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MINIMUM WAVE DRAG OF BODIES OF REVOLUTION WITH A CYLINDRICAL CENTER SECTION

By Franklyn B. Fuller and Benjamin R. Briggs

Ames Aeronautical Laboratory
Moffett Field, Calif.

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SUMMARY

The minimum wave-drag problem with auxiliary conditions is solved for axial flow about bodies of revolution consisting of two symmetrical ogival sections joined by a circular cylinder. The auxiliary conditions are that the total length, the length of the cylinder, the frontal area, and the volume are held constant. The results are related to similar results known for bodies of revolution without a cylindrical midsection, and it is found that the addition of small amounts of center section has little effect on the drag. The maximum thickness ratio leading to the least total of wave and friction drag is investigated briefly.

INTRODUCTION

The formula for the determination of the wave drag of a slender body of revolution in a supersonic free stream parallel to the axis of the body was first given by von Kármán and Moore (reference 1). In a later work (reference 2), von Kármán reformulated the problem and gave the form of the body of prescribed length and maximum cross section having a minimum wave drag. The bodies treated in both the above papers consisted of ogives at the upstream end of cylinders extending to infinity downstream. Somewhat later, Lighthill (reference 3) gave the solution to the problem of minimum drag with the auxiliary conditions of prescribed length and maximum thickness for a body consisting of two symmetrical ogives placed back to back. A paper by Haack (reference 4) gives a complete summary of all previous solutions, as well as some new solutions, for both symmetrical bodies and bodies of the type discussed in references 1 and 2. In reference 5, Busemann has attacked the problem of minimum drag of bodies of revolution by exploiting its analogy to the problem of wing-trailing-vortex systems of minimum energy. Sears (reference 6) discusses the body consisting of two ogives placed back to back but, in the case where length and maximum cross section are prescribed, he does not limit the analysis to the case of fore-and-aft symmetry. However, the results show that the least drag does occur for symmetrical bodies.

The present report offers an extension of the results outlined above by taking into consideration bodies consisting of two symmetrical ogives joined by a cylindrical center section.[1] (See fig. 1.) Since the stern section is pointed, the question of base drag does not arise. If it were desired to consider bodies with finite area at the stern, such as boattailed bodies, again without taking into account the base pressure, the method used in this report would be applicable, provided the meridian section of the body has zero slope at the end where the boattail occurs. The analysis then becomes considerably more complicated than that of the case treated here.

The introduction of the center section brings a new geometrical parameter into the problem, namely, the length of that center section. The minimum drag problem can be formulated as an isoperimetric problem, since the auxiliary conditions are expressible in integral form. It is solved under the conditions that frontal area (or maximum thickness), volume, length of cylinder, and total length are held constant. This rather restrictive set of conditions is then relaxed to include cases in which two of the geometric parameters are fixed while the other one is free to vary. In this way, three distinct minimum problems connected with the type of body considered here can be investigated systematically.

Finally, in the appendix, the frictional drag of a body of revolution is taken into account in an approximate manner to determine the thickness ratio of a body having the least value of combined wave and frictional drag.

LIST OF IMPORTANT SYMBOLS

B $\qquad \frac{1}{k^2}(E-k'^2K)$

$B(\sigma,k)$ $\qquad \frac{1}{k^2}\left[E(\sigma,k) - k'^2F(\sigma,k)\right]$

$C_D{}^*$ $\qquad$ wave drag coefficient, based on the area l^2 $\left(\frac{\text{drag}}{\frac{1}{2}\rho_o V_o{}^2 l^2}\right)$

C_{D_S} $\qquad$ wave drag coefficient, based on frontal area of body $\left(\frac{\text{drag}}{\frac{1}{2}\rho_o V_o{}^2 S_o}\right)$

[1] The present work generalizes particular cases of bodies with cylindrical midsections considered by Max. A. Heaslet and Harvard Lomax in the forthcoming series on High-Speed Aerodynamics and Jet Propulsion, Princeton University Press.

C_{D_V} wave drag coefficient, based on 2/3 power of volume of body $\left(\dfrac{\text{drag}}{\frac{1}{2}\rho_o V_o{}^2 V^{2/3}}\right)$

D $\dfrac{1}{k^2}(K-E)$

$D(\sigma,k)$ $\dfrac{1}{k^2}[F(\sigma,k) - E(\sigma,k)]$

E complete elliptic integral of second kind, modulus k

$E(\sigma,k)$ incomplete elliptic integral of second kind with argument σ and modulus k

$F(\sigma,k)$ incomplete elliptic integral of first kind with argument σ and modulus k

k modulus of elliptic integrals $\left[k = \sqrt{1-\left(\dfrac{L}{l}\right)^2}\right]$

k' complementary modulus $\left(k' = \sqrt{1-k^2} = \dfrac{L}{l}\right)$

K complete elliptic integral of first kind, modulus k

$2l$ total length of body of revolution

$2L$ length of cylindrical midsection of body of revolution

r_o maximum radius of body of revolution

$r(x)$ local radius of body of revolution

R wave drag divided by free-stream dynamic pressure $\left(\dfrac{\text{drag}}{\frac{1}{2}\rho_o V_o{}^2}\right)$

S_o frontal area of body of revolution $(\pi r_o{}^2)$

$S_o{}^*$ $\dfrac{S_o}{l^2}$

$S(x)$ local cross-sectional area of body of revolution $[\pi r^2(x)]$

t_o maximum thickness ratio of body (reciprocal of fineness ratio) $\left(\dfrac{r_o}{l}\right)$

t local thickness ratio of body

V total volume of body

V^* $\frac{V}{l^3}$

V_o velocity of the free stream

x coordinate along axis of body

$Z(\sigma,k)$ Jacobian Zeta function of argument σ and modulus k $\left[Z(\sigma,k) = E(\sigma,k) - \frac{E}{K} F(\sigma,k)\right]$

λ,μ Lagrange multipliers (equation (4b))

ξ $\frac{x}{l}$

σ argument of elliptic integrals $\left(\sigma = \sqrt{\frac{l^2-x^2}{l^2-L^2}} = \frac{1}{k}\sqrt{1-\xi^2}\right)$

