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TRANSONIC SIMILARITY RULES FOR LIFTING WINGS

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SUMMARY

Similarity rules for the transonic flow about lifting wings are derived by considering the change in the flow field due to angle of attack as a small perturbation to the nonlifting flow field. This approach has the advantage that the effects of angle of attack and airfoil geometry are partially separated.

It is found that the lift coefficient is proportional to the angle of attack as in other speed ranges. Other results are that the drag due to lift is proportional to the square of the lift coefficient as in other speed ranges and that the expression for the ratio of lift to drag is very similar to that obtained at supersonic speeds. It is found that the maximum value of the lift-to-drag ratio is approximately inversely proportional to the first power of the wing thickness ratio for cases in which the skin-friction drag is negligible compared with the pressure drag. For cases where the angle of attack is large compared with the thickness ratio, the lift coefficient is proportional to the angle of attack to the two-thirds power.

Since the effects of angle of attack and wing geometry are partially separated, the present form of the similarity rules is useful for correlation work. Experimental data indicate that such a correlation will be possible for a lift-coefficient range extending beyond the lift coefficient for maximum lift-to-drag ratio. Thus, many interesting results may be presented in terms of the similarity rules for low lift coefficients.

It is shown that the transonic similarity rules are valid at subsonic speeds but are more complicated in that range than the well-known rules of Prandtl-Glauert.

INTRODUCTION

A large amount of data on the transonic characteristics of wings have already been accumulated and more information may be expected from the research programs in progress and from those which are projected. As is well-known, the transonic characteristics of wings depend upon

the Mach number, thickness ratio, aspect ratio, and sweep, and to a lesser extent upon the airfoil family and taper ratio. One of the problems which the aerodynamist now faces is the correlation of these data in such form that the characteristics of an arbitrary wing may be estimated rapidly and with reasonable accuracy. Since it is convenient to use only two independent variables at a time, the task of correlating even straight-wing data is formidable unless there exists a guide to a particular combination of variables which will permit a more compact representation of the data. Such a guide is now available as the result of recent work in which the potentialities of the so-called transonic similarity rules have been explored more fully.

Similarity rules for two-dimensional transonic flows have been given by Von Kármán (ref. 1) and Kaplan (ref. 2). Although the results of references 1 and 2 provided the background and stimulus for the work to follow, the two-dimensional rules which were derived in these references were not of use for finite wings since aspect ratio has a strong effect at transonic speeds. Later, the similarity rules for finite wings were given by Spreiter (ref. 3). None of the sets of rules of references 1 to 3 is very convenient for lifting wings, however, since they require that the ratio of the angle of attack to the airfoil thickness ratio remain constant.

The similarity rules of the present paper are derived by considering the change in the flow field due to angle of attack as a small perturbation to the nonlifting flow field. This approach has the advantage that the effects of angle of attack and airfoil geometry are partially separated.

SYMBOLS

A	aspect ratio
a	speed of sound
b	wing span
c	wing chord
C_{Df}	skin-friction drag coefficient
C_{Dp}	zero-lift pressure-drag coefficient

ΔC_D	incremental drag due to lift
C_L	lift coefficient
C_m	pitching-moment coefficient
$D_p(H,K)$	function describing pressure drag at zero angle of attack
$D_{ph}(H,K)$	function describing pressure drag at zero angle of attack due to camber
$\Delta D, \Delta D_1, \Delta D_2$	functions describing drag due to angle of attack
ΔD_3	function describing drag due to camber at angle of attack
$g\left(\frac{x}{c}, \frac{y}{b}\right)$	wing thickness-distribution function
h	wing camber
$H = A \left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3}$	
$K = \frac{M_\infty^2 - 1}{\left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{2/3}}$	
$L(H,K), L_1\left(H, K, \frac{\alpha}{t/c}\right)$	lift functions
$L_h(H,K)$	function describing lift due to camber
M	local Mach number
$M(H,K)$	pitching-moment function
P	pressure coefficient $\left(\frac{p - p_\infty}{\rho U^2 / 2} \right)$

p	pressure
S	wing area
t	wing thickness
u,v,w	Cartesian velocity components in the x-, y-, and z-directions, respectively
U	free-stream velocity
x,y,z	Cartesian coordinates
X	aerodynamic-center function
x_{ac}	location of aerodynamic center measured rearward from leading edge
L/D	lift-to-drag ratio
$(L/D)_{max}$	maximum lift-to-drag ratio
α	angle of attack
$\beta = \sqrt{M^2 - 1}$	
γ	ratio of specific heats
δ	flow-deflection angle for Prandtl-Meyer flow
ρ	density
Φ	velocity potential
ϕ	perturbation velocity potential
Subscript:	
∞	undisturbed stream

DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

The equation for the velocity potential Φ governing the irrotational flow of a compressible fluid is

$$\left(1 - \frac{u^2}{a^2}\right)\phi_{xx} + \left(1 - \frac{v^2}{a^2}\right)\phi_{yy} + \left(1 - \frac{w^2}{a^2}\right)\phi_{zz} - \frac{2uv}{a^2}\phi_{xy} - \frac{2uw}{a^2}\phi_{xz} - \frac{2vw}{a^2}\phi_{yz} = 0 \quad (1)$$

where the velocity components u , v , and w in the x -, y -, and z -directions (x is in the direction of the free stream and y is along the span) are given by

$$\left. \begin{aligned} u &= \phi_x \\ v &= \phi_y \\ w &= \phi_z \end{aligned} \right\} \quad (2)$$

and a , the local speed of sound, is given by

$$\frac{a^2}{a_\infty^2} = 1 - \frac{\gamma - 1}{2} \left(\frac{u^2}{a_\infty^2} + \frac{v^2}{a_\infty^2} + \frac{w^2}{a_\infty^2} - M_\infty^2 \right) \quad (3)$$

where a_∞ is the velocity of sound corresponding to the free-stream velocity U and M_∞ is the stream Mach number U/a_∞ .

