{2\gamma_B(1+a_B\omega)}\left(1+\frac{\gamma_B}{\delta_B}a_B\omega\right)-\frac{(\nu+\nu')\gamma_B(1+a_B\omega)}{2\left(1+\frac{\gamma_B}{\delta_B}a_B\omega\right)}-\xi_o^{2}+\frac{\gamma_B^{2}(1+a_B\omega)^{2}}{\left(1+\frac{\gamma_B}{\delta_B}a_B\right)^{2}}\right]\frac{d\omega}{\sqrt{(1-k_B^{2}\omega^{2})(1-\omega^{2})}}$$

$$= \frac{H\beta}{2\pi\xi_o}\left\{\left[-\frac{\frac{\delta_B}{\gamma_B a_B}}{1+\frac{\gamma_B}{\delta_B}a_B\omega}\,\frac{1}{\sqrt{(1-k_B^{2}\omega^{2})(1-\omega^{2})}}\right]_{-1}^{1}+\int_{-1}^{1}\left[(\delta_B^{2}-\xi_o^{2})\left(1-\frac{\nu+\nu'}{2\delta_B}\right)+\frac{\xi_o^{2}(\nu+\nu')(\delta_B-\gamma_B)}{2\gamma_B\delta_B(1-a_B^{2}\omega^{2})}\right.\right.$$

$$\left.\left.-\frac{(\delta_B-\gamma_B)^{2}}{\left(1-\frac{\gamma_B^{2}a_B^{2}}{\delta_B^{2}}\right)(1-\omega^{2})}-\frac{(\delta_B-\gamma_B)^{2}}{\left(k_B^{2}-\frac{\gamma_B^{2}a_B^{2}}{\delta_B^{2}}\right)(1-k_B^{2}\omega^{2})}\right]\frac{d\omega}{\sqrt{(1-k_B^{2}\omega^{2})(1-\omega^{2})}}\right\} \tag{20}$$

When the transformation $\omega = \operatorname{sn} u$ is made, equation (20) is readily integrable (reference 3), and, after algebraic simplification, may be written

$$w_{W_B} = -\frac{H\beta}{\pi\xi_o}\,\frac{2a_B(\delta_B-\nu')^{2}}{\delta_B^{2}(\nu'^{2}-1)}\,\frac{\sqrt{(\nu-1)(\nu'+1)}}{(1+a_B)}\left\{(1-k_B)\,K_B-\frac{a_B E_B}{(a_B-K_B)}\right.$$

$$\left.-\frac{(1-k_B)\,(a_B^{2}+k_B)a_B\left[K_B E\left(\frac{1}{a_B},k_B\right)-E_B F\left(\frac{1}{a_B},k_B\right)\right]}{(a_B-k_B)\sqrt{(a_B^{2}-1)\,(a_B^{2}-k_B^{2})}}\right\} \tag{21}$$

Region C

In region C, $L_1 = \xi_o$ and $L_2 = -\xi_o$, and equation (15) is written

$$w_{W_C} = \frac{H\beta}{2\pi\xi_o}\int_{-\xi_o}^{\xi_o} \frac{\nu+\nu'-2\eta}{2\eta\sqrt{(\nu-\eta)\ (\eta-\nu')}}\sqrt{\xi_o^2-\eta^2}\ d\eta \tag{22}$$

The derivation of the expression for w_{W_C} is similar to that for w_{W_A} with the exception that in this case the substitution $\omega = \sqrt{\frac{\delta_C}{\gamma_C}}\ t$ is made, and equation (22) may be written

$$w_{W_C} = -\frac{H\beta}{\pi\xi_o}\left[(\xi_o^2-\nu^2)\ (\xi_o^2-\nu'^2)\right]^{1/4}\sqrt{1-k_C^2}\left\{\frac{-a_C}{\sqrt{(k_C^2-a_C^2)\ (1-a_C^2)}}\left[K_C E\left(\frac{a_C}{k_C},k_C\right)-E_C F\left(\frac{a_C}{k_C},k_C\right)\right]\right.$$

$$\left. + \frac{E_C}{1-k_C^2} - K_C\right\} \tag{23}$$

Solution in the xz Plane

Just as in the study of downwash in the xy plane, so also in its study in the xz plane it is convenient to consider separately the effects of the doublets distributed over the plan form and the wake. The subscript notation for w_W and w_P is the same as before and again w is equal to $w_W + w_P$.

Effect of doublets in the plan form.- In the xz plane two regions are indicated in figure 2. Region E lies between the Mach wedge from the trailing edge and the line of intersection of the two cones from the trailing-edge tips. Region D connects this region to infinity. Again the limits of integration form the basis of the division into the two regions.

The symbols listed below will be used in the derivation of the expressions for downwash in the xz plane, induced by the distribution of doublets over the plan form of the wing.

$E_4, E_5, E_{4_o}, E_{5_o}$	complete elliptic integrals of the second kind with moduli k_4, k_5, k_{4_o}, and k_{5_o}, respectively
$K_4, K_5, K_{4_o}, K_{5_o}$	complete elliptic integrals of the first kind with moduli k_4, k_5, k_{4_o}, and k_{5_o}, respectively
k_4	$\dfrac{\theta_o x_1}{\sqrt{(x-x_1)^2-\beta^2 z^2}}$
k_{4_o}	$\dfrac{\theta_o c_o}{\sqrt{(x-c_o)^2-\beta^2 z^2}}$
k_5	$\dfrac{\sqrt{(x-x_1)^2-\beta^2 z^2}}{\theta_o x_1}$
k_{5_o}	$\dfrac{\sqrt{(x-c_o)^2-\beta^2 z^2}}{\theta_o c_o}$

Region D

In region D, the first term in equation (7) is written

$$\Phi_{P_D} = -\frac{zH\beta^2}{\pi}\int_0^{c_o} dx_1 \int_0^{\frac{\theta_o x_1}{\beta}} \frac{\sqrt{\theta_o^2 x_1^2 - \beta^2 y_1^2}}{\left[(x-x_1)^2 - \beta^2 y_1^2 - \beta^2 z^2\right]^{3/2}}\, dy_1 \qquad (24)$$

Integrating with respect to y_1

$$\Phi_{P_D} = -\frac{zH\beta}{\pi}\int_0^{c_o} \frac{k_4}{\theta_o x_1}\left(K_4 - E_4\right) dx_1$$

Changing the variable of integration

$$\Phi_{P_D} = -\frac{zH\beta}{\pi}\int_0^{k_{4_o}} \frac{\theta_o\left[xk_4 - \sqrt{\theta_o^2 x^2 + \beta^2 z^2 (k_4^2 - \theta_o^2)}\right]}{\sqrt{\theta_o^2 x^2 + \beta^2 z^2 (k_4^2 - \theta_o^2)}} \left(\frac{K_4 - E_4}{k_4^2 - \theta_o^2}\right) dk_4$$

Taking the partial derivative of Φ_{P_D} with respect to z gives the expression for the downwash as

$$w_{P_D} = -\frac{H\beta}{\pi}\left\{\frac{k_{4_0}{}^2\beta^2 z^2}{\theta_0{}^2 c_0{}^2}\left(\frac{K_{4_0}-E_{4_0}}{k_{4_0}{}^2-\theta_0{}^2}\right)\frac{\theta_0}{\sqrt{\theta_0{}^2x^2+\beta^2z^2(k_{4_0}{}^2-\theta_0{}^2)}}\left[xk_{4_0}-\sqrt{\theta_0{}^2x^2+\beta^2z^2(k_{4_0}{}^2-\theta_0{}^2)}\right]\right.$$

$$\left.+\int_0^{k_{4_0}}\left(\frac{x^3k_4\theta_0{}^2}{[\theta_0{}^2x^2+\beta^2z^2(k_{4_0}{}^2-\theta_0{}^2)]^{3/2}}-1\right)\left(\frac{\theta_0}{k_4{}^2-\theta_0{}^2}\right)(K_4-E_4)\,dk_4\right\} \qquad (25)$$

Region E

In region E, a similar derivation yields the solution

$$w_{P_E} = -\frac{H\beta}{\pi}\left[\frac{\beta^2z^2}{\theta_0{}^2c_0{}^2k_{5_0}{}^2}\left(\frac{K_{5_0}-E_{5_0}}{1-k_{5_0}{}^2\theta_0{}^2}\right)\frac{\theta_0\left[x-\sqrt{x^2k_{5_0}{}^2\theta_0{}^2+\beta^2z^2(1-k_{5_0}{}^2\theta_0{}^2)}\right]}{\sqrt{x^2k_{5_0}{}^2\theta_0{}^2+\beta^2z^2(1-k_{5_0}{}^2\theta_0{}^2)}}\right.$$

$$+\int_{k_{5_0}}^{1}\frac{K_5-E_5}{k_5}\,\frac{\theta_0}{1-k_5{}^2\theta_0{}^2}\left\{\frac{x^3k_5{}^2\theta_0{}^2}{[x^2k_5{}^2\theta_0{}^2+\beta^2z^2(1-k_5{}^2\theta_0{}^2)]^{3/2}}-1\right\}dk_5$$

$$\left.+\int_0^1(K_4-E_4)\,\frac{\theta_0}{k_4{}^2-\theta_0{}^2}\left\{\frac{x^3k_4\theta_0{}^2}{[\theta_0{}^2x^2+\beta^2z^2(k_4{}^2-\theta_0{}^2)]^{3/2}}-1\right\}dk_4\right] \qquad (26)$$