ρ_o free-stream density

ANALYSIS

Nomenclature and Boundary Conditions

An example of the type of body to be considered in this report is shown in figure 1. Also in that figure is shown some of the notation to be used. If $S(x)$ denotes the cross-sectional area of the body at any point, then

$$S(x) = \pi r^2(x) \tag{1}$$

where $r(x)$ is the local radius of the body. It will be stipulated that the body is symmetrical about $x=0$, that it closes at each end, and that the ogival sections fair into the cylindrical section with vanishing slope. Analytically, these conditions become (see fig. 1)

$$r(\pm l) = 0$$

$$r(\pm L) = r_o$$

$$r'(\pm L) = 0$$

where a prime denotes differentiation with respect to x. In terms of the area function $S(x)$, these conditions become (since $S'=2\pi rr'$)

$$\left.\begin{aligned} S(\pm l) &= 0 \\ S(\pm L) &= S_o \\ S'(\pm L) &= 0 \end{aligned}\right\} \tag{2a}$$

where $S_o = \pi r_o^2$ is the cross-sectional area of the cylinder. On the cylinder the conditions on $S(x)$ are

$$\left.\begin{aligned} S(x) &= S_o \\ S'(x) &= 0 \\ S''(x) &= 0 \end{aligned}\right\} - L < x < L \tag{2b}$$

The cross-sectional area must always be positive or zero;

$$S(x) \geq 0 \tag{2c}$$

Finally, the restriction is made that the maximum cross section occurs at the cylindrical portion. Thus,

$$\left.\begin{aligned} &S'(x) \geq 0; \; -l \leq x \leq -L \\ &S'(x) \leq 0; \; L \leq x \leq l \end{aligned}\right\} \tag{2d}$$

The Variational Problem

On the basis of the work of reference 1 or 3, the wave drag of a body such as is illustrated in figure 1 is given by

$$R = \frac{\text{drag}}{\frac{1}{2}\rho_o V_o^2} = \frac{1}{2\pi} \int_{-l}^{l} S'(x)\,dx \int_{-l}^{l} \frac{S''(x_1)}{x-x_1}\,dx_1$$

In order to arrive at this approximation, it is assumed that the body is slender ($t_o << 1$), and that both $S(x)$ and $S'(x)$ are continuous and equal to zero at the ends of the body.

Because of the fore-and-aft symmetry of the body, the above expression for wave drag can be modified into one involving integration over either the nose or stern section alone. Thus, for integration over the stern ($x > 0$)

$$R = \frac{2}{\pi} \int_{L}^{l} S'(x)\,dx \int_{L}^{l} \frac{xS''(x_1)}{x^2-x_1^2}\,dx_1 \tag{3}$$

The variational problem to be solved is of the isoperimetric type, since the drag is to be minimized under the auxiliary conditions of constant length, frontal area, and volume. The body shapes determined as solutions to this problem will be referred to as optimum bodies. With the auxiliary conditions just mentioned, the quantity to be minimized can be written

$$T = R + \lambda_1 V + \mu_1 S_o \tag{4a}$$

where V is the total volume of the body, and λ_1 and μ_1 are undetermined constants, the so-called Lagrange multipliers. The volume V can be expressed as

$$V = 2 \int_L^l S(x)\, dx + 2L\, S_o = -2 \int_L^l x\, S'(x)\, dx$$

and the frontal area as

$$S_o = -\int_L^l S'(x)\, dx$$

Equation (4a) can now be written

$$T = \frac{2}{\pi} \int_L^l S'(x) \left[\int_L^l \frac{xS''(x_1)}{x^2-x_1{}^2}\, dx_1 + \lambda x + \mu \right] dx \tag{4b}$$

where λ and μ have replaced $-\pi\lambda_1$ and $-\frac{\pi}{2}\mu_1$, respectively.

In performing a variation of the quantity T, just defined, only so-called weak variations will be allowed. This means that the cross-section distribution $S(x)$ is deformed slightly, in such a way that the derivatives of the deformation function are of the same order of smallness as the deformation function itself. (See reference 7, p. 7.) The variation can be performed in any of a variety of ways, and the resulting necessary condition for a minimum (vanishing of the first variation) leads to the equation

$$\int_L^l \frac{2xS''(x_1)}{x^2-x_1{}^2}\, dx_1 + \lambda x + \mu = 0 \tag{5}$$

The function $S(x)$ obtained by solution of this integral equation, and two subsequent integrations, is the distribution of cross-sectional area which characterizes the required optimum body.

In order to show that equation (5) is the condition for a minimum instead of a maximum, the second variation can be examined. It is found that the second variation is proportional to the drag of the variation of the profile acting alone and, hence, is positive, by analogy to the result found by Munk in his work on minimum drag of wings (reference 8).

Determination of the Cross Section S(x)

Equation (5) can be written

$$\int_{L}^{l} \frac{2S''(x_1)}{x^2-x_1^2}\, dx_1 = -\left(\lambda + \frac{\mu}{x}\right) \tag{6}$$

It is only necessary to solve this equation for $x>0$ because of the symmetry of the body. Make the transformations

$$x_1^2 = \tau \; ; \; x^2 = t$$

in equation (6); it becomes

$$\int_{L^2}^{l^2} \frac{S''(x_1)}{t-\tau} \frac{d\tau}{\sqrt{\tau}} = -\left(\lambda + \frac{\mu}{\sqrt{t}}\right) \tag{6a}$$

Equation (6a) can be written in the form

$$w(t) = \int_a^b \frac{g(\tau)d\tau}{t-\tau}$$

which is the familiar airfoil equation. The general solution to the airfoil equation is known, being

$$g(t) = \frac{1}{\pi^2\sqrt{(b-t)(t-a)}}\left[\pi \int_a^b g(\tau)d\tau - \int_a^b \frac{w(\tau)\sqrt{(b-\tau)(\tau-a)}}{t-\tau}\, d\tau\right]$$