Equation (1) for the velocity potential is far too complicated to afford an insight into the properties of flow fields near Mach number 1. It is therefore important to replace equation (1) by a simpler approximate equation which still retains the essential features of transonic flow.

First, equation (1) for ϕ is replaced by an exact equation for a disturbance velocity potential ϕ defined by

$$\phi = U(x + \phi)$$

Then

$$\begin{aligned} & \phi_{xx} \left[1 - M_\infty^2 - (\gamma + 1)M_\infty^2 \phi_x - \frac{\gamma + 1}{2} M_\infty^2 \phi_x^2 - \frac{\gamma - 1}{2} M_\infty^2 (\phi_y^2 + \phi_z^2) \right] + \\ & \phi_{yy} \left[1 - \frac{\gamma + 1}{2} M_\infty^2 \phi_y^2 - \frac{\gamma - 1}{2} M_\infty^2 (2\phi_x + \phi_x^2 + \phi_z^2) \right] + \\ & \phi_{zz} \left[1 - \frac{\gamma + 1}{2} M_\infty^2 \phi_z^2 - \frac{\gamma - 1}{2} M_\infty^2 (2\phi_x + \phi_x^2 + \phi_y^2) \right] - \\ & 2M_\infty^2 \phi_{xy} (\phi_y + \phi_x \phi_y) - 2M_\infty^2 \phi_{xz} (\phi_z + \phi_x \phi_z) - 2M_\infty^2 \phi_{yz} \phi_y \phi_z = 0 \end{aligned} \quad (4)$$

If ϕ and its derivatives are assumed to be small compared with 1, the well-known Prandtl-Glauert differential equation valid at subsonic speeds is obtained by retaining only the first term from each of the first three lines. However, for stream Mach numbers near 1, the term $(\gamma + 1)M_\infty^2 \phi_x$ may be as large or larger than $1 - M_\infty^2$ and both of these terms must be retained at transonic speeds. The resulting differential equation is

$$\left[1 - M_\infty^2 - (\gamma + 1)M_\infty^2 \phi_x \right] \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad (5)$$

At first glance it may appear that the term involving ϕ_{xx} is of higher order than those involving ϕ_{yy} or ϕ_{zz} since the coefficient of the ϕ_{xx} term is small compared with 1. However, in general, ϕ_{xx} is much greater than ϕ_{xy} , ϕ_{xz} , ϕ_{yz} , ϕ_{yy} , and ϕ_{zz} for transonic flows. For example, the linearized theory of Prandtl-Glauert indicates that for stream Mach numbers near 1, a small disturbance is propagated practically unchanged to infinity in the y- and z-directions and is restricted to a small region in the x-direction.

Equation (5) is slightly more complicated than that given in references 1 to 3. This added complication is a result of the disturbance

velocities being referred to the free-stream velocity U rather than to the speed of sound corresponding to a local Mach number of 1 as in references 1 to 3. However, the added complication in the differential equation is compensated by simpler boundary conditions, since all disturbance velocities are required to vanish upstream at infinity.

The inclusion of the term $\phi_x \phi_{xx}$ in equation (5) is necessary at transonic speeds. For subsonic stream Mach numbers not near 1, this term should be neglected compared with $1 - M^2$ to give the well-known Prandtl-Glauert differential equation. Equation (5) is therefore not incorrect at subsonic speeds (where the term $\phi_x \phi_{xx}$ becomes second order) although in that range it is unnecessarily complicated. Thus, the similarity rules based upon equation (5) will be valid at subsonic speeds but the simpler Prandtl-Glauert similarity rules are to be preferred.

The appropriate boundary conditions at upstream infinity for a thin lifting wing are:

$$\phi_x = \phi_y = \phi_z = 0 \quad (6)$$

It is well-known that for thin bodies the effects of thickness and angle of attack may be separated in the boundary conditions to the first order. For symmetrical airfoils at small angles of attack, the boundary conditions on the body may be written

$$(\phi_z)_{z=0} = \pm \frac{t}{c} \frac{\partial}{\partial(x/c)} g\left(\frac{x}{c}, \frac{y}{b}\right) + \alpha \quad (7)$$

where the shape of the wing is given by

$$z = \pm \frac{t}{c} g\left(\frac{x}{c}, \frac{y}{b}\right) + \frac{x}{c} \alpha \quad (8)$$

where c is the wing chord, b , the span, and t/c , the thickness ratio.

Equation (8) defining the wing can be written in the form

$$z = \frac{t}{c} \left[\pm g\left(\frac{x}{c}, \frac{y}{b}\right) + \frac{\alpha}{t/c} \frac{x}{c} \right] \quad (9)$$

and, if the parameter $\frac{\alpha}{t/c}$ is kept constant, the family of airfoils depends only upon t/c . This is a requirement of the similarity rules presented in references 1 to 3. Some remarks concerning this requirement are given in appendix A.