Effect of doublets in the wake.- The limits of integration again necessitate the division of the portion of the plane behind the trailing-edge wave into two regions D and E. The following list of symbols will be used in the derivation of downwash in the xz plane, induced by the doublets distributed over the vortex wake:

a_D	$\dfrac{\sqrt{(x-c_o)^2-\beta^2z^2}}{\beta z}$
a'_D	$\dfrac{\sqrt{(x-c_o)^2-\beta^2z^2}}{(x-c_o)}$
a_E	$\dfrac{\theta_o c_o}{\beta z}$
a'_E	$\dfrac{\theta_o c_o}{\sqrt{\beta^2z^2+\theta_o{}^2c_o{}^2}}$
E_D, E_E	complete elliptic integrals of the second kind with moduli k_D and k_E, respectively
$E(a'_D,k'_D)$, $E(a'_E,k'_E)$	incomplete elliptic integrals of the second kind with arguments a'_D and a'_E and moduli k'_D and k'_E, respectively
K_D, K_E	complete elliptic integrals of the first kind with moduli k_D and k_E, respectively
$F(a'_D,k'_D)$, $F(a'_E,k'_E)$	incomplete elliptic integrals of the first kind with arguments a'_D and a'_E and moduli k'_D and k'_E, respectively
k_D	$\dfrac{\theta_o c_o}{\sqrt{(x-c_o)^2-\beta^2z^2}}$
k'_D	$\sqrt{1-k_D{}^2}$
k_E	$\dfrac{\sqrt{(x-c_o)^2-\beta^2z^2}}{\theta_o c_o}$
k'_E	$\sqrt{1-k_E{}^2}$

$$\operatorname{sn}^2(i\alpha_D) = a_D^2$$

$$\operatorname{sn}^2(i\alpha_E) = a_E'^2$$

Region D

In region D, the second term in equation (7) becomes

$$\Phi_{W_D} = -\frac{zH\beta^2}{\pi}\,\left\lceil \int_0^{\frac{\theta_0 c_0}{\beta}} \sqrt{\theta_0^2 c_0^2 - \beta^2 y_1^2}\, dy_1 \int_{c_0}^{x-\beta\sqrt{y_1^2+z^2}} \frac{dx_1}{\left[(x-x_1)^2-\beta^2(y_1^2+z^2)\right]^{3/2}} \right. \tag{27}$$

Integrating with respect to x_1, and using the definition of $\lceil$

$$\Phi_{W_D} = \frac{zH(x-c_0)}{\pi}\int_0^{\frac{\theta_0 c_0}{\beta}} \frac{\sqrt{\theta_0^2 c_0^2-\beta^2 y_1^2}}{(y_1^2+z^2)\sqrt{(x-c_0)^2-\beta^2 y_1^2-\beta^2 z^2}}\, dy_1$$

$$= \frac{H(x-c_0)}{\pi}\,\frac{\theta_0^2 c_0^2}{\beta z\sqrt{(x-c_0)^2-\beta^2 z^2}}\int_0^1 \frac{\sqrt{1-\omega^2}}{\left(1+\frac{\theta_0^2 c_0^2}{\beta^2 z^2}\,\omega^2\right)\sqrt{1-k_D^2\omega^2}}\, d\omega$$

where

$$\omega = \frac{\beta y_1}{\theta_0 c_0}$$

Applying the Jacobian transformation $\omega = \operatorname{sn} u$

$$\Phi_{W_D} = \frac{H(x-c_o)\theta_o^2 c_o^2}{\pi\beta z\sqrt{(x-c_o)^2-\beta^2 z^2}} \int_0^{K_D} \frac{1-\operatorname{sn}^2 u}{1+a_D^2 k_D^2 \operatorname{sn}^2 u}\, du \tag{28}$$

The expression for Φ_{W_D} as given in equation (28), is integrable (reference 3) and becomes

$$\Phi_{W_D} = \frac{H(x-c_o)\theta_o^2 c_o^2}{\pi\beta z\sqrt{(x-c_o)^2-\beta^2 z^2}} \left\{ K_D - \left(\frac{K_D\sqrt{1+a_D^2 k_D^2}}{k_D^2\sqrt{1+a_D^2}}\right)\left(\frac{1}{ia_D}\right)\left[iF(a'_D,k'_D) + \frac{ia_D\sqrt{1+a_D^2 k_D^2}}{\sqrt{1+a_D^2}} \right.\right.$$

$$\left.\left. - iE(a'_D,k'_D) - iF(a'_D,k'_D)\frac{E_D}{K_D}\right]\right\}$$

$$= -\frac{H\beta z}{\pi}\left\{a'_D k'^2_D K_D + \frac{\sqrt{1-a'^2_D k'^2_D}}{\sqrt{1-a'^2_D}}\left[K_D F(a'_D,k'_D) - E_D F(a'_D,k'_D) - K_D E(a'_D,k'_D)\right]\right\} \tag{29}$$

Since it may be shown that

$$\frac{\partial}{\partial z}\left[K_D\, F(a'_D,k'_D) - E_D\, F(a'_D,k'_D) - K_D\, E(a'_D,k'_D)\right] = -\frac{\beta}{\sqrt{\beta^2 z^2+\theta_o{}^2 c_o{}^2}}\left[(K_D-E_D)\, a'_D - K_D\,\frac{(1-a'_D{}^2 k'_D{}^2)}{a'_D}\right]$$

the expression for the downwash w_{W_D} obtained by taking the partial derivative of Φ_{W_D} with respect to z may be written

$$w_{W_D} = \frac{H\beta}{\pi}\left\{-\frac{E_D}{a'_D} + \frac{K_D(1-a'_D{}^2)}{a'_D} - \frac{\sqrt{1-a'_D{}^2}}{\sqrt{1-a'_D{}^2 k'_D{}^2}}\left[K_D\, F(a'_D,k'_D) - E_D\, F(a'_D,k'_D) - K_D\, E(a'_D,k'_D)\right]\right\} \quad (30)$$

<u>Region E</u>

In region E a similar derivation gives

$$w_{W_E} = \frac{H\beta}{\pi}\left\{\frac{\sqrt{1-a'_E{}^2 k'_E{}^2}}{a'_E}\left(\frac{K_E-E_E}{k_E{}^2} - a'_E{}^2\, K_E\right) - \sqrt{1-a'_E{}^2}\,\left[K_E\, F(a'_E,k'_E) - E_E\, F(a'_E,k'_E)\right.\right.$$

$$\left.\left. - K_E\, E(a'_E,k'_E)\right]\right\} \quad (31)$$

DISCUSSION

In reference 1, the downwash behind the triangular wing was computed along the x axis only; but it was indicated that only slight variations with y should be expected for some distance on either side of the axis. The value of w/w_o[1] along the x axis derived in reference 1 for various θ_o's is shown in figure 3.

The results presented in figure 4 can be used to assess the accuracy of the prediction of reference 1 in regards to the spanwise variation of downwash. The downwash in the xy plane is presented for various θ_o's and spanwise stations and for all values of x from the trailing edge to a point where the asymptotic value is closely approached. In figure 4 the curve for $y_o = 0$ represents the results of reference 1. The region covered in the y direction extends from the x axis out to about $\frac{1}{2}\theta_o$, where in the coordinate system used x_o equals x/c_o, y_o equals $\beta y/c_o$, and θ_o is the semispan of the wing. Within about a half span from the trailing edge of of the wing no general statement can be made as to the variation of w/w_o in the y direction except that in the region considered the increase or decrease in the values of downwash from those at $y_o = 0$ was less than 0.1. For distances greater than a half span from the trailing edge, however, the variation is quite uniform and w/w_o deviates from its value at $y=0$ only slightly for $-\frac{1}{2}\theta_o < y_o < \frac{1}{2}\theta_o$.

For $\theta_o=0.6$, a more extensive study was made of the variation of downwash with y. Figure 5 presents values of w/w_o across the span for several positions behind the trailing edge. Immediately behind the trailing edge the value of w/w_o falls and approaches $-\infty$ as the wing tip is approached. However, at 0.4 of a root chord behind the trailing edge $(x_o=1.4)$, w/w_o rises and reaches the value of 0.7 as the wing tip is reached. At $x_o=2.2$ the spanwise variation of w/w_o is essentially constant. The approximation of reference 1, that the downwash is independent of y in the neighborhood of the x axis, is also shown in figure 5. It is evident that this approximation is useful out to about a third of a semispan.

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[1]The variable w/w_o (i.e., $(w_P + w_W)/w_o$) represents the total downwash behind the wing divided by the induced vertical velocity on the wing itself. If ϵ is the downwash angle and α the angle of attack of the wing then

$$\frac{w}{w_o} = \frac{d\epsilon}{d\alpha}$$

The variation of downwash in the xz plane is presented in figure 6. The curves represent values of w/w_o from the trailing-edge wave downstream to a point where the asymptotic value is closely approached. In the immediate vicinity of the Mach cones from the trailing-edge tips (i.e., $x_o \approx 1 + \theta_o$) the curves were not continued because w/w_o becomes very large and approaches negative infinity as the Mach cone is approached. (Since this effect results from infinitely large values of the radial component of induced velocity at the Mach cone, it does not exist in the z_o=0 plane.) Such a behavior is consistent with the mathematical idealization of infinite pressures at the leading edge and of an abrupt fall of load at the trailing edge. However, in an actual flow field where these phenomena do not exist the flow will experience a milder change in passing across the Mach cone. Even in the theoretical results presented in this report the growth of the vertical induced velocity in the neighborhood of the Mach cone is logarithmic, and the interval in which w/w_o is appreciably distorted from the general trend is very small.

Some further insight into the behavior of w in the vicinity of the Mach cone from the trailing-edge tips can be obtained by studying a single vortex which extends infinitely far ahead from the origin at an oblique angle to the flow and infinitely far behind the origin parallel to the flow (fig. 7). The half of the vortex which extends ahead makes an angle with the free-stream direction less than the Mach angle so that the component of free-stream velocity normal to it will be subsonic. Thus, outside of the Mach cone originating at the sudden bend in the vortex at the origin, the flow will be exactly like that of a linearized compressible subsonic vortex with a superimposed uniform velocity parallel to the line of the vortex. Inside the Mach cone, however, the flow is completely changed. Figure 7 gives an indication of the change. The term "bent" vortex refers to the vortex along the x axis which is turned suddenly at the origin from the angle it had maintained from $-\infty$. The term "unbent" vortex on the other hand refers to a vortex which maintains the same angle from $-\infty$ to $+\infty$. The unbent vortex is included to show the effect of the sudden turn. The figure shows that on the z=0 plane (section AA) the downwash is finite and continuous in passing through the Mach cone, but that above the z=0 plane (section BB) the value of w becomes infinite as the cone surface is approached from the inside. This behavior at the Mach cone may aid in interpreting the discontinuity in the results for the complete wing as given in figure 6.

A comparison of the simplified study made in reference 1 with the results of this report in the xz plane is given in figure 8.

The conclusion of reference 1 was that to a first order the variation of w/w_o with z_o/θ_o was linear with a slope $-(1/E_o)$. The figure shows that the approximation is useful up to about a third of a semispan.

Values of w/w_o were not computed for points off of the xz and xy planes although the methods employed are quite capable of handling the problem. The results already given would apparently indicate that the approximations of reference 1 are valid in the vicinity of $-(1/3)\theta_o$ about the x axis. This assumption can easily be checked for large distances behind the trailing edge by considering the flow field as x_o approaches ∞. Thus figure 9 shows a comparison between the exact value of w/w_o derived by means of the linearized equation and the approximate value given in reference 1 for all points greater than about one chord length behind the trailing edge. Once again the agreement is fairly good out to about one-third of a semispan either vertically or horizontally from the x axis.

