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SECONDARY STRESSES IN THIN-WALLED BEAMS WITH CLOSED CROSS SECTIONS

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SUMMARY

An accurate method of determining secondary stresses in thin-walled, uniform beams of closed cross section is herein presented. The cross sections are assumed to be preserved by closely spaced rigid diaphragms. In section I the integrodifferential equation governing axial displacements is formulated and solved for a beam without longitudinal stiffeners. In section II the corresponding summation-difference equation is developed and solved for a beam with stiffeners (flanges and stringers). The cross section, loading distribution, and end conditions are assumed to be arbitrary.

By introducing generalized difference equations the mathematical analysis for the stiffened beam may be performed in a manner exactly analogous to the process used for the unstiffened beam. A separation of variables in the homogeneous equation leads to the natural stress or displacement modes for a cross section. The solution of the nonhomogeneous equation is then expressed as an expansion in terms of the natural stress modes. Particular attention is given to cross sections with single symmetry and double symmetry.

HISTORICAL NOTE

Reference will be made only to papers which give exact solutions for beams having rigid bulkheads with monocoque (unstiffened) or semi-monocoque (stiffened) sections. In most cases the solutions for semi-monocoque sections are those obtained by replacing the true section by an idealized section in which the normal stresses are carried on a finite number of stiffeners and uniformly distributed shear flows are carried on the connecting webs or wall elements.

If an idealized cross section has only three stiffeners, or flanges, the distribution of axial stress must be planar and there are no secondary stresses. The doubly symmetrical rectangular idealized section with four flanges was introduced and analyzed by Ebner in 1933

(reference 1). This cross section has one secondary stress mode which occurs under torsional loading. Ebner considered such modifications of the problem as the effects of web buckling, shear deformation of the ribs, and finite rib spacing. The solution for the four-flange doubly symmetrical section was also determined independently by Grzedzielski in 1934 (reference 2).[1] This author recognized the importance of idealizing the cross section for stress-analysis purposes. A comparison of the solution for the four-flange section in torsion with experimental tests on a box beam was given by Kuhn in 1935 (reference 4).

A four-flange section with curved webs of arbitrary shape was analyzed by Kirste in 1937 (reference 5). This author gave the location of the principal shear axes for any arbitrary cross section. He also gave the formulas for the associated shearing section properties. The torsional section constant was called the central moment of inertia, while the areas resistant to horizontal and vertical shear were called the reduced areas. This author also located the zero-warping axis for a four-flange beam. He showed that warping displacements in a four-flange beam are due entirely to the torque about the zero-warping axis.

The first exact solution for a secondary stress mode in bending was given by Kuhn in 1937 (reference 6). The cross section considered by him was a rectangular, five-flange, open section in which the bottom wall was missing. The section was symmetrical about a vertical axis and had a single central stringer on the top wall. Experimental test results were compared with the theoretical solution.

A general solution for stiffened beams of arbitrary cross section was given by Ebner and Köller in 1937 (reference 7). The beam was assumed to have rigid or elastic stiffening rings at finite spacing and to carry bending or torsional loads. The secondary self-equilibrating stress distributions were chosen in an arbitrary manner. Energy principles were used to determine the spanwise variation of the secondary stress distributions. The underlying viewpoint of the analysis consisted of regarding the structure as a space framework. The calculation effort required by the method is so extensive as to make the method impracticable except for a beam having a small number of stiffeners and stiffening rings. The specific cross sections considered in detail by Ebner and Köller were elliptical sections with four stiffeners and with six stiffeners having double symmetry. They also considered a circular section with twelve stiffeners and double symmetry. These authors did not introduce the principal shear axes or the associated section

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[1]The formulas of reference 2 are derived in a different manner in reference 3 where a corrected formula is given for the rate of twist at the root of a cantilever beam.

properties. A comparison was given between theory and experiment for a beam of circular cross section.

Following the method of analysis that had been given by Kirste, a more comprehensive treatment of the four-flange beam of arbitrary cross section was given by Drymael in 1941 (reference 8). This author gave many new formulas and equations for the three-flange section and the four-flange section. He gave a careful, detailed study of the geometric properties of the shear-carrying area of idealized stiffened sections. He showed that the "reduced areas" are a maximum and a minimum with respect to the principal shear axes. He also showed that the "central moment of inertia" is a minimum when computed with respect to the principal shear axes. Formulas applicable to arbitrary cross sections were given for lateral deflections due to shearing strains in thin-walled beams. Strangely enough, these formulas, even today, are not well-known even though the deflections due to shearing strains are of appreciable importance in aeroelastic and dynamic analysis of airplanes.

The first exact analysis for a beam of monocoque, or unstiffened, cross section was given by Von Kármán and Chien in 1946 (reference 9). Their solution is limited in its applicability to sections having double symmetry. They considered only the torsion case and assumed the beam to have closely spaced rigid diaphragms. The integrodifferential equations governing the axial displacements were formulated and solved for a semi-infinite cantilever beam acted upon by a torque at the tip. Stress distributions were obtained for a rectangular section and a rectanglelike section with rounded corners.

A very general and comprehensive exact analysis of beams with rigid diaphragms was given by Hadji-Argyris and Dunne in 1947 (reference 10). The end conditions, loading conditions, and cross section were considered to be arbitrary. Conical as well as uniform beams were analyzed with multicell or single-cell cross sections. Both open and closed cross sections were considered with and without stiffeners. It is impossible to summarize the contents of the paper here because of its great length. The method of analysis used by these authors is entirely different from that employed by Von Kármán and Chien. The differences between the methods are discussed in the text. These authors found a zero-warping axis to be associated with each secondary stress mode corresponding to the axis found by Kirste for the four-flange section. They illustrated clearly the large difference which may exist in the positions of the zero-warping axis and the shear center for a four-flange beam. A detailed analysis of a four-flange tapered wing with cut-outs was presented.

Specific formulas were given by Kempner in 1947 (reference 11) for a six-flange doubly symmetrical hexagonal section. A five-flange open section, obtained by removing the bottom webs and bottom stringer, was

also considered. The beam was subjected to vertical bending loads only. Theoretical solutions were compared with experimental results which had been previously published by Peterson in 1945 (reference 12).

A rather simple method of analysis was given by Levy in 1947 (reference 13) for beams with arbitrary stiffened cross sections but limited to beams with finite bulkhead spacing. The loading was assumed to consist of concentrated loads applied at the intersections of the ribs and spars. The solution involved the use of equilibrium conditions and an energy principle. The method is applicable to beams with cut-outs and sweepback. However, a comparison of theoretical solutions by Levy's method with experimental results, for a swept beam, was given by Bisplinghoff in 1949 (reference 14) and did not show satisfactory agreement. Deflections rather than stresses were compared. In the theory of Levy and the beams tested by Bisplinghoff the ribs of the swept beam were parallel to the plane of symmetry of the airplane.

The solution for a swept beam with closely spaced diaphragms was given by Wittrick in 1948 (reference 15). The method of analysis used was an extension of the theory of Hadji-Argyris and Dunne (reference 10) to include the effect of sweepback. The general integrodifferential equation for the stress function for a beam of arbitrary cross section was formulated. Specific solutions were given for uniform and exponentially varying section torque and bending moment.

An exact solution for a single-cell monocoque (unstiffened) beam with closely spaced diaphragms was given by Adadurov in 1948 (reference 16). The cross section and loading were assumed to be arbitrary. This author introduced a stress function having its first derivative proportional to the axial displacements and second derivatives proportional to the stresses. The method is actually a minor modification of the method of Von Kármán and Chien (reference 9) but is extended to arbitrary cross sections and loading. The author's method of solution is unwieldy because he failed to introduce the principal shear axes. No specific solutions were obtained.

Two specific solutions for swept beams were given by Thompson and Wittrick in 1949 (reference 17) using the theory that had been previously published by Wittrick (reference 15). The first solution was given for a singly symmetrical trapezoidal section with four flanges and closely spaced diaphragms located in a skewed position with respect to the axis of the beam. The walls were assumed to carry both normal and shearing stresses. The loading consisted of a concentrated moment and a concentrated torque at either end (the St. Venant problem). In the second example the cross section was a doubly symmetrical rectangle and the loading was assumed to produce a section torque varying exponentially.

A general analysis of stiffened beams with arbitrary cross sections was given by Duberg in a thesis submitted to the University of Illinois (Feb. 1948). An abstract of this thesis was published subsequently by Duberg in 1949 (reference 18). The problem treated by this author may be regarded as an extension of the four-flange problem treated by Kirste and Drymael to sections with more than four stiffeners. The method of solution used by Kirste and Drymael cannot be extended directly to more than four flanges and new methods of solution are required. Duberg's method of solution resembles that of Von Kármán and Chien in that the axial displacements were chosen as the unknown to be determined. Principal shear axes play an essential role in this method of analysis and convenient formulas were given for their calculation. The formulas correspond closely to those given by Kirste and Drymael. Numerical methods were employed to obtain two solutions for cantilever beams having rectangular cross sections with single symmetry. Both finite rib spacing and closely spaced ribs were considered.

The method of analysis to be given in this paper may be considered as an alternative to the method of Hadji-Argyris and Dunne. It corresponds to, and employs ideas contained in, the papers by Kirste, Drymael, Von Kármán and Chien, and Duberg.

This work was originally a thesis for the degree of doctor of philosophy at the California Institute of Technology and has been made available to the NACA for publication because of its general interest.

I - BEAMS HAVING THIN-WALLED UNSTIFFENED CROSS SECTIONS

Introduction

In section I the beam will be assumed to have a thin-walled, unstiffened cross section. It will be assumed to consist of a single cell without corners. This is the simplest and most convenient section to consider in developing a rational theory for the determination of secondary stresses in thin-walled beams of arbitrary cross section. The loading conditions and end conditions for the beam may be of any arbitrary nature. The beam is assumed to be of uniform section with no cut-outs.

The wall of the cross section is assumed to be sufficiently thin that one may consider the stresses to be uniformly distributed over the thickness of the wall. The thin shell then acts essentially as a membrane. The shell is assumed to be stabilized by closely spaced rigid diaphragms which preserve the cross-sectional shape of the beam under the action of loads. It will be assumed throughout the analysis that the diaphragms are infinitely stiff in their own planes but

completely free to warp out of their planes. This assumption is the only one which has a significant effect upon the final results and is, thus, the principal source of any error which may be contained in the solution.

Upon replacing the real, physical beam by a similar beam having rigid diaphragms, one obtains a well-defined problem in stress analysis for which exact solutions can be obtained. Such exact solutions of the similar beam may, in most cases, be regarded as "accurate" (but not exact) solutions for the real, physical beam. One may also obtain approximate solutions for the similar beam with rigid diaphragms. Such solutions may be regarded as "approximate" solutions for the real physical beam. Many such approximate solutions have been published for beams having simple cross sections.

Symbols

A	cell area
A_w	area of monocoque cross section
A_H	horizontal shear-resistant area
A_V	vertical shear-resistant area
$C_{T_i}, C_{H_i}, C_{V_i}$	section constants associated with ith stress mode
E	Young's modulus
$E' = \frac{E}{1 - \mu^2}$	
f	transverse distribution of axial displacements or normal stresses
F	stress function
g	spanwise variation of axial displacements
G	shearing modulus of elasticity
h	spanwise variation of normal stress
H	horizontal section shear

I_c	central moment of inertia
I_p	polar moment of inertia
I_x	moment of inertia about $\bar{x}$-axis
I_y	moment of inertia about $\bar{y}$-axis
J	torsion constant
M_x	bending moment about $\bar{x}$-axis
M_y	bending moment about $\bar{y}$-axis
p	normal-stress flow in monocoque sections
p_H	applied horizontal loading
p_V	applied vertical loading
P	axial force in beam
q	shear flow
r	radius to a tangent
s	tangential coordinate
t	applied torsional loading
t_w	wall thickness of monocoque sections
T	section torque
u	horizontal displacement
v	vertical displacement
V	vertical section shear
w	axial displacement
x,y,z	rectangular coordinates
$\bar{x},\bar{y}$	centroidal coordinates
α	angle between tangent and x-axis

γ shearing strain

ϵ normal strain

θ angle between principal bending axes and principal shear axes

λ characteristic number

$\mu = \lambda\sqrt{\frac{G}{E}}$; also Poisson's ratio

φ angle of twist (rotation)

ξ tangential displacement

σ normal stress

τ shearing stress

Fundamental Equations of Elasticity

The beam and coordinate system are shown in figure 1. Boundary conditions on stresses or displacements must be known at each end of the beam. A segment of the curved wall is shown in figure 2(a). It is convenient to introduce the coordinate s which is measured around the periphery of the beam along the center line of the wall. A point in the wall of the beam may be located by giving the values of s and z rather than the values of x, y, and z. The coordinate s and the shear flow q are considered to be positive in the counterclockwise direction as shown in figure 2(a).

The differential element $dsdz$ is shown in figure 2(b) with the forces which act on it in the axial direction. The remaining forces which act on the element are shown in figures 2(c) and 2(d). Assuming that Hooke's law is applicable, the stresses and strains are related by the following well-known equations:

$$\sigma_z - \mu\sigma_s = E\epsilon_z \tag{1}$$

$$\sigma_s - \mu\sigma_z = E\epsilon_s \tag{2}$$

$$\tau_{sz} = G\gamma_{sz} \tag{3}$$

Because of the assumption of infinitely stiff diaphragms, one may immediately set ϵ_s equal to zero, since such diaphragms would prevent any strain in the wall in the tangential direction. Using equation (2) the following equation is obtained:

$$\left.\begin{aligned} \epsilon_s &= 0 \\ \sigma_s &= \mu\sigma_z \end{aligned}\right\} \tag{4}$$

From this equation the tangential normal stress may be computed at any point after the axial normal stress distribution has been determined. Equation (4) may be substituted in equation (1) to obtain σ_z in terms of ϵ_z.

$$\sigma_z = \left(\frac{E}{1 - \mu^2}\right)\epsilon_z = E'\epsilon_z \tag{5a}$$

where

$$E' = \frac{E}{1 - \mu^2} \tag{5b}$$

The strains at a point may be expressed in terms of the axial displacement w and the tangential displacement ξ by the following well-known equations:

$$\epsilon_z = \frac{\partial w}{\partial z} \tag{6a}$$

$$\gamma_{sz} = \frac{\partial w}{\partial z} + \frac{\partial \xi}{\partial z} \tag{6b}$$

The axial displacement is measured as positive in the positive direction of the z-axis while the tangential displacement is positive in the positive direction of the s coordinate, that is, counterclockwise. The stress σ_s and the strain ϵ_s do not appear in the analysis henceforth and the subscripts of the remaining stresses and strains will be dropped with the following change of notation: $\sigma = \sigma_z$, $\epsilon = \epsilon_z$, $\tau = \tau_{sz}$,

elasticity gives the following formulas for stresses in terms of displacements:

$$\sigma = E' \frac{\partial w}{\partial z} \tag{7a}$$

$$\tau = G \frac{\partial w}{\partial s} + G \frac{\partial \xi}{\partial z} \tag{7b}$$

The last fundamental equation which is required for the analysis is that which expresses the law of equilibrium of forces acting on a differential wall element in the axial direction. These forces, as shown in figure 2(b), lead to the following well-known equation:

$$\frac{\partial \sigma}{\partial z} + \frac{\partial \tau}{\partial s} = 0 \tag{8}$$

Formation of Second-Order Equations

Equations (7a), (7b), and (8) must be solved simultaneously. These three equations contain the four unknowns σ, τ, ξ, and w. By combining these equations in various ways, three different second-order differential equations may be obtained. These three equations express relations between the displacement ξ and one of the three variables σ, τ, or w in the following forms:

$$\frac{\partial^2 w}{\partial s^2} + \frac{E'}{G} \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 \xi}{\partial s \, \partial z} = 0 \tag{9a}$$

$$\frac{\partial^2 \sigma}{\partial s^2} + \frac{E'}{G} \frac{\partial^2 \sigma}{\partial z^2} + E' \frac{\partial^3 \xi}{\partial s \, \partial z^2} = 0 \tag{9b}$$

$$\frac{\partial^2 \tau}{\partial s^2} + \frac{E'}{G} \frac{\partial^2 \tau}{\partial z^2} - E' \frac{\partial^3 \xi}{\partial z^3} = 0 \tag{9c}$$

The first of the above equations may be obtained immediately by substituting equations (7a) and (7b) into equation (8). Equation (9b) is most easily derived by differentiating equation (9a) with respect to z and multiplying each term by E'. Substitution of equation (7a) then gives equation (9b). In order to derive equation (9c) one may differentiate equation (7a) with respect to s and equation (7b) with respect to z to obtain

$$\frac{\partial \sigma}{\partial s} = E' \frac{\partial^2 w}{\partial s \, \partial z}$$

or

$$\frac{G}{E'} \frac{\partial \sigma}{\partial s} = G \frac{\partial^2 w}{\partial s \, \partial z}$$

and

$$\frac{\partial \tau}{\partial z} = G \frac{\partial^2 w}{\partial s \, \partial z} + G \frac{\partial^2 \xi}{\partial z^2}$$

Subtracting from the above equation the preceding one gives

$$\frac{\partial \tau}{\partial z} - \frac{G}{E'} \frac{\partial \sigma}{\partial s} = G \frac{\partial^2 \xi}{\partial z^2}$$

Differentiating the above equation with respect to z and equation (8) with respect to s gives

$$\frac{\partial^2 \tau}{\partial z^2} - \frac{G}{E'} \frac{\partial^2 \sigma}{\partial z \, \partial s} = G \frac{\partial^3 \xi}{\partial z^3}$$

and

$$\frac{G}{E'} \frac{\partial^2 \tau}{\partial s^2} + \frac{G}{E'} \frac{\partial^2 \sigma}{\partial s \, \partial z} = 0$$

Adding the above equations and multiplying by E'/G gives equation (9c).

The fact that equation (9b) may be obtained from equation (9a) indicates that there is a close mathematical association between the axial displacement w and the normal stress σ. A stress function F, introduced by Hadji-Argyris and Dunne (reference 10), bears a similar close relationship to the shearing stress. In extending the analysis to semimonocoque, or stiffened, sections it is advantageous to use shear flow as an unknown rather than shearing stress. The shear flow and normal-stress flow are defined by the equations

$$p = t_w \sigma \tag{10a}$$

$$q = t_w \tau \tag{10b}$$

where t_w is the wall thickness and will be considered to be a constant for convenience. The stress flows are related to the stress function by the equations

$$p = \frac{\partial F}{\partial s} \tag{11a}$$

$$q = q_o - \frac{\partial F}{\partial z} \tag{11b}$$

In equation (11b) the term q_o is the shear flow that may be computed from the section torque T with the formula from the torsion theory of St. Venant

$$q_o = \frac{T}{2A} \tag{12}$$

where A is the area of the cell formed by the center line of the wall. If equations (11) are substituted into equation (8), the equation of equilibrium, it will be seen that the equation is satisfied by any stress function.

It is now possible to obtain a second-order equation relating F to the displacement ξ. Multiplying equations (7) by t_w and substituting equations (11) into them gives

$$\frac{\partial F}{\partial s} = E' t_w \frac{\partial w}{\partial z}$$

$$q_o - \frac{\partial F}{\partial z} = G t_w \frac{\partial w}{\partial s} + G t_w \frac{\partial \xi}{\partial z}$$

or

$$\frac{E'}{G} \frac{\partial F}{\partial z} = -E' t_w \frac{\partial w}{\partial s} - E' t_w \frac{\partial \xi}{\partial z} + \frac{E'}{G} q_o$$

Differentiating the first of these equations with respect to s, the last with respect to z, and adding give

$$\frac{\partial^2 F}{\partial s^2} + \frac{E'}{G} \frac{\partial^2 F}{\partial z^2} + E' t_w \frac{\partial^2 \xi}{\partial z^2} = \frac{E'}{G} \frac{\partial q_o}{\partial z} \tag{13}$$

A second-order equation may be obtained for the shear flow by differentiating with respect to z.

$$\frac{\partial^2}{\partial s^2}\left(\frac{\partial F}{\partial z} - q_o\right) + \frac{E'}{G} \frac{\partial^2}{\partial z^2}\left(\frac{\partial F}{\partial z} - q_o\right) + E' t_w \frac{\partial^3 \xi}{\partial z^2} = 0$$

The quantity q_o may be inserted into the first term since it is a function of z only. Substituting equation (11b) gives

$$\frac{\partial^2 q}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 q}{\partial z^2} - E't_w\frac{\partial^3 \xi}{\partial z^3} = 0 \tag{14}$$

It should be noted that this equation is also obtained immediately by multiplying through equation (9c) by t_w.

It is now seen that the solution to the stress problem may be obtained by solving any one of the four equations (9a), (9b), (13), or (14). In each case the first step must be the elimination of the tangential displacement ξ. This elimination from equations (9a) or (9b) is obtained through considerations of equilibrium between internal stresses and external loads. On the other hand, the elimination of ξ from equations (13) and (14) is obtained through consideration of the necessity for continuity of the axial displacements. Consequently the process of solution for equations (9a) and (9b) is entirely different from that required for equations (13) and (14). Thus the two possible methods of solution differ, essentially, in the order in which one must introduce the concepts of equilibrium and continuity into the analysis. The analysis of Hadji-Argyris and Dunne (reference 10) involved the solution of equation (13) while the analysis of Von Kármán and Chien (reference 9), and also that of Duberg (reference 18), is based upon equation (9a). The analysis of this paper will be based on both equations (9a) and (9b). No further consideration will be given to equations (13) and (14).

Principal Shear Axes and Associated Section Properties

In the development of an exact analysis in which axial displacements, or normal stresses, are considered to be the unknowns, it is found to be very convenient to use coordinate axes in the cross section which have been called the principal shear axes. Convenient formulas for computing the location of these axes were given by Duberg (reference 18). At any given point of the wall a tangent to the center line may be drawn as shown in figure 3(a). The radius from the origin of the principal shear axes to the tangent is indicated as r while the angle made by the tangent with the positive direction of the x-axis, measured positively in the counterclockwise direction from the axis, is indicated as α. The location of the principal shear axes is defined in terms of r and α by the condition that the following three integrals must vanish:

$$\oint t_w r \cos\alpha \, ds = 0 \tag{15a}$$

$$\oint t_w r \sin \alpha \, ds = 0 \tag{15b}$$

$$\oint t_w \cos \alpha \sin \alpha \, ds = 0 \tag{15c}$$

The integration is to be carried over the area of the cross section. In the case of a single-cell section such an integration becomes a line integral and hence the symbol for a line integral has been used for convenience. (This integrating symbol would not be appropriate for a multicell section.)

The principal values of three section properties, which are associated with the principal shear axes, may be defined by the following formulas:

$$I_c = \oint t_w r^2 \, ds \tag{16a}$$

$$A_H = \oint t_w \cos^2 \alpha \, ds \tag{16b}$$

$$A_V = \oint t_w \sin^2 \alpha \, ds \tag{16c}$$

The first of these section properties was introduced by Kirste (reference 5) and called the central moment of inertia. The magnitude of this constant lies intermediate between that of the torsion constant J and the polar moment of inertia I_p

$$I_p \geq I_c \geq J$$

The constant A_H may be considered as the area resistant to horizontal shear and A_V may be considered as the area resistant to vertical shear. The sum of the "horizontal" and "vertical" shear areas is seen, from equations (16b) and (16c), to be equal to the total area of the wall of the beam

$$A_H + A_V = A_w$$

where

$$A_w = \oint t_w \, ds$$

The areas A_H and A_V were also introduced by Kirste (reference 5) and called reduced areas.