The quantity appearing in the solution as

$$\int_a^b g(\tau)d\tau$$

is of the nature of an arbitrary constant, and is to be evaluated from conditions of the problem. In the present case, the constant is

$$\int_{L^2}^{l^2} \frac{d^2}{dx_1^{\,2}} [S(x_1)] \frac{d\tau}{\sqrt{\tau}} = 2[S'(l) - S'(L)] = 0$$

by condition (2a) and the condition $S'(l) = 0$ imposed on equation (3). The solution to equation (6) can be written (in terms of x)

$$S''(x) = \frac{x}{\pi^2 \sqrt{(l^2-x^2)(x^2-L^2)}} \int_{L^2}^{l^2} \frac{\left(\lambda + \frac{\mu}{\sqrt{t_1}}\right)\sqrt{(l^2-t_1)(t_1-L^2)}}{x^2-t_1} dt_1 \tag{7}$$

The integrations of equation (7) can be performed, yielding

$$S''(x) = -\frac{\lambda}{2\pi} \frac{x(l^2+L^2-2x^2)}{\sqrt{(l^2-x^2)(x^2-L^2)}} + \frac{2\mu}{\pi^2}\left[x \frac{(x^2-L^2)K-l^2(K-E)}{l\sqrt{(l^2-x^2)(x^2-L^2)}} + KZ(\sigma,k)\right] \tag{8}$$

where

K, E complete elliptic integrals of first and second kind, respectively, modulus k

$Z(\sigma,k)$ Jacobian Zeta function of argument σ and modulus k $\left[Z(\sigma,k) = E(\sigma,k) - \frac{E}{K} F(\sigma,k)\right]$

$F(\sigma,k)$ $E(\sigma,k)$ incomplete elliptic integrals of first and second kinds, respectively, of argument σ and modulus k

k modulus of elliptic integrals $\left[k = \sqrt{1 - \left(\frac{L}{l}\right)^2}\right]$

σ argument of elliptic integrals $\left(\sigma = \sqrt{\frac{l^2-x^2}{l^2-L^2}}\right)$

Next, the first derivative, $S'(x)$, can be determined by

$$S'(x) = \int_L^x S''(x_1)\, dx_1$$

It is found that

$$S'(x) = -\frac{1}{2\pi}\left(\lambda + 4\frac{\mu}{\pi}\frac{K}{l}\right)\sqrt{(l^2-x^2)(x^2-L^2)} + \frac{2\mu}{\pi^2} K\, x\, Z(\sigma,k) \tag{9}$$

Finally, $S(x)$ is given by

$$S(x) = -\int_x^l S'(x_1)\,dx_1 = -\frac{\lambda l}{6\pi}\left[2L^2F(\sigma,k) - (l^2+L^2)E(\sigma,k) + \frac{x}{l}\sqrt{(l^2-x^2)(x^2-L^2)}\right] + \frac{\mu}{\pi^2}\left\{Kx^2\,Z(\sigma,k) + \left[l^2E\,E(\sigma,k) - L^2K\,F(\sigma,k)\right] - K\frac{x}{l}\sqrt{(l^2-x^2)(x^2-L^2)}\right\} \tag{10}$$

This function, $S(x)$, gives the cross-sectional area distribution of an optimum body of revolution of the type shown in figure 1, when the lengths, frontal area, and volume are prescribed. Since the solution appears in terms of the undetermined constants λ and μ, it is necessary to find these constants in terms of the prescribed quantities. This can be done by determining the frontal area and volume:

$$S_o = S(L) = \frac{\lambda l}{6\pi}\left[(l^2+L^2)E - 2L^2K\right] + \frac{\mu}{\pi^2}(l^2E^2 - L^2K^2) \tag{11}$$

$$\left.\begin{aligned} V &= 2L\,S_o + 2\int_L^l S(x)\,dx \\ &= \frac{\lambda}{16}(l^2-L^2)^2 + \frac{\mu l}{3\pi}\left[(l^2+L^2)E - 2L^2K\right] \end{aligned}\right\} \tag{12}$$

Thus, equations (11) and (12) serve to determine the constants λ and μ in terms of the prescribed quantities L, l, V, and S_o.

The remaining quantity to be evaluated is the drag. A combination of equations (3) and (6) yields

$$R = \frac{1}{\pi}\left(\frac{\lambda}{2}V + \mu\,S_o\right) \tag{13}$$

The solution obtained as equation (10) must now be examined to insure that it satisfies the boundary conditions. The conditions of equations (2a) have already been imposed in the analysis, as have the conditions of equations (2b). The other boundary conditions, (2c) and (2d), are more complex, however, and require some care in application. First, notice that if the conditions $S(L) = S_o$, $S(l) = 0$ are met, and $S'(x)$ is negative in the interval $L \leq x \leq l$, then certainly $S(x)$ cannot become negative in that interval. Thus if the condition (2d) on the derivative of $S(x)$ is satisfied, then condition (2c) is implicitly met. The remaining boundary condition can now be stated as

$$S'(x) = -\frac{1}{2\pi}\left(\lambda + \frac{4\mu}{\pi}\frac{K}{l}\right)\sqrt{(l^2-x^2)(x^2-L^2)} + \frac{2\mu}{\pi^2}KxZ(\sigma,k) \leq 0, \quad L \leq x \leq l$$

Analysis of this inequality at the end points shows that it implies the following two conditions on λ and μ:

$$\lambda l + \frac{4D}{\pi}\mu \geq 0 \tag{14a}$$

$$\lambda l + \frac{4B}{\pi}\mu \geq 0 \tag{14b}$$

where

$$D = \frac{K-E}{k^2}$$

$$B = \frac{E-k'^2K}{k^2}$$

It is interesting to note the meaning of the two equalities contained in expressions (14a) and (14b) in terms of the body geometry. Equation (8) can be written in the form

$$S''(x) = \frac{x}{2\pi l}\left[\left(\lambda l + \frac{4B}{\pi}\mu\right)\sqrt{\frac{x^2-L^2}{l^2-x^2}} - \left(\lambda l + \frac{4D}{\pi}\mu\right)\sqrt{\frac{l^2-x^2}{x^2-L^2}}\right] + \frac{2\mu}{\pi^2}KZ(\sigma,k) \tag{15}$$

which shows that $S''(x)$ is infinite at L unless the equality of expression (14a) holds, and is infinite at l unless the equality of (14b) holds. Since

$$S''(x) = 2\pi\left(r'^2 + rr''\right)$$

the singularity at L indicates that r'' is infinite there, while a singularity at l indicates that r' is infinite at l. On the other hand, if $S''(x)$ is zero at L, then r'' is zero, showing that the ogival section fairs into the cylinder with vanishing second derivative as well as vanishing first derivative. Similarly, the vanishing of $S''(x)$ at l gives a zero value of r' at the tip, so that the body is cusped. Since this only occurs when the equality

$$\lambda l + \frac{4B}{\pi}\mu = 0$$

holds, it is seen that, in general, the optimum bodies have vertical tangents at the tips.

