NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2651

SUPERSONIC CONICAL FLOW

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Washington
March 1952

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SUMMARY

A method is described for the solution of the nonlinear equations for supersonic conical flow. The procedure is mostly a numerical one based on the method of characteristics and the relaxation process. A procedure for calculating the position of the shock is inherent in the analysis. The method is applicable to any conical flow.

As an illustration, the flow about a triangular wing with supersonic edges is presented.

INTRODUCTION

A knowledge of the characteristics of bodies in supersonic flight is dependent on both theoretical and experimental information. Most theoretical aerodynamic data are obtained by the use of linearized theory which permits only very slender bodies and small angles of attack. In determining the value of such information and the practical range over which the parameters such as body diameter, wing thickness, and angle of attack may be allowed to vary, one must rely on comparison of these results with either experiment or a nonlinear theory. Because of the limited experimental facilities available and the expense involved in their operation and in model construction, the existence of a practical nonlinear theory becomes important. Further, if more accurate results are desired than may be obtained from a linear theory, a nonlinear one is required.

In the case of conical flows, the solutions which have been obtained up to the present time have been found mostly by the use of linearized theory. Busemann (reference 1) introduced this procedure. References 2 and 3, to mention only two investigations, present extensive studies of thin conical wings. Browne, Friedman, and Hodes (reference 4) have solved a special wing-body problem. In reference 5, a fairly general method of solution is given for fuselage-type conical bodies. Linearized solutions, satisfying the exact boundary conditions, have been discussed by Laporte and Bartels (reference 6). Moore (reference 7) and Broderick (reference 8) have obtained second-order linearized solutions for several conical flows. More general analyses applicable to conical problems include, for example, Evvard's work given in reference 9 (this is applicable in the lifting case only when there is a supersonic leading edge) and Spreiter's solutions (reference 10) for very slender bodies and low-aspect-ratio wings.

The application of nonlinear theory to conical flows has been limited mostly to Taylor and Maccoll's solutions for the right circular cone at zero angle of attack (references 11 and 12). Solutions for the yawed circular cone are given in references 13 to 15. Busemann (reference 16) and Ferrari (reference 17) have presented a general discussion of the nonlinear problem. Ferri (reference 18) has given some general analysis of the problem with particular reference to the yawed cone. However, until the present, no attempt has been made to present a detailed method for the solution of the general conical-flow problem. The solution presented herein is of necessity mostly a numerical one using the methods of characteristics and relaxation. In the latter case, reference to work by Southwell and Emmons (see, e.g., references 19 and 20) has been found valuable. A discussion, by Lighthill, of the shape of the shock about a conical obstacle (reference 21) also proved useful.

This investigation was conducted at Brown University under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

The author wishes to express his appreciation to Dr. W. D. Hayes for many valuable suggestions received in the course of discussions with him and to Dr. G. F. Carrier who read the manuscript critically and carefully.

SYMBOLS

A	rotation, defined by equation (10)
c	velocity of sound divided by maximum velocity
c'	velocity of sound
C_p	pressure coefficient, difference between local and free-stream static pressures divided by free-stream dynamic pressure
J	mechanical equivalent of heat
K	function of velocity and its derivatives, defined by equation (40)

M	Mach number of free stream
M_{N1}	component of free-stream Mach number taken normal to wing leading edge
M_{N2}	component of free-stream Mach number taken behind and normal to shock
Q	function of shock position, slope, and free-stream Mach number, defined by equation (33b)
R, θ, φ	spherical coordinates
R	gas constant
$S = JS'/\gamma R$	
S'	entropy
S_{N1}	speed normal to wing leading edge
S_{N2}	speed behind and normal to shock
S_T	speed tangent to wing leading edge
T	velocity component lying on surface of sphere through conical vertex $\left(\sqrt{v^2 + w^2}\right)$
u, v, w	ratio of velocity components in r-, θ-, and φ-directions, respectively, to maximum velocity
$\bar{u}$	ratio of free-stream velocity to maximum velocity
v_x, v_y, v_z	ratio of Cartesian velocity components to speed of sound
$\bar{V}'$	velocity vector
$\bar{V}$	$\bar{V}'$ divided by maximum velocity
W_{max}	maximum velocity
α	ratio of velocities in θ- and φ-directions $(-v/w)$
β	streamlines
γ	adiabatic exponent

δ arc-tangent of slope of shock on surface r = Constant

ϵ slope of body-tangent plane, defined in figure 4

η angle defined in figure 4 and equations (24)

θ_B equation of body

θ_W shock angle, measured in plane lying normal to leading edge of surface to which shock is attached

μ angle defined in figure 4 and equations (24)

ν angle defined in figure 4 and equations (24)

Subscripts:

θ, φ differentiation with respect to θ and φ, respectively

ANALYSIS

The equations of motion of a perfect isoenergetic fluid may be written according to reference 22 as

$$(\nabla \times \bar{V}') \times \bar{V}' = (c')^2 \, \nabla\left(\frac{JS'}{\gamma R}\right) \tag{1}$$

and

$$\nabla \cdot \left[\left(W_{max}^2 - \bar{V}' \cdot \bar{V}'\right)^{1/\gamma-1} \bar{V}'\right] = 0 \tag{2}$$

where

$$(c')^2 = \frac{\gamma - 1}{2}\left(W_{max}^2 - \bar{V}' \cdot \bar{V}'\right) \tag{3}$$

and where $\bar{V}'$ is the velocity vector; S', the entropy; J, the mechanical equivalent of heat; γ, the adiabatic exponent; R, the gas constant; W_{max}, the maximum speed obtainable on expansion into a vacuum (also constant); and c', the velocity of sound. A change of variables may now be made to nondimensionalize the quantities. Let

$$\left.\begin{aligned} S &= JS'/\gamma R \\ \overline{V} &= \overline{V}'/W_{max} \\ c &= c'/W_{max} \end{aligned}\right\} \tag{4}$$

Using these new variables, equations (1) and (2) were expanded in spherical coordinates r, θ, and φ (fig. 1). This was done for conical flow (so that neither the velocity nor entropy are functions of r). Noting that u, v, and w are the components of $\overline{V}$ in the r-, θ-, and φ-directions, respectively, one obtains

$$w\left(\frac{u_\varphi}{\sin\theta} - w\right) + v\left(u_\theta - v\right) = 0 \tag{5a}$$

$$u\left(v - u_\theta\right) - w\left(w_\theta - \frac{v_\varphi}{\sin\theta} + w\cot\theta\right) = c^2 S_\theta \tag{5b}$$

$$v\left(w_\theta - \frac{v_\varphi}{\sin\theta} + w\cot\theta\right) - u\left(\frac{u_\varphi}{\sin\theta} - w\right) = \frac{c^2 S_\varphi}{\sin\theta} \tag{5c}$$

and, using equation (5a),

$$\frac{c^2}{\sin\theta}\left[2u\sin\theta + (v\sin\theta)_\theta + w_\varphi\right] =$$

$$u\left(v^2 + w^2\right) + \frac{v}{2}\left(v^2 + w^2\right)_\theta + \frac{w}{2\sin\theta}\left(v^2 + w^2\right)_\varphi \tag{6}$$

Equations (5) yield

$$vS_\theta + \frac{wS_\varphi}{\sin\theta} = 0 \tag{7}$$

This says simply that the entropy is constant on the intersection of a stream surface with a sphere through the vertex of the conical flow. Henceforth, such an intersection will be termed, for convenience, a streamline. Denote by $\beta = \beta(\theta)$ the equation of such a streamline. Its equation, in terms of the velocity components, is

$$v\beta_{\theta} + \frac{w\beta_{\varphi}}{\sin\,\theta} = 0 \tag{8}$$

Thus the entropy is a function of β only and will hence be determined solely by conditions at the juncture between the disturbed and undisturbed flow.