Elimination of Tangential Displacement

In order to eliminate the tangential displacement ξ from equations (9a) and (9b), it is necessary to make further use of the assumption of rigid bulkheads. Because of this assumption a cross section of the beam experiences "rigid body" displacement under the action of load. The horizontal displacement u and the vertical displacement v are parallel to the principal shear axes and must in general be regarded as the translations of the origin of the principal shear axes. The rotation, or angle of twist, of the cross section is indicated by φ. These displacements, and the corresponding contribution that each makes to the tangential displacement ξ, are illustrated in figure 3. Because of rotation about the origin the tangential displacement is given by the product $r\varphi$. Because of horizontal translation of the cross section the tangential displacement is $u \cos \alpha$. Because of vertical translation of the cross section the tangential displacement is $v \sin \alpha$. These individual effects may be superimposed to obtain the formula

$$\xi = r\varphi + u \cos \alpha + v \sin \alpha \tag{17}$$

If the above equation were used to eliminate ξ, one would obtain a relation between the unknowns w or σ and the displacements φ, u, and v which are also unknown. It is necessary to find a relation between φ, u, and v and the applied loads in order to obtain a direct relation between w or σ and the applied loads. The required relations are obtained by equating the section torque and section shears, as computed by statics from the external loads, to the section torque and section shears as computed by an appropriate integration of the internal shearing stresses. (Secondary stresses in indeterminate beams will not be discussed.) Thus the section torque T, the horizontal shear H, and the vertical shear V are related to the internal shearing stresses by the following equations:

$$\oint t_w \tau r \, ds = T \tag{18a}$$

$$\oint t_w \tau \cos \alpha \, ds = H \tag{18b}$$

$$\oint t_w \tau \sin \alpha \, ds = V \tag{18c}$$

In equations (18) the shearing stress may be eliminated to obtain a relation between the tangential displacement and the loads by substituting equation (7b).

$$G\oint t_w r \frac{\partial w}{\partial s}\,ds + G\oint t_w r \frac{\partial \xi}{\partial z}\,ds = T$$

$$G\oint t_w \cos\alpha \frac{\partial w}{\partial s}\,ds + G\oint t_w \cos\alpha \frac{\partial \xi}{\partial z}\,ds = H$$

$$G\oint t_w \sin\alpha \frac{\partial w}{\partial s}\,ds + G\oint t_w \sin\alpha \frac{\partial \xi}{\partial z}\,ds = V$$

It is now necessary to eliminate the tangential displacement from the above equations by substituting equation (17). In making this substitution it will be found that in each equation two integrals vanish because of the applicability of equations (15) which define the location of the principal shear axes. After substitution of equations (17) and (15), the above equations become

$$G\oint t_w r \frac{\partial w}{\partial s}\,ds + G\frac{d\varphi}{dz}\oint t_w r^2\,ds = T$$

$$G\oint t_w \cos\alpha \frac{\partial w}{\partial s}\,ds + G\frac{du}{dz}\oint t_w \cos^2\alpha\,ds = H$$

$$G\oint t_w \sin\alpha \frac{\partial w}{\partial s}\,ds + G\frac{dv}{dz}\oint t_w \sin^2\alpha\,ds = V$$

In the above equations the second integral in each equation is seen to be one of the section properties which were previously introduced by equations (16). Because of the use of principal shear axes one may solve the above equations for the derivatives of φ, u, and v independently rather than having to solve a simultaneous system. This is the reason for employing the principal shear axes as reference axes. Inserting the section constants and solving for the cross-sectional displacements gives

$$\frac{d\varphi}{dz} = \frac{T}{GI_c} - \frac{1}{I_c}\oint t_w r \frac{\partial w}{\partial s}\,ds \qquad (19a)$$

$$\frac{du}{dz} = \frac{H}{GA_H} - \frac{1}{A_H}\oint t_w \cos\alpha \frac{\partial w}{\partial s}\,ds \tag{19b}$$

$$\frac{dv}{dz} = \frac{V}{GA_V} - \frac{1}{A_V}\oint t_w \sin\alpha \frac{\partial w}{\partial s}\,ds \tag{19c}$$

The above equations may be substituted into equation (17) if it is first differentiated to obtain

$$\frac{\partial \xi}{\partial z} = r\frac{d\varphi}{dz} + \cos\alpha \frac{du}{dz} + \sin\alpha \frac{dv}{dz}$$

Substituting equations (19) into the above equation gives

$$\frac{\partial \xi}{\partial z} = \frac{Tr}{GI_c} + \frac{H\cos\alpha}{GA_H} + \frac{V\sin\alpha}{GA_V} - \frac{r}{I_c}\oint t_w r \frac{\partial w}{\partial s}\,ds -$$

$$\frac{\cos\alpha}{A_H}\oint t_w \cos\alpha \frac{\partial w}{\partial s}\,ds - \frac{\sin\alpha}{A_V}\oint t_w \sin\alpha \frac{\partial w}{\partial s}\,ds$$

The above equation gives ξ as a function of the applied loads and w. Thus it is seen that this equation may be used to eliminate ξ from equation (9a) to obtain an equation relating the axial displacements to the applied loads. Substituting the above equation into equation (9a) gives

$$\frac{\partial^2 w}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w}{\partial z^2} - \frac{1}{I_c}\frac{dr}{ds}\oint t_w r \frac{\partial w}{\partial s}\,ds - \frac{1}{A_H}\frac{d\cos\alpha}{ds}\oint t_w \cos\alpha \frac{\partial w}{\partial s}\,ds -$$

$$\frac{1}{A_V}\frac{d\sin\alpha}{ds}\oint t_w \sin\alpha \frac{\partial w}{\partial s}\,ds = -\frac{T}{GI_c}\frac{dr}{ds} - \frac{H}{GA_H}\frac{d\cos\alpha}{ds} - \frac{V}{GA_V}\frac{d\sin\alpha}{ds} \tag{20a}$$

This partial integrodifferential equation, with the appropriate boundary conditions, defines the relation between the axial displacements and the loads. A corresponding equation for σ may be obtained by differentiating with respect to z and multiplying through by E'.

$$\frac{\partial^2\sigma}{\partial s^2} + \frac{E'}{G}\frac{\partial^2\sigma}{\partial z^2} - \frac{1}{I_c}\frac{dr}{ds}\oint t_w r \frac{\partial\sigma}{\partial s}\,ds - \frac{1}{A_H}\frac{d\cos\alpha}{ds}\oint t_w \cos\alpha \frac{\partial\sigma}{\partial s}\,ds -$$

$$\frac{1}{A_V}\frac{d\sin\alpha}{ds}\oint t_w \sin\alpha \frac{\partial\sigma}{\partial s}\,ds = \frac{E'}{G}\left(\frac{t}{I_c}\frac{dr}{ds} + \frac{p_H}{A_H}\frac{d\cos\alpha}{ds} + \frac{p_V}{A_V}\frac{d\sin\alpha}{ds}\right) \tag{20b}$$

On the right-hand side of equation (20b) t is the torsional load per unit of length, p_H is the horizontal component of resultant load per unit of length, and p_V is the vertical component of resultant load per unit of length.

At the root of a cantilever beam, or the end of a fixed-ended beam, the boundary condition is that w is zero. At the tip of a cantilever beam, or the end of a simply supported beam, the boundary condition is that σ is zero. This latter condition may be interpreted as meaning that $\partial w/\partial z$ is zero. A third type of boundary condition may arise when shearing or normal stresses of known distribution are being applied to the free ends of the beam.

Deflections Due to Shearing Strains

Before proceeding to a solution of the equation it is worth while to draw a few simple conclusions concerning the deflections of the beam from the equations of the previous section. Equation (17) for ξ may be substituted into equation (7b) for τ to obtain

$$\tau = G\frac{\partial w}{\partial s} + Gr\frac{d\varphi}{dz} + G\cos\alpha\frac{du}{dz} + G\sin\alpha\frac{dv}{dz} \tag{21}$$

It is convenient to consider, for the moment, that the beam is loaded in such a way as to produce rotations only without translations. Equation (21) would then become

$$r\frac{d\varphi}{dz} = \frac{\tau}{G} - \frac{\partial w}{\partial s}$$

$$= \gamma - \frac{\partial w}{\partial s}$$

This equation indicates that the angle of twist may be considered to consist of two parts, the first being due to shearing strains and the second due to axial displacements. If this equation is multiplied through by $t_w r\,ds$ and integrated over the cross section, equation (19a) will be obtained. This indicates immediately that the first

term on the right side of equation (19a) gives the rate of twist (change of rotation per unit of length) due to shearing strain, while the second term gives the rate of twist due to axial displacements. A similar interpretation can be given to the terms on the right-hand side of equations (19b) and (19c).

It is of considerable practical importance to be able to compute deflections of thin-walled beams due to shearing strains. The proper formulas for this purpose are obtained from equations (19) by setting w equal to zero.

$$\frac{d\varphi}{dz} = \frac{T}{GI_C} \tag{22a}$$

$$\frac{du}{dz} = \frac{H}{GA_H} \tag{22b}$$

$$\frac{dv}{dz} = \frac{V}{GA_V} \tag{22c}$$

These formulas were given by Drymael (reference 8). Equations (22) are exact formulas for the derivatives of the deflections due to shearing strain of a beam with rigid diaphragms. It is of considerable interest to note that the deflections due to shearing strains are independent of the distribution of the shearing stress on the cross sections.

In the case of torsion there is no practical need for computing the angle of twist due to shearing strains at various points along the span. However, equation (22a) is useful in giving the rate of twist at a fixed end where $w = 0$. This is useful in certain methods of deflection calculation. In the case of horizontal or vertical bending, equations (22b) and (22c) enable the designer to compute the deflections due to shearing strains at various points along the span. This gives an indication of the physical significance, as well as the practical utility, of the section constants I_C, A_H, and A_V.

When the axial displacement w is known it may be substituted into the integrals which appear in equations (19) in order to determine the deflections. By substituting an approximate solution for w into equations (19) one may obtain an approximate solution for the deflections. In order to show the relationship between equations (19b) and (19c) and the engineer's customary equations relating bending deflections and loads, the axial displacements may be assumed to have planar distribution. It is more convenient to substitute for normal

stresses than axial displacements. Equations (19b) and (19c) may be differentiated with respect to z to obtain

$$\frac{d^2u}{dz^2} = -\frac{p_H}{GA_H} - \frac{1}{E'A_H}\int t_w \cos\alpha \frac{\partial\sigma}{\partial s}\,ds \tag{23a}$$

$$\frac{d^2v}{dz^2} = -\frac{p_V}{GA_V} - \frac{1}{E'A_V}\int t_w \sin\alpha \frac{\partial\sigma}{\partial s}\,ds \tag{23b}$$

The normal stress in equations (23) may be assumed to be given by the flexure formula in the form

$$\sigma = \frac{P}{A_w} + \frac{M_y\bar{x}}{I_y} + \frac{M_x\bar{y}}{I_x} \tag{24}$$

In equation (24) the distances $\bar{x}$ and $\bar{y}$ are measured from the centroidal axes of the cross section. The moments M_x and M_y and the moments of inertia I_x and I_y are computed with respect to the centroidal axes. In general, the centroidal axes, or principal bending axes, do not coincide with the principal shear axes. Equation (24) may be substituted into equations (23) to obtain approximate formulas for the curvatures of the beam. This substitution is explained in detail in appendix A and leads to the following equations:

$$\frac{d^2u}{dz^2} = \frac{-p_H}{GA_H} - \frac{M_y\cos\theta}{E'I_y} - \frac{M_x\sin\theta}{E'I_x} \tag{25a}$$

$$\frac{d^2v}{dz^2} = \frac{-p_V}{GA_V} + \frac{M_y\sin\theta}{E'I_y} - \frac{M_x\cos\theta}{E'I_x} \tag{25b}$$

In these equations θ is the angular difference between the position of the principal shear axes and the principal bending axes. In the great majority of sections θ will be small and will have the value of zero when the cross section has an axis of symmetry. If θ is small enough that the terms containing $\sin\theta$ may be dropped and the $\cos\theta$ replaced by unity, equations (25) take the following well-known form:

$$\frac{d^2u}{dz^2} = -\frac{p_H}{GA_H} - \frac{M_y}{E'I_y} \tag{26a}$$

$$\frac{d^2v}{dz^2} = - \frac{p_V}{GA_V} - \frac{M_x}{E'I_x} \tag{26b}$$

In equations (26) the first term on the right-hand side is an exact formula for deflections due to shearing strains while the second term is an approximate formula for the deflections due to axial displacements or, as is more commonly stated, due to normal strains.

Simplifications Due to Symmetry of Cross Section

A maximum amount of simplification arises in the analysis when the cross section of the beam is symmetrical about both of the principal shear axes. In this case the centroidal axes are also the principal shear axes. The general integrodifferential equation for w may be replaced by three independent equations in which the actions of torsion, horizontal bending, and vertical bending become completely separated. From considerations of double symmetry it is found that equation (20a) must be replaced by the following three equations:

$$\frac{\partial^2 w}{\partial s^2} + \frac{E'}{G} \frac{\partial^2 w}{\partial z^2} - \frac{1}{I_c} \frac{dr}{ds} \oint t_w r \frac{\partial w}{\partial s} ds = - \frac{T}{GI_c} \frac{dr}{ds} \tag{27a}$$

$$\frac{\partial^2 w}{\partial s^2} + \frac{E'}{G} \frac{\partial^2 w}{\partial z^2} - \frac{1}{A_H} \frac{d \cos \alpha}{ds} \oint t_w \cos \alpha \frac{\partial w}{\partial s} ds = - \frac{H}{GA_H} \frac{d \cos \alpha}{ds} \tag{27b}$$

$$\frac{\partial^2 w}{\partial s^2} + \frac{E'}{G} \frac{\partial^2 w}{\partial z^2} - \frac{1}{A_V} \frac{d \sin \alpha}{ds} \oint t_w \sin \alpha \frac{\partial w}{\partial s} = - \frac{V}{GA_V} \frac{d \sin \alpha}{ds} \tag{27c}$$

Equation (27a) was derived and solved by Von Kármán and Chien (reference 9). The development of equations (27) is given in appendix B. A corresponding system of equations may be derived for the stress distribution in a beam with doubly symmetrical cross sections. These equations are as follows:

$$\frac{\partial^2 \sigma}{\partial s^2} + \frac{E'}{G} \frac{\partial^2 \sigma}{\partial z^2} - \frac{1}{I_c} \frac{dr}{ds} \oint t_w r \frac{\partial \sigma}{\partial s} ds = \frac{E't}{GI_c} \frac{dr}{ds} \tag{28a}$$

$$\frac{\partial^2\sigma}{\partial s^2} + \frac{E'}{G}\frac{\partial^2\sigma}{\partial z^2} - \frac{1}{A_H}\frac{d\cos\alpha}{ds}\oint t_w \cos\alpha \frac{\partial\sigma}{\partial s}\,ds = \frac{E'p_H}{GA_H}\frac{d\cos\alpha}{ds} \qquad (28b)$$

$$\frac{\partial^2\sigma}{\partial s^2} + \frac{E'}{G}\frac{\partial^2\sigma}{\partial z^2} - \frac{1}{A_V}\frac{d\sin\alpha}{ds}\oint t_w \sin\alpha \frac{\partial\sigma}{\partial s}\,ds = \frac{E'p_V}{GA_V}\frac{d\sin\alpha}{ds} \qquad (28c)$$

Every term of equation (27a) or (28a) is antisymmetrical about both the x-axis and the y-axis. Every term of equation (27b) or (28b) is symmetrical about the x-axis and antisymmetrical about the y-axis. Every term of equation (27c) or (28c) is symmetrical about the y-axis and antisymmetrical about the x-axis.

If a cross section has only one axis of symmetry, the general equation for w is replaced by two equations. For the case of a cross section which is symmetrical about the x-axis the equations take the following form:

$$\frac{\partial^2 w}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w}{\partial z^2} - \frac{1}{A_H}\frac{d\cos\alpha}{ds}\oint t_w \cos\alpha \frac{\partial w}{\partial s}\,ds = -\frac{H}{GA_H}\frac{d\cos\alpha}{ds} \qquad (29a)$$

$$\frac{\partial^2 w}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w}{\partial z^2} - \frac{1}{I_c}\frac{dr}{ds}\oint t_w r \frac{\partial w}{\partial s}\,ds - \frac{1}{A_V}\frac{d\sin\alpha}{ds}\oint t_w \sin\alpha \frac{\partial w}{\partial s}\,ds =$$

$$-\frac{T}{GI_c}\frac{dr}{ds} - \frac{V}{GA_V}\frac{d\sin\alpha}{ds} \qquad (29b)$$

Equations (29) are derived in appendix B. Equation (29a) indicates that the solution for axial displacements due to horizontal loads may be considered separately from the action of vertical and torsional loads. Every term of equation (29a) is symmetrical about the x-axis while every term of equation (29b) is antisymmetrical about the x-axis.

For the case of a cross section which is symmetrical only about the y-axis, the equations for w take the following form:

$$\frac{\partial^2 w}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w}{\partial z^2} - \frac{1}{A_V}\frac{d\sin\alpha}{ds}\oint t_w \sin\alpha \frac{\partial w}{\partial s}\,ds = -\frac{V}{GA_V}\frac{d\sin\alpha}{ds} \qquad (30a)$$

$$\frac{\partial^2 w}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w}{\partial z^2} - \frac{1}{I_c}\frac{dr}{ds}\oint t_w r \frac{\partial w}{\partial s}\,ds - \frac{1}{A_H}\frac{d\cos\alpha}{ds}\oint t_w \cos\alpha \frac{\partial w}{\partial s}\,ds =$$

$$-\frac{T}{GI_c}\frac{dr}{ds} - \frac{H}{GA_H}\frac{d\cos\alpha}{ds} \tag{30b}$$

Equations (30) are derived in appendix B. Every term of equation (30a) is symmetrical about the y-axis while every term of equation (30b) is antisymmetrical about the y-axis. It is obvious that equations for the normal stress, corresponding to equations (29) and (30), may be readily derived.

A few comments concerning the effects of symmetry upon deflections may be of some interest. Considering a cross section having double symmetry, the solution for w from equations (27) may be substituted into equations (19) to obtain the deflections of the beam. It will be found that torsional loads produce rotations only, horizontal loads produce horizontal translations only, and vertical loads produce vertical translations only. For the case of a cross section which is symmetrical about the x-axis, the solution of equations (29) may be substituted into equations (19) as before. It will be found that horizontal loads produce only horizontal displacements. However, vertical loads produce both vertical displacements and rotations, and torsional loads also produce both vertical displacements and rotations. In the case of a cross section which is symmetrical about the y-axis, vertical loads produce only vertical displacements but there is an interaction effect between horizontal loads and torsional loads. Such interaction effects do not occur in elementary beam theory wherein one considers only primary stresses.

Solution of Homogeneous Equation

The solution of the nonhomogeneous equation which governs the axial displacements, or the normal stresses, may be determined by assuming that the unknown may be expanded into an infinite series of appropriate orthogonal functions which individually satisfy the boundary conditions and the conditions of continuity. The orthogonal functions which are appropriate for the analysis may be determined by considering the solution of the homogeneous equation and the boundary conditions. In the case of cantilever beams, simple beams, or fixed-ended beams, the end conditions are homogeneous. The transverse distribution of the solution must be continuous and single-valued, or periodic.

The case which is most easily understood and capable of being given the simplest explanation of the method of solution is that of a beam

with doubly symmetrical cross section acted upon by torsional loading only. It was this case which was treated by Von Kármán and Chien, although their considerations were limited to a cantilever beam of semi-infinite length acted upon by a torque at the tip. In the present analysis the loading distribution and end conditions will remain arbitrary.

The homogeneous equation governing the axial displacements in a cross section with double symmetry is obtained by setting the right-hand side of equation (27a) equal to zero.

$$\frac{\partial^2 w}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w}{\partial z^2} - \frac{1}{I_c}\frac{dr}{ds}\oint t_w r \frac{\partial w}{\partial s}\,ds = 0 \tag{31a}$$

The corresponding equation for normal stresses becomes

$$\frac{\partial^2 \sigma}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 \sigma}{\partial z^2} - \frac{1}{I_c}\frac{dr}{ds}\oint t_w r \frac{\partial \sigma}{\partial s}\,ds = 0 \tag{31b}$$

A standard well-known method for the solution of homogeneous partial differential equations is to assume that the unknown may be expressed by a separation of the variables. The same method may be employed in solving the above integrodifferential equations. The displacement and stress may be assumed to have the following forms:

$$w = f(s)g(z) \tag{32a}$$

$$\sigma = f(s)h(z) \tag{32b}$$

where

$$h = E'\frac{dg}{dz} \tag{32c}$$

The relationship between g and h arises from the relationship between w and σ as expressed by equation (7a). It is also clear from equation (7a) that, if w and σ can be expressed by a separation of the variables, then the transverse distribution of both functions must be the same.

Substituting equation (32a) into equation (31a) gives

$$g\frac{d^2 f}{ds^2} + \frac{E'f}{G}\frac{d^2 g}{dz^2} - \frac{g}{I_c}\frac{dr}{ds}\oint t_w r \frac{df}{ds}\,ds = 0$$

Dividing through by the product fg and transposing the first and third terms gives

$$\frac{E'}{Gg}\frac{d^2g}{dz^2} = -\frac{1}{f}\frac{d^2f}{ds^2} + \frac{1}{fI_c}\frac{dr}{ds}\oint t_w r \frac{df}{ds}\,ds \tag{33}$$

Since the left side is a function of z only while the right side is a function of s only, each side may be equated to a constant independently. Equating the left side to λ^2 gives

$$\left.\begin{aligned} &\frac{d^2g}{dz^2} - \frac{G\lambda^2}{E'}g = 0 \\ \text{or} \\ &\frac{d^2g}{dz^2} - \mu^2 g = 0 \end{aligned}\right\} \tag{34a}$$

where

$$\mu^2 = \frac{G\lambda^2}{E'} \tag{34b}$$

The corresponding equation for the spanwise distribution of σ is obviously

$$\frac{d^2h}{dz^2} - \mu^2 h = 0 \tag{34c}$$

Equating the right-hand side of equation (33) to λ^2 gives the following equation for the transverse distribution of w or σ:

$$\frac{d^2f}{ds^2} + \lambda^2 f - \frac{1}{I_c}\frac{dr}{ds}\oint t_w r \frac{df}{ds} = 0 \tag{35}$$

It is convenient for the analysis to introduce the following integration by parts:

$$\oint t_w r \frac{df}{ds}\,ds = -\oint t_w f \frac{dr}{ds}\,ds \tag{36}$$

The bracketed term, which usually arises in an integration by parts, vanishes here because f must be continuous and periodic. Although

equation (35) is an integrodifferential equation it is convenient in certain phases of the analysis to consider the integral in the equation as a section constant for the beam. If the stress distribution were known it could be substituted into the integral which, in turn, could be evaluated to obtain a scalar constant that would be dependent only upon the properties of the cross section. This section constant, which is associated with torsional loading, may be indicated as C_T and defined as follows:

$$C_T = -\oint t_w r \frac{df}{ds} ds = \oint t_w f \frac{dr}{ds} ds \tag{37}$$

Using equation (37), equation (35) may be converted into either of the following forms:

$$\frac{d^2f}{ds^2} + \lambda^2 f + \frac{1}{I_c} \frac{dr}{ds} \oint t_w f \frac{dr}{ds} ds = 0 \tag{38a}$$

$$\frac{d^2f}{ds^2} + \lambda^2 f + \frac{C_T}{I_c} \frac{dr}{ds} = 0 \tag{38b}$$

It is of some interest and benefit to draw a comparison, at this point, between the present problem of determining transverse stress, or displacement, distributions, and the natural vibrations of a thin ring. The analogy is of a physical nature only, since the problems do not correspond mathematically. The ring that is to be considered lies in a plane and has a doubly symmetrical shape. It is to be considered as a free body vibrating in space without a gravitational field. The only vibrational displacements that are to be considered are those that are normal to the plane of the ring. There will be four infinite sets of modes that could be excited as follows: (a) Doubly symmetrical, (b) doubly antisymmetrical, (c) symmetrical about the x-axis only, and (d) symmetrical about the y-axis only, where the x-axis and y-axis are in the plane of the ring. In a similar manner there will arise four infinite sets of natural "displacement modes" or "stress modes" in a cross section having double symmetry. Each type of mode will be "excited" by a particular type of loading. The relationship between type of mode and loading is as follows (see appendix B):

Mode		Loading
(a) Doubly symmetrical	⟶	Axial force
(b) Doubly antisymmetrical	⟶	Torsion
(c) Symmetrical about x-axis	⟶	Horizontal bending
(d) Symmetrical about y-axis	⟶	Vertical bending

The analogy with a vibrating ring is useful in suggesting terminology, in visualizing the stress distributions that are being determined, and in suggesting calculation techniques for idealized stiffened cross sections that are to be treated in the second section of the paper.