It is convenient to have the formulas pertinent to the solution for optimum bodies in dimensionless terms. Introducing the following notation

$$\xi = \frac{x}{l}; \quad S_0^* = \frac{S_0}{l^2}, \; V^* = \frac{V}{l^3}, \; k' = \frac{L}{l}$$

and

$$\frac{S(x)}{l^2} = \pi t^2$$

where t is the local thickness ratio $\left(\frac{\text{diameter}}{2l}\right)$, the equations can be put into the following form

$$t^2(\xi) = \frac{\lambda l}{6\pi^2}\left\{k^2\left[B(\sigma,k) - k'^2\ D\ (\sigma,k)\right] - \xi\sqrt{(1-\xi^2)(\xi^2-k'^2)}\right\} +$$

$$\frac{\mu}{\pi^3}\left[\xi^2 KZ(\sigma,k) + E\ E(\sigma,k) - k'^2 KF(\sigma,k) - K\xi\sqrt{(1-\xi^2)(\xi^2-k'^2)}\right] \qquad (16)$$

where ξ varies from k' to 1, as x varies from L to l.

$$S_o^* = \pi t_o^2 = \frac{\lambda l}{6\pi} k^2(B-k'^2 D) + \frac{\mu}{\pi^2}(E^2-k'^2K^2) \qquad (17)$$

where t_o is the maximum thickness ratio of the body,

$$V^* = \frac{\lambda l}{16} k^4 + \frac{\mu}{3\pi} k^2(B-k'^2 D) \qquad (18)$$

$$C_D^* = \frac{\text{drag}}{\frac{1}{2}\rho_o V_o{}^2 l^2} = \frac{1}{\pi}\left(\frac{\lambda l}{2} V^* + \mu\ S_o^*\right) \qquad (19)$$

Equations (17) and (18) can be solved for λ and μ, resulting in

$$\lambda l = \frac{48}{k^4\Delta}\left[3(E^2-k'^2K^2)\ V^* - \pi k^2\ (B-k'^2 D)\ S_o^*\right] \qquad (20a)$$

$$\mu = \frac{3\pi}{k^4\Delta}\left[3\pi k^4 S_o^* - 8\ k^2(B - k'^2 D)\ V^*\right] \qquad (20b)$$

where

$$\Delta = 9(E^2-k'^2K^2) - 8\ (B - k'^2 D)^2$$

By use of the results of equations (20), the inequalities (14) can be expressed in terms of S_o^*, V^*, and k'; they become

$$\frac{\Phi_1}{\Psi_1} \leq \frac{S_o^*}{V^*} \leq \frac{\Phi_2}{\Psi_2} \qquad (21)$$

where

$$\Phi_1 = 2k^2 D\ (B-k'^2 D) - 3\ (E^2 - k'^2K^2)$$

$$\Psi_1 = \frac{\pi k^2}{4}\left[3k^2 D - 4\ (B - k'^2 D)\right]$$

$$\Phi_2 = 3\ (E^2 - k'^2K^2) - 2k^2\ B(B - k'^2 D)$$

$$\Psi_2 = \frac{\pi k^2}{4}\left[4\ (B - k'^2 D) - 3k^2 B\right]$$

Figure 2 shows the region defined by the inequalities (21) with $\frac{S_o^*}{V^*}$ plotted versus k'. The upper curve represents

$$\frac{S_o^*}{V^*} = \frac{\Phi_2}{\psi_2}$$

and the lower,

$$\frac{S_o^*}{V^*} = \frac{\Phi_1}{\psi_1}$$

The shaded region between these curves, which will be called the admissible region, defines the limits within which the parameters k' and S_o^*/V^* must lie in order that the solution for the optimum body satisfy the requirement given in expression (2d). That is, for a prescribed value of k', say, the prescribed values of S_o^* and V^* must be such that the ratio S_o^*/V^* falls in the shaded region of figure 2.

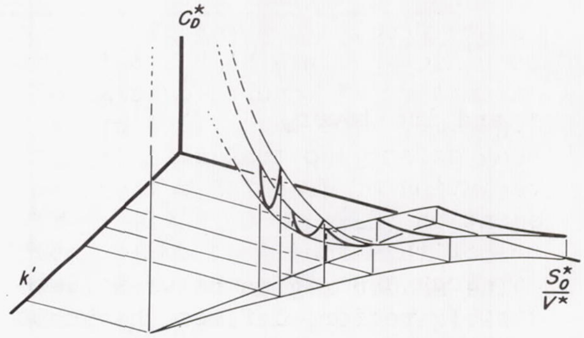
Finally, by using equations (20), the formula for the drag coefficient C_D^* (equation (19)) can be put in terms of the geometric parameters k', S_o^*, V^*;

$$C_D^* = \frac{3(S_o^*)^2}{\pi k^4 \Delta}\left[24(E^2-k'^2K^2\left(\frac{V^*}{S_o^*}\right)^2-16\pi k^2(B-k'^2D)\frac{V^*}{S_o^*} + 3\pi^2k^4\right] \quad (22)$$

DISCUSSION OF SOLUTION

From the results obtained in the previous section, one can find the characteristics of the body of revolution, of the type shown in figure 1, having minimum wave drag when the quantities total length ($2l$), length of cylinder ($2L$), frontal area (S_o) and volume (V) are fixed. Although the semitotal length l no longer appears in the formulas (equations (16), (17), (18), and (22)), having been absorbed into the dimensionless quantities C_D^*, k', S_o^*, V^*, it must be remembered that total length of the bodies is fixed. It was also found that when all four geometric quantities, l, L, S_o, V, were prescribed, certain limitations upon their magnitude must be observed in order to meet boundary conditions set forth in expressions (2c) and (2d). These limitations are most simply expressed in terms of the parameters k' and S_o/V^*, where k' is the ratio of the length of the cylindrical section to total length, and S_o^*/V^* is the ratio of the volume of the cylinder of radius r_o and length l to the volume of the body. The permissible range of values for k' and S_o^*/V^* is given in expressions (21), and is shown graphically in figure 2.