The following development shows that in some cases the effects of airfoil thickness ratio and angle of attack may be partially separated and that for these cases the parameter $\frac{\alpha}{t/c}$ need not be kept constant. Equation (7) for the boundary conditions suggests a solution of the form

$$\phi = \phi^t + \phi^\alpha \quad (10)$$

where ϕ^t is the thickness solution satisfying the boundary condition

$$\left(\phi^t_z\right)_{z=0} = \pm \frac{t}{c} \frac{\partial}{\partial(x/c)} g\left(\frac{x}{c}, \frac{y}{b}\right) \quad (11)$$

and ϕ^α is the angle-of-attack solution satisfying the boundary condition

$$\left(\phi^\alpha_z\right)_{z=0} = \alpha \quad (12)$$

Combining equations (5) and (10) gives

$$\left\{ \left[1 - M_\infty^2 - (\gamma + 1)M_\infty^2 \phi^t_x \right] \phi^t_{xx} + \phi^t_{yy} + \phi^t_{zz} \right\} + \left\{ \left[1 - M_\infty^2 - (\gamma + 1)M_\infty^2 \phi^\alpha_x \right] \phi^\alpha_{xx} + \phi^\alpha_{yy} + \phi^\alpha_{zz} - (\gamma + 1)M_\infty^2 \left(\phi^t_x \phi^\alpha_{xx} + \phi^t_{xx} \phi^\alpha_x \right) \right\} = 0 \quad (13)$$

where ϕ^t is defined as a solution of the equation obtained by setting the first expression in braces in equation (13) equal to zero and ϕ^α is defined as a solution of the equation obtained by setting the second expression in braces in equation (13) equal to zero. Thus the equation for ϕ^t is

$$\left[1 - M_\infty^2 - (\gamma + 1)M_\infty^2 \phi_x^t \right] \phi_{xx}^t + \phi_{yy}^t + \phi_{zz}^t = 0 \quad (14)$$

and the equation for ϕ^α is

$$\left[1 - M_\infty^2 - (\gamma + 1)M_\infty^2 \phi_x^\alpha \right] \phi_{xx}^\alpha + \phi_{yy}^\alpha + \phi_{zz}^\alpha - (\gamma + 1)M_\infty^2 \left(\phi_x^t \phi_{xx}^\alpha + \phi_{xx}^t \phi_x^\alpha \right) = 0 \quad (15)$$

It may be noted that equation (14) is identical to equation (5).

An important simplification may be made in equation (15) by assuming that $\phi_x^\alpha \ll \phi_x^t$ so that the term $\phi_x^\alpha \phi_{xx}^\alpha$ may be neglected compared with $\phi_x^t \phi_{xx}^\alpha$. That is, the angle-of-attack solution is considered to be a small perturbation to the thickness solution. The resulting equation for ϕ^α is linear with variable coefficients depending upon the thickness solution ϕ^t :

$$\left(1 - M_\infty^2 \right) \phi_{xx}^\alpha - (\gamma + 1)M_\infty^2 \left(\phi_x^t \phi_{xx}^\alpha + \phi_{xx}^t \phi_x^\alpha \right) + \phi_{yy}^\alpha + \phi_{zz}^\alpha = 0 \quad (16)$$

The assumption made to obtain equation (16) would appear to be valid only for $\alpha \ll \frac{t}{c}$; however, experimental data for many airfoils of practical interest indicate that similarity rules based on this equation are valid for a lift-coefficient range extending beyond C_L for $(L/D)_{\max}$. Equation (6) for the boundary conditions upstream at infinity becomes

$$\left. \begin{aligned} \phi_x^t &= \phi_y^t = \phi_z^t = 0 \\ \phi_x^\alpha &= \phi_y^\alpha = \phi_z^\alpha = 0 \end{aligned} \right\} \quad (17)$$

and, if the trailing edges are subsonic, the Kutta condition must be satisfied.

DERIVATION OF SIMILARITY RULES

An attempt will be made to express the solutions of equations (14) and (16) in the form

$$\begin{aligned} \phi^t &= A_1 f_1 \left(\frac{x}{c}, B_1 y, C_1 z \right) \\ &= A_1 f_1 (x_1, y_1, z_1) \end{aligned} \quad (18)$$

$$\begin{aligned} \phi^\alpha &= A_2 f_2 \left(\frac{x}{c}, B_2 y, C_2 z \right) \\ &= A_2 f_2 (x_2, y_2, z_2) \end{aligned} \quad (19)$$

where t , α , M_∞ , and b occur only in the parameters A_n , B_n , and C_n . The factor $1/c$ is included with x since the wing is in an unbounded fluid which extends infinitely far in every direction and hence the flow must be independent of the scale.