Solving equations (5a) and (8) for v and w as functions of β and u, one obtains

$$\left.\begin{aligned} v &= u_{\theta} - A \\ w &= \frac{u_{\varphi}}{\sin\,\theta} - \alpha A \end{aligned}\right\} \tag{9}$$

where

$$\alpha = -\frac{v}{w} = \frac{\beta_{\varphi}}{\beta_{\theta}\,\sin\,\theta}$$

and

$$A = \frac{u_{\theta} + \alpha u_{\varphi}/\sin\,\theta}{1 + \alpha^2} \tag{10}$$

The A is essentially the rotation. Then, using these relations, the two remaining ones (equations (5b) and (6)) become

$$uA + \left(\frac{u_{\varphi}}{\sin\,\theta} - \alpha A\right)\left[\frac{A_{\varphi}}{\sin\,\theta} - \frac{(\alpha A\,\sin\,\theta)_{\theta}}{\sin\,\theta}\right] = -c^2 S_{\beta}\beta_{\theta} \tag{11}$$

and

$$u\left(2 - \frac{u_\theta^2}{c^2} - \frac{u_\varphi^2}{c^2 \sin^2\theta}\right) + u_{\theta\theta}\left(1 - \frac{u_\theta^2}{c^2}\right) + \frac{u_{\varphi\varphi}}{\sin^2\theta}\left(1 - \frac{u_\varphi^2}{c^2 \sin^2\theta}\right) +$$

$$u_\theta \cot\theta\left(1 + \frac{u_\varphi^2}{c^2 \sin^2\theta}\right) - \frac{2u_\theta u_\varphi u_{\varphi\theta}}{c^2 \sin^2\theta} =$$

$$\frac{1}{\sin\theta}\left[(A \sin\theta)_\theta + (\alpha A)_\varphi\right] +$$

$$\frac{A}{c^2}\left\{\begin{array}{l} -2u_\theta u_{\theta\theta} + \dfrac{1}{\sin\theta}\left[\left(\dfrac{u_\varphi}{\sin\theta} + \alpha u_\theta\right)\left(u_\varphi \cot\theta - 2u_{\varphi\theta}\right) - u_\varphi u_\theta \alpha_\theta\right] - \\ \dfrac{1}{\sin^3\theta}\left[u_\varphi\left(\alpha_\varphi u_\varphi + 2\alpha u_{\varphi\varphi}\right)\right] \\ A\left[-u\left(1 + \alpha^2\right) + u_{\theta\theta} + \left(\dfrac{u_\varphi}{\sin\theta} + \alpha u_\theta\right)\alpha_\theta + \dfrac{\alpha}{\sin\theta}\left(2u_{\varphi\theta} - u_\varphi \cot\theta\right)\right] + \\ \dfrac{A\alpha}{\sin^2\theta}\left(2\alpha_\varphi u_\varphi + \alpha u_{\varphi\varphi}\right) - \alpha A^2\left(\alpha_\theta + \dfrac{\alpha\alpha_\varphi}{\sin\theta}\right) \end{array}\right. \tag{12}$$

Equation (12) can be written in this alternative form:

$$u\left[2 - \frac{\left(u_\theta - A\right)^2}{c^2} - \frac{\left(\dfrac{u_\varphi}{\sin\theta} - \alpha A\right)^2}{c^2}\right] + \left(u_\theta - A\right)_\theta\left[1 - \frac{\left(u_\theta - A\right)^2}{c^2}\right] +$$

$$\frac{\left(\dfrac{u_\varphi}{\sin\theta} - \alpha A\right)_\varphi}{\sin\theta}\left[1 - \frac{\left(\dfrac{u_\varphi}{\sin\theta} - \alpha A\right)^2}{c^2}\right] + \left(u_\theta - A\right)\cot\theta\left[1 + \frac{\left(\dfrac{u_\varphi}{\sin\theta} - \alpha A\right)^2}{c^2}\right] -$$

$$\frac{\left(\dfrac{u_\varphi}{\sin\theta} - \alpha A\right)\left(u_\theta - A\right)}{c^2}\left[\frac{\left(u_\varphi - \alpha \sin\theta\, A\right)_\theta}{\sin\theta} + \frac{\left(u_\theta - A\right)_\varphi}{\sin\theta}\right] = 0 \tag{13}$$

It is noted that, if the flow is irrotational, then $S_\beta \equiv 0$, $A \equiv 0$, and equation (11) is identically satisfied, and the right-hand side of equation (12) is also 0. In such a case, one need merely find u satisfying the left-hand side of equation (12) set equal to 0 and, of course, obeying the boundary conditions. It is worth noting that u may, in this case, be thought of as a velocity potential (actually, the velocity potential would be $ru(\theta,\varphi)$).

To solve a conical-flow problem equations (11) and (12) or (13) need to be solved, subject to the appropriate boundary conditions. There are two main difficulties which will arise. One comes from the boundary conditions from a shock surface, the location of which is initially unknown. The other difficulty is caused by the fact that, in general, the differential equations form a set which in every case has elliptic regions and in many cases will also have hyperbolic ones. The condition for the system to be elliptic is that

$$c^2 > v^2 + w^2 \tag{14}$$

One may note that in the case of small disturbances, such as linearized flow, this condition corresponds to whether the point in question is inside or outside the free-stream Mach cone extending downstream from the vertex. In configurations such as a body of revolution at angle of attack, the domain inside the shock will be entirely elliptic, while in the case, for example, of a wing with supersonic leading edges, as will be discussed in more detail later, there will be both elliptic and hyperbolic regions. Any part of a hyperbolic region which is completely independent of the elliptic region may be conveniently calculated by the method of characteristics, while any other region, either elliptic or hyperbolic, will be found in this report by a relaxation process.

To set up the method of characteristics for the systems, it is noted that there are two second-order quasi-linear partial differential equations, equations (11) and (13), for β and u, where their second derivatives appear only as the first derivatives of α, $\left(\dfrac{\dfrac{u_\varphi}{\sin\theta} - \alpha u_\theta}{\sqrt{1 + \alpha^2}}\right)$

Note that these two quantities are

$$\left.\begin{aligned} \alpha &= -v/w \\ T &\equiv \frac{\dfrac{u_\varphi}{\sin\theta} - \alpha u_\theta}{\sqrt{1 + \alpha^2}} \\ &= \sqrt{v^2 + w^2} \end{aligned}\right\} \tag{15}$$

Then equations (11) and (13) may be rewritten as

$$\sqrt{1 + \alpha^2}\left(T_\theta + \frac{\alpha T_\varphi}{\sin\theta}\right) + w\left(-\alpha\alpha_\theta + \frac{\alpha_\varphi}{\sin\theta}\right) =$$

$$- \frac{\left(1 + \alpha^2\right)}{w}\left(uA + c^2 S_\beta \beta_\theta + \frac{w^2}{c^2}\cot\theta\right) \tag{16}$$

and

$$\frac{\sqrt{1 + \alpha^2}}{\alpha}\left(\frac{v^2}{c^2} - \frac{\alpha^2}{1 + \alpha^2}\right)\left(T_\theta - \frac{T_\varphi}{\alpha\sin\theta}\right) -$$

$$\frac{w}{1 + \alpha^2}\left(\alpha_\theta + \frac{\alpha\alpha_\varphi}{\sin\theta}\right) = -u\left(2 - \frac{v^2}{c^2} - \frac{w^2}{c^2}\right) - v\cot\theta \tag{17}$$

The auxiliary equations required are

$$\left.\begin{aligned} dT &= T_\theta\, d\theta + T_\varphi\, d\varphi \\ d\alpha &= \alpha_\theta\, d\theta + \alpha_\varphi\, d\varphi \end{aligned}\right\} \tag{18}$$

The characteristics of this system are given by

$$\sin\theta\, d\varphi = \frac{d\theta}{1 - \frac{v^2}{c^2}}\left(-\frac{vw}{c^2} \pm \sqrt{\frac{v^2}{c^2} + \frac{w^2}{c^2} - 1}\right) \tag{19}$$