Using equation (22), the variation of the drag coefficient C_D^* with the variables k' and S_o^*/V^* can be found. The calculations were made for a value of S_o^* of $\pi/100$, corresponding to a maximum thickness ratio of 1/10. In the accompanying sketch, a three-dimensional view of the variation is shown. The two curves in the base plane are just those of figure 2, defining the admissible region for k' and S_o^*/V^*. For a given value of the length ratio k', the drag coefficient C_D^* varies parabolically with S_o^*/V^*; the minimum occurring between the extreme admissible values of S_o^*/V^*. With increasing k', C_D^* increases steadily and the rate of increase approaches infinity as k' approaches unity. A quantitative idea of the variation is afforded by figure 3, where the drag coefficient C_D^* is plotted against S_o^*/V^* for several values of the length ratio k'. The curve shown for $k'=0$ agrees with results of reference 4. The increase in drag coefficient with k' is seen to be slow for small values of k', indicating that the greater available volume resulting from the cylindrical center section may warrant acceptance of the slight increase in drag.



In order to obtain a more detailed view of the variation of drag with the geometric parameters, the variational problem that has been solved can be reinterpreted. The existence of the limits on the quantities S_o^*/V^* and k' (expressions (21)) suggests that two of the trio k', S_o^*, V^* might be fixed, while the third is left free to vary within the ascertainable limits determined by the two prescribed values. The total length, $2l$, of the bodies is also fixed, although it appears only implicitly. There are three such combinations possible:

1. l, k', S_o^* fixed; V^* free

2. l, V^*, k' fixed; S_o^* free

3. l, S_o^*, V^* fixed; k' free

These three problems can be stated in physical terms as follows:

1. Total length, cylinder length, and frontal area fixed; volume free

2. Total length, volume, and cylinder length fixed; frontal area free

3. Total length, frontal area, and volume fixed; cylinder length free

These three problems will be considered in order.

The Three Subcases

Case 1: Total length, cylinder length, and frontal area fixed; volume free.- In terms of the dimensionless parameters, this case applies to bodies having k' and S_o^* fixed. The admissible range of the volume parameter V^* can be determined either from expressions (21) or from figure 2. For each value of V^* in the range so found, an optimum body exists, so that a family of optimum bodies is now determined. The variation of drag for members of this family is readily found by use of equation (22), or an estimate may be obtained from figure 3. Now some one of this family of bodies must give rise to the least drag, and it is clear that this member is determined by the condition λ equals zero.[2] The formulas for thickness distribution and drag for the best optimum body in this family become (from equations (16) and (19))

$$t^2(\xi) = \frac{t_o^2}{E^2-k'^2K^2}\left\{\xi^2KZ(\sigma,k) + \left[E\,E(\sigma,k) - k'^2KF(\sigma,k)\right] - K\xi\sqrt{(1-\xi^2)(\xi^2-k'^2)}\right\} \tag{23}$$

$$C_D^* = \frac{\pi(S_o^*)^2}{E^2\,k'^2K^2} \tag{24}$$

The volume parameter for the optimum body is given by

$$V^* = \frac{\pi k^2}{3}\,\frac{B-k'^2D}{E^2-k'^2K^2}\,S_o^* \tag{25}$$

It is convenient to have drag coefficients based upon the frontal area and upon the volume (to the 2/3 power) of the body, rather than upon the area l^2. These are, respectively,

$$C_{D_S} = \frac{\text{drag}}{\frac{1}{2}\rho_oV_o^2S_o^*l^2} = \frac{\pi^2\,t_o^2}{E^2-k'^2K^2}$$

$$C_{D_V} = \frac{\text{drag}}{\frac{1}{2}\rho_oV_o^2(V^*)^{2/3}l^2} = \frac{9}{\pi}\,\frac{E^2-k'^2K^2}{k^4(B-k'^2D)^2}\,(V^*)^{4/3}$$

If the length L of the center section is allowed to vanish, k' approaches zero, and the last formulas become

$$C_{D_S}\Big|_{L=0} = \pi^2\,t_o^2$$

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[2]The condition that λ be zero corresponds to solving the original isoperimetric problem of minimum drag with fixed length and frontal area, the volume being unspecified. This problem will lead to the best body sought for the case 1 under consideration. This result could also be obtained from equation (22) by the ordinary method of differentiation. Similar remarks apply to the case when μ is taken to be zero.

$$C_{D_V}\Big|_{L=0} = \frac{9}{\pi}(V^*)^{4/3}$$

These results agree with those of references 4 and 6. The drag coefficients for the body with a center section can be expressed in terms of the above quantities, so that the effect of adding a center section is readily seen:

$$\frac{C_{D_S}}{\pi^2 t_o^2} = \frac{1}{E^2 - k'^2 K^2} \tag{26}$$

and

$$\frac{C_{D_V}}{\frac{9}{\pi}(V^*)^{4/3}} = \frac{E - k'^2 K^2}{k^4 (B - k'^2 D)^2} \tag{27}$$

Figure 4 shows a plot of the quantity $C_{D_S}/\pi^2 t_o^2$ against k'. This figure shows that the drag coefficient based on frontal area rises slowly when the ratio of length of cylindrical portion to total length is small (the drag coefficient having risen 10 percent when the cylinder makes up about 10 percent of the body), but goes up very rapidly for bodies on which the cylindrical section makes up more than about 50 percent of the body.