Solutions of equation (38a) exist only for certain characteristic values of the parameter λ. Ordinarily characteristic numbers are determined by boundary equations. In the present case the function $f(s)$ is governed by continuity and periodicity conditions which do not provide a characteristic equation. (In the case of a section with corners, the condition of continuity at the corners provides the characteristic equation.) The characteristic equation was obtained in a simple manner by Von Kármán and Chien by setting the line integral in equation (38a) equal to unity. The resulting equation determines an infinite set of values of λ. Inspection of equation (38a) shows that, if the line integral is replaced by any convenient constant, then the equation becomes a nonhomogeneous differential equation which can be readily solved. The solution for $f(s)$ will contain the number λ and hence there will be an infinite number of stress modes.

Unfortunately the simple method used by Von Kármán and Chien to form the characteristic equation cannot be extended directly to arbitrary cross sections. Consequently a method of forming the characteristic equation will be presented which can be extended to arbitrary cross sections. Equation (38b) suggests the following form of the general solution:

$$f = A \sin \lambda s + B \cos \lambda s + C_T \varphi(s,\lambda) \tag{39}$$

where φ remains to be determined. Substituting equation (39) into equation (38b) gives the following equation for φ:

$$\frac{d^2\varphi}{ds^2} + \lambda^2 \varphi = -\frac{1}{I_c} \frac{dr}{ds} \tag{40}$$

The function φ is the particular integral of equation (40) and is a function of the coordinate s and the parameter λ.

It is now necessary to show that the constants of integration A and B must vanish for noncircular sections. The stress mode f is doubly antisymmetrical. It is convenient to choose the origin of the coordinate s on one of the axes of symmetry. The term $\sin \lambda s$ may be made doubly antisymmetrical by a proper choice of values of λ. At first thought this would seem to be an appropriate method of determining the characteristic values of λ. Since dr/ds is antisymmetrical it is seen from equation (40) that φ must be antisymmetrical. However, the term $\cos \lambda s$ is symmetrical about the axis upon which the origin of s is chosen and, hence, B must be set equal to zero.

In order that the term $\sin \lambda s$ shall be antisymmetrical it is necessary that λ have one of the values λ_n given by the following formula:

$$\lambda_n = \frac{4n\pi}{L} \qquad n = 1, 2, 3, \ldots \tag{41}$$

where L is the circumference of the center line of the wall of the cross section. The geometric quantity dr/ds may be assumed to be expanded into a Fourier series as follows:

$$\frac{dr}{ds} = \sum_{i=1}^{\infty} a_i \sin \frac{4i\pi s}{L} \tag{42}$$

The function $\varphi(\lambda,s)$ may also be expressed as a Fourier series

$$\varphi_n(\lambda_n,s) = \sum_{i=1}^{\infty} b_i(\lambda_n) \sin \frac{4i\pi s}{L} \tag{43}$$

Substituting equations (42) and (43) into equation (40) leads to the following formula for b_i:

$$b_i = \frac{a_i}{\omega_i^2 - \lambda_n^2} \tag{44}$$

where

$$\omega_i = \frac{4i\pi}{L} \qquad i = 1, 2, 3, \ldots \tag{45}$$

For each value of φ_n, as defined by equation (43), there will be one term in the series for which ω_i equals λ_n and hence for which b_i

becomes infinite. Hence the only possibility for a finite solution for φ_n to exist is for all of the a_i values to be zero. This means that dr/ds must vanish, which would be true only for a circular section. An examination of equation (40) shows that if dr/ds vanishes the function φ must either be zero or be proportional to the quantity $\sin \lambda s$. In the latter case φ would add nothing to the formula for f as given by equation (39) and hence may be assumed to be zero for a circular cross section. For the noncircular section it is impossible to obtain a finite solution for φ by choosing λ to make $\sin \lambda s$ antisymmetrical. The only alternative in this case is to choose A equal to zero. Thus the following formulas for the stress distribution have been derived:

Circular section:

$$f = A \sin \lambda s \tag{46a}$$

Noncircular section:

$$f = C_T \varphi(\lambda, s) \tag{46b}$$

In the case of the circular section λ is given by equation (41). For the noncircular section a characteristic equation must now be derived.

Multiplying through equation (46b) by dr/ds and integrating over the cross section give

$$\oint t_w f \frac{dr}{ds} ds = C_T \oint t_w \varphi \frac{dr}{ds} ds$$

or

$$C_T = C_T \oint t_w \varphi \frac{dr}{ds} ds$$

or

$$\left(\oint t_w \varphi \frac{dr}{ds} ds - 1 \right) C_T = 0 \tag{47}$$

Equation (47) may be regarded as a homogeneous algebraic equation in which C_T is the unknown and thus remains arbitrary. The term C_T plays the role of an amplitude factor which must be determined subsequently by introducing a normalizing condition for the stress modes. The coefficient of C_T must vanish and hence

$$\oint t_w \varphi \frac{dr}{ds} ds = 1 \tag{48}$$

Equation (48) is the characteristic equation which determines the values of λ and agrees with that given by Von Kármán and Chien. The above derivation of the characteristic equation can be extended to arbitrary cross sections as will be shown in the next section.

The section constant is determined by introducing the following normalizing condition:

$$\oint t_w f^2 ds = 1 \tag{49}$$

This normalizing condition is more convenient in the analysis, for both monocoque and semimonocoque sections, than the engineering types of normalizing laws which might be suggested by the vibration analogy. The method of solution for the natural stress modes should now be clear. The function φ is determined in terms of λ by solving equation (40). Expansions in Fourier series should be useful for this purpose. The functions φ and dr/ds are then substituted into equation (48) to determine the characteristic values of λ. The stress modes are then known except for the coefficients C_T which are determined to satisfy equation (49).

Stress Modes for Arbitrary Cross Sections

The solution for the stress modes and characteristic numbers for arbitrary cross sections will be stated briefly. If the right-hand side of equations (20) is replaced by zero and a separation of variables is introduced, the following equation will be found to govern the transverse distribution of displacements or stresses:

$$\frac{d^2 f}{ds^2} + \lambda^2 f + \frac{C_T}{I_c} \frac{dr}{ds} + \frac{C_H}{A_H} \frac{d \cos \alpha}{ds} + \frac{C_V}{A_V} \frac{d \sin \alpha}{ds} = 0 \tag{50a}$$

where

$$C_T = \oint t_w f \frac{dr}{ds} ds \tag{50b}$$

$$C_H = \oint t_w f \frac{d \cos \alpha}{ds} ds \tag{50c}$$

$$C_V = \oint t_w f \frac{d \sin \alpha}{ds} ds \tag{50d}$$

The solution of equation (50a) may be written as follows:

$$f = C_T \varphi_T(\lambda,s) + C_H \varphi_H(\lambda,s) + C_V \varphi_V(\lambda,s) \tag{51a}$$

where φ_T, φ_H, and φ_V are governed by the following equations:

$$\frac{d^2\varphi_T}{ds^2} + \lambda^2 \varphi_T = -\frac{1}{I_c} \frac{dr}{ds} \tag{51b}$$

$$\frac{d^2\varphi_H}{ds^2} + \lambda^2 \varphi_H = -\frac{1}{A_H} \frac{d \cos \alpha}{ds} \tag{51c}$$

$$\frac{d^2\varphi_V}{ds^2} + \lambda^2 \varphi_V = -\frac{1}{A_V} \frac{d \sin \alpha}{ds} \tag{51d}$$

Terms containing constants of integration in the formula for f have been omitted since they may be shown to vanish for noncircular sections by arguments similar to those used in the previous section. Since, in the present case, bending as well as torsional action is being considered, the complete solution must include the planar stress distributions of ordinary engineering beam theory. These primary modes must satisfy equation (50a). The planar distribution may include the effect of an axial force as well as bending moments.

If a planar distribution of stress is assumed, it will be found that the section constant C_T has the value zero by evaluating the integral which it represents. The value of λ which is associated with the planar modes is also zero. If φ_H and φ_V are defined to have the linear distributions $a + x$ and $b + y$, respectively, in equation (51a) for f, this planar distribution will be found to satisfy equation (50c) if λ and C_T are set equal to zero. This existence of a solution for a zero value of λ is a feature of the present problem which is not found in the solution of homogeneous second-order differential equations with homogeneous boundary conditions (the Sturm-Liouville problem).

The characteristic equation may now be developed by the same method that was used in the previous section. Equation (51a) must be

multiplied through successively by the following quantities: dr/ds, $d \cos \alpha/ds$, and $d \sin \alpha/ds$. Integration over the cross section in each case gives the following three linear algebraic equations:

$$C_T = C_T \oint t_w \varphi_T \frac{dr}{ds} ds + C_H \oint t_w \varphi_H \frac{dr}{ds} ds + C_V \oint t_w \varphi_V \frac{dr}{ds} ds \qquad (52a)$$

$$C_H = C_T \oint t_w \varphi_T \frac{d \cos \alpha}{ds} ds + C_H \oint t_w \varphi_H \frac{d \cos \alpha}{ds} ds + C_V \oint t_w \varphi_V \frac{d \cos \alpha}{ds} ds \qquad (52b)$$

$$C_V = C_T \oint t_w \varphi_T \frac{d \sin \alpha}{ds} ds + C_H \oint t_w \varphi_H \frac{d \sin \alpha}{ds} ds + C_V \oint t_w \varphi_V \frac{d \sin \alpha}{ds} ds \qquad (52c)$$

Equations (52) are a homogeneous system of three algebraic equations in which C_T, C_H, and C_V are the unknowns. The integrals which appear are functions of λ and may be regarded as scalar coefficients of the equations. After transposing all terms to one side of the equations the determinant of the coefficients may be isolated and set equal to zero. This provides the characteristic equation which determines the values of λ. The calculation task of determining the characteristic values of λ is clearly a formidable one for arbitrary cross sections. For each value of λ a solution of equations (52) exists for relative values of C_T, C_H, and C_V. Their specific magnitudes must be determined from the normalizing condition as expressed by equation (49).

Orthogonality of Stress Modes

In order to obtain a solution to the general nonhomogeneous equation for stresses or displacements it is necessary to determine the orthogonality properties of the stress modes which have been defined. Since this argument follows customary well-known mathematical methods, it will be given only for the doubly antisymmetrical modes (the torsion case) in a beam having a cross section with double symmetry.

Equation (38b) may be written for the mth and nth stress modes as follows:

$$\frac{d^2 f_m}{ds^2} + \lambda_m{}^2 f_m + \frac{C_{Tm}}{I_c} \frac{dr}{ds} = 0 \qquad (53a)$$

$$\frac{d^2f_n}{ds^2} + \lambda_n{}^2 f_n + \frac{C_{Tn}}{I_c}\frac{dr}{ds} = 0 \tag{53b}$$

Equation (53a) is to be multiplied through by f_n, equation (53b) by f_m, and both equations are to be integrated over the cross section.

$$\oint f_n \frac{d^2f_m}{ds^2} t_w \, ds + \lambda_m{}^2 \oint f_n f_m t_w \, ds + \frac{C_{Tm}C_{Tn}}{I_c} = 0 \tag{54a}$$

$$\oint f_m \frac{d^2f_n}{ds^2} t_w \, ds + \lambda_n{}^2 \oint f_m f_n t_w \, ds + \frac{C_{Tn}C_{Tm}}{I_c} = 0 \tag{54b}$$

The first term in each of these equations may be integrated by parts to obtain

$$\oint f_n \frac{d^2f_m}{2} t_w \, ds = -\oint \frac{df_n}{ds}\frac{df_m}{ds} t_w \, ds$$

$$\oint f_m \frac{d^2f_n}{ds^2} t_w \, ds = -\oint \frac{df_m}{ds}\frac{df_n}{ds} t_w \, ds$$

The bracketed terms vanish since the stress mode and its derivative must be continuous and periodic. These formulas show that the first terms of equations (54a) and (54b) are identically equal. Hence, by subtracting equation (54b) from (54a), one obtains

$$\left(\lambda_m{}^2 - \lambda_n{}^2\right) \oint f_m f_n t_w \, ds = 0$$

Since the λ's form a discrete set of numbers, the first factor in the above equation does not vanish and the following orthogonality condition is obtained:

$$\oint f_m f_n t_w \, ds = 0 \tag{55}$$

A similar development of equation (55) can be formulated for arbitrary cross sections.

In deriving the solution of the nonhomogeneous equation for stresses or displacements it is convenient to have one additional

relationship which may now be derived. Making use of the normalizing condition as given by equation (49) it may be seen that, when $m = n$, the second term of equation (54b) is equal to λ_n^2 or λ_m^2. When m does not equal n, the second term of equation (54b) vanishes because of equation (55). Thus equation (54b) may be rewritten in the following manner:

$$\oint f_m \frac{d^2 f_n}{ds^2} t_w \, ds + \frac{C_{Tm} C_{Tn}}{I_c} = -\lambda_m^2 \delta_{mn} \tag{56a}$$

where

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \tag{56b}$$

Solution of Nonhomogeneous Equation

Again the action of torsional loads on a beam having cross sections with double symmetry will be considered first. The equations that must be solved are equations (27a) and (28a). The displacements and stresses may be assumed in series form as follows:

$$w = \sum_{n=1}^{\infty} f_n(s) g_n(z) \tag{57a}$$

$$\sigma = \sum_{n=1}^{\infty} f_n(s) h_n(z) \tag{57b}$$

Substituting equation (57a) into equation (27a) gives

$$\sum_n g_n \frac{d^2 f_n}{ds^2} + \frac{E'}{G} \sum_n f_n \frac{d^2 g_n}{dz^2} - \frac{1}{I_c} \frac{dr}{ds} \oint t_w r \left(\sum_n g_n \frac{df_n}{ds} \right) ds = - \frac{T}{G I_c} \frac{dr}{ds}$$

Interchanging the order of summation and integration in the third term of this equation and introducing C_{Tm} as defined by equation (37) give

$$\sum_n g_n \frac{d^2 f_n}{ds^2} + \frac{E'}{G} \sum_n f_n \frac{d^2 g_n}{dz^2} + \frac{1}{I_c} \frac{dr}{ds} \sum_n C_{Tn} g_n = - \frac{T}{G I_c} \frac{dr}{ds}$$

The preceding equation must now be multiplied through by f_m and integrated over the cross section.

$$\sum_n g_n \oint f_m \frac{d^2 f_n}{ds^2} t_w \, ds + \frac{E'}{G} \sum_n \frac{d^2 g_n}{dz^2} \oint f_m f_n t_w \, ds +$$

$$\frac{1}{I_c} \oint f_m \frac{dr}{ds} t_w \, ds \sum_n C_{Tn} g_n = - \frac{T}{GI_c} \oint f_m \frac{dr}{ds} t_w \, ds$$

The integrals which occur in the second series of this equation may be evaluated from the normalizing condition and the orthogonality condition for the stress modes. All terms of the series vanish except the mth term. After introducing C_{Tm} and rearranging terms the equation becomes

$$\frac{E'}{G} \frac{d^2 g_m}{dz^2} + \sum_n g_n \left(\oint f_m \frac{d^2 f_n}{ds^2} t_w \, ds + \frac{C_{Tm} C_{Tn}}{I_c} \right) = - \frac{C_{Tm} T}{GI_c}$$

The quantity in the parentheses has a finite value only when $m = n$. Substituting equation (56) into the above equation will give the equation governing the spanwise variation of the mth displacement mode. The subscript m may be changed to n for convenience to give the equation for the nth mode as follows:

$$\frac{E'}{G} \frac{d^2 g_n}{dz^2} - \lambda_n^2 g_n = - \frac{C_{Tn} T}{GI_c}$$

Multiplying through by G/E' and introducing μ as defined by equation (34b) gives

$$\frac{d^2 g_n}{dz^2} - \mu_n^2 g_n = - \frac{C_{Tn} T}{E' I_c} \tag{58a}$$

The corresponding equation for the stress distribution is

$$\frac{d^2 h_n}{dz^2} - \mu_n^2 h_n = \frac{C_{Tn} t}{I_c} \tag{58b}$$

The complete solution can now be obtained. After computing the transverse stress modes as previously explained the constants C_{Tn} and μ_n are known for each mode. Equation (58a) may then be solved

for the spanwise variation of the displacements due to the applied loads. The solution of equations (58a) or (58b) may be written in either of the following forms:

$$g_n = Ae^{\mu_n s} + Be^{-\mu_n s} + \psi(s) \tag{59a}$$

$$g_n = A \sinh \mu_n s + B \cosh \mu_n s + \psi(s) \tag{59b}$$

The coefficients A and B may be determined from the boundary conditions after the particular integral $\psi(s)$ has been determined for the given loading.

In the case of a beam of arbitrary cross section without symmetry the same mathematical method may be applied to equations (20). The resulting equation for the spanwise variation of axial displacements is

$$\frac{d^2 g_n}{dz^2} - \mu_n^2 g_n = - \frac{C_{Tn} T}{E' I_c} - \frac{C_{Hn} H}{E' A_H} - \frac{C_{Vn} V}{E' A_V} \tag{60}$$

A corresponding equation may be written for the stress distribution. Equations (59) also give the form of solution for equation (60). It is of interest to note that a zero-warping axis for the nth mode may be located by setting the right-hand side of equation (60) equal to zero. However, it is not apparent that there is any practical value to be gained by locating this axis in the present method of analysis.

Associated with each normal stress mode there is also a shearing stress distribution. These shearing stresses are computed for each normal stress mode independently by integrating equation (8), the equation of equilibrium. This integration is also required in primary stress analysis and hence is well-known. The constant of integration for the secondary shear flow is determined from the condition that the resultant torque must be zero.

II - BEAMS HAVING THIN-WALLED STIFFENED CROSS SECTIONS

Introduction

In section II a method of analysis is to be presented which is directly applicable to beams of practical cross sections such as occur in airplane wings and fuselages. The beam will be assumed to have a thin-walled stiffened cross section which may be of single-cell or

multicell design. The type of cross section which is actually to be analyzed is the idealized stiffened section which results from the customary engineering idealization that is used in primary stress analysis of airplanes. The loading distribution and end conditions will again be considered as arbitrary.

As before the beam will be assumed to contain closely spaced internal rigid diaphragms which stabilize the shell and preserve the cross-sectional shape. There are two sources of error in the present analysis. The first is the idealization of the cross section and the second is the assumption of rigid bulkheads. As explained in section I, it is possible to obtain "approximate" solutions or "accurate" solutions for stresses and displacements. Only accurate solutions will be considered herein.

The introduction of generalized difference equations permits the analysis of a semimonocoque, or stiffened, section to be developed in a manner that is exactly analogous to the method of solution which has been previously given in terms of analytic functions for monocoque sections. The use of generalized difference equations provides certain conveniences and simplifications which are not found in the customary algebraic treatments of the stiffened beam. The differencing symbols have been specifically defined for application to the stress-analysis problem of idealized multicell cross sections.

Symbols

a_j	area of jth stiffener
A_k	central area associated with kth web
A_s	total area of stiffeners
j	stiffener number
k	web number
L_k	length of kth web
P_j	axial force in jth stiffener
β_k	warping stiffness of kth web
$\gamma_{Tj},\gamma_{Hj},\gamma_{Vj}$	geometric discontinuities at jth stiffener
$[A]$	matrix of stiffener areas

[B] matrix of warping stiffnesses

[Γ] matrix of geometric discontinuities

Difference and Summation Definitions

The idealized semimonocoque section which is to be analyzed has a finite number of stiffeners which carry all of the normal stress and the shear flows are distributed uniformly over the connecting webs. The idealization is illustrated in figure 4 where a small portion of the cross section of a beam is shown. The flange and stringers of the true section are replaced by idealized stiffeners centered on the webs. The idealized stiffeners also include an "effective width" of web material as is commonly employed in primary stress analysis. In the idealized section it is assumed that the webs which connect the stiffeners are straight. The curvature of the webs is neglected. The idealization of a cross section for secondary stress analysis need not be highly accurate for design purposes although for research studies some improvement in the accuracy of the idealization may be found desirable.

The normal stresses on a cross section act over the area of the stiffeners only. The normal stress σ_j, acting on the jth stiffener, may be multiplied by the stiffener area a_j to obtain the axial force p_j on the stiffener. The axial forces acting on three stiffeners on a straight wall are shown in figure 5(a). The shearing stresses are distributed uniformly along each individual web. The shearing stress τ_k, acting on the kth web, may be multiplied by the wall thickness t_k to obtain the shear flow q_k acting on the web. The distribution of shear flow over a cross section is given by a stepped diagram as illustrated in figure 5(b).

The above definitions in regard to stresses are equally useful in primary or secondary stress analysis. In the secondary stress problem displacements must also be considered. Because of the fact that each idealized web has been assumed to be straight, the tangential displacement ξ_k of points on the kth web will be a constant for the web and the transverse distribution of ξ_k will be a stepped diagram. The function ξ_k will thus behave in the analysis in a manner similar to the stress τ_k and the product $t_k\xi_k$ may be regarded as a "flow-type" quantity. The axial displacements must be considered to be defined at the stiffeners, the displacement w_j occurring at the jth stiffener. It must be assumed that the axial displacements have a linear variation over each web. The resulting distribution of axial displacements is given by a broken-line diagram as shown in figure 5(c). The distribution of axial displacements must be a continuous function from physical considerations.

All quantities entering into the analysis which are dependent upon the transverse coordinates are completely defined by giving their values either at the stiffeners or at one point on each web. Those functions which are defined over the webs are uniformly distributed over each web. Those quantities which are defined by their values at the stiffeners are given a subscript j. Those quantities which are defined by their values over the webs are given a subscript k.

Just as two types of functions arise in the analysis, so it is necessary to define two types of differencing symbols. The symbol Δ_j indicates a differencing operation which produces a function that is defined by its values at the stiffeners. The symbol Δ_k indicates a differencing operation which produces a function that is defined over the webs. The differencing symbols may be defined most conveniently by applying them to particular functions. The difference $\Delta_k w_j$ may be defined as the difference between the value of w_j at the forward stiffener and the value at the rearward stiffener relative to the kth web. If the forward stiffener is stiffener a and the rearward stiffener is stiffener b, the difference $\Delta_k w_j$ is defined by

$$\Delta_k w_j = w_a - w_b \tag{61}$$

This definition is illustrated in figure 5(c). The above difference must be divided by the length L_k of the kth web to obtain the slope, or derivative, of the function. The correspondence between derivatives and differences is as follows:

$$\frac{\partial w}{\partial s} \longrightarrow \frac{1}{L_k} \Delta_k w_j$$

Throughout the analysis for multicell sections the positive direction for the coordinate s and all "flow-type" quantities is counterclockwise along the outer shell and in the positive direction of the coordinate axes on the interior webs.