Also, the streamlines are given by

$$d\theta + \alpha\sin\theta\, d\varphi = 0 \tag{20}$$

From equations (16) through (18), the condition to be fulfilled by $d\alpha$ and dT on the characteristics is also obtained. This is, for the characteristics defined by equation (19),

$$\pm \sqrt{\frac{T^2}{c^2} - 1}\sqrt{1 + \alpha^2}\, dT + \frac{T\, d\alpha}{\sqrt{1 + \alpha^2}} =$$

$$-d\theta\left(\frac{1 \pm \alpha\sqrt{\frac{T^2}{c^2} - 1}}{1 - \frac{v^2}{c^2}}\right)\left[u\left(\frac{T^2}{c^2} - 2\right) - v \cot \theta \pm\right.$$

$$\left.\frac{\sqrt{\frac{T^2}{c^2} - 1}}{w}\left(uA + c^2 S_\beta \beta_\theta + \frac{w^2}{c^2} \cot \theta\right)\right] \tag{21}$$

The ± signs in equations (19) and (21) coincide. Thus, referring to figure 2, to continue a solution from a curve abc (not a characteristic curve) to a point d, one constructs the straight line segments ad and cd by equation (19) and bd by equation (20). Then equation (21) enables one to find α and T (and hence v and w by equation (15)) at point d. Then u is obtained by

$$du = u_\theta\, d\theta + u_\varphi\, d\varphi$$

or, along the streamline bd, using equations (15) and (20),

$$du = \left(T \sin \theta \sqrt{1 + \alpha^2}\right) d\varphi \tag{22}$$

Finally β_θ is needed. Since bd is a streamline, $\beta_d = \beta_b$. Then, on the characteristic ad,

$$\begin{aligned} d\beta &= \beta_b - \beta_a \\ &= \beta_\theta\, d\theta + \beta_\varphi\, d\varphi \\ &= \beta_\theta(d\theta + \alpha \sin \theta\, d\varphi) \\ &= \frac{1}{2}\left(\beta_{\theta d} + \beta_{\theta a}\right)(d\theta + \alpha \sin \theta\, d\varphi) \end{aligned} \tag{23}$$

which gives β_θ at d and this completes the determination of the conditions at point d. Recall that, of course, S_β is constant along bd.

In general, the construction of the solution in the hyperbolic region by the method of characteristics is complicated by the fact that the boundary from which the solution is begun will be an initially unknown shock surface. Construction of the shock solution may be carried out in a manner similar to that given, for example, in reference 23, for the case of plane or axially symmetric flow.

Referring to figure 3, suppose that the shock is to start at point D. Its initial angle may be found by assuming that near D there is a plane two-dimensional shock. This may be easily calculated from the usual oblique-shock theory (see, e.g., reference 24), where one must first calculate the components of the free-stream flow normal and parallel to the ray passing from the vertex of the cone through the point D. Then the component of the Mach number normal to this ray is used as the initial Mach number in the oblique-shock relations with the angle of flow deflection also measured perpendicular to that ray. One can then determine conditions immediately behind the shock. So long as the shock remains plane, the Cartesian velocity components, in the region influenced only by the shock, will remain constant.

Thus, referring to figure 4(a) suppose that the boundary surface to which the shock is attached is tangent to OAH at its leading edge OA. The shock will be tangent to the plane OAJ at OA. One then seeks to determine the equation of the shock tangent to plane OAJ and the Cartesian velocity components immediately behind the shock, in terms of the free-stream Mach number M, the position of the leading edge μ,ν, and the slope of the body-tangent plane ϵ. Thus,

$$\left.\begin{aligned} M_{N1} &= M\sqrt{1 - \cos^2\mu\ \cos^2\nu} \\ \eta = \angle HKC &= \tan^{-1}\left[\frac{\sin\mu + \tan\nu\ \tan\epsilon}{\cos\mu\ \cos\nu(\tan\nu - \sin\mu\ \tan\epsilon)}\right] \end{aligned}\right\} \qquad (24)$$

where M_{N1} is the component of the free-stream Mach number normal to the leading edge and η is the angle of flow deflection through the shock. Using M_{N1} and η one may obtain from oblique-shock tables (e.g., reference 24) the shock angle θ_w (angle JKC in fig. 4(a)) and the Mach number M_{N2} behind the shock, normal to it. Then the velocity component normal to the shock, behind it, is

$$\left.\begin{aligned} S_{N2} &= \sqrt{\left(1 - S_T^2\right)\frac{\frac{\gamma - 1}{2} M_{N2}^2}{1 + \frac{\gamma - 1}{2} M_{N2}^2}} \\ \text{where} \\ S_T &= \cos\mu \cos\nu \sqrt{\frac{\frac{\gamma - 1}{2} M^2}{1 + \frac{\gamma - 1}{2} M^2}} \end{aligned}\right\} \tag{25}$$

is the tangential component.

Then the Cartesian velocity components v_x, v_y, and v_z behind the shock are

$$\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = S_T \begin{Bmatrix} \cos\mu \cos\nu \\ \sin\nu \\ \cos\nu \cdot \sin\mu \end{Bmatrix} +$$

$$\frac{S_{N2} \cos\eta}{\sqrt{1 - \cos^2\mu \cos^2\nu}} \begin{Bmatrix} 1 - \cos^2\mu \cos^2\nu \\ -\cos\nu(\sin\mu \tan\eta + \cos\mu \sin\nu) \\ \sin\nu \tan\eta - \sin\mu \cos\mu \cos^2\nu \end{Bmatrix} \tag{26}$$

The equation of the plane portion of the shock is, in spherical coordinates,

$$\tan\theta = \frac{\tan\theta_w\left(\frac{1 - \cos^2\mu \cos^2\nu}{\cos\mu \cos\nu}\right)}{\left(\tan\theta_w \sin\mu \cos\nu + \frac{\tan\nu}{\cos\mu}\right)\cos\varphi - \left(\tan\theta_w \sin\nu - \tan\mu\right)\sin\varphi} \tag{27}$$

Now consider again figure 3. Suppose that the shock is known up to a point A and the conditions at a point B, lying on a characteristic from A, are also known. One then seeks the conditions at point C on the shock. The position of C may be found from the intersection of the characteristic from B, found from equation (19), and the line segment AC, having the slope $\tan \delta_A$. Then, the conditions at C may be determined from the differential expression (21) along BC and the conditions at the shock. Let subscripts 1 and 2 refer to conditions upstream (to the left of the shock in fig. 3) and downstream of the shock, respectively. Then, since the components of velocity tangent to the shock are continuous across it, there hold

$$\left.\begin{aligned} u_2 &= u_1 \\ \left(v_2 \sin \delta - w_2 \cos \delta\right) &= \left(v_1 \sin \delta - w_1 \cos \delta\right) \end{aligned}\right\} \qquad (28)$$

and, for the jump in normal velocity across the shock (recall that u, v, and w are the ratios of physical velocity to maximum velocity), there holds

$$\left(v_1 \cos \delta + w_1 \sin \delta\right)\left(v_2 \cos \delta + w_2 \sin \delta\right) = \frac{\gamma - 1}{\gamma + 1}\left[1 - u_1^2 - \left(v_1 \sin \delta - w_1 \cos \delta\right)^2\right] \qquad (29)$$

But, because u_1, v_1, and w_1 are simply the components of the free stream which lies parallel to the $\theta = 0$ axis, one obtains,

$$\left.\begin{aligned} u_1 &= \bar{u} \cos \theta \\ v_1 &= -\bar{u} \sin \theta \\ w_1 &= 0 \end{aligned}\right\} \qquad (30)$$

where

$$\bar{u} = \sqrt{\frac{\frac{\gamma - 1}{2} M^2}{1 + \frac{\gamma - 1}{2} M^2}}$$