In order that some idea of the variation of downwash with Mach number at various positions downstream could be obtained, figure 10 was prepared. This figure shows values of w/w_o on the x axis plotted as a function of M_o for various values of x_o. The value of the sweepback angle is 45° and the Mach number range covered could be extended to 1.4 and the leading edge of the wing would still be subsonic.

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APPENDIX A

EVALUATION OF SPECIAL INTEGRALS

Integral I_1

Since there are no singularities in I_1, the finite part sign may be discarded. The linear term in the radical is eliminated by the transformation $\eta = (\gamma_1+\delta_1 t)/(1+t)$ and the integral becomes

$$I_1 = \sqrt{\frac{(\mu'-\gamma_1)\,(\gamma_1-\mu)}{(\delta_1-\gamma_1)\gamma_1^{\,3}}}\int_{\frac{\mu'-\gamma_1}{\delta_1-\mu'}}^{\frac{\mu-\gamma_1}{\delta_1-\mu}} \frac{\sqrt{1-\left(\frac{\delta_1-\mu'}{\mu'-\gamma_1}\right)^2 t^2}}{\left[1-\left(\frac{\delta_1}{\gamma_1}\right)^2 t^2\right]^{3/2}}\,dt \tag{A1}$$

The expressions for δ_1 and γ_1 may be combined to give the useful identities

$$\xi^2 = \gamma_1\delta_1$$

and

$$(\gamma_1-\mu)\,(\delta_1-\mu') = (\mu'-\gamma_1)\,(\delta_1-\mu)$$

Noting that the integrand is an even function, equation (A1) may be reduced to the canonical form

$$I_1 = 2\sqrt{\frac{(\mu'-\gamma_1)\,(\gamma_1-\mu)}{(\delta_1-\gamma_1)\gamma_1^{\,3}}}\left(\frac{\mu'-\gamma_1}{\delta_1-\mu'}\right)\int_0^1 \frac{\sqrt{1-\omega^2}}{(1-k_1^{\,2}\omega^2)^{3/2}}\,d\omega \tag{A2}$$

by the substitution

$$\omega = \frac{\delta_1-\mu'}{\mu'-\gamma_1}\,t$$

By the introduction of the Jacobian elliptic functions (reference 3) in the transformation $\omega = \operatorname{sn} u$, the integration may be completed, and

$$I_1 = 2\sqrt{\frac{(\mu'-\gamma_1)\,(\gamma_1-\mu)}{(\delta_1-\gamma_1)\,\gamma_1^{s}}}\left(\frac{\mu'-\gamma_1}{\delta_1-\mu'}\right)\int_0^{K_1} \mathrm{cd}^2\,u\,du = \frac{1}{\xi^2}\sqrt{\frac{2\xi(\mu'-\mu)}{k_1}}\left(K_1-E_1\right) \qquad \text{(A3)}$$

where $\mathrm{cd}\,u \equiv \mathrm{cn}\,u/\mathrm{dn}\,u$.

Integral I_2

As the first step in reducing I_2 to canonical form the integral is written

$$I_2 = \int_\mu^\xi \left[1 + \frac{\xi^2-\xi(\mu+\mu')+\mu\mu'}{2\xi(\eta-\xi)} - \frac{\xi^2+\xi(\mu+\mu')+\mu\mu'}{2\xi(\eta+\xi)}\right]\frac{d\eta}{\sqrt{(\xi+\eta)\,(\mu'-\eta)}\,\sqrt{(\xi-\eta)\,(\eta-\mu)}} \qquad \text{(A4)}$$

In this case the following identities may be obtained directly from the definitions of γ_2 and δ_2.

$$(\gamma_2-\mu)(\xi-\delta_2)+(\gamma_2-\xi)(\mu-\delta_2) = 0$$

and

$$(\xi+\gamma_2)(\mu'-\delta_2)+(\mu'-\gamma_2)(\xi+\delta_2) = 0$$

The transformations $\eta = \frac{\gamma_2+\delta_2 t}{1+t}$ and $\omega = \frac{\delta_2-\mu}{\gamma_2-\mu}\, t$ are made, and after algebraic simplifications equation (A4) becomes

$$I_2 = \frac{\delta_2-\gamma_2}{(\mu'-\gamma_2)(\delta_2-\mu)}\sqrt{\frac{\mu'-\mu}{2\xi}}\left\{\left[1 - \frac{(\mu'-\gamma_2)(\mu+\xi)(\mu'+\xi)}{(\delta_2-\mu')(\delta_2+\xi)(2\xi)} - \frac{(\gamma_2-\mu)(\xi-\mu)(\xi-\mu')}{(\delta_2-\mu)(\gamma_2-\xi)(2\xi)}\right]\left[\int_{-1}^{1}\frac{d\omega}{\sqrt{(1-k_2{}^2\omega^2)(1-\omega^2)}}\right]\right.$$

$$-\left[\frac{(\mu+\xi)(\mu'+\xi)}{2\xi(\gamma_2+\xi)}\left(1-\frac{\mu'-\gamma_2}{\delta_2-\mu'}\right)\right]\int_{-1}^{1}\frac{(1+k_2\omega)}{(1-k_2{}^2\omega^2)^{3/2}\sqrt{1-\omega^2}}\,d\omega$$

$$\left. + \frac{(\xi-\mu)(\xi-\mu')}{2\xi(\gamma_2-\xi)}\left(1+\frac{\gamma_2-\mu}{\delta_2-\mu}\right)\left|\overline{\int_{-1}^{1}\frac{(1+\omega)}{(1-\omega^2)^{3/2}\sqrt{1-k_2{}^2\omega^2}}\,d\omega}\right.\right\} \qquad \text{(A5)}$$

By applying the fundamental properties of even and odd functions, the first two integrals in equation (A5) can readily be integrated. The procedure for handling the finite part sign over the third integral will be considered in detail. Since,

$$\left|\overline{\int_{-1}^{1}\frac{(1+\omega)}{(1-\omega^2)\sqrt{(1-k_2{}^2\omega^2)(1-\omega^2)}}\,d\omega}\right. = 2\left|\overline{\int_{0}^{1}\frac{d\omega}{(1-\omega^2)^{3/2}\sqrt{1-k_2{}^2\omega^2}}}\right.$$

and if

$$f(\omega) = \frac{1}{(1+\omega)^{3/2}\sqrt{1-k_2{}^2\omega^2}}$$

and

$$f(1) = \frac{1}{2^{3/2}\sqrt{1-k_2{}^2}}$$

then by equation (6)

$$2\left|\overline{\int_0^1 \frac{d\omega}{(1-\omega^2)^{3/2}\sqrt{1-k_2^2\omega^2}}}\right. = 2\left[\int_0^1 \frac{d\omega}{(1-\omega^2)^{3/2}\sqrt{1-k_2^2\omega^2}} - \int_0^1 \frac{d\omega}{2^{3/2}(1-\omega)^{3/2}\sqrt{1-k_2^2}} - \frac{1}{\sqrt{2(1-k_2^2)}}\right]$$

$$= 2\left[\int_0^1 \frac{d\omega}{(1-\omega^2)^{3/2}\sqrt{1-k_2^2\omega^2}} - \lim_{\omega\to 1} \frac{1}{\sqrt{2(1-\omega)\,(1-k_2^2)}}\right]$$

$$= 2\left[\int_0^{K_2} \frac{du}{\mathrm{cn}^2 u} - \lim_{\omega\to 1} \frac{1}{\sqrt{2(1-\omega)\,(1-k_2^2)}}\right]$$

$$= 2\left\{\left[K_2 + \lim_{\omega\to 1} \frac{\sqrt{1-k_2^2\omega^2}}{(1-k_2^2)\sqrt{1-\omega^2}}\,\omega - \frac{E_2}{1-k_2^2}\right] - \lim_{\omega\to 1} \frac{1}{\sqrt{2(1-\omega)\,(1-k_2^2)}}\right\}$$

$$= 2\left(K_2 - \frac{E_2}{1-k_2^2}\right)$$

The solution of equation (A5) becomes, after algebraic simplification,

$$I_2 = 2\left\{\sqrt{\frac{k_2}{(\xi-\mu)\,(\xi-\mu')}}\left[\frac{\delta_2(\mu-\mu'+2\xi)-2\mu\xi}{\xi^2}\right] K_2 - \frac{1}{2\xi^2}\sqrt{\frac{(\xi-\mu)\,(\xi+\mu')}{k_2}}\,E_2\right\} \qquad \text{(A6)}$$

Integral I_3

The procedure for integrating I_3 is similar to that previously discussed in connection with I_1. In this case, the integral is canonicalized by the substitution $\omega = \sqrt{\frac{\delta_3}{\gamma_3}}\, t$ and the solution may be written

$$I_3 = \frac{1}{\xi^2}\sqrt{\frac{2\xi(\mu' - \mu)}{k_3}}\,(K_3 - E_3) \tag{A7}$$

REFERENCES

1. Heaslet, Max. A., and Lomax, Harvard: The Calculation of Downwash Behind Supersonic Wings with an Application to Triangular Plan Forms. NACA TN No. 1620, 1948.

2. Heaslet, Max. A., and Lomax, Harvard: The Use of Source-Sink and Doublet Distributions Extended to the Solution of Arbitrary Boundary Value Problems in Supersonic Flow. NACA TN No. 1515, 1948.

3. Whittaker, E. T., and Watson, G. N.: A Course of Modern Analysis. Cambridge, The University Press, 4th ed., 1940, ch. 22.