There is another limiting case of some interest for the body with k' and S_o^* prescribed, besides the one in which k' vanishes. That is the case in which the cylindrical section becomes infinitely long while the nose and stern sections have a prescribed length. Thus, both l and L become infinite while $(l-L)$ remains fixed. The drag resulting from such a configuration is given by

$$R = \frac{\text{drag}}{\frac{1}{2}\rho_o V_o^2} = \frac{8\,S_o^2}{\pi(l-L)^2}$$

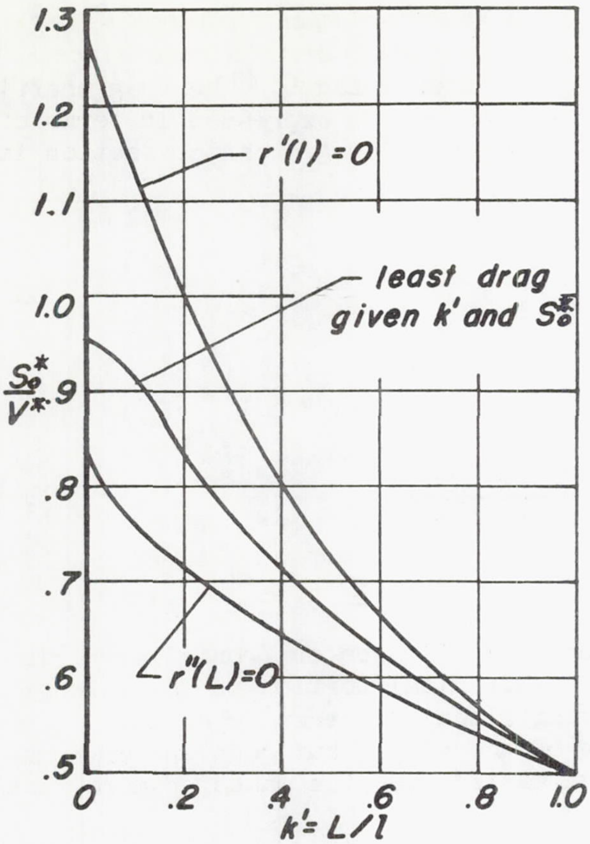
which agrees with a result of reference 2 for an ogive of given caliber at the end of a semi-infinite cylinder.

The shape of the best body of a family can be computed by means of equation (23). Since the thickness t is given by the ratio of diameter to total length, the ratio of the local radius r to the maximum radius r_o is

$$\left(\frac{r}{r_o}\right)_1 = \left\{\frac{\xi^2 KZ(\sigma,k) + [EE(\sigma,k) - k'^2\, KF(\sigma,k)] - K\xi\sqrt{(1-\xi^2)(\xi^2-k'^2)}}{E^2 - k'^2 K^2}\right\}^{1/2}$$

where

$$\xi = \frac{x}{l} \tag{28}$$



Plots of this thickness distribution for several values of k' are shown in figure 5. The accompanying sketch shows a plot of equation (25) in relation to the extreme admissible values of S_o^*/V^* versus k'. The upper curve corresponds to the equality in expression (14b), and represents the relation for a body with cusped tips, while the lowest curve corresponds to the equality in (14a), and gives zero curvature where the end sections join to the cylinder. Therefore, the best bodies of the present families never have cusped tips, nor zero curvature where the ogival sections and center section join.

<u>Case 2: Total length, volume, and cylinder length fixed; frontal area free.</u>- In this case, which can be analyzed in the same way as the one just preceding, the volume and length-ratio parameters V^* and k' serve to determine a range of permissible values of the frontal area parameter S_o^*. This range is again obtainable either from expressions (21), or from figure 2. Figure (3), showing variation of drag with k' and S_o^*/V^*, can once more be consulted for a general view of the behavior of the drag as the parameters change.

The optimum body in a given family is now determined by the vanishing of μ (see footnote 2), and the equations for thickness distribution and drag become

$$t^2(\xi) = \frac{t_o^2}{k^2(B-k'^2D)}\left\{k^2\left[B(\sigma,k) - k'^2D(\sigma,k)\right] - \xi\sqrt{(1-\xi^2)(\xi^2-k'^2)}\right\} \tag{29}$$

$$C_D^* = \frac{3\pi\ t_o^2V^*}{k^2(B-k'^2D)} \tag{30}$$

The relation between S_o^* and V^* for the best member of a family is

$$S_o^* = \frac{8}{3}\,\frac{B-k'^2D}{\pi\ k^2}\,V^* \tag{31}$$

Again the drag coefficient can be based either upon $(\text{volume})^{2/3}$ or upon frontal area, the former being the more useful in this case since volume is fixed. The drag coefficient based on $(\text{volume})^{2/3}$ is

$$C_{D_V} = \frac{8}{\pi}\left(V^*\right)^{4/3}\left[\frac{1}{k^4}\right] \tag{32}$$

and that based on the frontal area is

$$C_{D_S} = \frac{9}{8}\pi^2 t_o{}^2\left[\frac{1}{(B-k'^2D)^2}\right] \tag{33}$$

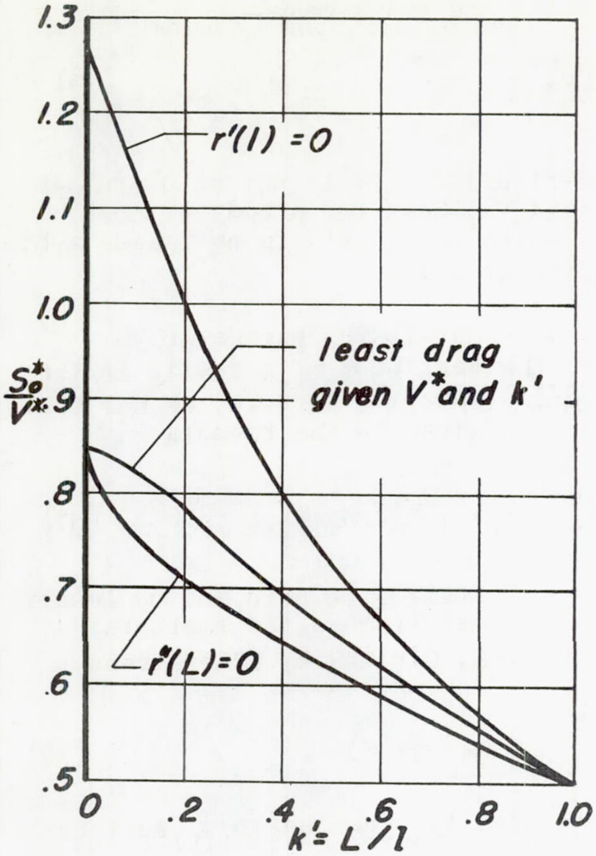
In both cases, the function of k' in square brackets reduces to unity as k' vanishes, and the resulting expressions for bodies without center sections agree with results of references 4 and 6. Figure 6 shows a plot of the quantity

$$\frac{C_{D_V}}{\frac{8}{\pi}(V^*)^{4/3}} = \frac{1}{k^4}$$

as a function of the length ratio k'. The behavior is qualitatively the same as found in case 1 for the drag coefficient based on frontal area, but the increase of drag coefficient with k' is somewhat slower in the present case.