Then, the solution of equations (28) to (30) yields

$$u_2 = \overline{u} \cos \theta \tag{31}$$

$$\begin{aligned} \alpha_2 &\equiv -v_2/w_2 \\ &= -\cot \delta\left(\frac{Q + \sin^2\delta}{Q - \cos^2\delta}\right) \end{aligned} \tag{32}$$

$$\begin{aligned} T^2 &\equiv {v_2}^2 + {w_2}^2 \\ &= \overline{u}^2 \sin^2\theta\left(\frac{Q^2}{\cos^2\delta} + \sin^2\delta\right) \end{aligned} \tag{33a}$$

where

$$Q = \frac{\gamma - 1}{\gamma + 1}\left(\frac{1 + \frac{\gamma - 1}{2} M^2 \sin^2\theta \cos^2\delta}{\frac{\gamma - 1}{2} M^2 \sin^2\theta}\right) \tag{33b}$$

hence, one may consider that, from equations (32) and (33),

$$T_C = T(\alpha_C, \theta_C, M) \tag{34}$$

or expanding in a Taylor's series along the characteristic BC (fig. 3),

$$T_C = T_B + \left(\frac{\partial T}{\partial \alpha}\right)_B(\alpha_C - \alpha_B) + O(\Delta\alpha^2) \tag{35}$$

Then, using equations (21) and (35), there is obtained

$$d\alpha\left(\pm \frac{\partial T}{\partial \alpha}\sqrt{\frac{T^2}{c^2} - 1}\sqrt{1 + \alpha^2} + \frac{T}{\sqrt{1 + \alpha^2}}\right)_B =$$

$$- d\theta\left(\frac{1 \pm \alpha\sqrt{\frac{T^2}{c^2} - 1}}{1 - v^2/c^2}\right)\left[-u\left(2 - \frac{T^2}{c^2}\right) - v \cot \theta \pm\right.$$

$$\left.\frac{\sqrt{\frac{T^2}{c^2} - 1}}{w}\left(uA + c^2 S_{\beta}\beta_{\theta} + \frac{w^2}{c^2} \cot \theta\right)\right] \tag{36}$$

Hence one finds α_C from equation (36), then T_C from equation (35), and hence v and w. The quantity δ_C follows from equation (32), u_C follows from equation (31), and the remaining quantities are obtained as in the general case already discussed. This procedure will enable one to calculate the shape of the shock and the physical quantities on it. However, the reader is cautioned to note that in most cases the shock will be in nearly the same direction as the characteristics immediately downstream of it. Hence the point B should be taken close to point A.

In some cases, such as the low-pressure side of a lifting wing, there will be no shock, but rather an expansion of the flow. In such a case the solution is simplified because the flow is irrotational and the velocities vary continuously. The solution may be started by a Prandtl-Meyer expansion in a manner analogous to that used where there was a shock. As in the shock case, the normal component of the Mach number M_{N1} and the angle of flow deflection η are found by equations (24). Then, referring to figure 4(b), angle CKJ_1 is the Mach angle corresponding to M_{N1}. Hence,

$$\angle CKJ_1 = \sin^{-1}\left(1/M_{N1}\right) \tag{37}$$

This defines the plane of the start of the expansion fan $\left(OKAJ_1\right)$. Then, using M_{N1} as the initial Mach number and η as the deflection angle, one can immediately obtain from tables of Prandtl-Meyer flows (see reference 24) the normal Mach number M_{N2} behind the flow. Then S_T and S_{N2} are found by equations (25) and the Cartesian velocity components, by equation (26). Then angle $\eta + CKJ_2$ is the Mach angle corresponding to M_{N2}. Hence

$$\angle CKJ_2 = \sin^{-1}\left(1/M_{N2}\right) - \eta \tag{38}$$

The equations of the planes of the start $\left(OKAJ_1\right)$ and end $\left(OKAJ_2\right)$ of the fan of the expansion are then found by using equation (27), where, instead of θ_w, one uses angle CKJ_1 (equation 37) and angle CKJ_2 (equation (38)) for the beginning and end, respectively. To find the velocities inside the fan, simply take a series of deflection angles between 0 and η and calculate, as already described, the corresponding v_x, v_y, and v_z, and the angle, corresponding to angle CKJ_2, of the plane where those velocities apply.

Since the end of the expansion fan is not a characteristic of the flow, the solution may be continued from it in the manner already described.

The remaining kind of point which can arise is at the surface of the body in the flow (in fig. 3, point E). Here there is specified the equation of the body, $\theta_B = \theta_B(\varphi)$. Hence

$$\alpha_E = \left(\frac{d\theta_B}{d\varphi}\right)_E \tag{39}$$

and therefore, in equation (21), $d\alpha$ is known and dT may be solved for immediately. This completes the solution of a purely hyperbolic region.

If the flow in a given domain is such that the system (equations (11) and (12)) of differential equations defining the flow is elliptic, the solution must be found in some other manner. Also, if a hyperbolic region is bounded by an unknown elliptic region, that hyperbolic region will interact with the elliptic one so that both must be solved concurrently. In either case, the relaxation procedure (see, e.g., references 19 and 20) appears to be the best for this portion. In most conical-flow problems, the shock will be fairly weak, at least as compared with a normal shock, because it must remain attached to the body in order to have conical flow. Furthermore, even if it is of appreciable strength, that strength will probably vary fairly slowly about the body. Hence, except at singular points (see reference 18), it appears that the entropy gradients will be small and, to a first approximation, may be ignored. In such a case, one need solve only equation (12) with the right-hand side set equal to 0. The effect of rotation may then be found by an iteration process. For this, u, u_φ, α, S_β, and β_θ in equation (11) are as found for the irrotational case and A is then determined. These values determine then the right-hand side of equation (12) and the left-hand side may then be relaxed again. The process may be repeated if necessary.

To apply the relaxation procedure to the system, one must first write the differential equation in finite difference form. Referring to figure 5, at point E there holds approximately,

$$\left.\begin{aligned}
(u_\theta)_E &= \frac{u_B - u_H}{2\Delta\theta} \\
\left(\frac{u_\varphi}{\sin\theta}\right)_E &= \frac{u_D - u_F}{2\Delta\varphi \sin\theta_E} \\
(u_{\theta\theta})_E &= \frac{u_B - 2u_E + u_H}{(\Delta\theta)^2} \\
\left(\frac{u_{\varphi\theta}}{\sin\theta}\right)_E &= \frac{u_A - u_G + u_J - u_C}{4\Delta\theta\,\Delta\varphi \sin\theta_E} \\
\left(\frac{u_{\varphi\varphi}}{\sin^2\theta}\right)_E &= \frac{u_D - 2u_E + u_F}{(\Delta\varphi \sin\theta_E)^2}
\end{aligned}\right\} \tag{40}$$

However, in solving the (irrotational) problem, it appears easiest for computing purposes to relax u by using $u_{\theta\theta}$ and $u_{\varphi\varphi}$ in finite difference form and to correct periodically the other terms. Thus one gets, for point E

$$\left(u_B - 2u_E + u_H\right)\left(\frac{1 - \frac{u_\theta^2}{c^2}}{\Delta\theta^2}\right)_E + \left(u_D - 2u_E + u_F\right)\left(\frac{1 - \frac{u_\varphi^2}{(c \sin \theta)^2}}{(\Delta\varphi \sin \theta)^2}\right)_E = -K \quad (41)$$

where

$$K = u\left(2 - \frac{u_\theta^2}{c^2} - \frac{u_\varphi^2}{c^2 \sin^2\theta}\right) + u_\theta \cot \theta\left(1 + \frac{u_\varphi^2}{c^2 \sin^2\theta}\right) - 2\frac{u_\theta u_\varphi u_{\varphi\theta}}{c^2 \sin^2\theta}$$

Also it is seen that, since the difference equation (41) has variable coefficients, there is no particular advantage in making $\Delta\theta$ and $\Delta\varphi$ everywhere the same. In particular, the range of variation of φ will usually be much greater than that of θ, so that it appears reasonable to take $\Delta\varphi$ greater than $\Delta\theta$.