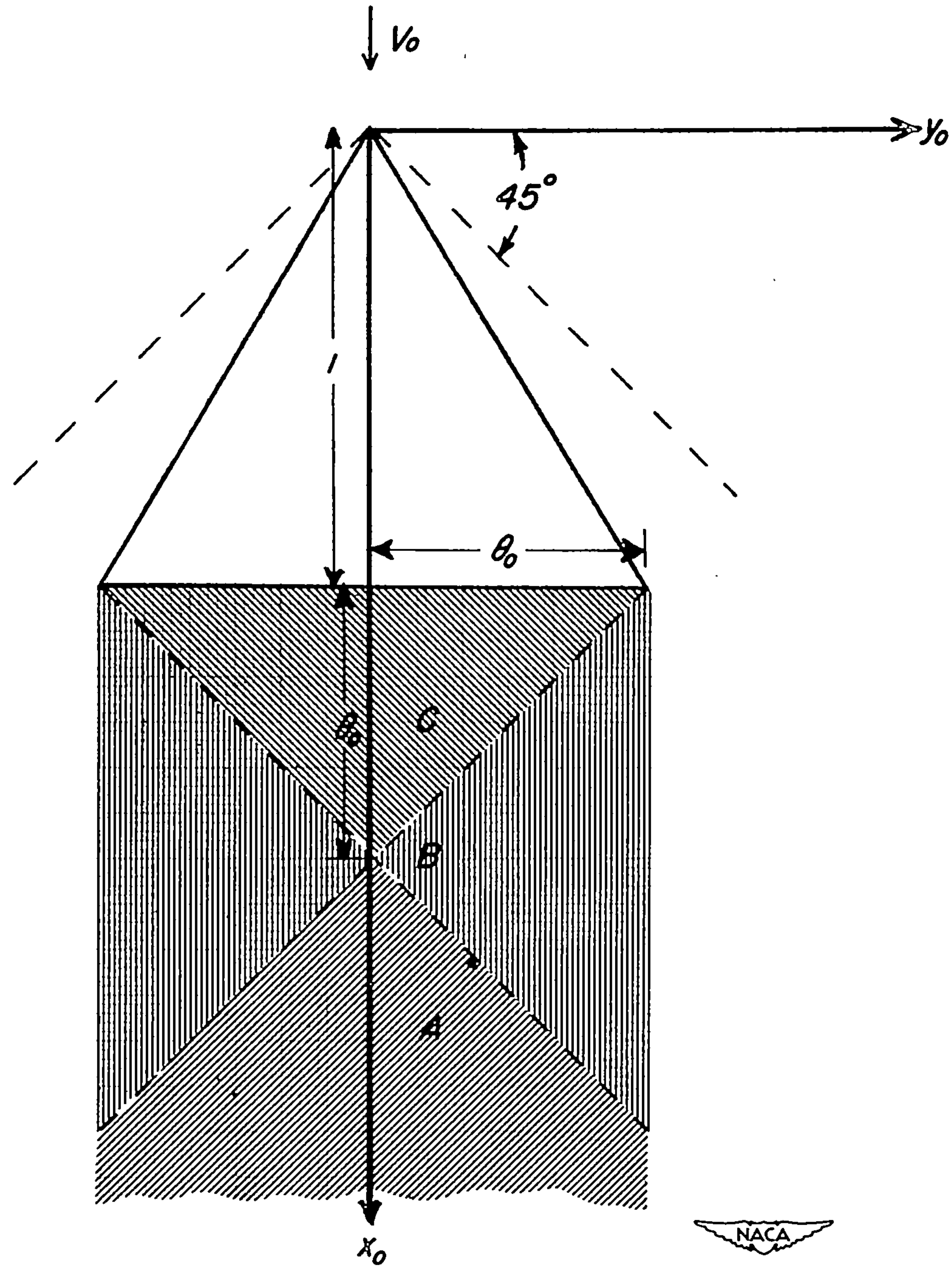


Figure 1.- Regions A, B and C for triangular wing in x_0y_0 plane.

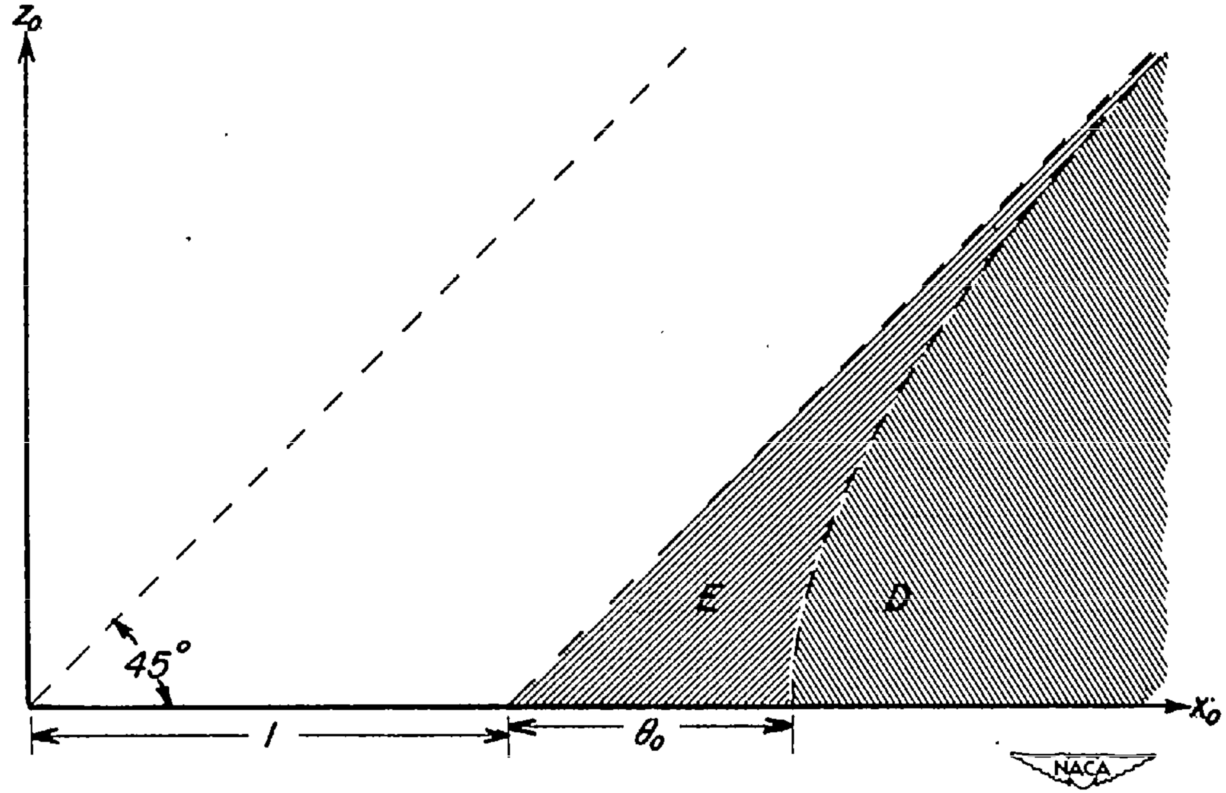


Figure 2. - Regions D and E in the $x_0 z_0$ plane for a triangular wing.