In the present instance, where V^* and k' are given, the thickness distribution function is, from equation (29),

$$\left(\frac{r}{r_o}\right)_2 = \left\{\frac{k^2[B(\sigma,k) - k'^2D(\sigma,k)] - \xi\sqrt{(1-\xi^2)(\xi^2-k'^2)}}{k^2\ (B-k'^2D)}\right\}^{\frac{1}{2}} \tag{34}$$



Plots of this thickness distribution are shown in figure 7, where shapes for several values of k' are given. The sketch shows in this case that again the best body of a family is one having neither cusped nose and stern sections nor zero curvature at the junctions of end and center sections; the best body at $k' = 0$ has zero curvature at the juncture of the ogival sections.

Case 3: Total length, frontal area, and volume fixed; cylinder length free.- In this case the dimensionless parameters S_o^* and V^* are fixed, while the length ratio k' is free to vary. It is the most difficult case to analyze because the parameter k' appears implicitly in the functions involved. By differentiating equation (22) with respect to k', holding S_o^* and V^* fixed, it is found that the derivative $\partial C_D^*/\partial k'$ is never negative, and vanishes for a value of k' such that

$$\frac{\Phi_1}{\Psi_1} = \frac{S_o^*}{V^*}$$

where Φ_1 and ψ_1 are defined after expressions (21). In terms of the admissible range of the length ratio k' for fixed values of the frontal-area and volume parameters S_o^* and V^*, this equation means that the member of the family for which k' has the least admissible value (fig. 2) is the one with least wave drag in that family. It can be seen from figure 2, however, that if the ratio of the given parameters S_o^*/V^* is greater than $8/3\pi = 0.849$, the least admissible value of the length ratio is always k' equal to zero.

In the cases where the ratio S_o^*/V^* is less than $8/3\pi$, the drag coefficient of the best body of a family is given by

$$C_D^* = \frac{3(S_o^*)^2}{\pi k^4}\left[24(E^2-k'^2K^2)\left(\frac{V^*}{S_o^*}\right)^2 - 16\pi k^2(B-k'^2D)\frac{V^*}{S_o^*} + 3\pi^2k^4\right] \tag{35}$$

where the value of k' to be used is the one which makes

$$\frac{\Phi_1}{\Psi_1} = \frac{S_o^*}{V^*} \tag{36}$$

This value of k' is best found from figure 2. It is not so useful in this case to refer the drag coefficients to those for a body without center section, for the value of the length ratio k' is no longer arbitrary and hence cannot be made to vanish at will.

For the cases in which the ratio of frontal-area parameter to volume parameter is greater than $8/3\pi$, the best body of a family is the one for which the length of the cylindrical section is zero, as has been noted. The drag coefficient C_D^* is then given by the formula

$$C_D^* = \frac{3}{\pi}\left(24\ V^{*2} - 16\pi\ V^*\ S_o^* + 3\pi^2\ S_o^{*2}\right) \tag{37}$$

Since the length of the cylindrical center section is zero and no longer enters as a parameter, it is convenient to reintroduce the semitotal length l into the formulas. The wave drag, divided by free-stream dynamic pressure, is

$$R = \frac{3}{\pi l^4}\left(24\ V^2 - 16\pi\ l\ VS_o + 3\pi^2 l^2 S_o^2\right) \tag{38}$$

and this agrees with the result of reference 4. The length l must be between the limits

$$\frac{8}{3\pi}\frac{V}{S_o} \le l \le \frac{4}{\pi}\frac{V}{S_o} \tag{39}$$

Since for a given cross-sectional area, the body of least wave drag will be the one with the longest admissible length, it is clear that for the present case, where S_o and V are prescribed, and no center section exists, the best body is the one for which

$$l = \frac{4}{\pi}\frac{V}{S_o} \tag{40}$$

as stated in reference 4. Bodies, the length of which is greater than the value of equation (40), do not fall within the admissible region and hence violate condition (2c).

Shapes of the bodies for the present case are shown in figure 8. The body for which S_o^*/V^* equals $4/\pi$ and $k' = 0$ is shown at the top of the figure. This body is the best one of all those without a midsection, having the relation between l, S_o, and V given in equation (40). The other body shapes shown are each the best body for the prescribed value of the parameter ratio S_o^*/V^*, the value of k' being chosen from the lower curve of figure 2. The equations for the shapes are

for $0.5 \quad \frac{S_o^*}{V^*} \leq \frac{8}{3\pi}$

$$\left(\frac{r}{r_o}\right)_3 = \frac{1}{k^2\sqrt{\Delta}}\left\{8k^2(B-k'^2D)\left[\frac{3}{\pi}\left(E^2-k'^2K^2\right)\frac{V^*}{S_o^*} - k^2(B-k'^2D)\right]\left(\frac{r}{r_o}\right)_2^2 + 3(E^2-k'^2K^2)\left[3k^4 - 8\,\frac{k^2(B-k'^2D)}{\pi}\,\frac{V^*}{S_o^*}\right]\left(\frac{r}{r_o}\right)_1^2\right\}^{1/2} \tag{41a}$$

where $(r/r_o)_1$ and $(r/r_o)_2$ are defined in equations (28) and (34), respectively,

for $$\frac{8}{3\pi} \leq \frac{S_o^*}{V^*} \leq \frac{4}{\pi}$$

$$\left(\frac{r}{r_o}\right)_3 = \left[8\left(\frac{3}{\pi}\,\frac{V^*}{S_o^*} - 1\right)\left(\frac{r}{r_o}\right)_2^2 + 3\left(3 - \frac{8}{\pi}\,\frac{V^*}{S_o^*}\right)\left(\frac{r}{r_o}\right)_1^2\right]^{1/2} \tag{41b}$$