The boundary conditions in this region may appear in several forms. At a solid boundary, the condition is

$$v = -w \frac{d\theta_B}{d\varphi} \quad (42)$$

where $\theta_B = \theta_B(\varphi)$ is the equation of the boundary. On a line where the region abuts a known hyperbolic region, u will be specified. On a line of symmetry, the normal derivative of u will be 0.

The final kind of boundary condition will appear at a shock. The essential difficulty here is that the location of the shock is initially unknown. However, the following method may be used to specify this condition. First, the position of the shock is estimated to as great accuracy as possible. This may be done in a number of ways. If the body is nearly the shape of another for which the shock is known (as a circular cone), the shock will be close to that known shock. If part of the shock is already known from calculations in an independent hyperbolic region, the remainder may probably be estimated fairly accurately. Finally, the method given by Lighthill in reference 21 may be used to approximate the position. In any case, the more accurately the shock position is known at the start, the easier will be the solution.

Suppose now that the shock has been estimated in some manner. One then uses as the shock boundary condition the (known) values of the radial velocity u corresponding to that shock. Then the flow is calculated roughly and the variation of u_θ, or approximately, v, is obtained. At the shock this velocity must satisfy the known jump condition in the normal velocity. From equations (31) to (33), this is

$$v_2 = -\bar{u} \sin \theta \left(Q + \sin^2\delta\right) \tag{43}$$

Then, assuming δ remains constant, the variation of v_2 with θ may be found (fig. 6). If one then plots also the variation of v from the calculated velocity field, extrapolating if necessary, a new shock position may be found. This relaxation of the shock position can readily be carried out concurrently with the relaxation of the field.

Once this irrotational solution is found, the entropy distribution at the shock is given by

$$\gamma(\gamma - 1)S = \log_e\left[\frac{2\gamma M^2 \sin^2\theta \cos^2\delta - (\gamma - 1)}{\gamma + 1}\right] - \gamma \log_e\left[\frac{(\gamma + 1)M^2 \sin^2\theta \cos^2\delta}{2 + (\gamma - 1)M^2 \sin^2\theta \cos^2\delta}\right] \tag{44}$$

This completes the description of the method of solution of the general conical-flow problem. It should be noted that, although the relaxation procedure is applicable without having any initial idea of the solution, it will be difficult to solve unless a reasonable estimate of the field can be made at the start.

EXAMPLE

Consider a flat-plate, zero-thickness, triangular wing (fig. 7) at a -12° angle of attack, having as the included angle of the leading edges, 90°, and moving in an otherwise uniform stream at a Mach number of 3. Since the leading edges are supersonic $\left(\text{the Mach angle is } \sin^{-1} 1/3 \approx 20^\circ\right)$, the top and bottom surfaces are independent. The lower surface is an expansion surface and has no shock and hence the flow on this side is irrotational. The solution on this side will be discussed first (fig. 8).

From equation (24), $M_{N1} = 2.1667$ and $\eta = 16.731^{\circ}$. Then, from tables of Prandtl-Meyer expansions, $M_{N2} = 2.8902$. Thus

$$\left.\begin{aligned} v_x &= 0.8388 \\ v_y &= -0.0732 \\ v_z &= 0.1783 \end{aligned}\right\} \tag{45}$$

These are the Cartesian velocity components in region CDF. Also, the equation of the start of the expansion BF is

$$-\cot\theta = 2.6614\cos(\varphi - 11.740) + 0.9577\sin(\varphi - 11.740) \tag{46}$$

and the end of the expansion fan CF is given by

$$-\cot\theta = 22.5608\cos(\varphi - 11.740) + 0.9577\sin(\varphi - 11.740) \tag{47}$$

The values for the Cartesian velocity components, given by equations (45), are constant in CDF. This is not the entire hyperbolic region. However, by using the defining equation of the characteristics (equation (19)) and values of the velocity components calculated in the expansion fan, there was calculated that characteristic starting at B, which is the point of tangency of the Mach cone from the wing vertex and the start of the expansion. This is BCD. Although there is a hyperbolic region to the left of BCD, it is dependent on the unknown elliptic one. Hence the remaining domain ABCDEA must be solved by relaxation. The boundary conditions are:

$$\left.\begin{aligned} &\text{ABCD:}\quad u \ \text{specified} \\ &\text{DE:}\quad u_\theta = -\frac{u_\varphi}{\sin\theta}\sqrt{\left(\frac{\sin\theta}{\sin\mu}\right)^2 - 1} \\ &\text{EA:}\quad u_\varphi = 0 \end{aligned}\right\} \tag{48}$$

To get an initial estimate of the solution, the linearized solution of the problem was found. The perturbation component of the radial velocity δu is given (in the elliptic region, that is, inside the Mach cone from the wing vertex) by

$$\delta u = -\frac{\bar{u}\alpha \cos\theta}{\pi} \times$$

$$\left\{ \begin{aligned} & \frac{-\pi}{\sqrt{M^2 - 2}} + \frac{1 - \tan\theta \sin\varphi}{\sqrt{M^2 - 2}} \sin^{-1}\left[\frac{1 - (M^2 - 1)\tan\theta \sin\varphi}{\sqrt{M^2 - 1}\sqrt{(1 - \tan\theta \sin\varphi)^2 + \tan^2\theta \cos^2\varphi(2 - M^2)}}\right] \\ & + \frac{1 + \tan\theta \sin\varphi}{\sqrt{M^2 - 2}} \sin^{-1}\left[\frac{1 + (M^2 - 1)\tan\theta \sin\varphi}{\sqrt{M^2 - 1}\sqrt{(1 + \tan\theta \sin\varphi)^2 + \tan^2\theta \cos^2\varphi(2 - M^2)}}\right] \\ & + \tan\theta \cos\varphi \left(\tan^{-1}\left\{\frac{2 \tan\theta \cos\varphi\sqrt{1 - (M^2 - 1)\tan^2\theta}}{-1 + \tan^2\theta\left[1 + (M^2 - 1)\cos^2\varphi\right]}\right\} - \pi\right) \end{aligned} \right.$$

Then, the linearized solution is

$$u = \bar{u} \cos\theta + \delta u \tag{49}$$

The initial guesses for the values of u were the values obtained from equation (49), modified in an arbitrary manner to fit the known boundary values and the actual shape of the elliptic (and dependent hyperbolic) region.

The solution was then completed by a relaxation process. Figure 8 shows the configuration together with the various regions encountered. In figure 9, the surface pressure coefficient, given by

$$C_p = \frac{\Delta P}{q} = \frac{2}{\gamma M^2}\left\{\left[\frac{1 - (u^2 + v^2 + w^2)}{1 - \bar{u}^2}\right]^{\gamma/\gamma-1} - 1\right\} \tag{50}$$

is shown. For comparison purposes, the linearized pressure coefficient is also given. Because the ratio of angle of attack to Mach angle is high (0.62), the linearized theory can hardly be expected to yield accurate results. This completes the solution on the expansion side of the wing.

On the compression side, the solution in region FGH (fig. 8) was found first. As before, $M_{N1} = 2.1667$ and $\eta = 16.731^{\circ}$. Then, from the shock tables $M_{N2} = 1.524$. Thus

$$\left.\begin{aligned} v_x &= 0.7076 \\ v_y &= 0.0607 \\ v_z &= 0.1504 \end{aligned}\right\} \tag{51}$$

The equation of the first part of the shock (FG) is then

$$-\cot \theta = -1.4318 \cos(\varphi - 11.740) + 0.9577 \sin(\varphi - 11.740) \tag{52}$$

As in the expansion case, this analytic solution does not cover the entire hyperbolic region, but only that part limited by the characteristic GH, starting from the point immediately downstream of the shock where $v^2 + w^2 = c^2$. As before, there is a small hyperbolic region which is dependent on the elliptic one. The remaining domain GHDEKG must be solved by relaxation. First, however, the shock position is needed. The Mach angle is 19.5° and the shock angle corresponding to a plane wedge at a 12° angle of attack and a Mach number of 3 is 29.5°. Hence the shock angle at the middle was chosen arbitrarily as 27.5° and a smooth curve fitted in. Then, for the domain, these boundary conditions hold

$$\left.\begin{aligned} &\text{KGH:} \quad u \ \text{specified} \\ &\text{HDE:} \quad u_\theta = -\frac{u_\varphi}{\sin \theta}\sqrt{\left(\frac{\sin \theta}{\sin \mu}\right)^2 - 1} \\ &\text{EK:} \quad u_\varphi = 0 \end{aligned}\right\} \tag{53}$$

An initial estimate of the solution was made in the same manner as in the previous case. Then, after the relaxation had been carried out so that the residuals were fairly small, the shock position was recalculated in the manner already described. This procedure was continued and the solution obtained. No particular difficulties were encountered and, in this case at least, the shock position did not have to be changed a second time.