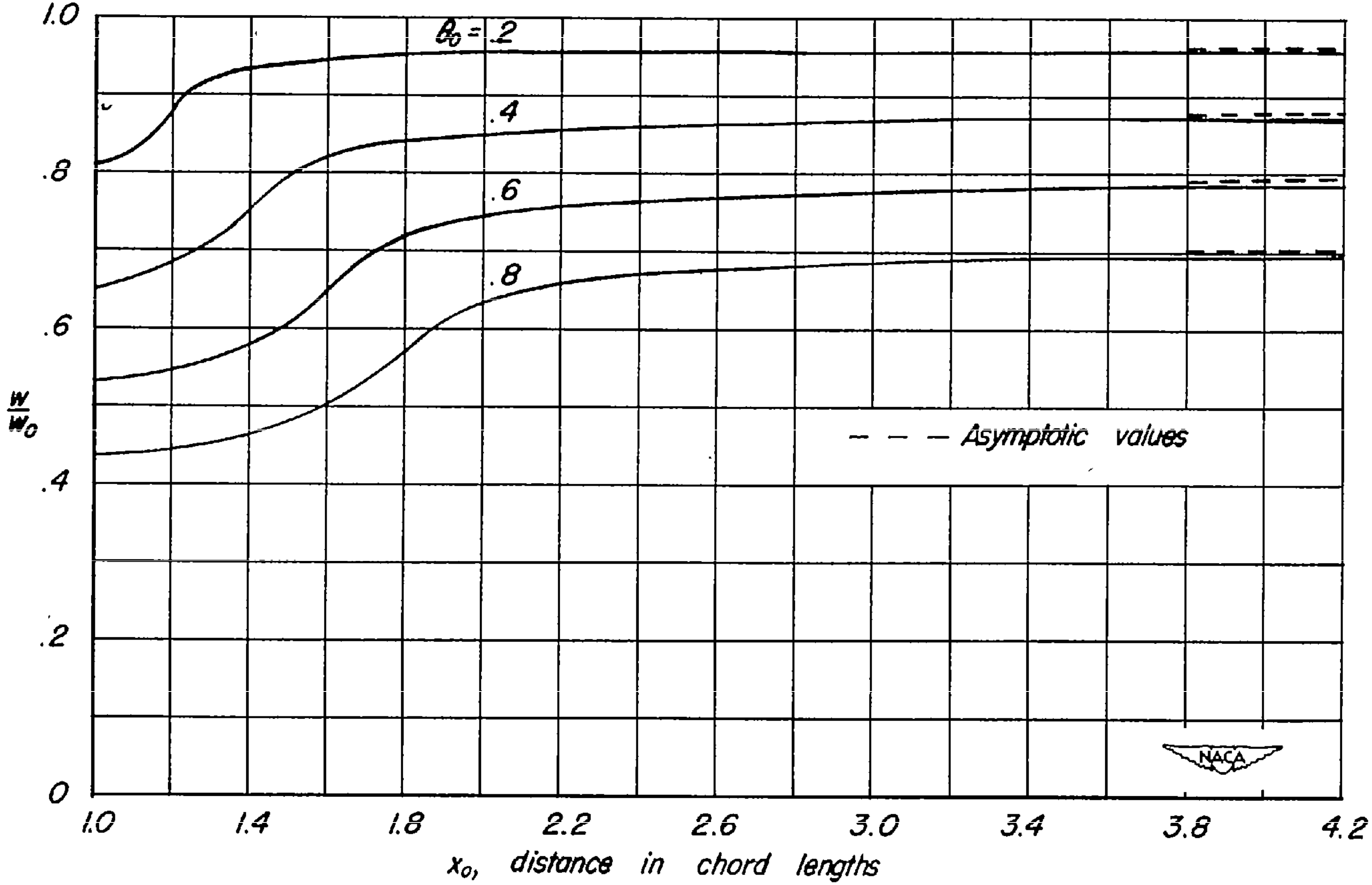
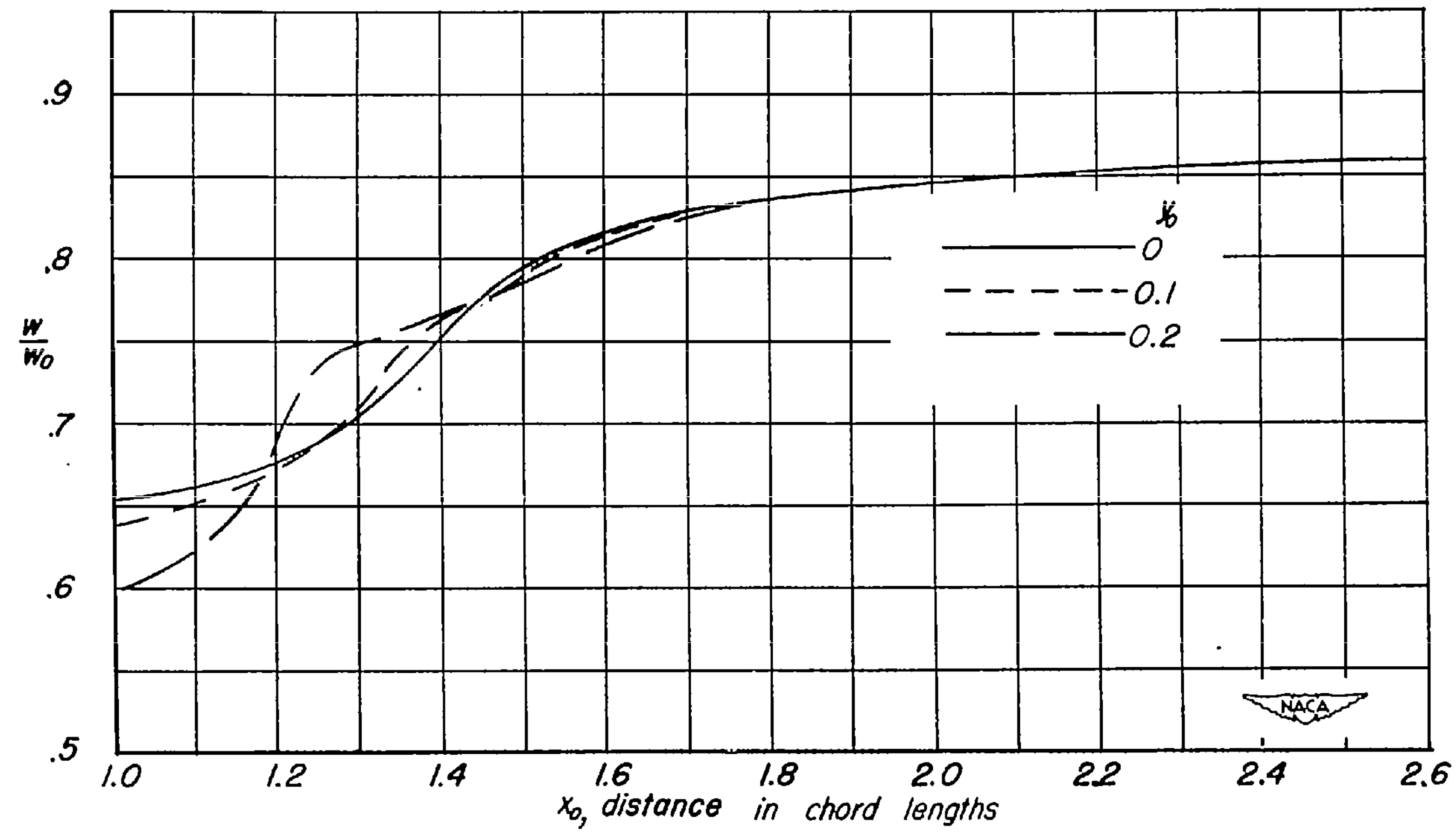
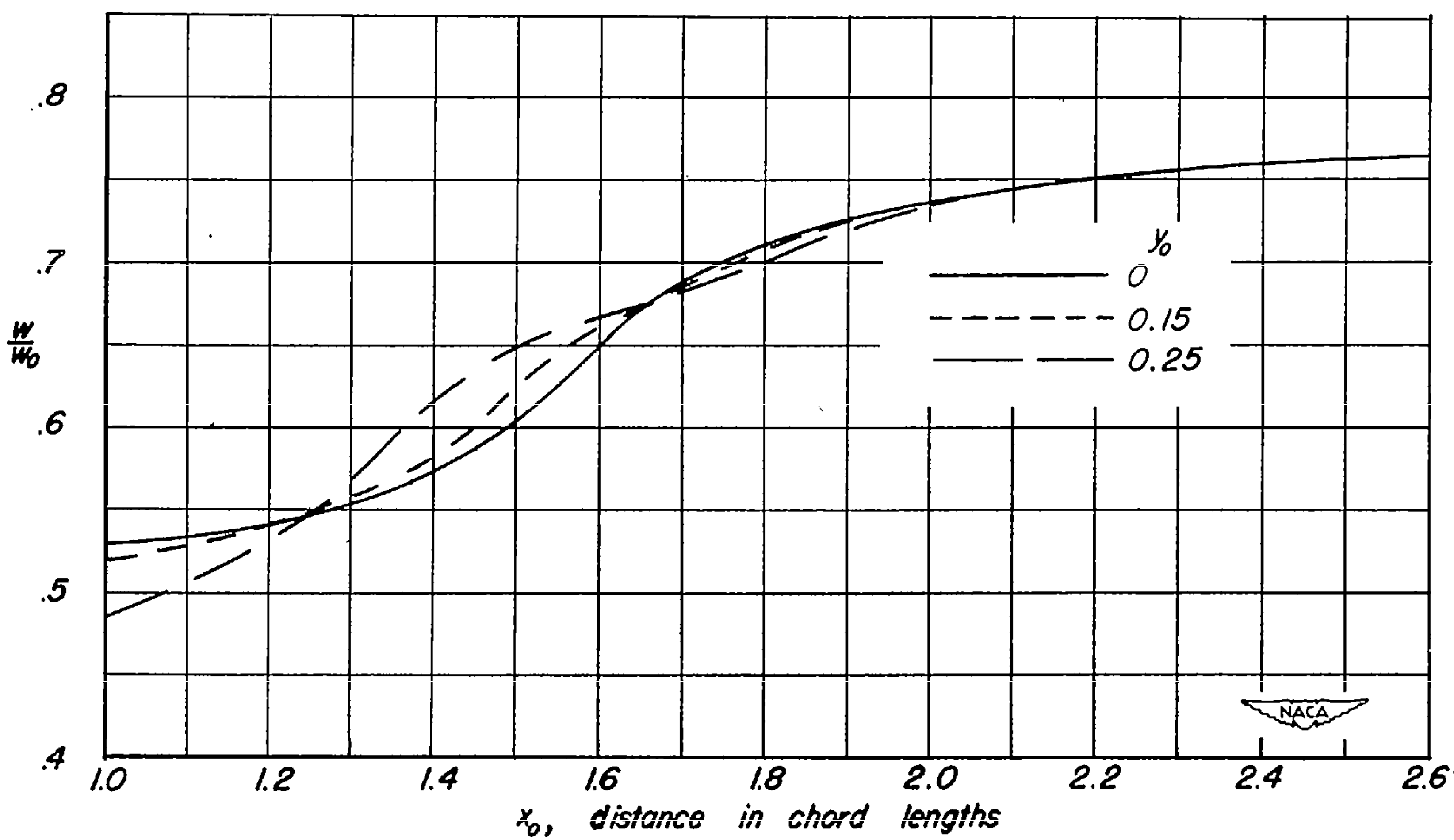


Figure 3. - Variation of the downwash on the x axis downstream from the trailing edge (Reference 1).



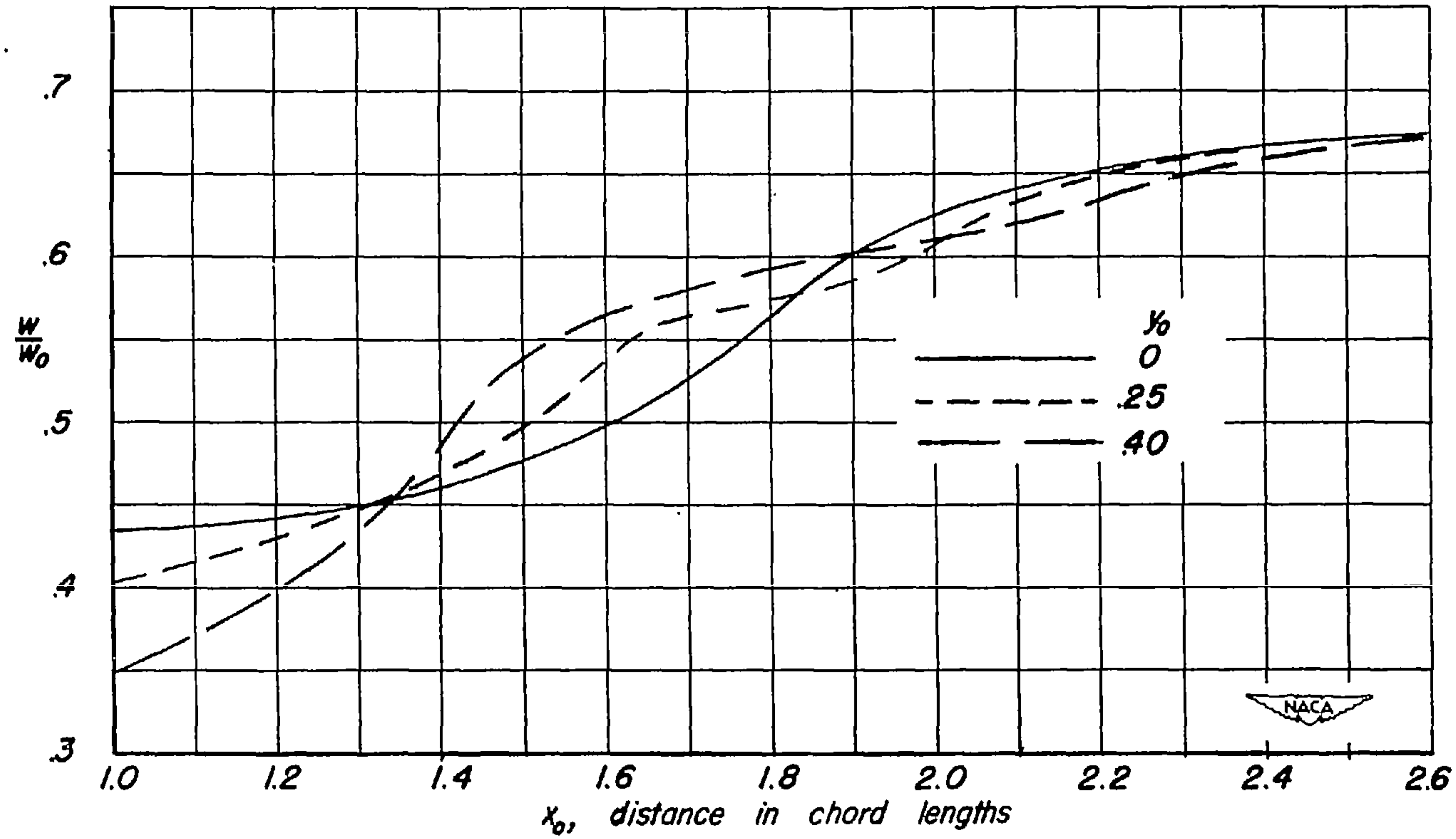
(a) $\theta_0 = 0.4$.