Equation (14) is first considered. Inserting equation (18) into equation (14) gives

$$(1 - M_\infty^2) \frac{A_1}{c^2} f_{1x_1 x_1} - (\gamma + 1) M_\infty^2 \frac{A_1^2}{c^3} f_{1x_1} f_{1x_1 x_1} + A_1 B_1^2 f_{1y_1 y_1} +$$

$$A_1 C_1^2 f_{1z_1 z_1} = 0$$

It has been assumed that f_1 is a function only of x_1 , y_1 , and z_1 . This assumption holds if

$$\left. \begin{aligned} A_1 &= \frac{c(1 - M_\infty^2)}{(\gamma + 1)M_\infty^2} \\ B_1 = C_1 &= \frac{\sqrt{1 - M_\infty^2}}{c} \end{aligned} \right\} \quad (20)$$

The differential equation then takes the form

$$(1 - f_{1x_1})f_{1x_1x_1} + f_{1y_1y_1} + f_{1z_1z_1} = 0$$

and does not depend explicitly upon A_n , B_n , or C_n .

The boundary condition at the surface, equation (11), yields

$$A_1 C_1 f_{1z_1}(x_1, y_1, 0) = \pm \frac{t}{c} \frac{\partial}{\partial(x/c)} g\left(\frac{x}{c}, \frac{y}{b}\right) \quad (21)$$

from which

$$y_1 = \frac{y}{b} = B_1 y$$

or

$$B_1 = \frac{1}{b} \quad (22)$$

and

$$A_1 C_1 \propto \frac{t}{c} \quad (23)$$

It should be noted that the function $g\left(\frac{x}{c}, \frac{y}{b}\right)$ cannot change if the flows are to be similar. The geometrical differences between wings are associated with differences in their aspect ratio and thickness ratio. For example, if the function g is to be the same for swept wings, the parameter $A \tan \Lambda$ must remain constant, where Λ is the sweep angle of a constant-percent-chord line. Also, the taper ratios are the same for similar wings.

Equations (20) and (23) are four equations for the three unknowns A_1 , B_1 , and C_1 . In general, such a set of equations will not be consistent for arbitrary values of the coefficients. Equations (20) and (23) will be consistent provided the coefficients satisfy the

relation $K = \frac{M_\infty^2 - 1}{\left[(\gamma + 1)M_\infty^2 \frac{t}{c}\right]^{2/3}}$ where K is a constant. The quan-

tity K is one of the similarity parameters and corresponds to the form first given by Von Kármán (ref. 1). From equations (20), (22), and (23) $\frac{b}{c} \left[(\gamma + 1)M_\infty^2 \frac{t}{c}\right]^{1/3}$ must be kept constant and, since $\frac{b}{c}$ is proportional to the aspect ratio A for wings of the same chord and shape, the parameter $H = A \left[(\gamma + 1)M_\infty^2 \frac{t}{c}\right]^{1/3}$ must also be kept constant.¹ Equation (21) for the boundary condition thus becomes

$$K^{3/2} f_{1z_1}(x_1, y_1, 0) = \frac{\partial}{\partial x_1} g(x_1, y_1)$$

Thus, the expression for ϕ^t may be written

$$\phi^t = \frac{(t/c)^{2/3}}{\left[(\gamma + 1)M_\infty^2\right]^{1/3}} f^t\left(\frac{x}{c}, \frac{y}{b}, \frac{z}{b}\right)$$

¹Similarity rules involving the parameter $(\gamma + 1)M_\infty^2$ were first formulated by A. Busemann (ref. 4). The factor M_∞^2 arose as a consequence of referring the disturbance velocities to the free-stream velocity rather than to the speed of sound for a local Mach number of 1 as in references 1 to 3.

provided $H = A \left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3}$ and $K = \frac{M_\infty^2 - 1}{\left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{2/3}}$ are held constant. For a sonic stream, $K = 0$ and the only parameter occurring is $H = A \left[(\gamma + 1) \frac{t}{c} \right]^{1/3}$. Thus a difficulty encountered by Spreiter (ref. 3) who gave the two parameters as $\frac{M_\infty^2 - 1}{\left[(\gamma + 1) \frac{t}{c} \right]^{2/3}}$ and $A \sqrt{M_\infty^2 - 1}$

is eliminated. The difficulty was that there appeared to be some question concerning the combination of these two parameters for $M_\infty = 1$, because of the possibility of indeterminate forms.

The derivation for the lifting case follows an analysis parallel to that for thickness. Inserting equations (18) and (19) into (16) yields

$$\left((1 - M_\infty^2) \frac{A_2^2}{c^2} f_{2x_2x_2} - (\gamma + 1) M_\infty^2 \left(\frac{A_1 A_2}{c^3} f_{1x_1} f_{2x_2x_2} + \frac{A_1 A_2}{c^3} f_{1x_1x_1} f_{2x_2} \right) + A_2 B_2^2 f_{2y_2y_2} + A_2 C_2^2 f_{2z_2z_2} \right) = 0 \quad (24)$$

which can be rendered independent of A_n , B_n , and C_n if

$$B_2 = C_2 = \frac{\sqrt{1 - M_\infty^2}}{c}$$

and

$$A_1 = \frac{c(1 - M_\infty^2)}{(\gamma + 1) M_\infty^2} \quad (25)$$

The equation for f_2 then becomes

$$(1 - f_{1x_1}) f_{2x_2x_2} - f_{1x_1x_1} f_{2x_2} + f_{2y_2y_2} + f_{2z_2z_2} = 0$$

independent explicitly of A_n , B_n , and C_n . From the boundary condition (12),

$$A_2 C_2 f_{2z_2} (x_2, y_2, 0) = \alpha$$

or

$$A_2 C_2 \propto \alpha$$

and

$$A_2 \propto \frac{c\alpha}{\sqrt{1 - M_\infty^2}} = \frac{c\alpha}{K^{1/2} [(\gamma + 1) M_\infty^2 \frac{t}{c}]^{1/3}} \quad (26)$$

Thus, the expression for ϕ^α may be written for $\alpha \ll \frac{t}{c}$

$$\phi^\alpha = \frac{c\alpha}{K^{1/2} [(\gamma + 1) M_\infty^2 \frac{t}{c}]^{1/3}} f^\alpha \left(\frac{x}{c}, \frac{y}{b}, \frac{z}{b} \right) \quad (27)$$

provided H and K are kept constant.