The entropy distribution along the shock was found to vary fairly slowly and hence no attempt was made to correct the solution for this variation.

The pressure distribution on this surface is shown in figure 9 and compared with the result predicted by linearized theory.

In table I, the velocity field for the wing is presented. In the regions where the flow is plane, the Cartesian velocity components are given and, in the remaining regions, the radial component is given.

DISCUSSION

The method described in this report for the solution of conical flows appears to be applicable to any such flow. With one possible exception, the relaxation procedure should converge fairly readily. That exception might arise in the case where a large hyperbolic region where the flow properties varied rapidly is embedded in an elliptic region. An example of such a case is the tip region of a lifting wing with subsonic leading edges. In such a case there will be a very large upwash velocity at the tip (the linearized theory predicts infinite upwash).

It also should be remembered that, as was found in the example discussed previously, the solution obtained for an independent hyperbolic region will not be valid up to the parabolic line, but, rather, a portion of the hyperbolic domain will depend on the elliptic one.

Regarding the calculated example, several things are noteworthy. First the pressure on the shock and expansion sides are both considerably higher than the linearized theory indicates. Thus the lift of the wing is larger than that predicted by the linear theory. However, one would expect that there will actually be separation on the expansion side of the airfoil and that therefore some of this added lift will not be realized. A second observation is that the extent of the constant-pressure region has been drastically changed.

CONCLUSIONS

A numerical method has been described for the calculation of the supersonic flow about cones for the case where the shock, if any, is attached to the cone so that the flow is conical. The procedure employs a combination of the method of characteristics and the relaxation method. Also, an iterative process for obtaining the position of the shock is described. The method can be applied to any conical flow. The main problem in practical application is the convergence of the relaxation solution. No difficulty on this score should be anticipated unless there is a large embedded hyperbolic region where the flow velocities are expected to vary rapidly.

Brown University
Providence, R. I., August 23, 1951

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TABLE I

VELOCITY FIELD IN DISTURBED REGION[1]

(a) In region FGH where $v_x = 0.7076$, $v_y = 0.0607$, and $v_z = 0.1504$.

θ \ φ	0	-10	-20	-30	-40	-50	-60	-70
15	0.7438	0.7433	0.7418	0.7396	------	------	------	------
18.5	.7393	.7388	.7371	.7344	0.7303	0.7250	------	------
22	.7318	.7313	.7296	.7265	.7219	.7159	------	------
25.5	.7211	.7208	.7189	.7158	.7109	.7043	0.6958	------
29	------	------	------	------	.6975	.6905	.6813	------
32.5	------	------	------	------	------	.6752	.6654	0.6545

(b) In region CDF where $v_x = 0.8388$, $v_y = -0.0732$, and $v_z = 0.1783$.

θ \ φ	0	-30	-60	-90	-120	-150	-180
0	0.8278	0.8278	0.8278	0.8278	0.8278	0.8278	0.8278
5	.8365	.8346	.8299	.8242	.8189	.8155	.8144
10	.8405	.8352	.8252	.8120	.8029	.7985	.7973
15	------	------	.8141	.7912	.7809	.7778	.7774
20	------	------	------	------	.7536	.7554	.7554

(c) On characteristic BC.[2]

φ	v_x	v_y	v_z
-90	0.8378	-0.0682	0.1607
-105	.8304	-.0490	.1000
-120	.8178	-.0255	.0470
-135	.8059	-.0055	.0100
-148	.8018	0	0

[1]All velocities are radial unless otherwise specified.

[2]Values given are constant on rays extending through wing tip.

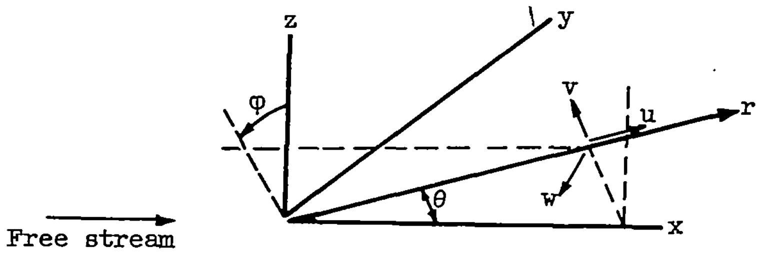


Figure 1.- Coordinate systems and velocity components used in general analysis of supersonic conical flow.

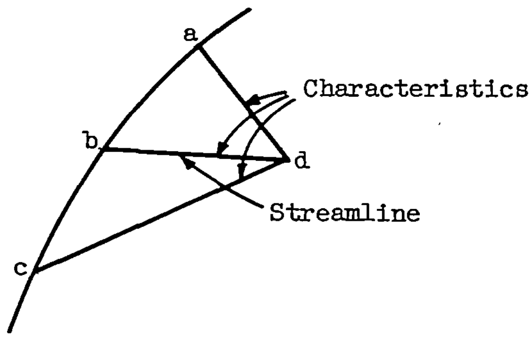


Figure 2.- Construction of solution for conditions at point in hyperbolic region by method of characteristics.