Figure 4. - Variation of the downwash in the $x_0 y_0$ plane downstream from the trailing edge for various span stations.



(b) $\theta_0 = 0.6$.

Figure 4. - Continued.



(c) $\theta_0 = 0.8$.

Figure 4. - Concluded.

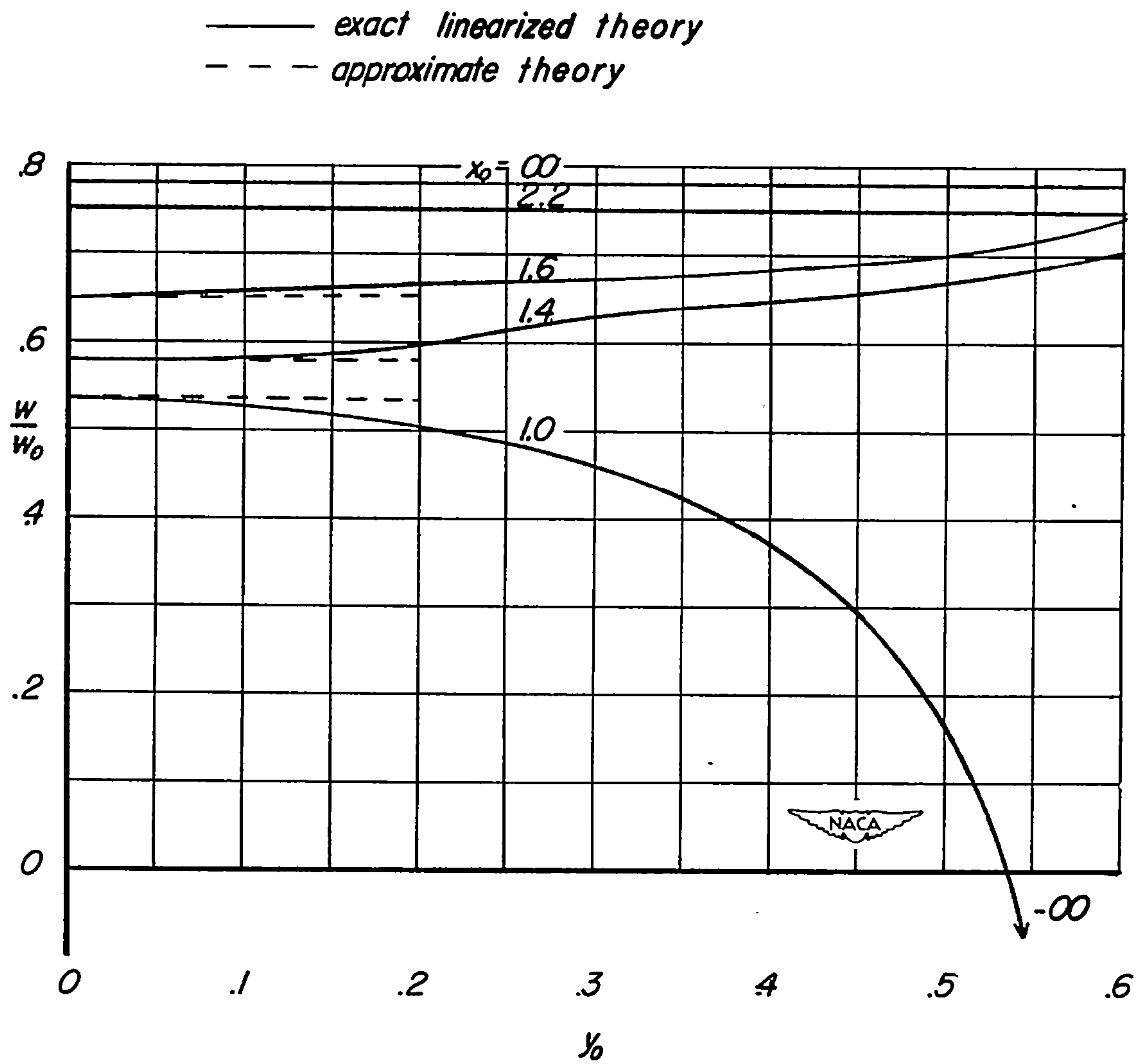
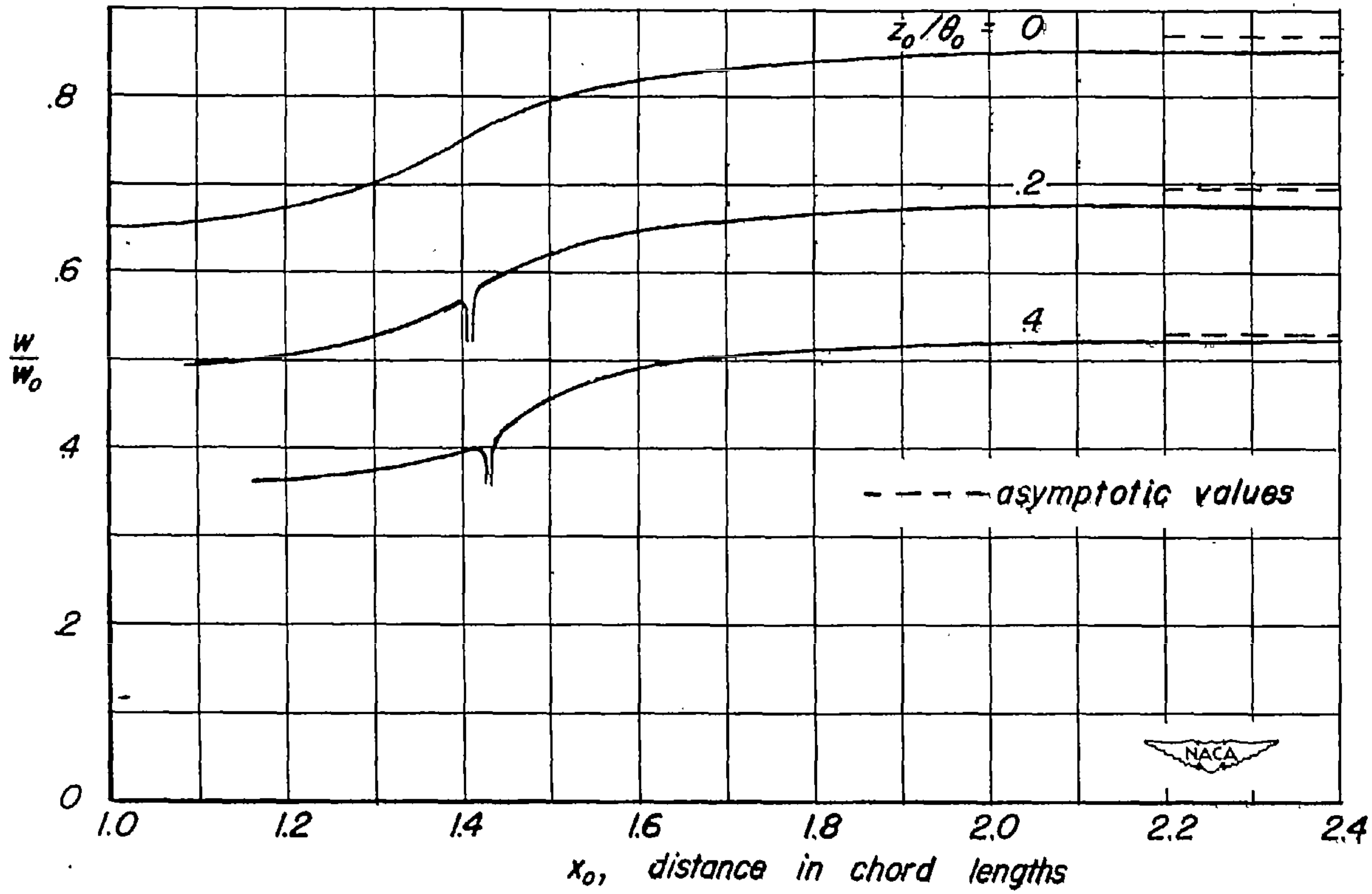
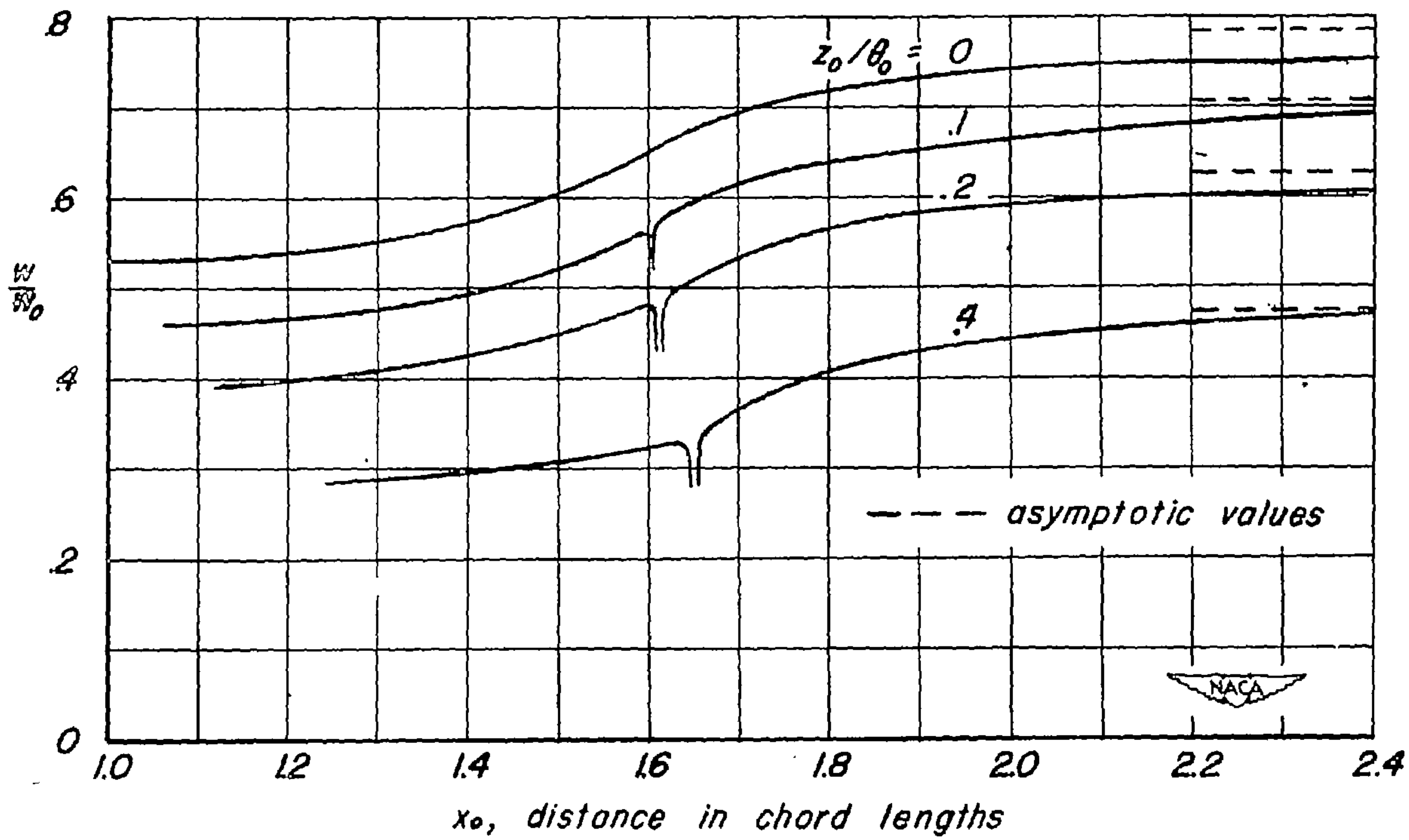