In order to evaluate the forces on the wing, the approximate expression for the pressure coefficient $P \propto \phi_x$ given by first-order theory is used. Combining some of the previous results yields the following expression for the pressure coefficient:

$$P = \frac{(t/c)^{2/3}}{\left[(\gamma + 1) M_\infty^2 \right]^{1/3}} P^t \left(H, K; \frac{x}{c}, \frac{y}{b}, \frac{z}{b} \right) + \frac{\alpha}{\left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3}} P^\alpha \left(H, K; \frac{x}{c}, \frac{y}{b}, \frac{z}{b} \right) \quad (28)$$

This result differs from the conception that at transonic speeds the pressure coefficient and hence the lift are proportional to $(t/c)^{2/3}$.

For cases where $\alpha \gg \frac{t}{c}$, the thickness solution is considered as a small perturbation to the angle-of-attack solution. For this case, the differential equations (14) and (16) for t/c and α are interchanged to give

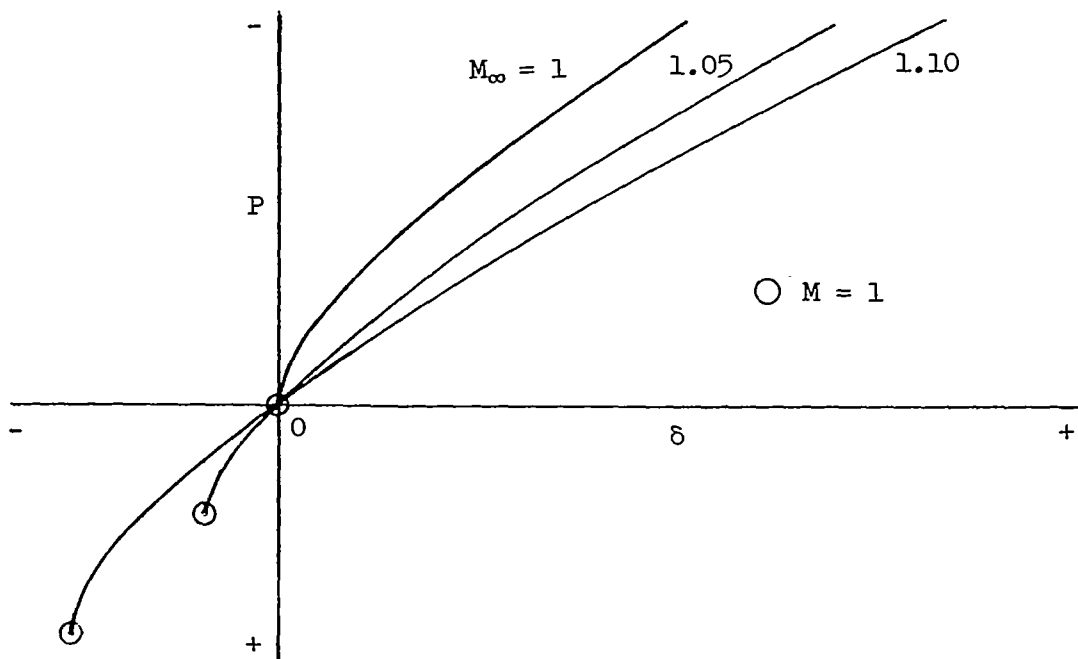
$$P = \frac{\alpha^{2/3}}{\left[(\gamma + 1) M_\infty^2 \right]^{1/3}} P_1^\alpha \left(H^\alpha, K^\alpha; \frac{x}{c}, \frac{y}{b}, \frac{z}{b} \right) + \frac{t/c}{\left[(\gamma + 1) M_\infty^2 \alpha \right]^{1/3}} P_1^t \left(H^\alpha, K^\alpha; \frac{x}{c}, \frac{y}{b}, \frac{z}{b} \right)$$

where H^α and K^α are the parameters H and K with α and t/c interchanged. Similar flows exist only in the approximation of small-disturbance theory which requires that both α and t/c be small. Thus, the form of the similarity rule for $\alpha \gg \frac{t}{c}$ is applicable only to very thin airfoils at small angles of attack.

SOME PHYSICAL CONSIDERATIONS CONCERNING THE LIFT OF TWO-DIMENSIONAL WINGS

If only the terms which influence the lift are considered, $P \propto \frac{\alpha}{(t/c)^{1/3}}$ for $\alpha \ll \frac{t}{c}$ and $P \propto \alpha^{2/3} g(K^\alpha)$ for $\alpha \gg \frac{t}{c}$. The

physical reason for this difference is believed to be the following: Consider the pressure coefficient for Prandtl-Meyer flow due to a flow deflection δ as shown in the following sketch for several values of the stream Mach number:



If the stream is supersonic, both positive and negative values of δ are permitted. An increase in δ corresponds to an increase in the local Mach number M and a decrease in δ corresponds to a decrease in local Mach number. Examination of the sketch illustrates the known result for Prandtl-Meyer flow that $P \propto \delta$ for local Mach numbers much larger than 1.