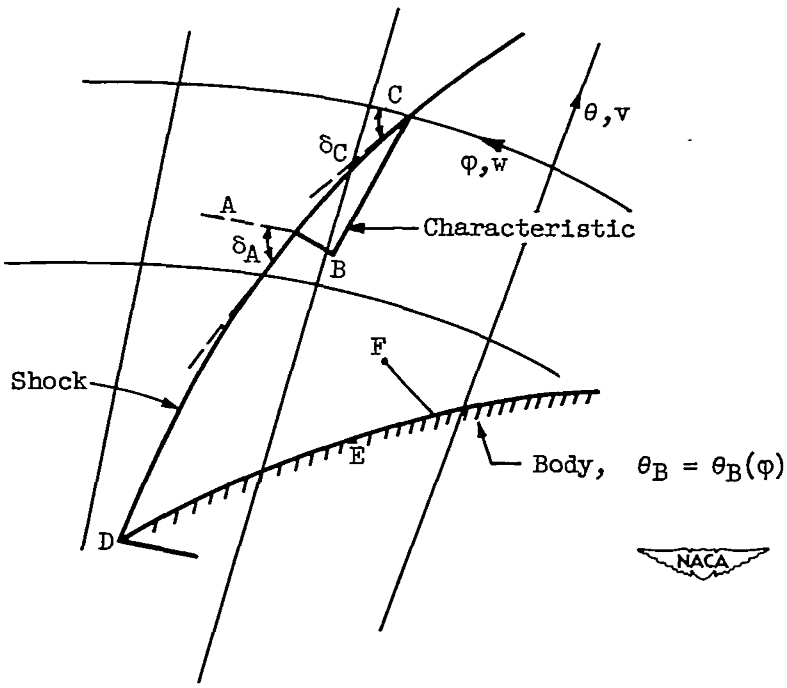
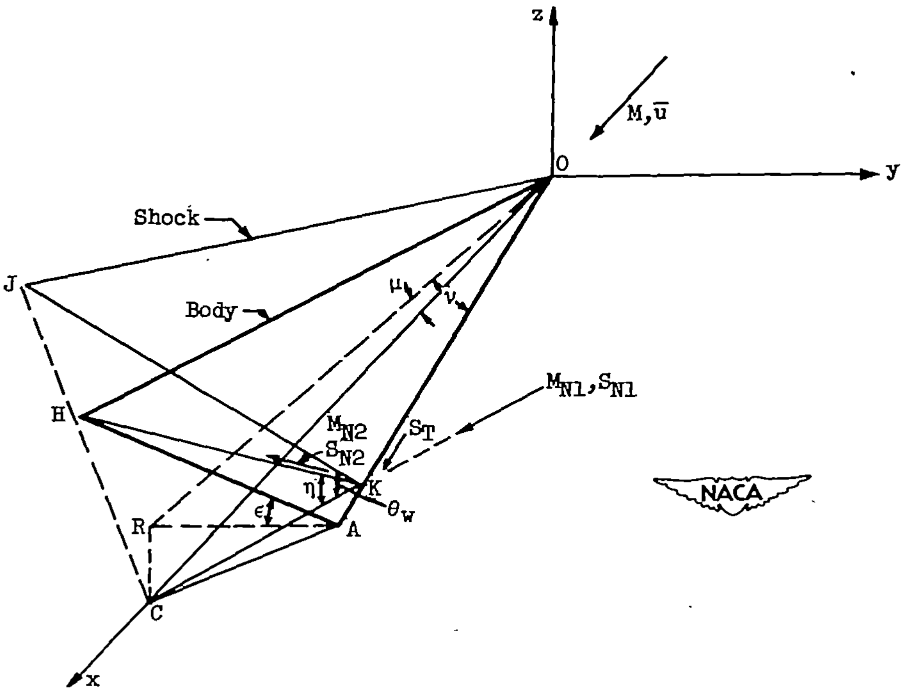
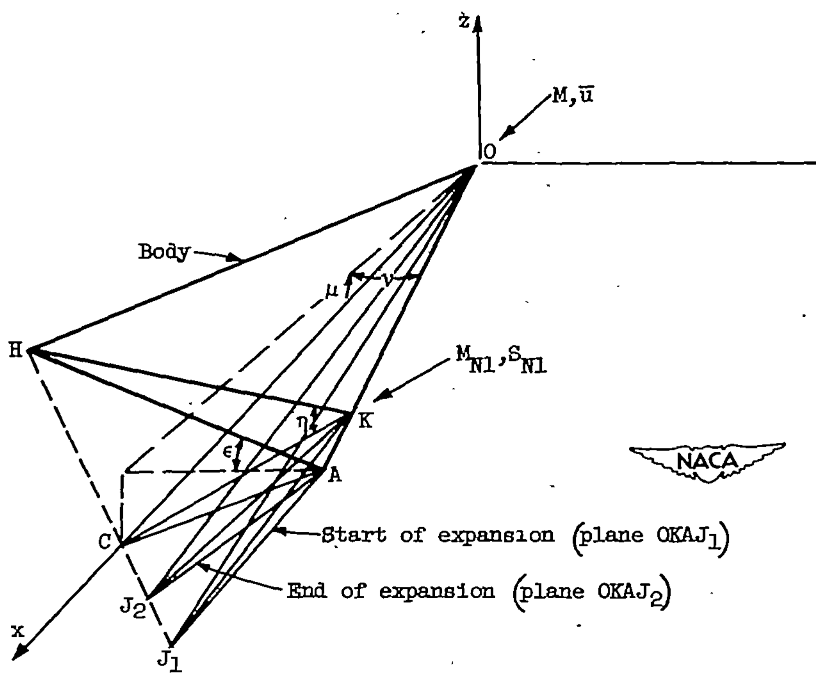


Figure 3.- Construction of solution for plane two-dimensional shock in hyperbolic region by method of characteristics.



(a) For shock attached to boundary surface tangent to body surface. Plane JACH is orthogonal to OC; plane JKCH is orthogonal to OK.



(b) For expansion of flow (no shock). Plane HAJ_1J_2C is orthogonal to OC; plane HKJ_1J_2C is orthogonal to OK.

Figure 4.- Construction of solutions by method of characteristics for shock attached to body-tangent boundary surface and for expansion of flow in hyperbolic region.

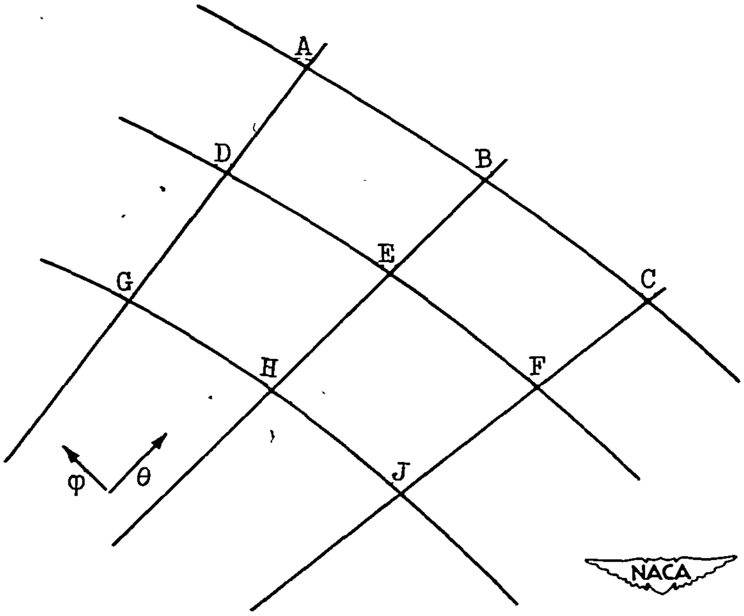


Figure 5.- Construction of solution for elliptic region by relaxation process.

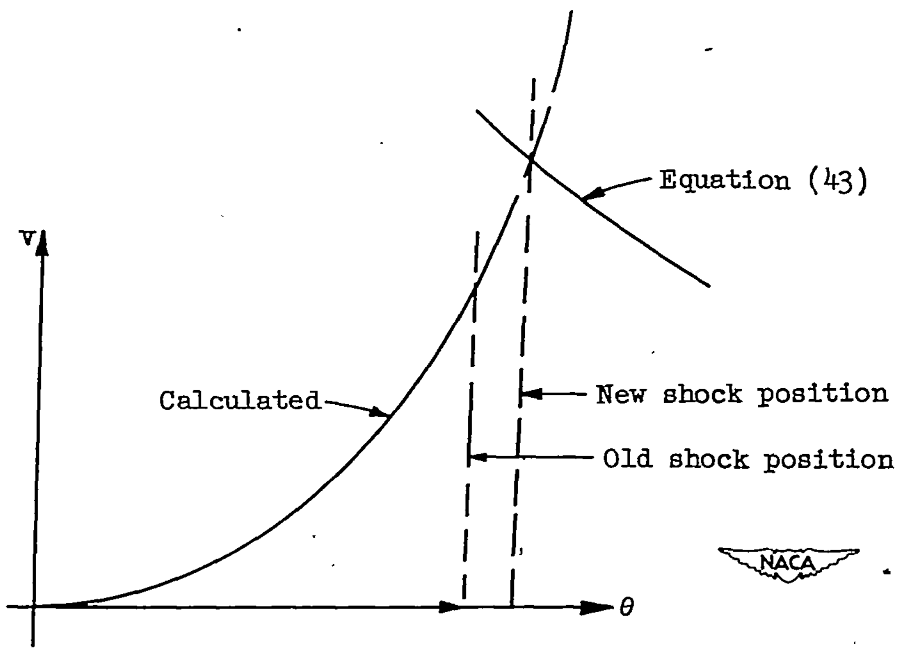


Figure 6.- Variation of v with θ when δ is assumed constant.