Figure 5. - Variation of downwash across the span at various stations downstream of wing trailing edge for $\theta_0 = 0.6$.



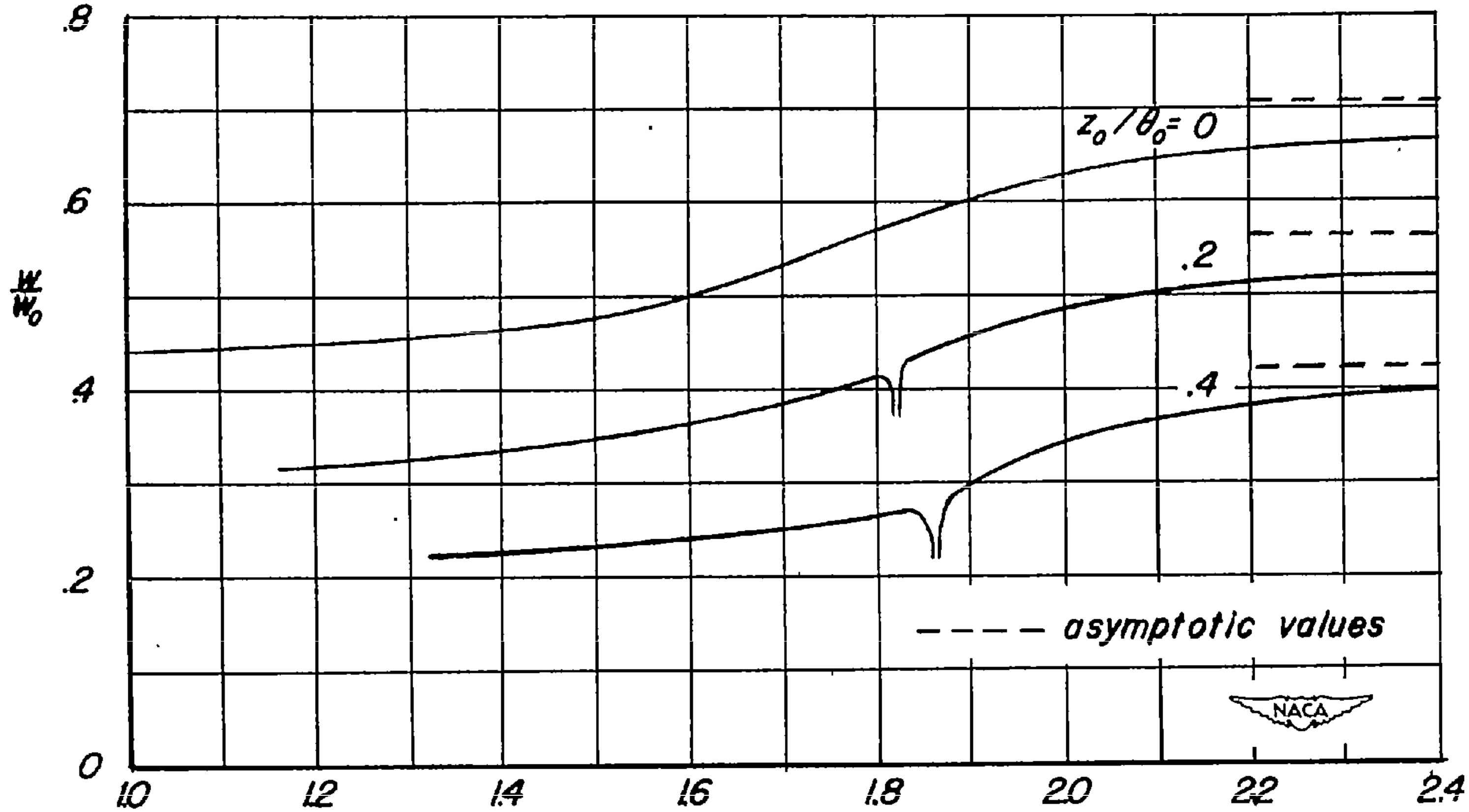
(a) $\theta_0 = 0.4$.

Figure 6.- Variation of the downwash in the x_0z_0 plane downstream from the trailing edge.



(b) $\theta_0 = 0.6$.

Figure 6. - Continued.



x_0, distance in chord lengths

(c) $\theta_0 = 0.8$.

Figure 6. - Concluded.

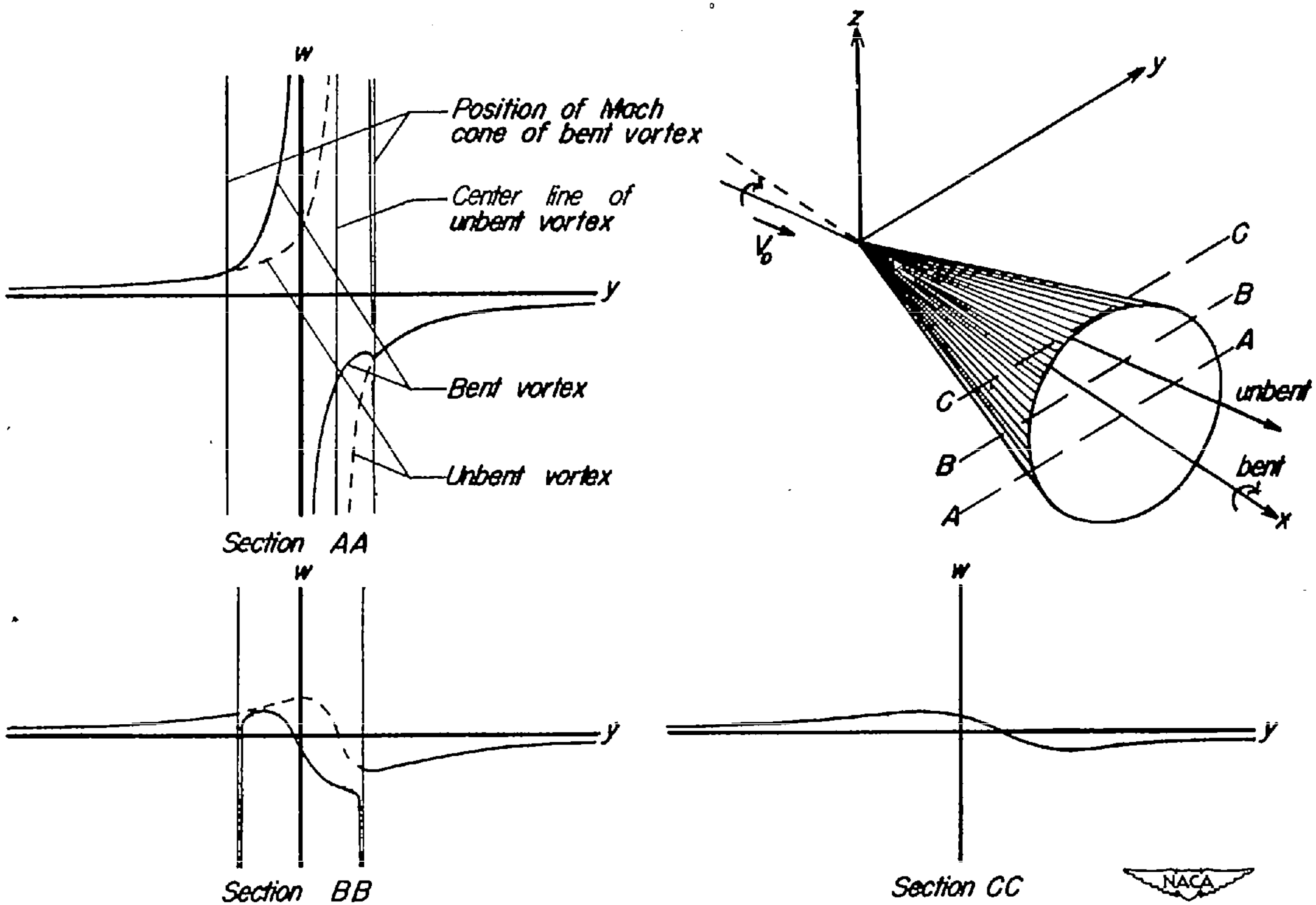
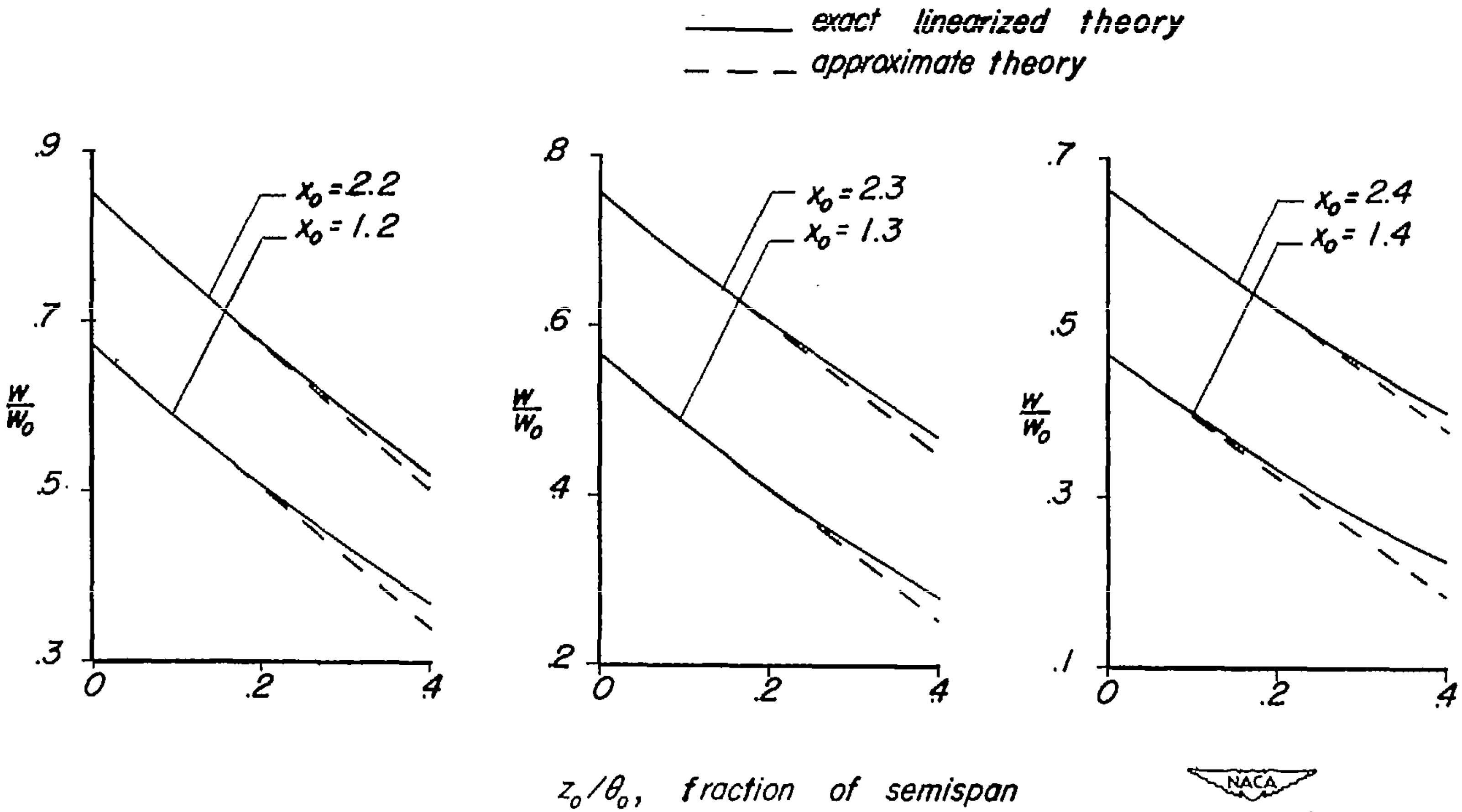