It is shown in appendix B that $P \propto \frac{\delta^{2/3}}{[(\gamma + 1)M_\infty^2]^{1/3}} g(K\delta)$ for Prandtl-Meyer flow if the local Mach numbers are near 1.

Prandtl-Meyer flow is intimately related to the method of characteristics (two-dimensional flow) which may be used to determine supersonic flow fields. Consider an airfoil with thickness at zero angle of attack in a slightly supersonic stream. The flow over the rear part is supersonic and, since the stream is supersonic, there must be a Mach line originating at the surface which does not intersect the sonic line. The change in the flow field due to a small change (within limits) in the shape (or slope) of the airfoil downstream of this point may be determined by the method of characteristics.

If the flow deflection δ for Prandtl-Meyer flow is considered as the local slope of the airfoil, the change in the pressure coefficient is proportional to the change in the local slope provided the local Mach numbers are much larger than 1. Thus, for an airfoil with thickness, the change in the pressure coefficient is proportional to the change in the local slope in regions where the local Mach numbers are much larger than 1. Since a change in slope may be considered to arise from putting that portion of the airfoil at an angle of attack, the change in the pressure coefficient is proportional to the change in angle of attack provided the local Mach numbers are not near 1. This reasoning is based upon the concepts of Prandtl-Meyer flow which require that the local Mach numbers be greater than 1. However, the transonic similarity rules contain no restrictions requiring the local Mach numbers to be greater than 1. Therefore, it is believed that these considerations concerning the importance of the local Mach numbers are also of physical significance when the local Mach numbers are less than 1. This concept has been used in reference 5 to obtain a velocity correction formula for airfoils with a fixed sonic point.

This line of reasoning is applied to the complete airfoil to obtain the result that the lift coefficient is proportional to the angle of attack when there are no large regions of near-sonic flow at the surface. Analogous reasoning leads to the result that the lift coefficient is proportional to the angle of attack to the two-thirds power times a function of K^α when large regions of near-sonic flow exist at the surface. An example of an airfoil with large regions of near-sonic flow at the surface is the flat-plate airfoil at a small angle of attack in a slightly supersonic stream with an attached shock wave. This flow may be determined by the use of shock tables to obtain the result that the surface pressure coefficients and hence the lift are proportional to the angle of attack to the two-thirds power times a function of K^α . Presumably, the preceding reasoning concerning the importance of the local Mach number for two-dimensional wings will not be appreciably altered for finite wings provided the tip effects are small.

RESULTS AND APPLICATIONS

Aerodynamic characteristics.- For the symmetrical airfoils considered, p^t is an even function of z and p^α is an odd function of z near $z = 0$. Thus, the only contribution to lift comes from p^α .

The lift coefficient is given approximately by

$$\begin{aligned}
 C_L &= \frac{\alpha}{\left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3}} \frac{1}{S} \int p^\alpha dx dy \\
 &= \frac{\alpha}{\left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3}} L(H, K)
 \end{aligned} \tag{29}$$

and C_L is proportional to the angle of attack as in other speed ranges. Presumably then, the similarity rules presented herein are applicable throughout the angle-of-attack range for which the lift coefficient varies linearly with angle of attack. The zero-angle-of-attack pressure drag due to thickness is given by

$$\begin{aligned}
 C_{Dp} &= \frac{(t/c)^{2/3}}{\left[(\gamma + 1) M_\infty^2 \right]^{1/3}} \frac{1}{S} \int p^t \frac{\partial z}{\partial x} dx dy \\
 &= \frac{(t/c)^{5/3}}{\left[(\gamma + 1) M_\infty^2 \right]^{1/3}} D_p(H, K)
 \end{aligned} \tag{30}$$

The drag due to angle of attack ΔC_D is given by

$$\begin{aligned}
 \Delta C_D &= \frac{\alpha^2}{\left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3}} \frac{1}{S} \int p^\alpha dx dy \\
 &= \frac{\alpha^2}{\left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3}} \Delta D_1(H, K) \\
 &= \left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3} C_L^2 \Delta D(H, K)
 \end{aligned} \tag{31}$$

or, since $A \left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3}$ must be kept constant,

$$\Delta C_D = \frac{C_L^2}{A} \Delta D_2(H, K) \quad (32)$$

This form is similar to the well-known result for incompressible flow. The moment coefficient depends upon the lift distribution and

$$C_m = \frac{\alpha}{\left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3}} M(H, K) \quad (33)$$

Keeping H and K constant means that the pressure distributions will be similar in shape. Thus the position of the aerodynamic center $\frac{x_{ac}}{c}$ will depend only upon these two parameters, that is,

$$\frac{x_{ac}}{c} = X(H, K) \quad (34)$$

From equations (29), (30), and (31), the expression for L/D is (including the friction-drag coefficient C_{Df})

$$\begin{aligned} \frac{L}{D} &= \frac{\alpha L}{\alpha^2 \Delta D_1 + \left(\frac{t}{c} \right)^2 D_p + \left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3} C_{Df}} \\ &= \frac{C_L}{\left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3} C_L^2 \Delta D + \frac{(t/c)^{5/3}}{\left[(\gamma + 1) M_\infty^2 \right]^{1/3}} D_p + C_{Df}} \end{aligned} \quad (35)$$

It is interesting to note that expression (35) for L/D is similar to that for supersonic flow although the functions involved are more complicated.