The symbol Δ_j is defined as the difference between outflow and inflow at the jth stiffener. This definition of a differencing symbol is of a rather general nature since it may be applied at a flange having any number of connecting webs. The difference $\Delta_j q_k$ is illustrated in figure 5(b) for a stringer with two connecting webs. If the web ahead of the jth stiffener is web c and the web behind the jth stiffener is web d, the difference $\Delta_j q_k$ is given by

$$\Delta_j q_k = q_c - q_d \tag{62a}$$

In this case no clear correspondence between derivatives and differences is apparent. A second example of the difference $\Delta_j q_k$ is shown in figure 6 where the jth stiffener is connected to webs a, b, and c. The positive directions of the shear flows are indicated. The difference $\Delta_j q_k$ becomes

$$\Delta_j q_k = q_b - q_a - q_c \tag{62b}$$

In addition it is necessary to introduce summations for the idealized stiffened sections corresponding to the various integrals which occurred in the analysis of section I. Two types of summations must be introduced to indicate summation over all of the stiffener areas or summation over all of the web areas. Three simple examples of summations over the stiffener areas are as follows:

$$P = \sum_j a_j \sigma_j = \sum_j p_j \tag{63a}$$

$$M_y = \sum_j a_j \sigma_j \bar{x}_j = \sum_j p_j \bar{x}_j \tag{63b}$$

$$M_x = \sum_j a_j \sigma_j \bar{y}_j = \sum_j p_j \bar{y}_j \tag{63c}$$

It is apparent that these summations give the axial force and bending moments on the cross section. Three simple examples of the second type of summation are as follows:

$$T = \sum_k t_k \tau_k L_k r_k = \sum_k q_k L_k r_k \tag{64a}$$

$$H = \sum_k t_k \tau_k L_k \cos \alpha_k = \sum_k q_k L_k \cos \alpha_k \tag{64b}$$

$$V = \sum_k t_k \tau_k L_k \sin \alpha_k = \sum_k q_k L_k \sin \alpha_k \tag{64c}$$

These summations give the section torque and shears. The geometric quantities r_k, $\cos \alpha_k$, and $\sin \alpha_k$ are constant over the kth web since the web is assumed to be straight in the idealized section.

It may be worth while to note that the introduction of generalized difference equations in one-dimensional problems would be of no practical value. In such problems difference equations become useful only when the regularity of the structural layout leads to "standard difference equations" such as those treated in chapter XII of reference 19. In problems of two or more dimensions, however, it is found that generalized difference equations are of considerable value. They not only provide a mathematical method of complete generality for deriving the equations governing the axial displacements or stresses from the basic equations of elasticity but, more importantly, they make clear the possibility of using the method of separation of variables in spite of complete irregularity of the structural layout which may occur. The method of analysis for an idealized stiffened multicell section may be carried out, step by step, in a manner exactly parallel to the method used in section I for the single-cell monocoque section which is more easily understood. The method of separation of variables leads automatically to the natural stress modes for the cross section. The possible utility of generalized difference equations in other planar or spatial structures, such as gridworks, remains to be investigated.

Fundamental Equations of Elasticity

Since the present analysis is exactly parallel to that given in section I, it need be stated only briefly. The analysis will be made clearer by frequent comparison with corresponding equations in section I. The stresses are related to the strains by the formulas

$$\sigma_j = E\epsilon_j \tag{65a}$$

$$\tau_k = G\gamma_k \tag{65b}$$

The strains are related to the displacements by the following formulas:

$$\epsilon_j = \frac{\partial w_j}{\partial z} \tag{66a}$$

$$\gamma_k = \frac{1}{L_k} \Delta_k w_j + \frac{\partial \xi_k}{\partial z} \tag{66b}$$

Equations (66) are comparable with equations (6) of section I. From

equations (65) and (66) it is seen that the stresses may be expressed in terms of the displacements by the formulas

$$\sigma_j = E \frac{\partial w_j}{\partial z} \tag{67a}$$

$$\tau_k = \frac{G}{L_k} \Delta_k w_j + G \frac{\partial \xi_k}{\partial z} \tag{67b}$$

The stress flows are given by the following formulas:

$$p_j = a_j \sigma_j = E a_j \frac{\partial w_j}{\partial z} \tag{68a}$$

$$q_k = t_k \tau_k = \frac{G t_k}{L_k} \Delta_k w_j + G t_k \frac{\partial \xi_k}{\partial z}$$

$$= G \beta_k \Delta_k w_j + G t_k \frac{\partial \xi_k}{\partial z} \tag{68b}$$

where

$$\beta_k = \frac{t_k}{L_k} \tag{68c}$$

The quantity $G\beta_k$ may be called the warping stiffness of the kth web. The forces which act on a differential length of a stiffener are shown in figure 7. The equation of equilibrium for the stiffener becomes

$$\frac{dp_j}{dz} + \Delta_j q_k = 0 \tag{69}$$

This equation corresponds to equation (8) of section I.

Development of Equation for Axial Displacements

In the present analysis the equation will be derived only for the axial displacement. Equations for the other functions may be readily derived. Substituting equations (68) into equation (69) gives the following equation relating the displacements.

$$Ea_j \frac{d^2 w_j}{dz^2} + G\Delta_j\left(\beta_k \Delta_k w_j\right) + G\Delta_j\left(t_k \frac{\partial \xi_k}{\partial z}\right) = 0 \tag{70}$$

This equation corresponds to equation (9a). The second term in this equation is a second-order difference quantity.

Before eliminating the tangential displacement it is necessary to introduce the definition of the principal shear axes and the associated section properties. The principal shear axes are located by the condition that the following three summations must vanish:

$$\sum_k t_k r_k L_k \cos \alpha_k = 0 \tag{71a}$$

$$\sum_k t_k r_k L_k \sin \alpha_k = 0 \tag{71b}$$

$$\sum_k t_k L_k \cos \alpha_k \sin \alpha_k = 0 \tag{71c}$$

The associated section properties are as follows:

$$I_c = \sum_k t_k L_k r_k^2 \tag{72a}$$

$$A_H = \sum_k t_k L_k \cos^2 \alpha_k \tag{72b}$$

$$A_V = \sum_k t_k L_k \sin^2 \alpha_k \tag{72c}$$

Convenient methods for calculating these section properties are given in appendix C.

Corresponding to equation (17) of section I the tangential displacement of all points on the kth web is related to the cross-sectional displacements by the formula

$$\xi_k = r_k\varphi + u \cos \alpha_k + v \sin \alpha_k \tag{73}$$

The section torque and shears have been expressed in terms of the shear flows by equations (64). If equation (68b) for the shear flow is substituted into equations (64), the following equations are obtained:

$$G \sum_k t_k r_k \Delta_k w_j + G \sum_k t_k r_k L_k \frac{\partial \xi_k}{\partial z} = T$$

$$G \sum_k t_k \cos \alpha_k \Delta_k w_j + G \sum_k t_k L_k \cos \alpha_k \frac{\partial \xi_k}{\partial z} = H$$

$$G \sum_k t_k \sin \alpha_k \Delta_k w_j + G \sum_k t_k L_k \sin \alpha_k \frac{\partial \xi_k}{\partial z} = V$$

Substituting equation (73) and omitting those summations which vanish according to equations (71), the above equations become

$$G \sum_k t_k r_k \Delta_k w_j + G \frac{d\varphi}{dz} \sum_k t_k L_k r_k^2 = T$$

$$G \sum_k t_k \cos \alpha_k \Delta_k w_j + G \frac{du}{dz} \sum_k t_k L_k \cos^2 \alpha_k = H$$

$$G \sum_k t_k \sin \alpha_k \Delta_k w_j + G \frac{dv}{dz} \sum_k t_k L_k \sin^2 \alpha_k = V$$

Introducing the section properties as defined by equations (72) gives, after rearranging terms,

$$\frac{d\varphi}{dz} = \frac{T}{GI_c} - \frac{1}{I_c} \sum_k t_k r_k \Delta_k w_j \tag{74a}$$

$$\frac{du}{dz} = \frac{V}{GA_H} - \frac{1}{A_H} \sum_k t_k \cos \alpha_k \Delta_k w_j \tag{74b}$$

$$\frac{dv}{dz} = \frac{V}{GA_V} - \frac{1}{A_V}\sum_k t_k \sin \alpha_k \Delta_k w_j \tag{74c}$$

These equations were given by Drymael (reference 8).

The required formula for ξ_k which will permit its elimination from equation (70) may now be obtained by differentiating equation (73) with respect to z and then substituting equations (74).

$$\frac{\partial \xi_k}{\partial z} = \frac{Tr_k}{GI_c} + \frac{H \cos \alpha_k}{GA_H} + \frac{V \sin \alpha_k}{GA_V} - \frac{r_k}{I_c}\sum_k t_k r_k \Delta_k w_j -$$

$$\frac{\cos \alpha_k}{A_H}\sum_k t_k \cos \alpha_k \Delta_k w_j - \frac{\sin \alpha_k}{A_V}\sum_k t_k \sin \alpha_k \Delta_k w_j$$

This formula may be substituted into equation (70) to obtain the following equation governing w_j:

$$\Delta_j\left(\beta_k \Delta_k w_j\right) + \frac{E}{G} a_j \frac{d^2 w_j}{dz^2} - \frac{1}{I_c}\Delta_j \rho_{Tk}\sum_k \rho_{Tk}\Delta_k w_j - \frac{1}{A_H}\Delta_j \rho_{Hk}\sum_k \rho_{Hk}\Delta_k w_j -$$

$$\frac{1}{A_V}\Delta_j \rho_{Vk}\sum_k \rho_{Vk}\Delta_k w_j = -\frac{T}{GI_c}\Delta_j \rho_{Tk} - \frac{H}{GA_H}\Delta_j \rho_{Hk} - \frac{V}{GA_V}\Delta_j \rho_{Vk} \tag{75}$$

In equation (75) the flow-type quantities ρ_{Tk}, ρ_{Hk}, and ρ_{Vk} have been introduced in accord with the following definitions:

$$\rho_{Tk} = t_k r_k \tag{76a}$$

$$\rho_{Hk} = t_k \cos \alpha_k \tag{76b}$$

$$\rho_{Vk} = t_k \sin \alpha_k \tag{76c}$$

Equation (75), which is for arbitrary cross sections, may be replaced by simplified groups of equations for sections having single or double symmetry just as in section I.

In section I an integration by parts was introduced after the variables had been separated. This integration by parts could have been introduced earlier in the original partial integrodifferential

equation but there was no apparent reason for so doing. In the present case an appreciable simplification of the form of the equation may be obtained by performing a summation by parts (corresponding to an integration by parts) providing one simultaneously introduces new symbols for such geometric differences as $\Delta_j \rho_{Tk}$. Consequently the following definitions are introduced:

$$\gamma_{Tj} = \Delta_j \rho_{Tk} = \Delta_j \left(t_k r_k \right) \tag{77a}$$

$$\gamma_{Hj} = \Delta_j \rho_{Hk} = \Delta_j \left(t_k \cos \alpha_k \right) \tag{77b}$$

$$\gamma_{Vj} = \Delta_j \rho_{Vk} = \Delta_j \left(t_k \sin \alpha_k \right) \tag{77c}$$

Convenient methods for computing these geometric discontinuities are given in appendix D. The operation of summation by parts may now be performed on the summations which occur in equation (75) to obtain the following formulas:

$$\sum_k \rho_{Tk} \Delta_k w_j = - \sum_j w_j \Delta_j \rho_{Tk} = - \sum_j \gamma_{Tj} w_j \tag{78a}$$

$$\sum_k \rho_{Hk} \Delta_k w_j = - \sum_j w_j \Delta_j \rho_{Hk} = - \sum_j \gamma_{Hj} w_j \tag{78b}$$

$$\sum_k \rho_{Vk} \Delta_k w_j = - \sum_j w_j \Delta_j \rho_{Vk} = - \sum_j \gamma_{Vj} w_j \tag{78c}$$

The validity of the above summations by parts is demonstrated in appendix E. Equation (75) may now be written in the following form:

$$\Delta_j \left(\beta_k \Delta_k w_j \right) + \frac{E}{G} a_j \frac{d^2 w_j}{dz^2} + \frac{\gamma_{Tj}}{I_c} \sum_j \gamma_{Tj} w_j + \frac{\gamma_{Hj}}{A_H} \sum_j \gamma_{Hj} w_j + \frac{\gamma_{Vj}}{A_V} \sum_j \gamma_{Vj} w_j =$$

$$- \frac{T}{GI_c} \gamma_{Tj} - \frac{H}{GA_H} \gamma_{Hj} - \frac{V}{GA_V} \gamma_{Vj} \tag{79}$$

Simplifications Due to Symmetry of Cross Section

The arguments which permit the replacement of the general equation for axial displacements by groups of simplified equations are identically the same as those used in section I. Hence it is not necessary to repeat the arguments but merely to state the equations. In the case of a cross section having double symmetry the four components of axial displacement are governed by the following four equations:

$$\Delta_j\left(\beta_k\Delta_k w_j\right) + \frac{E}{G} a_j \frac{d^2 w_j}{dz^2} = 0 \tag{80a}$$

$$\Delta_j\left(\beta_k\Delta_k w_j\right) + \frac{E}{G} a_j \frac{d^2 w_j}{dz^2} + \frac{\gamma_{Tj}}{I_c} \sum_j \gamma_{Tj} w_j = -\frac{T}{GI_c} \gamma_{Tj} \tag{80b}$$

$$\Delta_j\left(\beta_k\Delta_k w_j\right) + \frac{E}{G} a_j \frac{d^2 w_j}{dz^2} + \frac{\gamma_{Hj}}{A_H} \sum_j \gamma_{Hj} w_j = -\frac{H}{GA_H} \gamma_{Hj} \tag{80c}$$

$$\Delta_j\left(\beta_k\Delta_k w_j\right) + \frac{E}{G} a_j \frac{d^2 w_j}{dz^2} + \frac{\gamma_{Vj}}{A_V} \sum_j \gamma_{Vj} w_j = -\frac{V}{GA_V} \gamma_{Vj} \tag{80d}$$

In the case of a cross section which is symmetrical about the x-axis the two components of displacement are governed by the following two equations:

$$\Delta_j\left(\beta_k\Delta_k w_j\right) + \frac{E}{G} a_j \frac{d^2 w_j}{dz^2} + \frac{\gamma_{Hj}}{A_H} \sum_j \gamma_{Hj} w_j = -\frac{H}{GA_H} \gamma_{Hj} \tag{81a}$$

$$\Delta_j\left(\beta_k\Delta_k w_j\right) + \frac{E}{G} a_j \frac{d^2 w_j}{dz^2} + \frac{\gamma_{Tj}}{I_c} \sum_j \gamma_{Tj} w_j + \frac{\gamma_{Vj}}{A_V} \sum_j \gamma_{Vj} w_j = -\frac{T}{GI_c} \gamma_{Tj} - \frac{V}{GA_V} \gamma_{Vj} \tag{81b}$$

In the case of a cross section which is symmetrical about the y-axis the two components of displacement are governed by the following two equations:

$$\Delta_j\left(\beta_k\Delta_k w_j\right) + \frac{E}{G}\,a_j\,\frac{d^2w_j}{dz^2} + \frac{\gamma_{Vj}}{A_V}\sum_j \gamma_{Vj}w_j = -\frac{V}{GA_V}\,\gamma_{Vj} \tag{82a}$$

$$\Delta_j\left(\beta_k\Delta_k w_j\right) + \frac{E}{G}\,a_j\,\frac{d^2w_j}{dz^2} + \frac{\gamma_{Tj}}{I_c}\sum_j \gamma_{Tj}w_j + \frac{\gamma_{Hj}}{A_H}\sum_j \gamma_{Hj}w_j =$$

$$-\frac{T}{GI_c}\,\gamma_{Tj} - \frac{H}{GA_H}\,\gamma_{Hj} \tag{82b}$$

Solution of Homogeneous Equation

The solution of the nonhomogeneous problem may again be expressed in terms of the natural displacement modes which arise from the homogeneous equation after a separation of the variables. The axial displacement may be assumed as the product of a function of j by a function of z.

$$w_j = f_j g(z) \tag{83}$$

For convenience the doubly symmetrical cross section will be considered with torsional loading (the antisymmetrical modes). The homogeneous equation is obtained from equation (80b) by setting the right-hand side equal to zero.

$$\Delta_j\left(\beta_k\Delta_k w_j\right) + \frac{E}{G}\,a_j\,\frac{d^2w_j}{dz^2} + \frac{\gamma_{Tj}}{I_c}\sum_j \gamma_{Tj}w_j = 0 \tag{84}$$

If equation (83) is substituted into equation (84), the variables may be separated in the same manner as explained in section I. The spanwise and transverse distributions are found to be governed by the following two equations:

$$\frac{d^2g}{dz^2} - \mu^2 g = 0 \tag{85a}$$

$$\Delta_j\left(\beta_k\Delta_k f_j\right) + \lambda^2 a_j f_j + \frac{\gamma_{Tj}}{I_c}\sum_j \gamma_{Tj}f_j = 0 \tag{85b}$$

where $\mu^2 = \frac{\lambda^2 G}{E}$. Unfortunately the method of solution for the stress modes which was used in section I is not particularly convenient here. The solution of equation (85b) may be obtained by several different methods. Equation (85b) involves only one independent variable. In such cases, as has been previously noted, the use of generalized differences offers no assistance in the solution of the problem although they do permit the writing of the equation in a condensed form.

In order to solve equation (85b) it is necessary to recognize that it represents a homogeneous system of linear algebraic equations which must be solved simultaneously. For simple cross sections having only a few stiffeners the equations may be solved by direct algebraic methods. For sections having a large number of stiffeners numerical methods of solution must be employed. In order to proceed to the solution of the original nonhomogeneous problem, it will be assumed that the natural stress modes have been computed.

The equations which govern the stress modes for other types of loading and symmetry conditions will now be stated briefly. For a beam with a doubly symmetrical cross section acted upon by horizontal loads, the stress modes are symmetrical about the x-axis and are governed by the following equation:

$$\Delta_j\left(\beta_k \Delta_k f_j\right) + \lambda^2 a_j f_j + \frac{\gamma_{Hj}}{A_H} \sum_j \gamma_{Hj} f_j = 0 \tag{86a}$$

For a beam with a doubly symmetrical cross section acted upon by vertical loads, the stress modes are symmetrical about the y-axis and are governed by the following equation:

$$\Delta_j\left(\beta_k \Delta_k f_j\right) + \lambda^2 a_j f_j + \frac{\gamma_{Vj}}{A_V} \sum_j \gamma_{Vj} f_j = 0 \tag{86b}$$

If the cross section of the beam has single symmetry about the x-axis, the symmetrical modes are governed by equation (86a) while antisymmetrical modes are governed by the following equation:

$$\Delta_j\left(\beta_k \Delta_k f_j\right) + \lambda^2 a_j f_j + \frac{\gamma_{Tj}}{I_c} \sum_j \gamma_{Tj} f_j + \frac{\gamma_{Vj}}{A_V} \sum_j \gamma_{Vj} f_j = 0 \tag{87a}$$

When the cross section of the beam has single symmetry about the y-axis, the symmetrical modes are governed by equation (86b) and the

antisymmetrical modes are governed by the following equation:

$$\Delta_j\left(\beta_k\Delta_k f_j\right) + \lambda^2 a_j f_j + \frac{\gamma_{Tj}}{I_c}\sum_j \gamma_{Tj} f_j + \frac{\gamma_{Hj}}{A_H}\sum_j \gamma_{Hj} f_j = 0 \tag{87b}$$

If the cross section is completely unsymmetrical the stress modes are governed by the following equation:

$$\Delta_j\left(\beta_k\Delta_k f_j\right) + \lambda^2 a_j f_j + \frac{\gamma_{Tj}}{I_c}\sum_j \gamma_{Tj} f_j + \frac{\gamma_{Hj}}{A_H}\sum_j \gamma_{Hj} f_j + \frac{\gamma_{Vj}}{A_V}\sum_j \gamma_{Vj} f_j = 0 \tag{88}$$

Solution of Nonhomogeneous Equation

Before solving the nonhomogeneous equation it is necessary to prove the orthogonality of the stress modes and to introduce a law of normalization as in section I. Since the proof of orthogonality follows the same method used in section I it will be omitted. The condition of orthogonality and law of normalization may be stated in one equation by using a Kronecker delta.

$$\sum_j a_j f_{jm} f_{jn} = \delta_{mn} \tag{89a}$$

where

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{89b}$$

In equation (89a) f_{jm} and f_{jn} are the mth and nth modes, respectively. Equation (85b) may be written for the nth mode to obtain

$$\Delta_j\left(\beta_k\Delta_k f_j\right) + \lambda_n^2 a_j f_{jn} + \frac{\gamma_{Tj}}{I_c} C_{Tn} = 0 \tag{90a}$$

where

$$C_{Tn} = \sum_j \gamma_{Tj} f_{jn} \tag{90b}$$

Multiplying through equation (90a) by f_{jm} and summing over all stiffener areas give, after introducing equations (89),

$$\sum_j f_{mn}\Delta_j\left(\beta_k\Delta_k w_j\right) + \frac{C_{Tm}C_{Tn}}{I_c} = -\lambda_m{}^2\delta_{mn} \tag{91}$$

Equations (89) and (91) may now be employed in solving the nonhomogeneous equation.

A beam having a cross section with double symmetry and acted upon by torsional loading will again be considered. The axial displacement may be assumed to be given by a finite series as follows:

$$w_j = \sum_n f_{jn}g_n(z) \tag{92}$$

Substituting this formula into equation (80b) gives, after interchanging the order of summation in the third term,

$$\sum_n g_n\Delta_j\left(\beta_k\Delta_k f_{jn}\right) + \frac{E}{G}a_j\sum_n f_{jn}\frac{d^2g_n}{dz^2} + \frac{\gamma_{Tj}}{I_c}\sum_n C_{Tn}g_n = -\frac{T}{GI_c}\gamma_{Tj} \tag{93}$$

Equation (93) must now be multiplied through by f_{jm} and summed over all stiffeners. Reversing the order of summation in the first two terms gives

$$\sum_n\sum_j g_n f_{jm}\Delta_j\left(\beta_k\Delta_k f_{jn}\right) + \frac{E}{G}\sum_n\sum_j \frac{d^2g_n}{dz^2}a_j f_{jm}f_{jn} +$$

$$\frac{C_{Tm}}{I_c}\sum_n C_{Tn}g_n = -\frac{T}{GI_c}C_{Tm}$$

Introducing equations (89) and rearranging terms gives

$$\frac{E}{G}\frac{d^2g_m}{dz^2} + \sum_n g_n\left[\sum_j f_{jm}\Delta_j\left(\beta_k\Delta_k f_{jm}\right) + \frac{C_{Tm}C_{Tn}}{I_c}\right] = -\frac{C_{Tm}T}{GI_c}$$

Substituting equation (91) and changing the subscripts m to n give

$$\left.\begin{aligned} \frac{E}{G}\frac{d^2g_n}{dz^2} - \lambda_n{}^2 g_n &= -\frac{C_{Tn}T}{GI_c} \\ \text{or} \\ \frac{d^2g_n}{dz^2} - \mu_n{}^2 g_n &= -\frac{C_{Tn}T}{EI_c} \end{aligned}\right\} \tag{94a}$$

Equation (94a), with the end conditions, gives the spanwise distribution of each stress mode and thus completes the solution for the displacements.

Similar equations may be developed for the spanwise distribution of stress modes with other types of loading and symmetry conditions. Since the mathematical method of development of the equations is the same in all cases, the equations will be merely stated without proof. For a doubly symmetrical section subjected to horizontal loading the stress modes are determined by

$$\frac{d^2g_n}{dz^2} - \mu_n{}^2 g_n = -\frac{C_{Hn}H}{EA_H} \tag{94b}$$

where

$$C_{Hn} = \sum_j \gamma_{Hj} f_{jn} \tag{94c}$$

For a doubly symmetrical section subjected to vertical loading the stress modes are determined by

$$\frac{d^2g_n}{dz^2} - \mu_n{}^2 g_n = -\frac{C_{Vn}V}{EA_V} \tag{94d}$$

where

$$C_{Vn} = \sum_j \gamma_{Vj} f_{jn} \tag{94e}$$

When a cross section has single symmetry about the x-axis, the symmetrical stress modes are governed by equation (94b) while the

antisymmetrical modes are governed by the equation

$$\frac{d^2g_n}{dz^2} - \mu_n{}^2 g_n = -\frac{C_{Tn}T}{EI_c} - \frac{C_{Vn}V}{EA_V} \tag{94f}$$

If the section has single symmetry about the y-axis, the symmetrical modes are governed by equation (94d) while the antisymmetrical modes are governed by the equation

$$\frac{d^2g_n}{dz^2} - \mu_n{}^2 g_n = -\frac{C_{Tn}T}{EI_c} - \frac{C_{Hn}H}{EA_H} \tag{94g}$$

When the cross section is completely unsymmetrical, the stress modes are governed by the following equation:

$$\frac{d^2g_n}{dz^2} - \mu_n{}^2 g_n = -\frac{C_{Tn}T}{EI_c} - \frac{C_{Hn}H}{EA_H} - \frac{C_{Vn}V}{EA_V} \tag{94h}$$

A convenient method for calculating the section constants C_{Tn}, C_{Hn}, and C_{Vn} is given in appendix D.