Figure 7.- Induced vertical velocity field for bent and unbent supersonic oblique vortex making an angle with the free stream less than the Mach angle.



(a) $\theta_0 = 0.4$. (b) $\theta_0 = 0.6$. (c) $\theta_0 = 0.8$.

Figure 8. - Variation of downwash in the $x_0 z_0$ plane at various positions on x_0 axis.

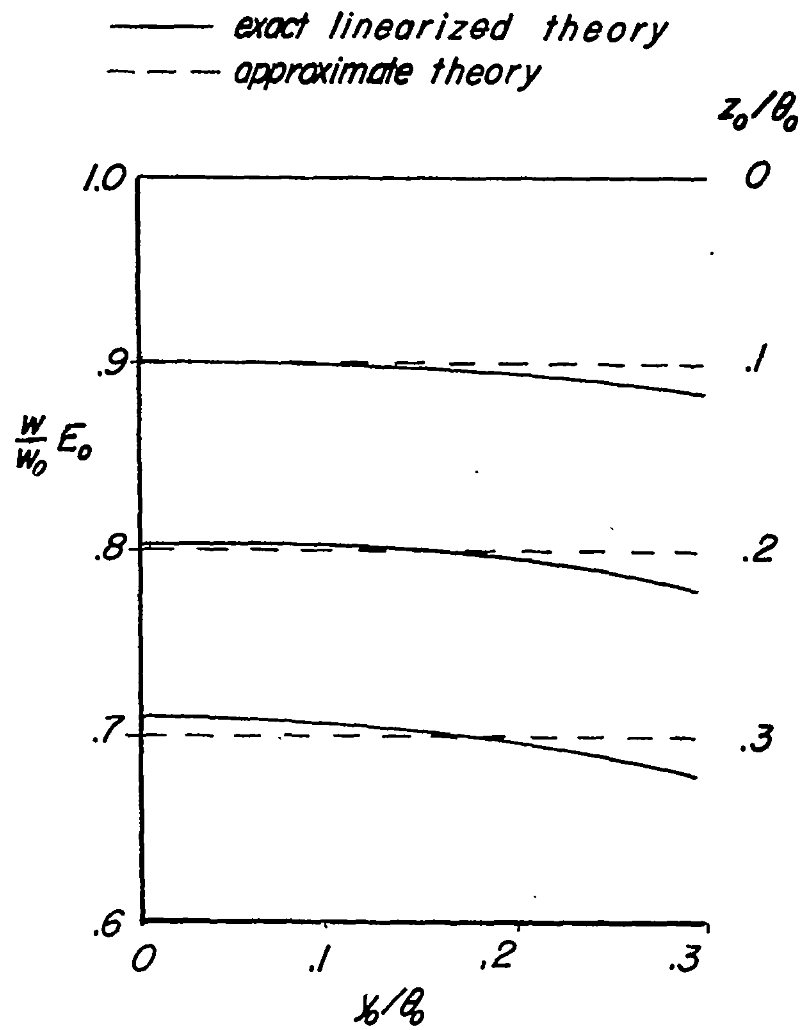


Figure 9. - Downwash at a large distance behind triangular wing.

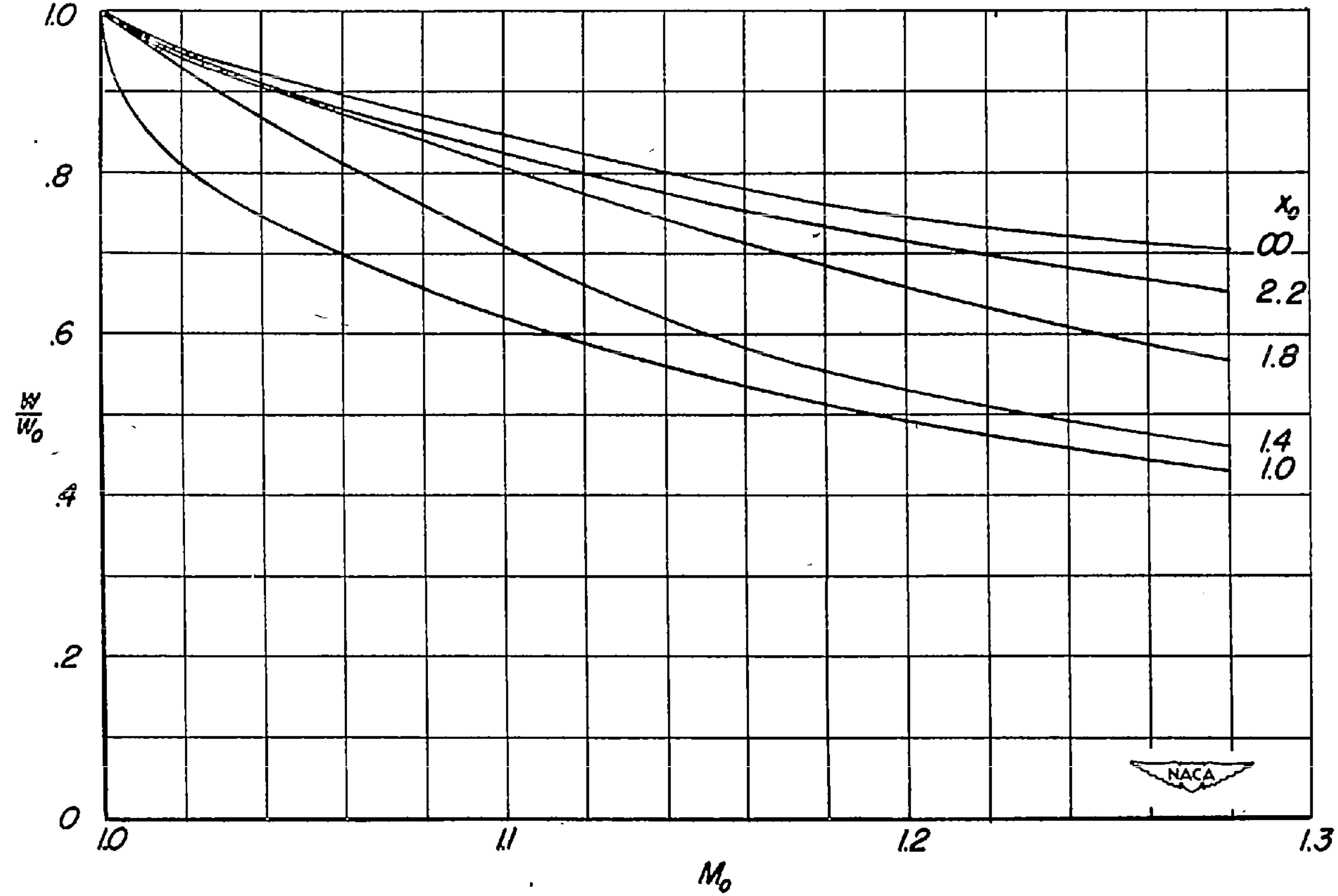


Figure 10.- Variation of downwash on x axis with Mach number at various positions downstream of trailing edge. $\psi = 45°$