From equation (35), C_L for $(L/D)_{\max}$ is

$$(C_L)_{(L/D)_{\max}} = \frac{(t/c)^{2/3}}{[(\gamma + 1)M_\infty^2]^{1/3}} \sqrt{\frac{D_p + \frac{[(\gamma + 1)M_\infty^2]^{1/3}}{(t/c)^{5/3}} C_{Df}}{\Delta D}} \quad (36)$$

and $(L/D)_{\max}$ is

$$\left(\frac{L}{D}\right)_{\max} = \frac{1}{2 \frac{t}{c}} \frac{1}{\sqrt{\Delta D \left\{ D_p + \frac{[(\gamma + 1)M_\infty^2]^{1/3}}{(t/c)^{5/3}} C_{Df} \right\}}} \quad (37)$$

Equation (37) shows that $(L/D)_{\max}$ increases with decreasing thickness at transonic speeds and is almost inversely proportional to t/c for small values of the friction-drag coefficient for two-dimensional airfoils at $M_\infty = 1$.

Similarity rules may be formulated for camber by the same method used for angle of attack. The resulting forms for the aerodynamic coefficients for wings having small angles of sweep and small camber ratios, at low angles of attack, are

$$C_{Dp} = \frac{(t/c)^{5/3}}{[(\gamma + 1)M_\infty^2]^{1/3}} D_p(H,K) + \frac{(h/c)^2}{[(\gamma + 1)M_\infty^2 \frac{t}{c}]^{1/3}} D_{ph}(H,K)$$

$$C_L = \frac{\alpha}{[(\gamma + 1)M_\infty^2 \frac{t}{c}]^{1/3}} L(H,K) + \frac{h/c}{[(\gamma + 1)M_\infty^2 \frac{t}{c}]^{1/3}} L_h(H,K)$$

$$\Delta C_D = \frac{\alpha^2}{[(\gamma + 1)M_\infty^2 \frac{t}{c}]^{1/3}} \Delta D_1(H,K) + \frac{\alpha(h/c)}{[(\gamma + 1)M_\infty^2 \frac{t}{c}]^{1/3}} \Delta D_3(H,K)$$

Comparison with subsonic theory.- The lift predicted by subsonic lifting-line theory is given by the formula

$$C_L = \frac{2\pi A \alpha}{2 + A \sqrt{1 - M_\infty^2}}$$

or in terms of the transonic similarity parameters as

$$C_L = \frac{\alpha}{\left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3}} \frac{2\pi A \left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3}}{2 + A \left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3} \sqrt{\frac{1 - M_\infty^2}{\left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{2/3}}}}$$

so that at subsonic speeds

$$L(H, K) = \frac{2\pi H}{2 + H\sqrt{K}}$$

Similar results are obtained for other aerodynamic coefficients determined by subsonic wing theory.

Correlation studies.- The transonic similarity rules previously presented (ref. 3) require that the parameters H , K , and $\frac{\alpha}{t/c}$ be kept constant for similar flows. Since the aerodynamic coefficients depend upon three parameters, this form of the rules is inconvenient for correlation work on lifting wings. The present derivation indicates that, for low lift coefficients, the parameter $\frac{\alpha}{t/c}$ need not be kept constant and single charts may be prepared for each aerodynamic coefficient. For

example, from equation (29), $\left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3} \frac{C_L}{\alpha}$ may be plotted against $\frac{M_\infty^2 - 1}{\left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{2/3}}$ with lines of constant $A \left[(\gamma + 1) M_\infty^2 \frac{t}{c} \right]^{1/3}$

to give a single chart for the lift of a family of wings of varying aspect ratio and thickness ratio for the transonic Mach number range.

CONCLUDING REMARKS

Similarity rules for the transonic flow about lifting wings have been derived by considering the change in the flow field due to angle of attack as a small perturbation to the nonlifting flow field. This approach has the advantage that the effects of angle of attack and airfoil geometry are partially separated.

It was found that the lift coefficient is proportional to the angle of attack as in other speed ranges. Other results are that the drag due to lift is proportional to the square of the lift coefficient as in other speed ranges and that the expression for the ratio of lift to drag is very similar to that obtained at supersonic speeds. It was found that the maximum value of the lift-to-drag ratio is approximately inversely proportional to the first power of the wing thickness ratio for cases in which the skin-friction drag is negligible compared with the pressure drag for two-dimensional airfoils at sonic velocity. It is believed that for large-aspect-ratio wings the lift coefficient is proportional to the angle of attack when there are no large regions of near-sonic flow at the surface and is proportional to the angle of attack to the two-thirds power times a function of the similarity parameter K^α when large regions of near-sonic flow exist at the surface.