Evaluation of Difference Term

In each of the equations for transverse stress, or displacement, modes there appears a difference term of second order. This term must be expressed in algebraic form before the stress modes can be determined. The difference term has the form $\Delta_j\left(\beta_k\Delta_k f_j\right)$. In converting this difference quantity to algebraic form it is necessary to express all subscripts in terms of stiffener numbers. The subscripts referring to webs must be eliminated.

In figure 4(b) there is shown a segment of an idealized stiffened section in which web a and web b are connected to the jth stiffener. The first-order difference quantity $\Delta_k f_j$ is expressed, over web a, in the following manner:

$$\left(\Delta_k f_j\right)_{k=a} = f_j - f_{j-1}$$

The quantity $\beta_k\Delta_k f_j$ becomes, for web a,

$$\left(\beta_k\Delta_k f_j\right)_{k=a} = \beta_{j(j-1)}\left(f_j - f_{j-1}\right)$$

For web b this quantity becomes

$$\left(\beta_k \Delta_k f_j\right)_{k=b} = \beta_{j(j+1)}\left(f_{j+1} - f_j\right)$$

In the above equations it should be noted that the single subscript on the relative warping stiffness β has been replaced by a double subscript corresponding to the stiffeners at either end of the particular web being considered.

The second-order difference quantity may now be expressed in the following algebraic form:

$$\Delta_j\left(\beta_k \Delta_k f_j\right) = \left(\beta_k \Delta_k f_j\right)_{k=b} - \left(\beta_k \Delta_k f_j\right)_{k=a}$$

$$= \beta_{j(j+1)} f_{j+1} - \left(\Sigma_j \beta\right) f_j + \beta_{j(j-1)} f_{j-1} \tag{95a}$$

where

$$\Sigma_j \beta = \beta_{j(j-1)} + \beta_{j(j+1)} \tag{95b}$$

The quantity $\Sigma_j \beta$ is the sum of the relative warping stiffnesses of the webs which connect to the jth stiffener.

It may be of some interest to note the algebraic form of the difference term when more than two webs connect to the stiffener as may occur in multicell sections. In figure 6 there are three webs connected to the jth stiffener. The assumed positive directions for all flow-type quantities are indicated on the webs. The difference term, for this case, becomes

$$\Delta_j\left(\beta_k \Delta_k f_j\right) = \left(\beta_k \Delta_k f_j\right)_{k=b} - \left(\beta_k \Delta_k f_j\right)_{k=a} - \left(\beta_k \Delta_k f_j\right)_{k=c}$$

$$= \beta_{jm} f_m + \beta_{j(j+1)} f_{j+1} - \left(\Sigma_j \beta\right) f_j + \beta_{j(j-1)} f_{j-1} \tag{96a}$$

where

$$\Sigma_j \beta = \beta_{j(j-1)} + \beta_{j(j+1)} + \beta_{jm} \tag{96b}$$

Modified Stiffnesses for Symmetrical Cross Sections

In any idealized stiffened cross section the number of stress modes will be equal to the number of stiffeners. For a section with n stiffeners there will be three primary planar modes and n - 3 secondary

self-equilibrating modes. For an unsymmetrical section equation (88) corresponds to n linear algebraic equations which may be solved to obtain the stress modes. If a cross section has an axis of symmetry, the symmetrical and antisymmetrical modes may be computed from two independent smaller systems of equations. The conditions of symmetry may be conveniently incorporated into the equations by using modified stiffness values for webs or stiffeners which lie on an axis of symmetry.

As an example a section may be considered which is symmetrical about the y-axis and which has a web on the y-axis as shown in figure 8. The symmetrical modes are governed by equation (86b) which may be written for every stiffener in the cross section. However, it is only necessary to write this equation for the stiffeners on one side of the axis of symmetry. For a symmetrical mode the axial displacements will be uniform over the central web as shown in figure 8(a). The axial displacement and normal stress will be the same in stiffeners 3 and 4.

If equation (86b) is written for stiffener 3, and equation (95a) is substituted, the following equation may be obtained:

$$-\beta_{34}f_4 + (\Sigma_3\beta)f_3 - \beta_{32}f_2 - \frac{\gamma_{V3}}{A_V}\sum_j \gamma_{Vj}f_j = \lambda^2 a_3 f_3 \tag{97}$$

Assuming that a system of equations would be formed for all stiffeners on the right side of the axis of symmetry, it becomes essential to eliminate f_4 from equation (97). Since f_4 is equal to f_3 it may be replaced by f_3 to obtain

$$\beta_{32}f_3 - \beta_{32}f_2 - \frac{\gamma_{V3}}{A_V}\sum_j \gamma_{Vj}f_j = \lambda^2 a_3 f_3 \tag{98a}$$

since

$$\Sigma_3\beta = \beta_{34} + \beta_{32} \tag{98b}$$

Equation (98a) could have been written immediately by omitting the term containing the stress value on the left side of the axis and assuming the warping stiffness of web 34 to be zero as shown on the right-hand side of figure 8(a).

In a similar manner the antisymmetrical mode may be considered as illustrated in figure 8(b). In this case it is found that the standard stiffness value must be multiplied by a factor of 2 as indicated on the right-hand side of figure 8(b).

The cross sections may have a stiffener on the axis of symmetry as shown in figure 9. The equation which is associated with this stiffener may be formed, for the symmetrical mode, by multiplying the stiffener area by one-half as shown in figure 9(a) and omitting the term containing the stress in the stiffener on the left side of the axis. If there is a geometric discontinuity at this stiffener a factor of one-half must be applied to this quantity. If there is an internal web lying on the axis its stiffness must be multiplied by one-half. Although the use of modified stiffnesses does provide some convenience, the advantage of their use is not large since the symmetry conditions are easy to apply. For a section with double symmetry it is only necessary to consider the material of the cross section that is contained within the first quadrant.

Algebraic Equations for Stress Modes in a Doubly Symmetrical Section

The bending and torsion modes for a doubly symmetrical section are governed by equations (85b), (86a), and (86b). An equation for the doubly symmetrical modes, which are due to axial force, may be obtained from equation (80a) by a separation of the variables. An algebraic form of the equations for the stress modes may be obtained by replacing the second-order difference term by its algebraic equivalent as given by equations (95) or (96).

For purposes of illustration a single-cell section may be considered in which each stiffener will be connected to two webs. Typical equations for the jth stiffener in each stress mode appear as follows:

$$-\beta_{j(j+1)}f_{j+1} + \left(\Sigma_j\beta - a_j\lambda^2\right)f_j - \beta_{j(j-1)}f_{j-1} = 0 \qquad (99a)$$

$$-\beta_{j(j+1)}f_{j+1} + \left(\Sigma_j\beta - a_j\lambda^2\right)f_j - \beta_{j(j-1)}f_{j-1} + \frac{\gamma_{Tj}}{I_c}\sum_j \gamma_{Tj}f_j = 0 \qquad (99b)$$

$$-\beta_{j(j+1)}f_{j+1} + \left(\Sigma_j\beta - a_j\lambda^2\right)f_j - \beta_{j(j-1)}f_{j-1} + \frac{\gamma_{Hj}}{A_H}\sum_j \gamma_{Hj}f_j = 0 \qquad (99c)$$

$$-\beta_{j(j+1)}f_{j+1} + \left(\Sigma_j\beta - a_j\lambda^2\right)f_j - \beta_{j(j-1)}f_{j-1} + \frac{\gamma_{Vj}}{A_V}\sum_j \gamma_{Vj}f_j = 0 \qquad (99d)$$

Each of the above equations is a typical equation of a system of equations governing a particular type of mode. The number of equations in each system depends upon the number of stiffeners in the first quadrant having finite values of stress. The process of formation of similar equations for sections having single symmetry or without symmetry is obvious. Examples of the application of these equations to specific cross sections are given in a subsequent section.

Matrix Forms of Equations for Stress Modes

The stress modes for an unsymmetrical section are governed by equation (88). In order to express the corresponding system of algebraic equations it is convenient to introduce the notation of matrix algebra. In order to make clear the meaning of each matrix symbol it is desirable to show the matrices in expanded form. In order to avoid the necessity of writing out large matrices the expanded matrices will be written for an unsymmetrical four-flange section. The contracted matrix equations, however, will have complete generality.

Before writing the equations it is useful to note that the summations which occur in equation (88) may be expressed as scalar products of vectors. If $[f]$ indicates a column matrix, or column vector, then $[f]'$ will indicate a row matrix or row vector. The prime is used to indicate the transpose of a matrix. If $[f]$ is a vector of stress-mode ordinates, then $[f]$ and $[f]'$ for a four-flange section are given in expanded form as

$$[f] = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}, \qquad [f]' = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} \tag{100}$$

Vectors of the geometric discontinuities associated with the flanges are defined by

$$[\gamma_T]' = \begin{bmatrix} \gamma_{T1} & \gamma_{T2} & \gamma_{T3} & \gamma_{T4} \end{bmatrix} \tag{101a}$$

$$[\gamma_H]' = \begin{bmatrix} \gamma_{H1} & \gamma_{H2} & \gamma_{H3} & \gamma_{H4} \end{bmatrix} \tag{101b}$$

$$[\gamma_V]' = \begin{bmatrix} \gamma_{V1} & \gamma_{V2} & \gamma_{V3} & \gamma_{V4} \end{bmatrix} \tag{101c}$$

The summations which are contained in equation (88) may now be expressed by the following scalar products:

$$\sum_j \gamma_{Tj} f_j = \left[\gamma_T\right]' \left[f\right] \tag{102a}$$

$$\sum_j \gamma_{Hj} f_j = \left[\gamma_H\right]' \left[f\right] \tag{102b}$$

$$\sum_j \gamma_{Vj} f_j = \left[\gamma_V\right]' \left[f\right] \tag{102c}$$

Equation (88) may now be written in the following form:

$$\Delta_j\left(\beta_k \Delta_k f_j\right) + \lambda^2 a_j f_j + \frac{\gamma_{Tj}}{I_c}\left[\gamma_T\right]'\left[f\right] + \frac{\gamma_{Hj}}{A_H}\left[\gamma_H\right]'\left[f\right] + \frac{\gamma_{Vj}}{A_V}\left[\gamma_V\right]'\left[f\right] = 0 \tag{103}$$

Substituting equation (95a) and transposing the term containing λ^2 gives

$$-\beta_{j(j+1)} f_{j+1} + \left(\Sigma_j \beta\right) f_j - \beta_{j(j-1)} f_{j-1} - \frac{\gamma_{Tj}}{I_c}\left[\gamma_T\right]'\left[f\right] -$$

$$\frac{\gamma_{Hj}}{A_H}\left[\gamma_H\right]'\left[f\right] - \frac{\gamma_{Vj}}{A_V}\left[\gamma_V\right]'\left[f\right] = \lambda^2 a_j f_j \tag{104}$$

In this equation the coefficients of the first three terms are warping stiffnesses. These coefficients, for a four-flange beam, may be arranged to form the following matrix:

$$\left[B\right] = \begin{bmatrix} \Sigma_1\beta & -\beta_{12} & 0 & -\beta_{14} \\ -\beta_{21} & \Sigma_2\beta & -\beta_{23} & 0 \\ 0 & -\beta_{32} & \Sigma_3\beta & -\beta_{34} \\ -\beta_{41} & 0 & -\beta_{43} & \Sigma_4\beta \end{bmatrix} \tag{105}$$

The rows of this matrix contain the warping stiffnesses as they would appear in the four algebraic equations for the stress modes. It may also be noted that the subscripts correspond to the position of the element in the matrix. This will be true for any cross section. The diagonal element of the jth row is the sum of the stiffnesses of all webs which connect to the jth stiffener. The nondiagonal nonzero elements of the jth row are the individual stiffnesses of the webs connecting to the jth stiffener. From these considerations it is seen that this matrix can be easily formed for any cross section from the web dimensions.

On the right-hand side of equation (104) there appears the stiffener area. These areas may be formed into the following diagonal matrix:

$$\left[A\right] = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix} \tag{106}$$

Using the definitions of equations (105) and (106), the system of equations, corresponding to equation (104), may be written in the following matrix form:

$$\left[B\right]\left[f\right] - \frac{1}{I_c}\left[\gamma_T\right]\left[\gamma_T\right]'\left[f\right] - \frac{1}{A_H}\left[\gamma_H\right]\left[\gamma_H\right]'\left[f\right] - \frac{1}{A_V}\left[\gamma_V\right]\left[\gamma_V\right]'\left[f\right] = \lambda^2\left[A\right]\left[f\right] \tag{107}$$

When a column vector is premultiplied by a row vector the result is a scalar number. Hence the product may be said to be a scalar multiplication. Examples of such products are given in equations (102). In equation (107) it is seen that there are three terms in which column vectors of geometric discontinuities are postmultiplied by row vectors of the same geometric quantities. When a column vector is postmultiplied by a row vector the result is a square matrix and the product may be called a matrix product of two vectors. It is convenient to introduce matrix symbols for these products as follows:

$$\left[\Gamma_T\right] = \left[\gamma_T\right]\left[\gamma_T\right]' \tag{108a}$$

$$\left[\Gamma_H\right] = \left[\gamma_H\right]\left[\gamma_H\right]' \tag{108b}$$

$$\left[\Gamma_V\right] = \left[\gamma_V\right]\left[\gamma_V\right]' \tag{108c}$$

In expanded form these matrices appear as follows for a four-flange beam:

$$\left[\Gamma_T\right] = \begin{bmatrix} \gamma_{T1}^2 & \gamma_{T1}\gamma_{T2} & \gamma_{T1}\gamma_{T3} & \gamma_{T1}\gamma_{T4} \\ \gamma_{T2}\gamma_{T1} & \gamma_{T2}^2 & \gamma_{T2}\gamma_{T3} & \gamma_{T2}\gamma_{T4} \\ \gamma_{T3}\gamma_{T1} & \gamma_{T3}\gamma_{T2} & \gamma_{T3}^2 & \gamma_{T3}\gamma_{T4} \\ \gamma_{T4}\gamma_{T1} & \gamma_{T4}\gamma_{T2} & \gamma_{T4}\gamma_{T3} & \gamma_{T4}^2 \end{bmatrix} \tag{109a}$$

$$\left[\Gamma_H\right] = \begin{bmatrix} \gamma_{H1}^2 & \gamma_{H1}\gamma_{H2} & \gamma_{H1}\gamma_{H3} & \gamma_{H1}\gamma_{H4} \\ \gamma_{H2}\gamma_{H1} & \gamma_{H2}^2 & \gamma_{H2}\gamma_{H3} & \gamma_{H2}\gamma_{H4} \\ \gamma_{H3}\gamma_{H1} & \gamma_{H3}\gamma_{H2} & \gamma_{H3}^2 & \gamma_{H3}\gamma_{H4} \\ \gamma_{H4}\gamma_{H1} & \gamma_{H4}\gamma_{H2} & \gamma_{H4}\gamma_{H3} & \gamma_{H4}^2 \end{bmatrix} \tag{109b}$$

$$\left[\Gamma_V\right] = \begin{bmatrix} \gamma_{V1}^2 & \gamma_{V1}\gamma_{V2} & \gamma_{V1}\gamma_{V3} & \gamma_{V1}\gamma_{V4} \\ \gamma_{V2}\gamma_{V1} & \gamma_{V2}^2 & \gamma_{V2}\gamma_{V3} & \gamma_{V2}\gamma_{V4} \\ \gamma_{V3}\gamma_{V1} & \gamma_{V3}\gamma_{V2} & \gamma_{V3}^2 & \gamma_{V3}\gamma_{V4} \\ \gamma_{V4}\gamma_{V1} & \gamma_{V4}\gamma_{V2} & \gamma_{V4}\gamma_{V3} & \gamma_{V4}^2 \end{bmatrix} \tag{109c}$$

The formation of these matrices is obvious by inspection since the subscripts correspond to the position of the element in the matrix.

Equation (107) may now be written in the following form:

$$[B][f] - \frac{1}{I_c}[\Gamma_T][f] - \frac{1}{A_H}[\Gamma_H][f] - \frac{1}{A_V}[\Gamma_V][f] = \lambda^2[A][f]$$

The vector [f] may be factored out on the left-hand side to obtain

$$\left\{[B] - \frac{1}{I_c}[\Gamma_T] - \frac{1}{A_H}[\Gamma_H] - \frac{1}{A_V}[\Gamma_V]\right\}[f] = \lambda^2[A][f]$$

The matrices within the braces may be combined to give this equation in the form

$$[C][f] = \lambda^2[A][f] \tag{110a}$$

where

$$[C] = [B] - \frac{1}{I_c}[\Gamma_T] - \frac{1}{A_H}[\Gamma_H] - \frac{1}{A_V}[\Gamma_V] \tag{110b}$$

The matrices [A] and [C] are dependent solely upon the geometric properties of the cross section. Convenient methods for computing the elements of $[\Gamma_T]$, $[\Gamma_H]$, and $[\Gamma_V]$ are shown in appendix C. The only information that is required for the computation of these latter matrices is the coordinates of the stiffeners with respect to the principal shear axis. Thus the properties of the cross section which enter into the matrices [C] and [A] are the areas of the stiffeners, the coordinates of the stiffeners, and the warping stiffnesses of the webs.

A few comments concerning numerical methods of solution of equations (110) may be worth while. It would seem, by analogy with vibration problems, that the method of matrix iteration could be applied after multiplying through the equation by $[A]^{-1}$. However, unfortunately, the process proves to be divergent. If the matrices are reduced to eliminate the planar modes, the iteration process will continue to be divergent. There are several numerical methods which may be used to solve these equations. However, a study of the relative merits of such methods is an extensive project in itself and will not be treated herein.

Equations governing the stress modes in sections having single or double symmetry will now be considered. From arguments similar to those used to derive equations (110), two matrix equations may be obtained for the symmetrical and antisymmetrical modes for a cross section which is symmetrical about the x-axis as follows:

$$[C_1][f] = \lambda^2[A][f] \qquad \text{(Symmetrical)} \qquad (111a)$$

$$[C_2][f] = \lambda^2[A][f] \qquad \text{(Antisymmetrical)} \qquad (111b)$$

where

$$[C_1] = [B] - \frac{1}{A_H}[\Gamma_H] \qquad (111c)$$

$$[C_2] = [B] - \frac{1}{I_c}[\Gamma_T] - \frac{1}{A_V}[\Gamma_V] \qquad (111d)$$

If the cross section is symmetrical about the y-axis, equations (111a) and (111b) may again be used with the following definitions of the coefficient matrices:

$$[C_1] = [B] - \frac{1}{A_V}[\Gamma_V] \qquad (112a)$$

$$[C_2] = [B] - \frac{1}{I_c}[\Gamma_T] - \frac{1}{A_H}[\Gamma_H] \qquad (112b)$$

If the cross section has double symmetry, there are four types of modes as determined by the following matrix equations:

$$[C_s][f] = \lambda^2[A][f] \qquad \text{(Symmetrical)} \qquad (113a)$$

$$[C_a][f] = \lambda^2[A][f] \qquad \text{(Antisymmetrical)} \qquad (113b)$$

$$[C_x][f] = \lambda^2[A][f] \qquad \text{(Symmetrical about x-axis)} \qquad (113c)$$

$$[C_y][f] = \lambda^2[A][f] \qquad \text{(Symmetrical about y-axis)} \qquad (113d)$$

where

$$[C_s] = [B] \qquad (113e)$$

$$[C_a] = [B] - \frac{1}{I_c}[\Gamma_T] \qquad (113f)$$

$$[C_x] = [B] - \frac{1}{A_H}[\Gamma_H] \qquad (113g)$$

$$\left[C_y\right] = \left[B\right] - \frac{1}{A_V}\left[\Gamma_V\right] \tag{113h}$$

For simple cross sections algebraic solutions of equations (111), (112), and (113) may be readily derived. A few examples are given in the following section.

Examples of Stress Modes and Section Constants

A few simple examples will be solved for the stress modes and section constants. Four rectangular sections, having various numbers of stiffeners, and also a trapezoidal section, will be considered. The rectangular sections will have double symmetry while the trapezoidal section will have single symmetry.

Four-flange section with double symmetry.- The dimensions of the section are indicated in figure 10(a) and the web stiffnesses are shown in figure 10(b). Since the section is doubly symmetrical it is only necessary to consider the first quadrant with modified stiffnesses as shown in figure 10(c). In the case of a four-flange section there are three planar modes and one secondary mode. The secondary mode is governed by equation (99b) which takes the following form:

$$\left(\Sigma_1\beta\right)f_1 - \frac{4\gamma_{T1}}{I_c}\left(\gamma_{T1}f_1\right) = \lambda^2 a_1 f_1 \tag{114}$$

The coefficient of the first term is given by

$$\Sigma_1\beta = 2\left(\beta_h + \beta_b\right) \tag{115}$$

The geometric discontinuity is given by (see appendix D)

$$\gamma_{T1} = \Delta_1\left(2A_k\beta_k\right)$$

$$= 2A_b\beta_b - 2A_h\beta_h = \frac{A}{2}\left(\beta_b - \beta_h\right) \tag{116}$$

The central moment of inertia is given by (see appendix C)

$$I_c = \sum_k \left(2A_k\right)^2\beta_k = \frac{A^2}{2}\left(\beta_b + \beta_h\right) \tag{117}$$

Inspection of equation (114) shows that the stress f_1 is arbitrary. Its value must be determined by the normalizing condition as expressed by equations (89). The following formula is obtained:

$$f_1 = \frac{1}{\sqrt{A_s}} \tag{118}$$

The remaining stresses are obvious from the conditions of symmetry. The only remaining unknown in equation (114) is the characteristic number λ. Substituting equations (115), (116), and (117) into equation (114) leads to the following formula for λ^2:

$$\lambda^2 = \frac{8\beta_b\beta_h}{a\left(\beta_b + \beta_h\right)} \tag{119}$$

This formula was originally given by Grzedzielski (reference 2).