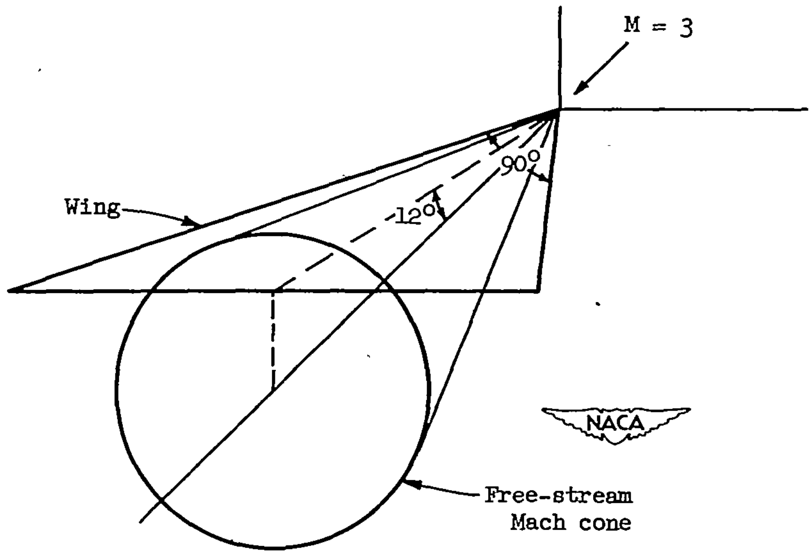


Figure 7.- Flat-plate, zero-thickness, triangular wing with supersonic leading edges in uniform stream at Mach number of 3.

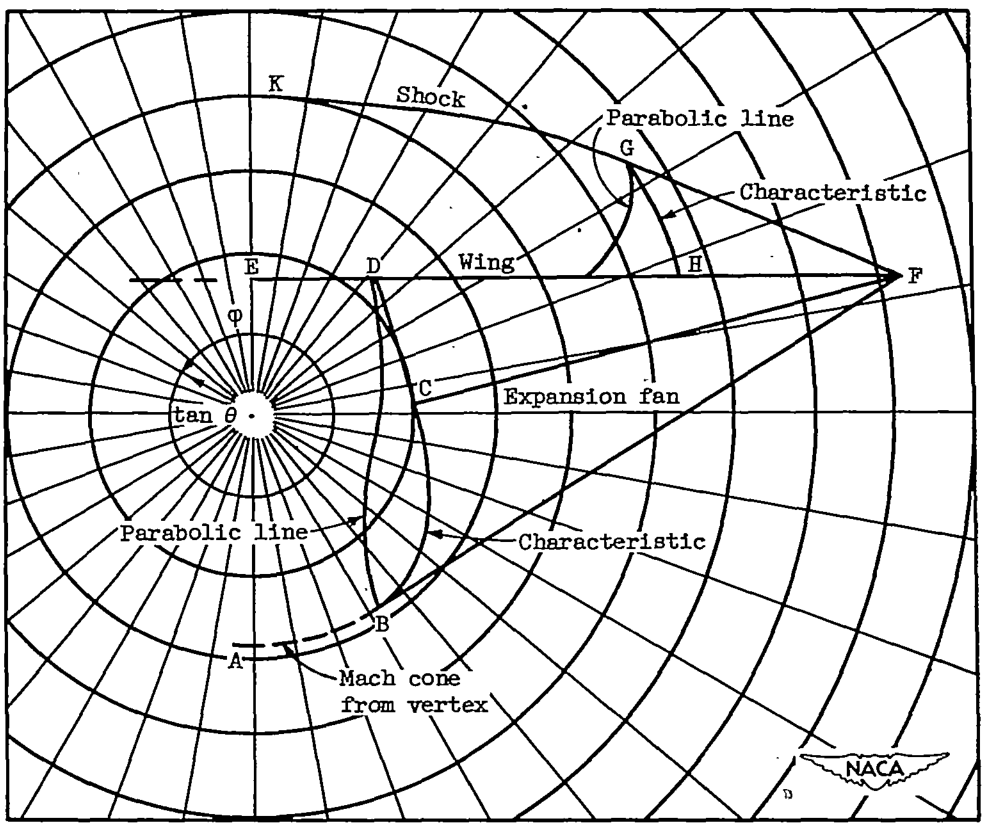


Figure 8.- Construction of solution for flow about triangular wing with supersonic leading edges.

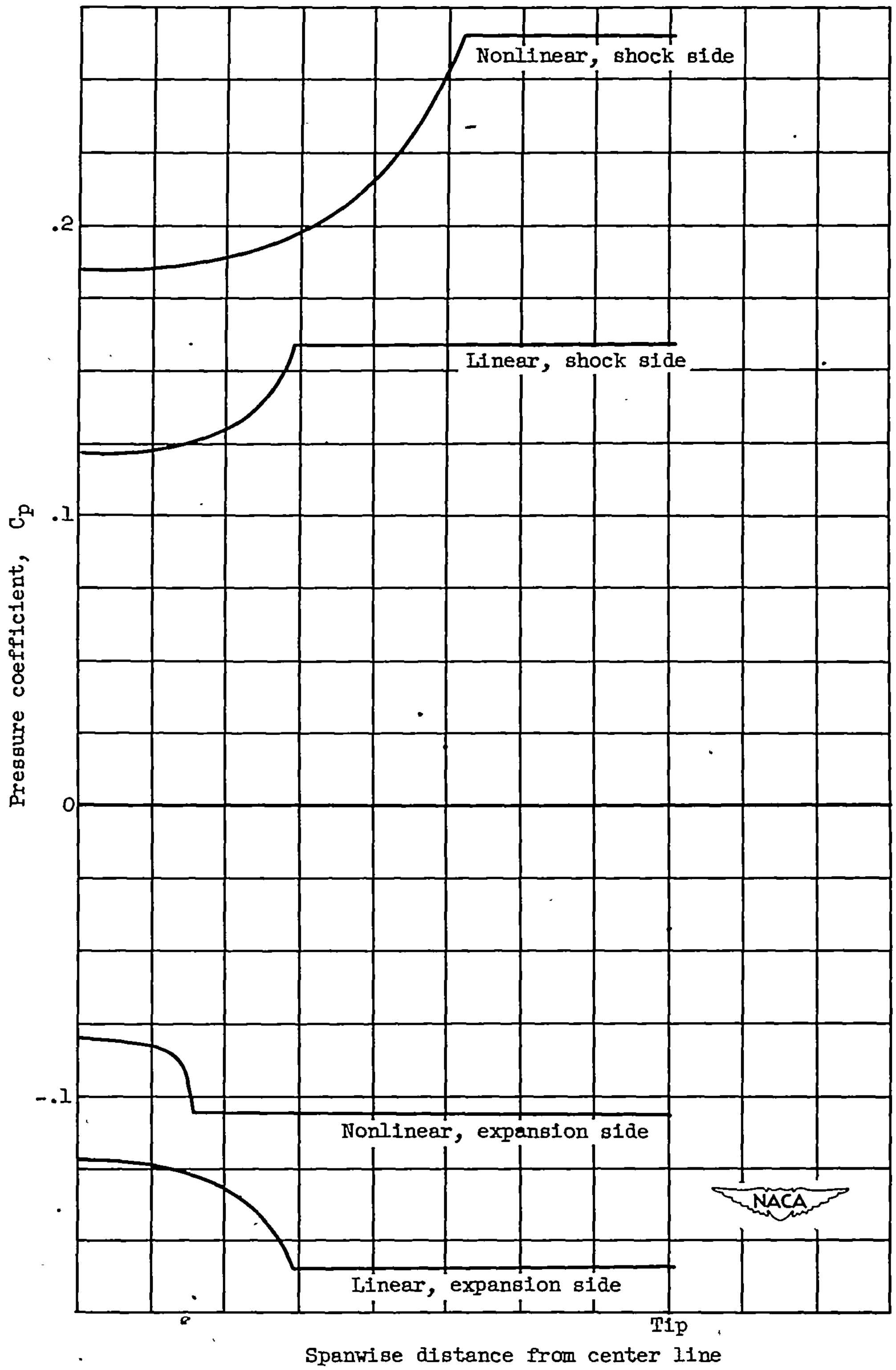


Figure 9.- Surface pressure coefficient.

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