The present form of the similarity rules permits each aerodynamic coefficient for a family of wings of varying aspect ratio and thickness ratio to be presented in a single chart for the transonic range.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., March 24, 1952

APPENDIX A

SOME REMARKS CONCERNING THE APPLICATION OF TRANSONIC
SIMILARITY RULES TO LIFTING WINGS

The similarity rules of references 1, 2, and 3 were derived for a family of airfoils whose geometrical differences could be described in terms of a single parameter (t/c in eq. (9)). However, the geometry of a family of airfoils with thickness at angle of attack depends upon the two parameters α and t/c . If such a family is to be described by a single parameter, α and t/c cannot be independent. The proper relationship for thin airfoils at small angles of attack is $\frac{\alpha}{t/c} = \text{Constant}$. The solution for the flow past a lifting wing with a one-parameter boundary condition is similar to the form for ϕ^t and may be expressed as

$$\phi = \frac{(t/c)^{2/3}}{[(\gamma + 1)M^2]^{1/3}} f\left(\frac{x}{c}, \frac{y}{b}, \frac{z}{b}, H, K, \frac{\alpha}{t/c}\right)$$

where the parameter $\frac{\alpha}{t/c}$ has been included explicitly since it appears explicitly in the body shape (eq. (9)). The expression for the lift coefficient is

$$C_L = \frac{(t/c)^{2/3}}{[(\gamma + 1)M^2]^{1/3}} L_1\left(H, K, \frac{\alpha}{t/c}\right)$$

from which the slope of the lift curve at zero lift has the form

$$\left(\frac{\partial C_L}{\partial \alpha}\right)_{\alpha=0} = \frac{1}{[(\gamma + 1)M^2]^{1/3} \frac{t}{c}} \left[\frac{\partial L_1}{\partial \left(\frac{\alpha}{t/c}\right)} \left(H, K, \frac{\alpha}{t/c}\right) \right]_{\alpha=0}$$

Equation (29) is in agreement with this form with

$$L(H, K) = \left[\frac{\partial L_1}{\partial \left(\frac{\alpha}{t/c}\right)} \right]_{\alpha=0}$$

APPENDIX B

DERIVATION OF TRANSONIC APPROXIMATION FOR PRANDTL-MEYER FLOW

The exact equation for Prandtl-Meyer flow may be written

$$\theta = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \left(\sqrt{\frac{\gamma - 1}{\gamma + 1}} \sqrt{M^2 - 1} \right) - \tan^{-1} \sqrt{M^2 - 1} \quad (\text{B1})$$

where θ is the expansion angle required to accelerate the flow from $M_\infty = 1$ to M . The transonic approximation for Prandtl-Meyer flow is derived by expanding equation (B1) in a Taylor series about the point $M = M_\infty$ for M and M_∞ both near 1. With $\sqrt{M^2 - 1}$ replaced by β , the Taylor series is

$$\begin{aligned} \theta(\beta) = & \theta(\beta_\infty) + (\beta - \beta_\infty) \left(\frac{d\theta}{d\beta} \right)_{\beta=\beta_\infty} + \frac{(\beta - \beta_\infty)^2}{2!} \left(\frac{d^2\theta}{d\beta^2} \right)_{\beta=\beta_\infty} + \\ & \frac{(\beta - \beta_\infty)^3}{3!} \left(\frac{d^3\theta}{d\beta^3} \right)_{\beta=\beta_\infty} + \dots \end{aligned} \quad (\text{B2})$$

The quantity $\theta(\beta) - \theta(\beta_\infty)$ is the flow deflection from the stream direction and will be denoted by δ . The first three terms of equation (B2) are all of the order β^3 and to this order

$$\frac{3}{2}(\gamma + 1)M_\infty^2\delta = \beta^3 - \beta_\infty^3 \quad (\text{B3})$$

In the small-disturbance approximation $P \propto \phi_x$ which is now expressed in terms of β and β_∞ . Inserting $\Phi = U(x + \phi)$ into equation (3) gives

$$\begin{aligned} \frac{a^2}{a_\infty^2} &= 1 - \frac{\gamma - 1}{2} M_\infty^2 (2\phi_x + \phi_x^2 + \phi_y^2 + \phi_z^2) \\ &\approx 1 - (\gamma - 1)M_\infty^2\phi_x \end{aligned}$$

$$M^2 = \frac{u^2 + v^2 + w^2}{a^2} \approx M_\infty^2 \left[1 + (\gamma + 1)\phi_x \right]$$

or

$$\beta^2 = \beta_\infty^2 + (\gamma + 1)M_\infty^2 \phi_x \quad (B4)$$

Combining equations (B3) and (B4) yields

$$\begin{aligned} \phi_x &= \frac{1}{(\gamma + 1)M_\infty^2} \left\{ -\beta_\infty^2 + \left[\beta_\infty^3 + \frac{3}{2}(\gamma + 1)M_\infty^2 \delta \right]^{2/3} \right\} \\ &= \frac{\delta^{2/3}}{\left[(\gamma + 1)M_\infty^2 \right]^{1/3}} \left\{ -K^\delta + \left[(K^\delta)^{3/2} + \frac{3}{2} \right]^{2/3} \right\} \end{aligned}$$

and, since $P \propto \phi_x$, the approximate expression for the pressure coefficient for Prandtl-Meyer flow for transonic speeds may be expressed in the form

$$P \propto \frac{\delta^{2/3}}{\left[(\gamma + 1)M_\infty^2 \right]^{1/3}} g(K^\delta)$$

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