In order to determine the spanwise variation of the stress mode it is necessary to determine the section constant C_T. This constant is given by the following formula:

$$C_T = \sum_j \gamma_{Tj} f_j$$

$$= 4\gamma_{T1} f_1 = \frac{2A}{\sqrt{A_s}} \left(\beta_b - \beta_h\right) \tag{120}$$

It is also of interest to note the following section properties which are obvious in this case:

$$\left.\begin{aligned} A_H &= 2bt_b \\ A_V &= 2ht_h \end{aligned}\right\} \tag{121}$$

Six-flange section with double symmetry.- This section is illustrated in figure 11. There are six stress modes of which three are primary and three are secondary. The doubly symmetrical modes will be first considered. Only the first quadrant, as shown in figure 11(b), need be considered with modified stiffnesses as shown. Corresponding to equation (99a) two equations may be written as follows:

$$\left(\beta_{12} - a_1\lambda^2\right)f_1 - \beta_{12}f_2 = 0 \tag{122a}$$

$$-\beta_{12}f_1 + \left(\beta_{12} - \frac{a_2}{2}\lambda^2\right)f_2 = 0 \tag{122b}$$

The determinant of the coefficients must vanish.

$$\begin{vmatrix} \left(\beta_{12} - a_1\lambda^2\right) & -\beta_{12} \\ -\beta_{12} & \left(\beta_{12} - \frac{a_2}{2}\lambda^2\right) \end{vmatrix} = 0 \tag{123}$$

Since the primary planar mode has the characteristic value $\lambda = 0$, the expansion of the above determinant must lead to an equation which is linear in λ^2. This equation may be immediately solved to give

$$\lambda^2 = \beta_{12}\left(\frac{1}{a_1} + \frac{2}{a_2}\right) \tag{124}$$

The determination of the secondary stress mode consists merely of solving for f_2 in terms of f_1. This may be done by substituting equation (124) into equation (122a) or (122b). It may also be done by using the orthogonality condition between the primary and secondary modes. After normalization the stresses thus obtained are

$$f_1 = \frac{1}{\sqrt{A_s}}\sqrt{\frac{a_2}{2a_1}} \tag{125a}$$

$$f_2 = -\frac{1}{\sqrt{A_s}}\sqrt{\frac{2a_1}{a_2}} \tag{125b}$$

Four stress modes remain to be determined. There will be one doubly antisymmetrical mode. In this mode the stresses in stiffeners 2 and 5 will be zero. The section will thus behave as a four-flange section under torsional loading and the formulas which have been previously given may be used to determine this mode. If the beam is subjected to horizontal bending there will be one planar mode which may be computed from the flexure formula.

Under the action of vertical bending two stress modes will arise. These modes will be symmetrical about the y-axis and antisymmetrical about the x-axis. The essential stiffnesses are shown in figure 11(c). The stresses are governed by the following two equations:

$$\left(\Sigma_1\beta\right)f_1 - \beta_{12}f_2 - \frac{4\gamma_{V1}^2}{A_V} f_1 = \lambda^2 a_1 f_1 \tag{126a}$$

$$-\beta_{12}f_1 + \beta_{12}f_2 = \lambda^2 \frac{a_2}{2} f_2 \tag{126b}$$

When these equations are solved the normalized values of f_1 and f_2 for the secondary mode are found to be the same as given by equations (125). The value of λ^2 for the secondary mode is given by equation (124). In order to calculate the spanwise variation of the secondary mode it is necessary to determine the section constant C_{V2}. Using equation (94e) this constant is found to have the value

$$C_{V2} = -\frac{4t_{16}}{\sqrt{A_s}}\sqrt{\frac{a_2}{2a_1}} \tag{127}$$

Six-flange sections with double symmetry and two cells.- This cross section is illustrated in figure 12. It differs from the previous cross section by having an internal vertical shear web. It may be readily determined from symmetry conditions that five of the six stress modes for this section are identically the same as for the previous section. The mode which is different is the secondary mode due to vertical bending. Since the mathematical methods of solution are the same as before, the final formulas will be stated without derivation for the secondary vertical bending mode only.

The characteristic number is given by

$$\lambda^2 = \frac{1}{a_1}\left(\Sigma_1\beta - \frac{4\gamma_{V1}^2}{A_V}\right) + \frac{2}{a_2}\left(\Sigma_2\beta - \frac{\gamma_{V2}^2}{A_V}\right) \tag{128}$$

The stiffness quantities $\Sigma_1\beta$ and $\Sigma_2\beta$ are modified stiffness values as shown in figure 12(c). They are given by the formulas

$$\Sigma_1\beta = \beta_{12} + 2\beta_{16}$$

$$\Sigma_2\beta = \beta_{12} + \beta_{25}$$

The normalized formulas for the stresses are again given by equations (125). The section constant C_{V2} is given by

$$C_{V2} = -\frac{2}{A_s}\left(2t_{16}\sqrt{\frac{a_2}{2a_1}} - t_{25}\sqrt{\frac{2a_1}{a_2}}\right) \tag{129}$$

Eight-flange section with double symmetry.- This cross section is illustrated in figure 13(a). There are two modes having each type of symmetry. The stresses and characteristic number for the secondary mode having double symmetry are obtained from equations (124) and (125) by substituting a_2 for $\frac{a_2}{2}$. The two doubly antisymmetrical modes are governed by the following equations:

$$\left(\Sigma_1\beta\right)f_1 - \beta_{12}f_2 - \frac{\gamma_{T1}}{I_c}\sum_j \gamma_{Tj}f_j = \lambda^2 a_1 f_1 \tag{130a}$$

$$-\beta_{12}f_1 + \left(\Sigma_2\beta\right)f_2 = \lambda^2 a_2 f_2 \tag{130b}$$

The characteristic equation is as follows:

$$\begin{vmatrix} \left(\Sigma_1\beta - \frac{4\gamma_{T1}^2}{I_c} - a_1\lambda^2\right) & -\beta_{12} \\ -\beta_{12} & \left(\Sigma_2\beta_2 - a_2\lambda^2\right) \end{vmatrix} = 0 \tag{131}$$

Both modes are secondary and it is necessary to solve a quadratic equation in λ^2. Consequently no simple formulas can be given for these two modes.

In the case of vertical bending there is one primary mode and one secondary mode. For this secondary mode the stresses and characteristic numbers are again obtained from equations (124) and (125) by substituting a_2 for $\frac{a_2}{2}$. The associated section constant C_{V2} is given by

$$C_{V2} = -\frac{4t_{18}}{\sqrt{A_s}}\sqrt{\frac{a_2}{a_1}} \tag{132}$$

The equations which govern the horizontal bending modes are as follows:

$$\left(\Sigma_1\beta\right)f_1 - \beta_{12}f_2 - \frac{\gamma_{H1}}{A_H}\sum_j \gamma_{Hj}f_j = \lambda^2 a_1 f_1 \tag{133a}$$

$$-\beta_{12}f_1 + \left(\Sigma_2\beta\right)f_2 = \lambda^2 a_2 f_2 \tag{133b}$$

The characteristic equation is linear in λ^2 and leads immediately to the following formula:

$$\lambda^2 = \frac{1}{a_1}\left(\Sigma_1\beta - \frac{4\gamma_{H1}^2}{A_H}\right) + \frac{1}{a_2}\Sigma_2\beta \tag{134}$$

Orthogonality between the primary and secondary mode is expressed by the following equation:

$$a_1x_1f_1 + a_2x_2f_2 = 0$$

or

$$f_2 = -\left(\frac{a_1x_1}{a_2x_2}\right)f_1$$

If the above formula is substituted into the normalizing equation, it is found that the normalized stress formulas are as follows:

$$f_1 = \frac{x_2}{\sqrt{I_y}}\sqrt{\frac{a_2}{a_1}} \tag{135a}$$

$$f_2 = -\frac{x_1}{\sqrt{I_y}}\sqrt{\frac{a_1}{a_2}} \tag{135b}$$

The associated section constant C_{H2} is given by

$$C_{H2} = -\frac{4t_{12}x_2}{\sqrt{I_y}}\sqrt{\frac{a_2}{a_1}} \tag{136}$$

Trapezoidal section.- This section is assumed to have single symmetry as shown in figure 14. The section properties A_H and A_V may be immediately determined to be as follows:

$$A_H = 2t_{12}L_{12}\cos^2\alpha_{12} \tag{137a}$$

$$A_V = t_{14}L_{14} + t_{23}L_{23} + 2t_{12}L_{12}\sin^2\alpha_{12} \tag{137b}$$

The remaining section properties cannot be determined until the principal shear axes are located. Because of symmetry it is only necessary to locate the origin of the axes. This is determined from equation (71b). This equation provides the following formula for the coordinate of the first flange:

$$\frac{x_1}{b} = \frac{1}{A_V}\left(t_{23}L_{23} - t_{12}L_{14}\sin\alpha_{12}\right) \tag{138}$$

This formula may be obtained immediately by direct substitution into the formulas given by Kirste (reference 5), Drymael (reference 8), or Duberg (reference 18). The radii to the wall segments and the geometric discontinuities may now be computed.

Two of the stress modes are symmetrical about the x-axis. These are primary modes due to axial force and horizontal bending. The remaining two modes are antisymmetrical about the x-axis. One of these is a primary mode and the other a secondary mode. The antisymmetrical modes are governed by the following equations:

$$\left(\Sigma_1\beta\right)f_1 - \beta_{12}f_2 - \frac{\gamma_{T1}}{I_c}\sum_j \gamma_{Tj}f_j - \frac{\gamma_{V1}}{A_V}\sum_j \gamma_{Vj}f_j = a_1\lambda^2 f_1 \tag{139a}$$

$$-\beta_{12}f_1 + \left(\Sigma_2\beta\right)f_2 - \frac{\gamma_{T2}}{I_c}\sum_j \gamma_{Tj}f_j - \frac{\gamma_{V2}}{A_V}\sum_j \gamma_{Vj}f_j = a_2\lambda^2 f_2 \tag{139b}$$

The characteristic equation is linear in λ^2 and yields the following formula:

$$\lambda^2 = \frac{1}{a_1}\left(\Sigma_1\beta - \frac{2\gamma_{T1}^2}{I_c} - \frac{2\gamma_{V1}^2}{A_V}\right) + \frac{1}{a_2}\left(\Sigma_2\beta - \frac{2\gamma_{T2}^2}{I_c} - \frac{2\gamma_{V2}^2}{A_V}\right) \tag{140}$$

In the above equation the summations $\Sigma_1\beta$ and $\Sigma_2\beta$ contain modified stiffnesses as illustrated in figure 14(b).

The stress values are most easily obtained from the orthogonality condition which is expressed as follows:

$$a_1 y_1 f_1 + a_2 y_2 f_2 = 0$$

or

$$f_2 = -\left(\frac{a_1 y_1}{a_2 y_2}\right) f_1 \tag{141}$$

The normalizing condition then gives the following specific stress formulas:

$$f_1 = \frac{y_2}{\sqrt{I_x}}\sqrt{\frac{a_2}{a_1}} \tag{142a}$$

$$f_2 = -\frac{y_1}{\sqrt{I_x}}\sqrt{\frac{a_1}{a_2}} \tag{142b}$$

The section properties C_{V2} and C_{T2} are given by the formulas

$$C_{V2} = \frac{2}{\sqrt{I_x}}\left(\gamma_{V1} y_2 \sqrt{\frac{a_1}{a_2}} - \gamma_{V2} y_1 \sqrt{\frac{a_2}{a_1}}\right) \tag{143a}$$

$$C_{T2} = \frac{2}{\sqrt{I_x}}\left(\gamma_{T1} y_2 \sqrt{\frac{a_1}{a_2}} - \gamma_{T2} y_1 \sqrt{\frac{a_2}{a_1}}\right) \tag{143b}$$

where

$$\gamma_{V1} = t_{12} \sin \alpha_{12} - t_{14} \tag{143c}$$

$$\gamma_{V2} = -t_{23} - t_{12} \sin \alpha_{12} \tag{143d}$$

$$\gamma_{T1} = r_{12}t_{12} - r_{14}t_{14} \tag{143e}$$

$$\gamma_{T2} = r_{23}t_{23} - r_{12}t_{12} \tag{143f}$$

CONCLUDING REMARKS

A method of solution has been obtained for determining secondary stresses in thin-walled beams of arbitrary cross section. Monocoque sections with corners require additional conditions of continuity at the corners and have not been considered. The introduction of generalized difference equations for the semimonocoque sections appears to offer definite advantages and may prove to be useful in the analysis of other types of structures. By using these difference equations the analysis for stiffened beams can be carried out in a manner exactly parallel to the method of analytic functions for a single-cell unstiffened beam.

Future research study should consider elastic ribs for wings, elastic stiffening rings for noncircular fuselages and the thick-walled wing without ribs. It is possible that the natural stress modes will be useful in such problems. An extension of the present theory to swept beams is also a currently important problem. Although of lesser practical importance, the effect of secondary stresses on vibration frequencies may now be determined. Practical numerical methods of computing the natural stress modes of stiffened sections should be developed. A comparison of all available test data with accurate, as well as approximate, solutions for secondary stresses would be of considerable interest.

California Institute of Technology
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APPENDIX A

DEFLECTIONS DUE TO PLANAR STRESS DISTRIBUTION

Equations (23) give the following relations between stresses and deflections:

$$\frac{d^2u}{dz^2} = -\frac{p_H}{GA_H} - \frac{1}{E'A_H}\oint t_w \cos\alpha \frac{\partial\sigma}{\partial s}\,ds \tag{A1}$$

$$\frac{d^2v}{dz^2} = -\frac{p_V}{GA_V} - \frac{1}{E'A_V}\oint t_w \sin\alpha \frac{\partial\sigma}{\partial s}\,ds \tag{A2}$$

The flexure formula defines a planar distribution of stresses as follows:

$$\sigma = \frac{P}{A_w} + \frac{M_y\bar{x}}{I_y} + \frac{M_x\bar{y}}{I_x} \tag{A3}$$

It is necessary to express the centroidal coordinates $\bar{x}$ and $\bar{y}$ in equation (A3) in terms of the coordinates x and y with respect to the principal shear axes. The relationship between the two coordinate axes is shown in figure 15. From the geometry of this figure it may be shown that the coordinates of a point are related by the following equations:

$$\bar{x} = \bar{x}_o + x\cos\theta - y\sin\theta \tag{A4}$$

$$\bar{y} = \bar{y}_o + x\sin\theta + y\cos\theta \tag{A5}$$

The differentials dx, dy, and ds are related to the angle α by the following formulas:

$$\left.\begin{aligned}\frac{dy}{ds} &= \sin\alpha\\ \frac{dx}{ds} &= \cos\alpha\end{aligned}\right\} \tag{A6}$$

It is necessary to determine $\partial\sigma/\partial s$ to substitute into equations (A1) and (A2). Differentiating equation (A3) gives

$$\frac{\partial \sigma}{\partial s} = \frac{M_y}{I_y}\frac{d\bar{x}}{ds} + \frac{M_x}{I_x}\frac{d\bar{y}}{ds} \tag{A7}$$

Differentiating equations (A4) and (A5) gives

$$\frac{d\bar{x}}{ds} = \frac{dx}{ds}\cos\theta - \frac{dy}{ds}\sin\theta \tag{A8}$$

$$\frac{d\bar{y}}{ds} = \frac{dx}{ds}\sin\theta + \frac{dy}{ds}\cos\theta \tag{A9}$$

Substituting equations (A6) into equations (A8) and (A9) gives

$$\frac{d\bar{x}}{ds} = \cos\alpha\cos\theta - \sin\alpha\sin\theta \tag{A10}$$

$$\frac{d\bar{y}}{ds} = \cos\alpha\sin\theta + \sin\alpha\cos\theta \tag{A11}$$

Substituting equations (A10) and (A11) into equation (A7) gives

$$\frac{\partial \sigma}{\partial s} = \frac{M_y}{I_y}(\cos\alpha\cos\theta - \sin\alpha\sin\theta) + \frac{M_x}{I_x}(\cos\alpha\sin\theta + \sin\alpha\cos\theta) \tag{A12}$$

The integral contained in equation (A1) may now be evaluated as follows:

$$\oint t_w \cos\alpha \frac{\partial \sigma}{\partial s}\,ds = \frac{M_y}{I_y}\oint t_w \cos\alpha(\cos\alpha\cos\theta - \sin\alpha\sin\theta)\,ds + \frac{M_x}{I_x}\oint t_w \cos\alpha(\cos\alpha\sin\theta + \sin\alpha\cos\theta)\,ds \tag{A13}$$

Two of the four integrals on the right-hand side vanish because of the definition of the location of the principal shear axes as expressed by equation (15c). The integral takes the following value:

$$\oint t_w \cos\alpha \frac{\partial \sigma}{\partial s}\,ds = \frac{M_y}{I_y} A_H \cos\theta + \frac{M_x}{I_x} A_H \sin\theta \tag{A14}$$

Similarly the integral in equation (A2) may be shown to have the following value:

$$\oint t_w \sin \alpha \frac{\partial \sigma}{\partial s} ds = -\frac{M_y}{I_y} A_V \sin \theta + \frac{M_x}{I_x} A_V \cos \theta \tag{A15}$$

Equations (A14) and (A15) may now be substituted into equations (A1) and (A2) to obtain the following formulas for deflections:

$$\frac{d^2u}{dz^2} = -\frac{p_H}{GA_H} - \frac{M_y \cos \theta}{E'I_y} - \frac{M_x \sin \theta}{E'I_x} \tag{A16}$$

$$\frac{d^2v}{dz^2} = -\frac{p_V}{GA_V} + \frac{M_y \sin \theta}{E'I_y} - \frac{M_x \cos \theta}{E'I_x} \tag{A17}$$

These equations are given in the text as equations (25).

APPENDIX B

EFFECTS OF SYMMETRY OF CROSS SECTION

The axial displacements in an unsymmetrical cross section are governed by equation (20a). From considerations of symmetry it is possible to replace this equation by four equations of simpler form. It is well-known that solutions to structural problems involving one independent variable can be divided into symmetrical and antisymmetrical parts when the structure is symmetrical. In the case of a beam with a cross section having double symmetry, since the transverse distribution of stresses or displacements on a cross section depends on two independent variables, the solutions may be divided into four parts or components. The four components of a function will be indicated, in the present appendix only, by subscripts defined as follows:

a	antisymmetrical about both axes
s	symmetrical about both axes
x	symmetrical about x-axis
y	symmetrical about y-axis

The stresses and displacements which occur on a cross section may be expressed in terms of their components as follows:

$$\left.\begin{aligned} \tau &= \tau_a + \tau_s + \tau_x + \tau_y \\ \sigma &= \sigma_a + \sigma_s + \sigma_x + \sigma_y \\ \xi &= \xi_a + \xi_s + \xi_x + \xi_y \\ w &= w_a + w_s + w_x + w_y \end{aligned}\right\} \tag{B1}$$

Three geometrical quantities r, $\cos\alpha$, and $\sin\alpha$, enter into the analysis. For a section with double symmetry these quantities have the following symmetry properties:

$r \longrightarrow$ Symmetrical about both axes

$\cos\alpha \longrightarrow$ Symmetrical about y-axis

$\sin\alpha \longrightarrow$ Symmetrical about x-axis

The symmetry properties of the derivatives of these geometric quantities are readily seen to be as follows:

$dr/ds \longrightarrow$ Antisymmetrical about both axes

$d \cos \alpha/ds \longrightarrow$ Symmetrical about x-axis

$d \sin \alpha/ds \longrightarrow$ Symmetrical about y-axis

It is now necessary to determine the symmetry properties of the second derivative of w with respect to s. The four components of the second derivative may be indicated as follows:

$$\frac{\partial^2 w}{\partial s^2} = \left(\frac{\partial^2 w}{\partial s^2}\right)_a + \left(\frac{\partial^2 w}{\partial s^2}\right)_s + \left(\frac{\partial^2 w}{\partial s^2}\right)_x + \left(\frac{\partial^2 w}{\partial s^2}\right)_y \tag{B2}$$

If a function having double symmetry is differentiated once, a function having double antisymmetry is produced, and vice versa. If a function which is symmetrical about the x-axis is differentiated once, a function is obtained which is symmetrical about the y-axis, and vice versa. However, if a function has a particular type of symmetry, the second derivative of the function will have the same type of symmetry. Consequently the four components of $\frac{\partial^2 w}{\partial s^2}$ can be written in terms of the four components of w as follows:

$$\left.\begin{aligned} \left(\frac{\partial^2 w}{\partial s^2}\right)_a &= \frac{\partial^2 w_a}{\partial s^2} & \left(\frac{\partial^2 w}{\partial s^2}\right)_s &= \frac{\partial^2 w_s}{\partial s^2} \\ \left(\frac{\partial^2 w}{\partial s^2}\right)_x &= \frac{\partial^2 w_x}{\partial s^2} & \left(\frac{\partial^2 w}{\partial s^2}\right)_y &= \frac{\partial^2 w_y}{\partial s^2} \end{aligned}\right\} \tag{B3}$$

In addition it is necessary to consider the integrals which appear in equation (20a). After an integration by parts the integrand of each integral contains the product of a geometric property and the axial displacement. The only component of the axial displacement which contributes a finite quantity to the value of a given integral is that component which has the same symmetry as the geometric property appearing in the integral. From this consideration the three integrals are seen to have the following values:

$$\left.\begin{aligned}
\oint t_w r \frac{\partial w}{\partial s}\,ds &= \oint t_w r \frac{\partial w_a}{\partial s}\,ds \\
\oint t_w \cos\alpha \frac{\partial w}{\partial s}\,ds &= \oint t_w \cos\alpha \frac{\partial w_x}{\partial s}\,ds \\
\oint t_w \sin\alpha \frac{\partial w}{\partial s}\,ds &= \oint t_w \sin\alpha \frac{\partial w_y}{\partial s}\,ds
\end{aligned}\right\} \tag{B4}$$

It is now possible to consider the symmetry properties of each term in equation (20a). The first two terms have four components as defined by equations (B1), (B2), and (B3). Each of the remaining terms has only one type of symmetry which may be determined from the symmetry properties of the derivatives of the geometric properties of the cross section. The sum of the terms on the left side of equation (20a) having a particular type of symmetry must be equal to the term on the right side having that same type of symmetry. This permits the writing of four equations corresponding to the four types of symmetry as shown below.

Symmetrical about both axes:

$$\frac{\partial^2 w_s}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w_s}{\partial z^2} = 0 \tag{B5}$$

Antisymmetrical about both axes:

$$\frac{\partial^2 w_a}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w_a}{\partial z^2} - \frac{1}{I_c}\frac{dr}{ds}\oint t_w r \frac{\partial w_a}{\partial s}\,ds = -\frac{T}{GI_c}\frac{dr}{ds} \tag{B6}$$

Symmetrical about x-axis:

$$\frac{\partial^2 w_x}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w_x}{\partial z^2} - \frac{1}{A_H}\frac{d\cos\alpha}{ds}\oint t_w \cos\alpha \frac{\partial w_x}{\partial s}\,ds = -\frac{H}{GA_H}\frac{d\cos\alpha}{ds} \tag{B7}$$

Symmetrical about y-axis:

$$\frac{\partial^2 w_y}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w_y}{\partial z^2} - \frac{1}{A_V}\frac{d\sin\alpha}{ds}\oint t_w \sin\alpha \frac{\partial w_y}{\partial s}\,ds = -\frac{V}{GA_V}\frac{d\sin\alpha}{ds} \tag{B8}$$

A corresponding set of four equations for normal stresses is readily obtainable.

The displacement distribution governed by equation (B5) is due to the action of an axial force only. Methods for the solution of this elliptic equation are well-known. The problem is mathematically equivalent to a flat rectangular plate with transverse stiffeners of infinite stiffness and with sidesway prevented. Equations (B6), (B7), and (B8) are given in the text as equations (27) with the subscripts omitted for convenience. The subscripts which appear in equations (B5) to (B8) show clearly the particular component of the displacement which is associated with each type of external loading. From the differential equations which relate τ, σ, and ξ to w it is possible to determine which component of τ, σ, or ξ occurs with a particular type of loading on a beam with doubly symmetrical cross section.

When a cross section is symmetrical about one axis only, the solution for an unknown may be divided into symmetrical and antisymmetrical parts only. For a cross section which is symmetrical about the x-axis only, the geometrical properties of the cross section have the following symmetry properties:

$r \longrightarrow$ Symmetrical about x-axis

$\cos \alpha \longrightarrow$ Antisymmetrical about x-axis

$\sin \alpha \longrightarrow$ Symmetrical about x-axis

$dr/ds \longrightarrow$ Antisymmetrical about x-axis

$d \cos \alpha/ds \longrightarrow$ Symmetrical about x-axis

$d \sin \alpha/ds \longrightarrow$ Antisymmetrical about x-axis

By using arguments similar to those used for the doubly symmetrical section, it is found that equation (20a), for the general cross section, separates into the following two equations for sections which are symmetrical about the x-axis:

$$\frac{\partial^2 w_{sx}}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w_{sx}}{\partial z^2} - \frac{1}{A_H}\frac{d \cos \alpha}{ds}\oint t_w \cos \alpha \frac{\partial w_{sx}}{\partial s}\, ds = -\frac{H}{GA_H}\frac{d \cos \alpha}{ds} \qquad \text{(B9)}$$

$$\frac{\partial^2 w_{ax}}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w_{ax}}{\partial z^2} - \frac{1}{I_c}\frac{dr}{ds}\oint t_w r \frac{\partial w_{ax}}{\partial s}\,ds - \frac{1}{A_V}\frac{d \sin \alpha}{ds}\oint t_w \sin \alpha \frac{\partial w_{ax}}{\partial s}\,ds =$$

$$-\frac{T}{GI_c}\frac{dr}{ds} - \frac{V}{GA_V}\frac{d \sin \alpha}{ds} \qquad \text{(B10)}$$

In these equations the subscripts sx and ax indicate symmetry and antisymmetry, respectively, about the x-axis. Equations (B9) and (B10) are given without subscripts as equations (29) in the text.

In a similar manner it may be shown that, for the case of a section which is symmetrical about the y-axis, the general equation separates into the following two equations:

$$\frac{\partial^2 w_{sy}}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w_{sy}}{\partial z^2} - \frac{1}{A_V}\frac{d \sin \alpha}{ds}\oint t_w \sin \alpha \frac{\partial w_{sy}}{\partial s}\,ds = -\frac{V}{GA_V}\frac{d \sin \alpha}{ds} \qquad \text{(B11)}$$

$$\frac{\partial^2 w_{ay}}{\partial s^2} + \frac{E'}{G}\frac{\partial^2 w_{ay}}{\partial z^2} - \frac{1}{I_c}\frac{dr}{ds}\oint t_w r \frac{\partial w_{ay}}{\partial s}\,ds - \frac{1}{A_H}\frac{d \cos \alpha}{ds}\oint t_w \cos \alpha \frac{\partial w_{ay}}{\partial s}\,ds =$$

$$-\frac{T}{GI_c}\frac{dr}{ds} - \frac{H}{GA_H}\frac{d \cos \alpha}{ds} \qquad \text{(B12)}$$

These equations are given without subscripts in the text as equations (30).

APPENDIX C

SHEARING SECTION PROPERTIES FOR STIFFENED SECTIONS

Referring to figure 16(a), the kth web is shown to extend from stiffener a to stiffener b. The length of this web l_k has a horizontal component given by $x_a - x_b$, or $-\Delta_k x_j$ and a vertical component $y_b - y_a$, or $\Delta_k y_j$. The trigonometric functions of the angle α_k may thus be written as

$$\cos \alpha_k = \frac{1}{l_k} \Delta_k x_j \tag{C1}$$

$$\sin \alpha_k = \frac{1}{l_k} \Delta_k y_j \tag{C2}$$

The shear resistant areas are defined by equations (72b) and (72c). They may thus be computed from the following formulas:

$$A_H = \sum_k t_k l_k \cos^2 \alpha_k = \sum_k \left(\Delta_k x_j\right)^2 \beta_k \tag{C3}$$

$$A_V = \sum_k t_k l_k \sin^2 \alpha_k = \sum_k \left(\Delta_k y_j\right)^2 \beta_k \tag{C4}$$

In order to develop a convenient formula for computing the central moment of inertia it is necessary to define the "central area" which may be associated with the kth web. The central area A_k is illustrated in figure 16(b). It may be defined as the area of the triangle formed by the kth web and the two radial lines drawn to the origin of the principal shear axes from the ends of the kth web. The radius r_k is related to the coordinates of the stiffeners by the following equations:

$$r_k = x_a \sin \alpha_k - y_a \cos \alpha_k$$

Substituting equations (C1) and (C2) gives

$$r_k = \frac{x_a}{L_k}\Delta_k y_j - \frac{y_a}{L_k}\Delta_k x_j$$

$$= \frac{x_a}{L_k}\left(y_b - y_a\right) - \frac{y_a}{L_k}\left(x_b - x_a\right)$$

$$= \frac{1}{L_k}\left(x_a y_b - y_a x_b\right) \tag{C5}$$

The area A_k may now be related to the coordinates of the stiffeners by the following equations:

$$2A_k = r_k L_k = x_a y_b - y_a x_b \tag{C6}$$

This formula may be written as a determinant.

$$2A_k = \begin{vmatrix} x_a & x_b \\ y_a & y_b \end{vmatrix} \tag{C7}$$

The formula for the central moment of inertia is given by equation (72a) and may now be computed from the following formula:

$$I_c = \sum_k t_k L_k r_k^2 = \sum_k \left(2A_k\right)^2 \beta_k \tag{C8}$$

APPENDIX D

GEOMETRIC DISCONTINUITIES OF CROSS SECTION

Reference may be made to figure 4(b) where the jth stiffener lies between web a and web b. The geometric discontinuites, which are defined at the stiffeners, are given by equations (77). The discontinuity γ_{Hj} is defined by

$$\gamma_{Hj} = \Delta_j \rho_{Hk} = \Delta_j \left(t_k \cos \alpha_k \right) \tag{D1}$$

This may be written as

$$\gamma_{Hj} = t_b \cos \alpha_b - t_a \cos \alpha_a$$

Substituting equation (C1) gives

$$\gamma_{Hj} = \frac{t_b}{L_b}\left(x_{j+1} - x_j\right) - \frac{t_a}{L_a}\left(x_j - x_{j-1}\right)$$

$$= \beta_{j(j+1)} x_{j+1} - \left(\Sigma_j \beta\right) x_j + \beta_{j(j-1)} x_{j-1} \tag{D2}$$

From equation (D2) it is seen that the column vector $\left[\gamma_H\right]$ as defined by equation (101b) may be computed from the following matrix formula:

$$\left[\gamma_H\right] = -\left[B\right]\left[x\right] \tag{D3}$$

where

$$\left[x\right]' = \left[x_1 x_2 \cdots x_n\right]$$

The matrix $\left[B\right]$ is defined for a four-flange section in equation (105). Equation (D3) is also applicable to multicell sections. From similar arguments the discontinuity γ_{Vj} is found to be given by the following formula:

$$\gamma_{Vj} = \beta_{j(j+1)} y_{j+1} - \left(\Sigma_j \beta\right) y_j + \beta_{j(j-1)} y_{j-1} \tag{D4}$$

The column vector $\left[\gamma_V\right]$ is given by the equation

$$\left[\gamma_V\right] = -\left[B\right]\left[y\right] \tag{D5}$$

where

$$\left[y\right]' = \left[y_1 y_2 \cdots y_n\right]$$

The discontinuity γ_{Tj} is defined by the following formula:

$$\gamma_{Tj} = \Delta_j \rho_{Tk} = \Delta_j\left(t_k r_k\right) \tag{D6}$$

Introducing the central area A_k gives

$$\gamma_{Tj} = \Delta_j\left(2\beta_k A_k\right) \tag{D7}$$

The square matrices $\left[\Gamma_T\right]$, $\left[\Gamma_H\right]$, and $\left[\Gamma_V\right]$ are most readily computed from their definitions as given by equations (108) after the column vectors $\left[\gamma_T\right]$, $\left[\gamma_H\right]$, and $\left[\gamma_V\right]$ have been computed.

The section constants C_{Tn}, C_{Hn}, and C_{Vn}, corresponding to the nth stress mode, may be immediately computed from the formulas

$$C_{Tn} = \sum_j \gamma_{Tj} f_{jn}$$

$$C_{Hn} = \sum_j \gamma_{Hj} f_{jn}$$

$$C_{Vn} = \sum_j \gamma_{Vj} f_{jn}$$

APPENDIX E

SUMMATION BY PARTS FOR STIFFENED SECTIONS

The summation by parts which is to be justified is given by equations (78) and has the following form:

$$\sum_k \rho_k \Delta_k w_k = -\sum_j w_j \Delta_j \rho_k \tag{E1}$$

The proof will be given for a very simple multicell section. The section to be considered is shown in figure 17. It has two cells and four stiffeners. The webs and stiffeners are numbered and the positive direction of flow-type quantities is shown.

The first summation of equation (E1) may be expanded in the following manner:

$$\sum_k \rho_k \Delta_k w_j = \rho_1 \Delta_1 w_j + \rho_2 \Delta_2 w_j + \rho_3 \Delta_3 w_j + \rho_4 \Delta_4 w_j + \rho_5 \Delta_5 w_j =$$

$$\rho_1(w_1 - w_4) + \rho_2(w_2 - w_1) + \rho_3(w_3 - w_2) + \rho_4(w_4 - w_3) + \rho_5(w_1 - w_3)$$

The terms on the right-hand side may now be arranged in the following manner:

$$\sum_k \rho_k \Delta_k w_j = w_1(\rho_1 - \rho_2 + \rho_5) + w_2(\rho_2 - \rho_3) +$$

$$w_3(\rho_3 - \rho_4 - \rho_5) + w_4(\rho_4 - \rho_1) \tag{E2}$$

The summation which occurs on the right-hand side of equation (E1) may be expanded as follows:

$$\sum_j w_j \Delta_j \rho_k = w_1 \Delta_1 \rho_k + w_2 \Delta_2 \rho_k + w_3 \Delta_3 \rho_k + w_4 \Delta_4 \rho_k$$

$$= w_1(\rho_2 - \rho_1 - \rho_5) + w_2(\rho_3 - \rho_2) +$$

$$w_3(\rho_4 + \rho_5 - \rho_3) + w_4(\rho_1 - \rho_4) \qquad \text{(E3)}$$

If equation (E2) is now compared with equation (E3), it is seen that equation (E1) is valid. It is clear that the argument could be applied to sections having any finite number of cells or stiffeners since no new element would be brought into the problem by increasing the number of cells or stiffeners.

REFERENCES

1. Ebner, Hans: Torsional Stresses in Box Beams with Cross Sections Partially Restrained against Warping. NACA TM 744, 1934.

2. Grzedzielski, A.: Sur un cas particulier de coopération des longerons d'ailes. Sprawozdania, Inst. Rech. de l'Aero. (Warsaw), vol. 13, no. 1, 1934, pp. 5-19.

3. Billewicz, W., and Grzedzielski, A.: General Methods of Calculation of Two-Spar Wings under Torsion. Proc. Fifth Int. Cong. Appl. Mech. (Sept. 1938, Cambridge, Mass.), John Wiley & Sons, Inc., 1939, pp. 151-158.

4. Kuhn, Paul: Bending Stresses Due to Torsion in Cantilever Box Beams. NACA TN 530, 1935.

5. Kirste, L.: Sur le calcul des poutres en caisson. L'Aéronautique (L'Aérotechnique), vol. 19, no. 212, Jan. 1937, pp. 1-6; no. 216, May 1937, pp. 57-58.

6. Kuhn, Paul: Stress Analysis of Beams with Shear Deformation of the Flanges. NACA Rep. 608, 1937.

7. Ebner, H., and Köller, H.: Calculation of Load Distribution in Stiffened Cylindrical Shells. NACA TM 866, 1938.

8. Drymael, Jean: Some Theoretical Considerations on Box Beams. Jour. Roy. Aero. Soc., vol. XLV, no. 361, Jan. 1941, pp. 5-28.

9. Von Kármán, Theodore, and Chien, Wei-Zang: Torsion with Variable Twist. Jour. Aero. Sci., vol. 13, no. 10, Oct. 1946, pp. 503-510.

10. Hadji-Argyris, J., and Dunne, P. C.: The General Theory of Cylindrical and Conical Tubes under Torsion and Bending Loads. Jour. Roy. Aero. Soc., vol. 51, 1947, pp. 199-269, 757-784, and 884-930; vol. 53, 1949, pp. 461-483 and 558-620.

11. Kempner, Joseph: Recurrence Formulas and Differential Equations for Stress Analysis of Cambered Box Beams. NACA TN 1466, 1947.

12. Peterson, James P.: Shear-Lag Tests of a Box Beam with a Highly Cambered Cover in Tension. NACA ARR L5F27b, 1945.

13. Levy, Samuel: Computation of Influence Coefficients for Aircraft Structures with Discontinuities and Sweepback. Jour. Aero. Sci., vol. 14, no. 10, Oct. 1947, pp. 547-560.

14. Bisplinghoff, R. L.: An Investigation of Stresses in Aircraft Structures under Dynamic Loading. Contract No. N0a(s) 8790, M.I.T. and Bur. Aero., Navy Dept., Jan. 1949.

15. Wittrick, W. H.: Preliminary Analysis of a Highly Swept Cylindrical Tube under Torsion and Bending. Aero. Res. Rep. ACA-39, Australian Council for Scientific and Industrial Research, May 1948.

16. Adadurov, R. A.: Strains and Deformations in a Cylindrical Shell Stiffened by Transverse Membranes. Doklady Akad. Nauk SSSR (N.S.), vol. 62, no. 2, Sept. 1948, pp. 183-186. (Russian).

17. Thompson, J. J., and Wittrick, W. H.: The Stresses in Certain Cylindrical Swept Tubes under Torsion and Bending. Aero. Res. Rep. ACA-43, Australian Council for Scientific and Industrial Research, Jan. 1949.

18. Duberg, J. È.: A Numerical Procedure for the Stress Analysis of Stiffened Shells. Jour. Aero. Sci., vol. 16, no. 8, Aug. 1949, pp. 451-462.

19. Von Kármán, Theodore, and Biot, Maurice A.: Mathematical Methods in Engineering. McGraw-Hill Book Co., Inc., 1940.

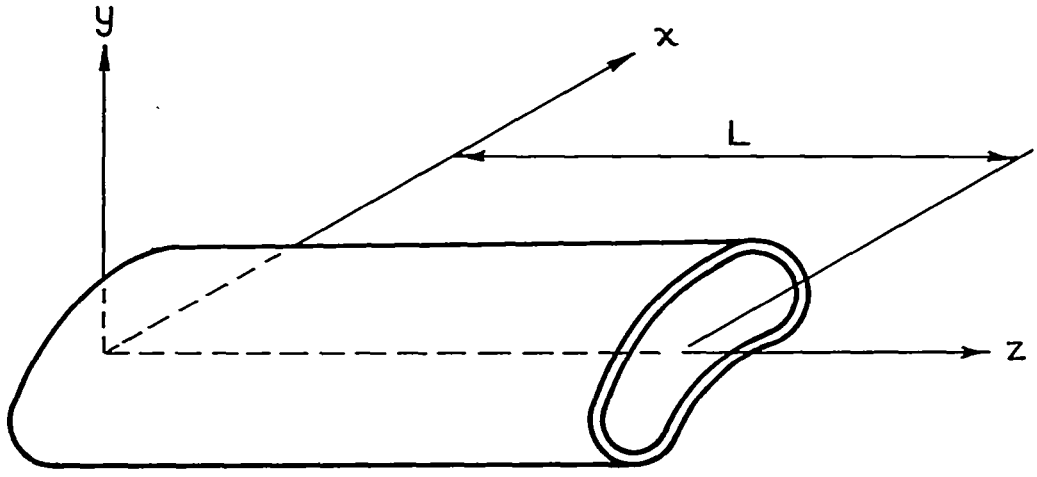
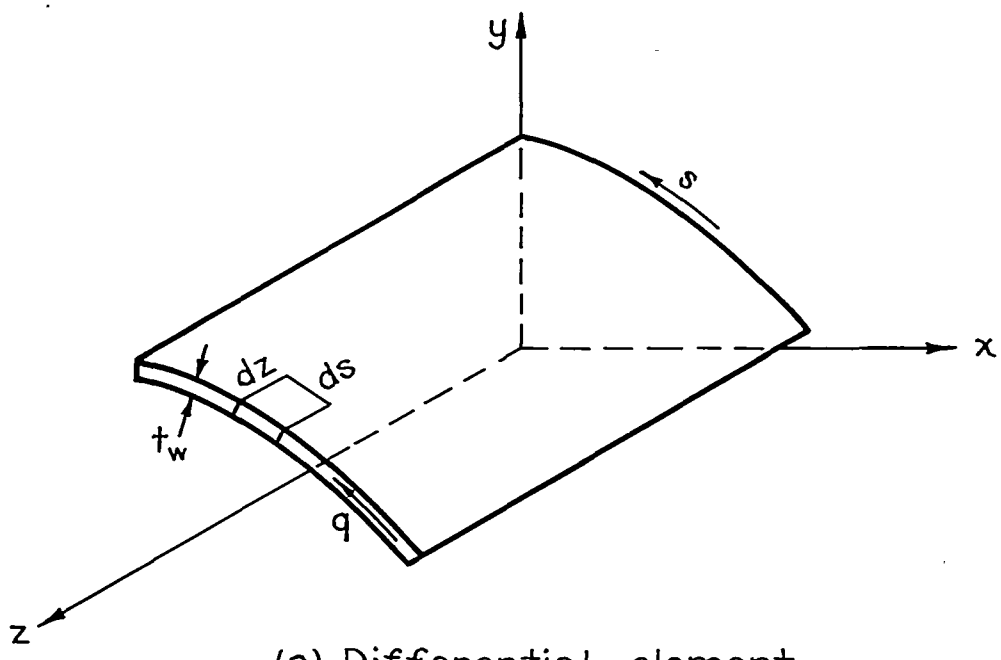
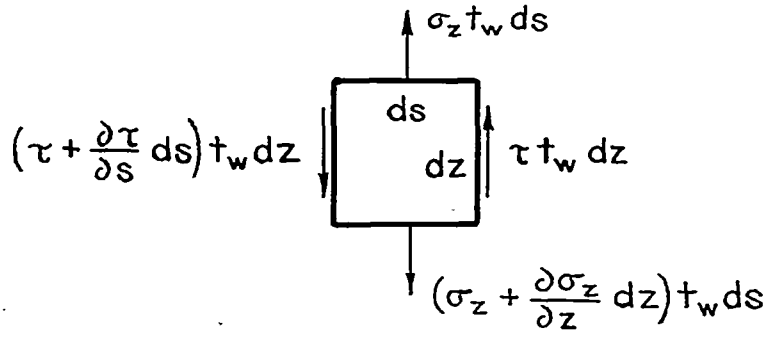


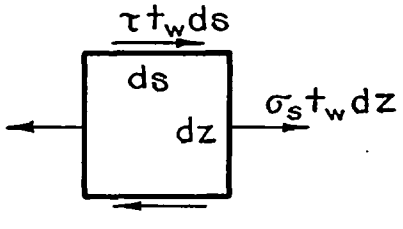
Figure 1.- Coordinate system.



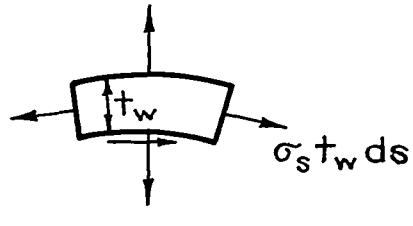
(a) Differential element.



(b) Axial forces.

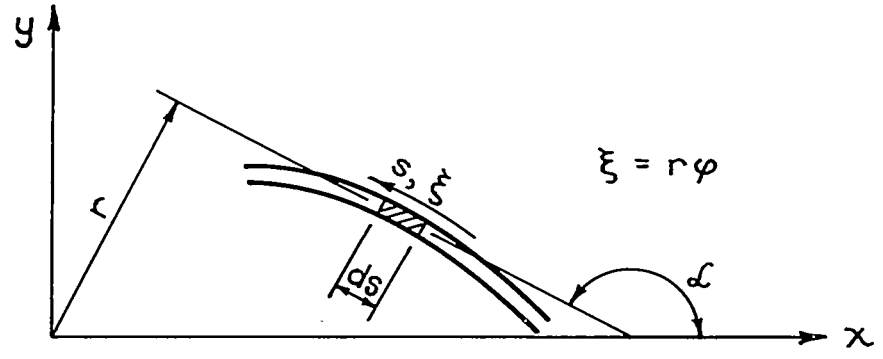


(c) Tangential forces.

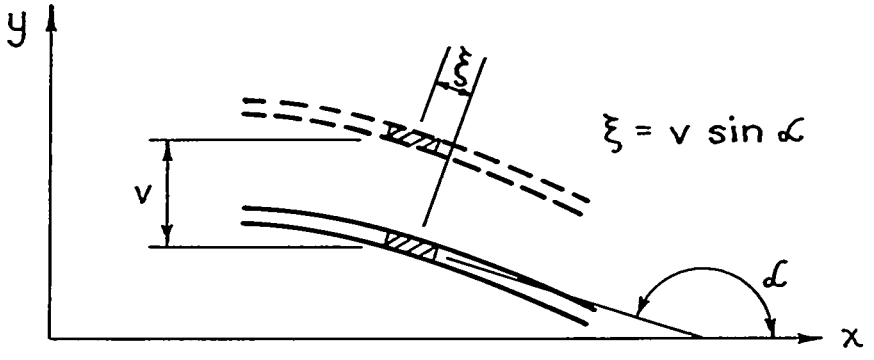


(d) Normal forces.

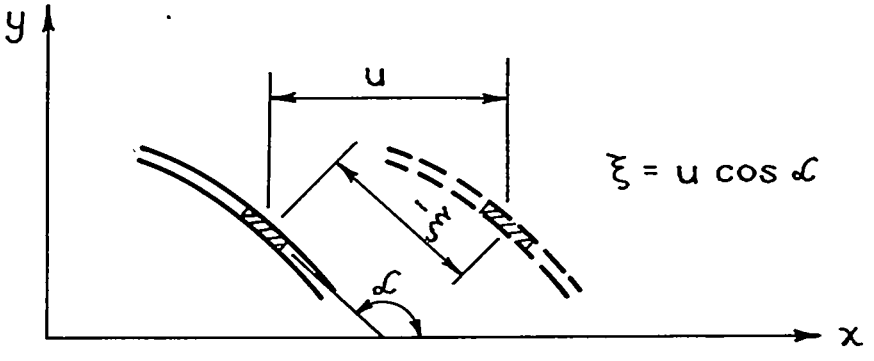
Figure 2.- Forces acting on a wall element.



(a) Due to rotation.

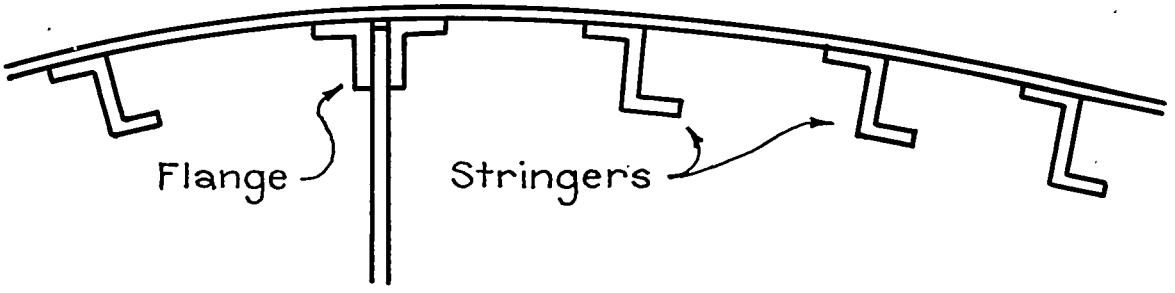


(b) Due to vertical translation.

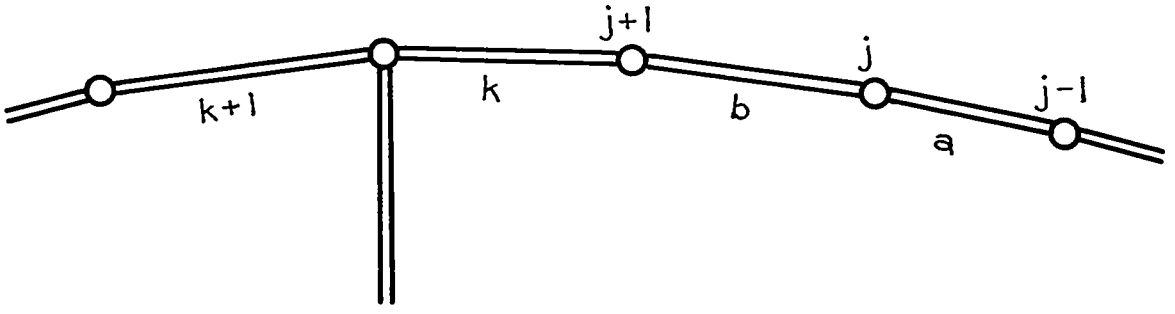


(c) Due to horizontal translation

Figure 3.- Tangential displacements.

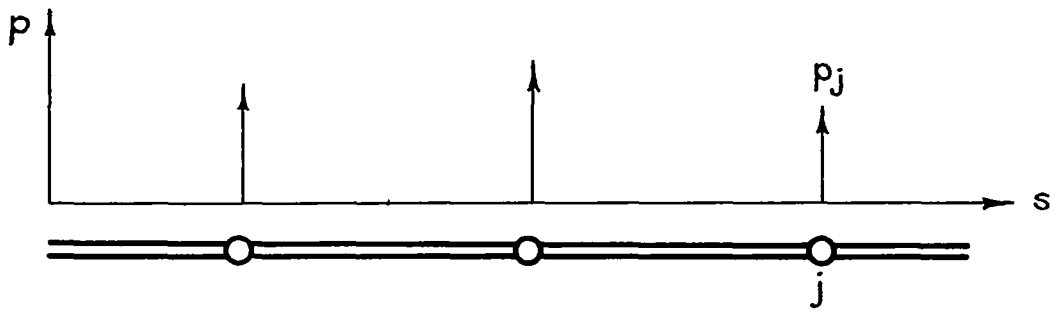


(a) True section.

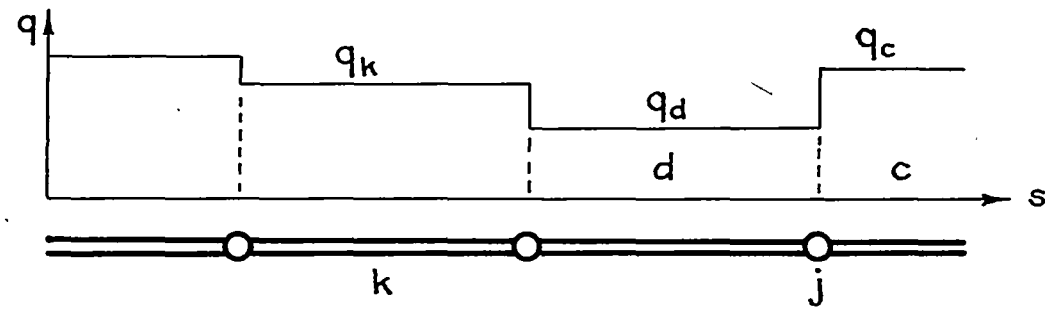


(b) Idealized section.

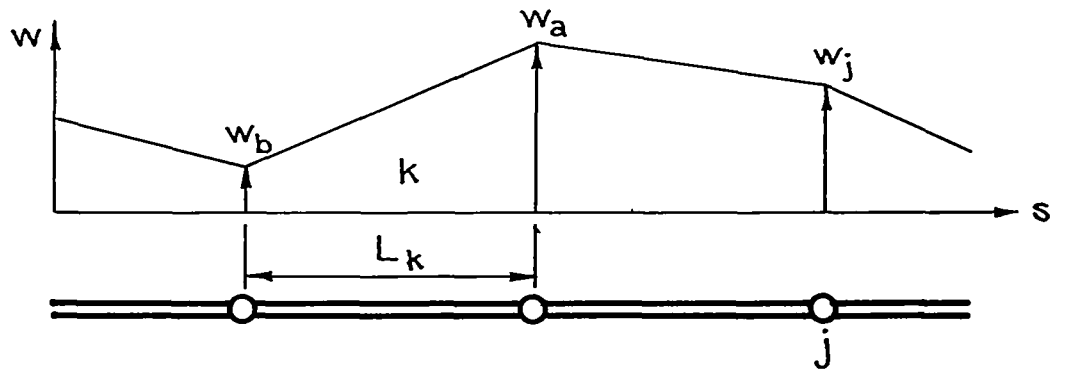
Figure 4.- Idealization of beam cross section.



(a) Singularities.



(b) Step function.



(c) Broken-line function.

Figure 5.- Typical functions for idealized sections.

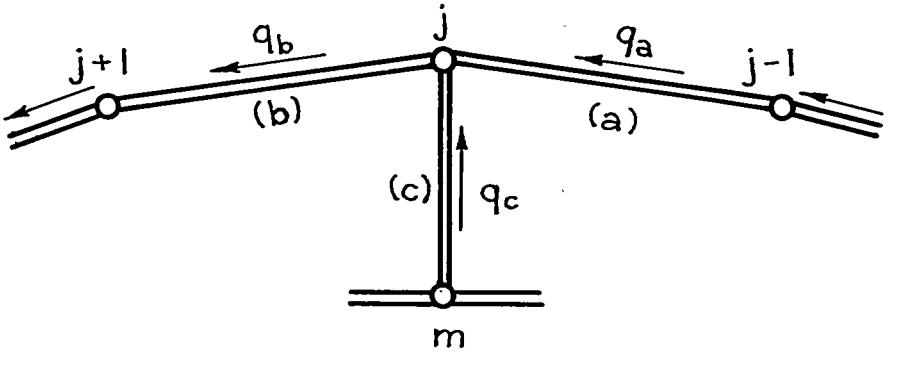


Figure 6.- Shear flows at jth stiffener.

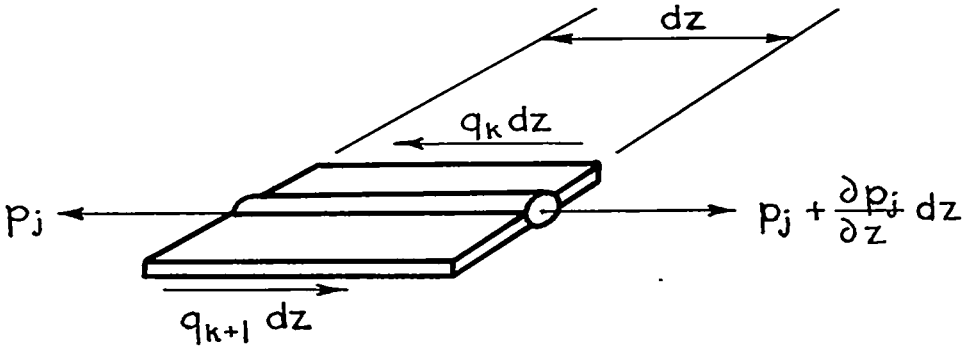
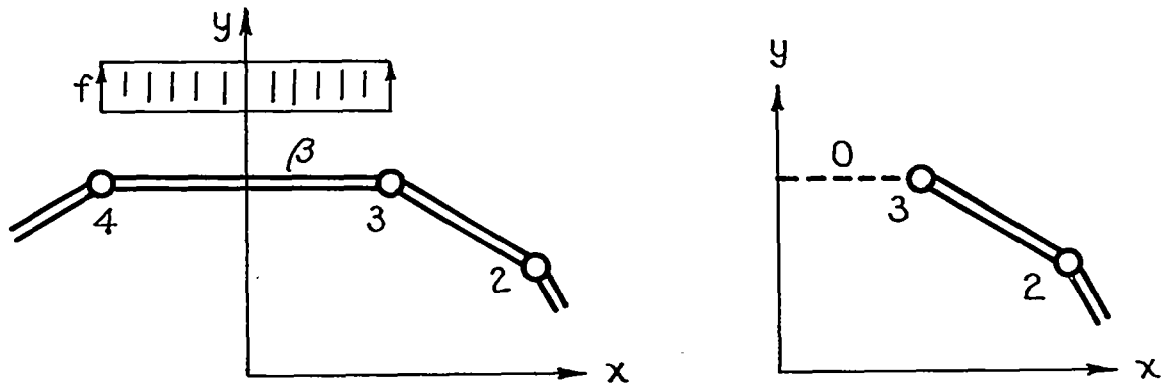
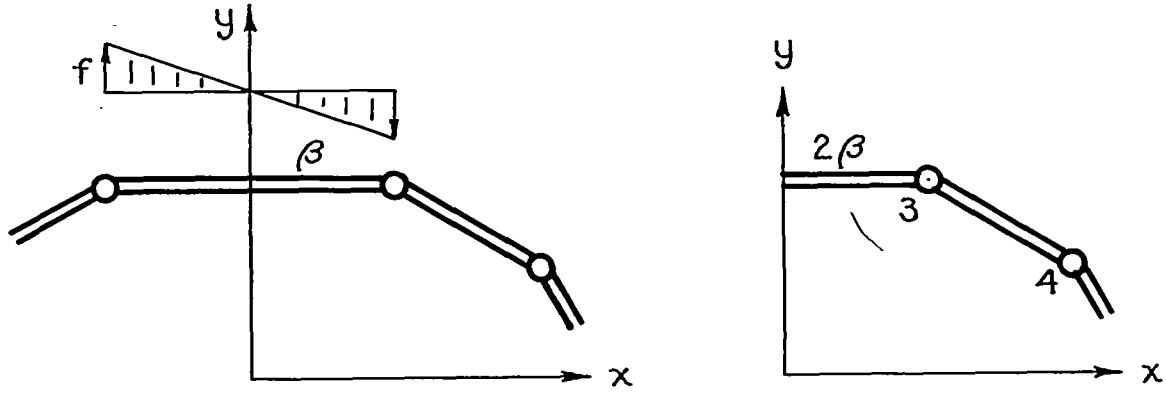


Figure 7.- Forces acting on a differential length of a stiffener.

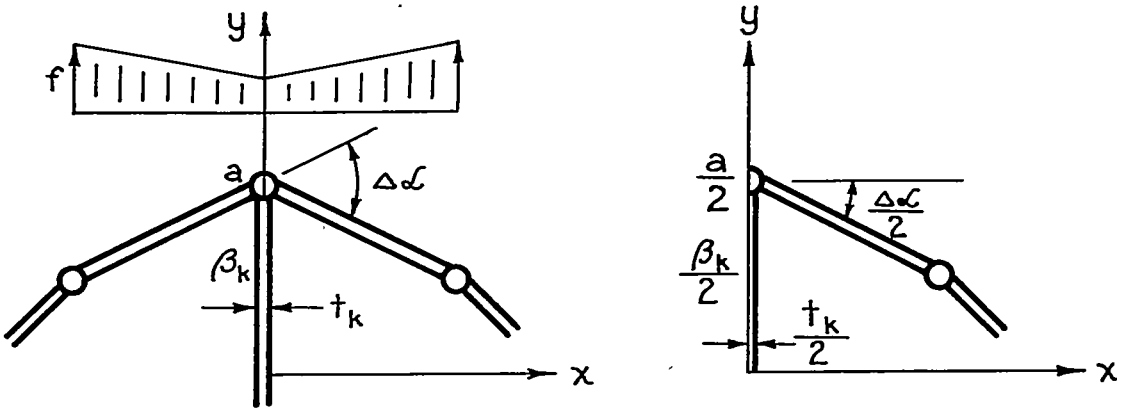


(a) Symmetrical case.

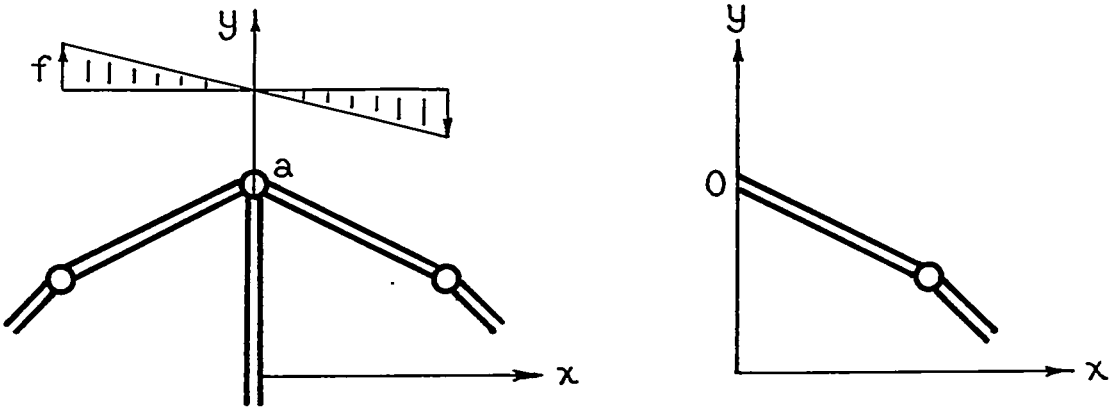


(b) Antisymmetrical case.

Figure 8.- Modified warping stiffness.



(a) Symmetrical case.



(b) Antisymmetrical case.

Figure 9.- Stiffness modifications for a central flange.

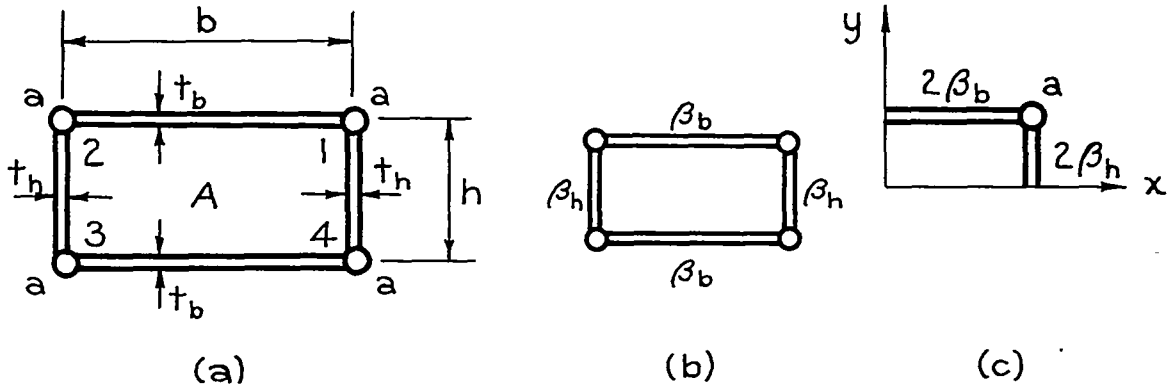
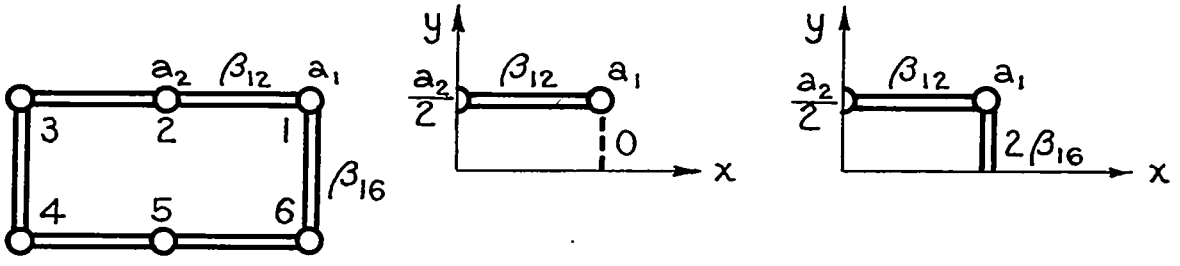


Figure 10.- Four-flange section.

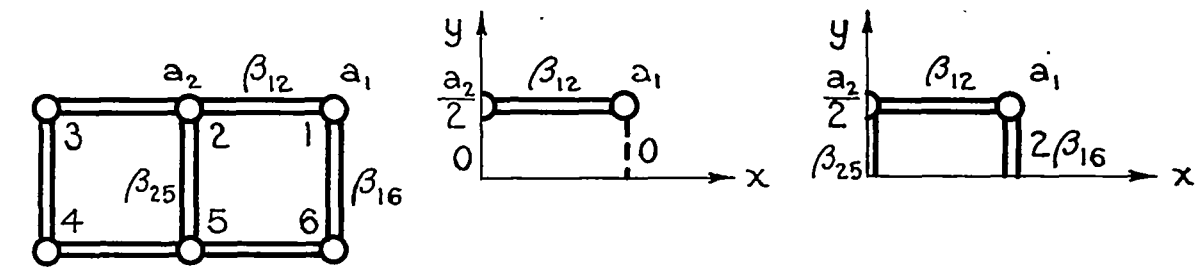


(a) Stiffnesses.

(b) Double symmetry.

(c) Single symmetry about y-axis.

Figure 11.- Six-flange section.

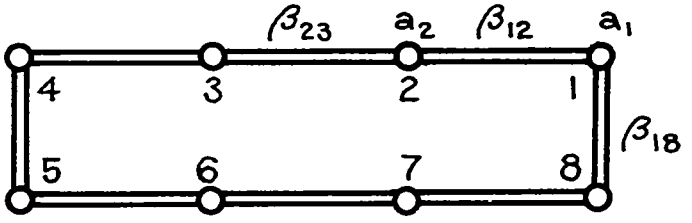


(a) Stiffnesses.

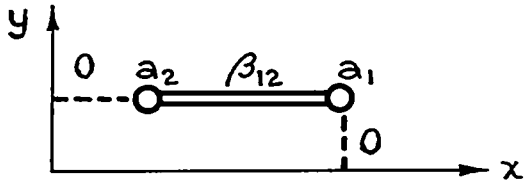
(b) Double symmetry.

(c) Single symmetry about y-axis.

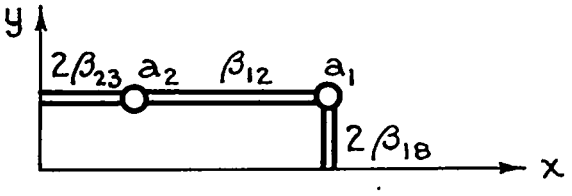
Figure 12.- Six-flange section with two cells.



(a) Stiffnesses.



(b) Double symmetry.



(c) Double Antisymmetry

Figure 13.- Eight-flange section.

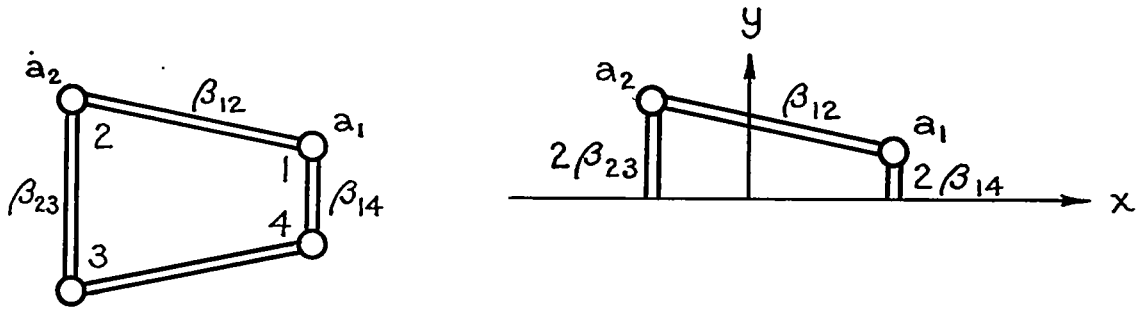


Figure 14 - Trapezoidal section.

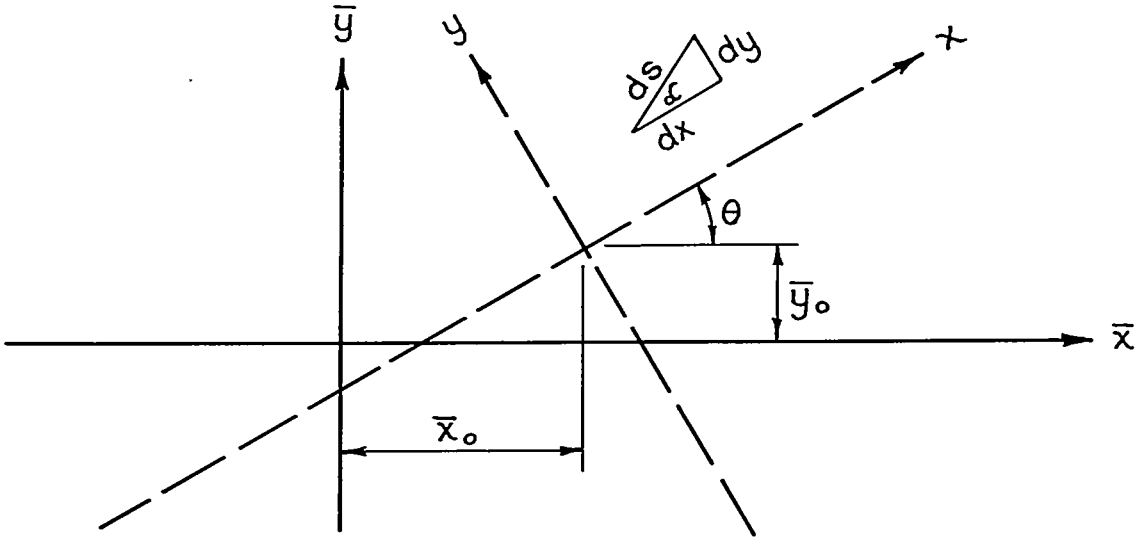
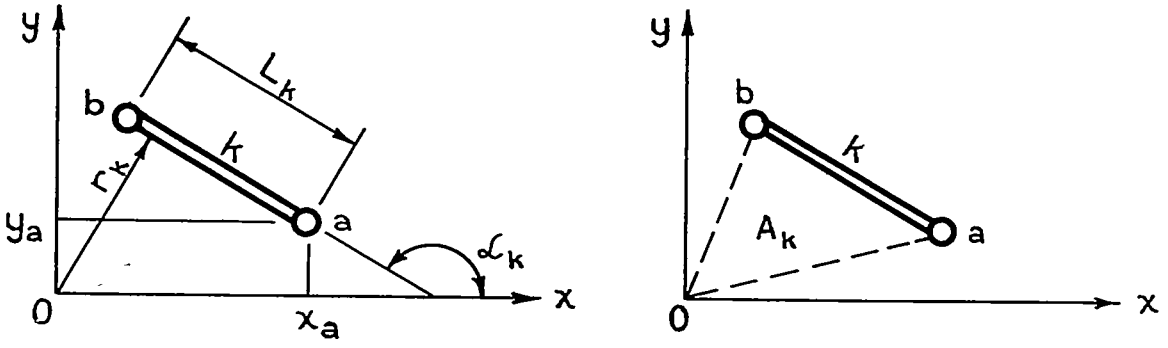


Figure 15.- Relative positions of coordinate axes.



(a) Coordinates of web. (b) Central area.

Figure 16. - Geometric properties associated with kth web.

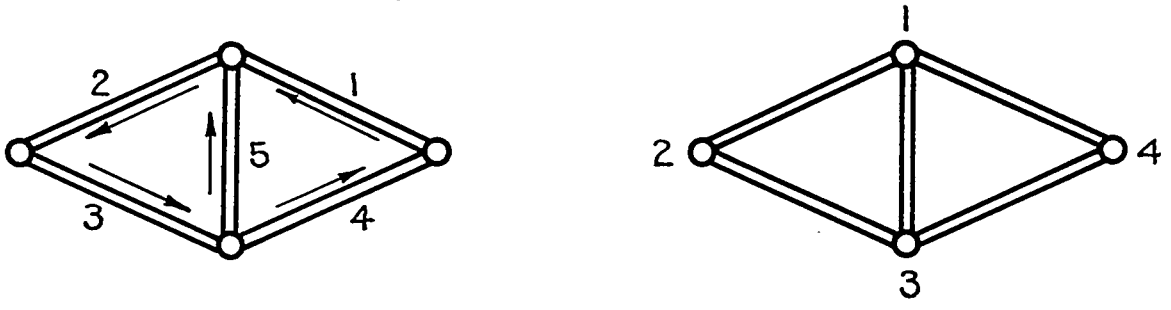


Figure 17.- Elementary section with two cells.