In order to obtain the best body of the family described by equation (41a), the value of k' to be used can be found from equation (36), or from the lower curve of figure 2. For the family described by equation (41b), the best member is the one for which

$$\frac{S_o^*}{V^*} = \frac{4}{\pi}$$

SUMMARY OF RESULTS

For convenience, the important drag formulas of the preceding analysis have been gathered together in the present section. The equations are numbered just as they appear in the text. The formulas are given in terms of the dimensionless parameters k', S_o^*, and V^*, which are related to the total length $2l$, the cylinder length $2L$, the frontal area S_o, and the volume V by means of the equations

$$k' = \frac{L}{l}$$

$$S_o^* = \frac{S_o}{l^2}$$

$$V^* = \frac{V}{l^3}$$

Case 1: Total Length, Cylinder Length, and
Frontal Area Fixed; Volume Free

The cylinder length and frontal area determine the dimensionless parameters k' and S_o*, while the volume is represented by the parameter $V*$. The prescribed values of k' and S_o* determine an admissible range of values of $V*$ (see fig. 2), and the optimum body of this family has the following characteristics:

$$V* = \frac{\pi k^2}{3} \frac{B-k'^2 D}{E^2-k'^2 K^2} S_o* \tag{25}$$

Drag coefficient based on frontal area,

$$\frac{C_{D_S}}{\pi^2 t_o^2} = \frac{1}{E^2-k'^2 K^2} \text{ (See fig. 4)} \tag{26}$$

Drag coefficient based on (volume)$^{2/3}$

$$\frac{C_{D_V}}{\frac{9}{\pi}(V*)^{4/3}} = \frac{E^2-k'^2 K^2}{k^4(B-k'^2 D)^2} \tag{27}$$

Case 2: Total Length, Volume, and Cylinder
Length Fixed; Frontal Area Free

The given volume and cylinder length determine the dimensionless parameters $V*$ and k', and these values lead to a range of admissible values for S_o*. (See fig. 2.) The best body of the family so determined has the following characteristics:

$$S_o* = \frac{8}{3\pi} \frac{B-k'^2 D}{k^2} V* \tag{31}$$

Drag coefficient based on (volume)$^{2/3}$,

$$\frac{C_{D_V}}{\frac{8}{\pi}(V*)^{4/3}} = \frac{1}{k^4} \text{ (See fig. 6)} \tag{32}$$

Drag coefficient based on frontal area,

$$\frac{C_{D_S}}{\frac{9}{8}\pi^2 t_o^2} = \frac{1}{(B-k'^2 D)^2} \tag{33}$$

Case 3: Total Length, Frontal Area, and Volume Fixed; Cylinder Length Free

The frontal area and volume determine S_o^* and V^*, and these in turn determine a range of admissible values of k'. (See fig. 2.) If the quotient S_o^*/V^* is less than $8/3\pi = 0.849$, the best body is one with a value of k' such that

$$\frac{\Phi_1}{\psi_1} = \frac{S_o^*}{V^*}$$

where Φ_1 and ψ_1 are defined in expressions (21). This value of k' is most easily found from figure 2. The drag coefficient for the optimem body is then

$$C_D^* = \frac{3}{\pi k^4 \Delta}\left[24(E^2-k'^2K^2)(V^*)^2-16\pi k^2(B-k'^2D)V^* S_o^* + 3\pi^2 k_4 (S_o^*)^2\right] \qquad (35)$$

In case the value of the ratio S_o^*/V^* is such that

$$\frac{8}{3\pi} \leq \frac{S_o^*}{V^*} \leq \frac{4}{\pi}$$

the best body is one with no center section ($k' = 0$), and the drag coefficient is

$$C_D^* = \frac{3}{\pi}\left[24(V^*)^2-16\pi V^* S_o^* + 3\pi^2(S_o^*)^2\right] \qquad (37)$$

or, in terms of the remaining three parameters l, S_o, V,

$$R = \frac{\text{drag}}{\frac{1}{2}\rho_o V_o{}^2} = \frac{3}{\pi l^4}\left(24 V^2-16\pi l V S_o + 3\pi^2 l^2 S_o{}^2\right) \qquad (38)$$

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif. August 24, 1951

APPENDIX

A MINIMUM DRAG PROBLEM WHICH INCLUDES AN APPROXIMATION TO THE EFFECT OF SKIN FRICTION

It is possible to include the effects of skin friction in the analysis of minimum drag of bodies of revolution, provided the surface area of the bodies is known. If it be assumed that this surface area is expressible in the form

$$\text{Area} = 2L(2\pi r_o) + 2(l - L)(2\gamma r_o) \tag{A1}$$

where the second term represents the area of the ogival end sections, and γ is an unspecified constant, then the drag due to skin friction is

$$\frac{D_f}{\frac{1}{2}\rho_o V_o{}^2 l^2} = 4\gamma\left[1 + \left(\frac{\pi}{\gamma} - 1\right)k'\right] C_{D_f}\, t_o \tag{A2}$$

where C_{D_f} is the friction-drag coefficient, t_o is the maximum thickness ratio, and k' is again the ratio of length of cylindrical section to total length.

Consider now a body of revolution with prescribed length ratio k' and frontal-area parameter $S_o{}^*$. The wave-drag coefficient based on frontal area for the best such body is, from equation (26) of the text,

$$C_{D_S} = \frac{\pi^2\, t_o{}^2}{E^2 - k'^2 K^2}$$

The total-drag coefficient C_{D_T}, based on frontal area, is then

$$C_{D_T} = \frac{\pi^2\, t_o{}^2}{E^2 - k'^2 K^2} + \frac{4\gamma}{\pi\, t_o}\left[1 + \left(\frac{\pi}{\gamma} - 1\right)k'\right]C_{D_f} \tag{A3}$$

Now the total drag can be minimized with respect to maximum thickness ratio t_o; there results for the optimum, $t_o{}'$,

$$t_o{}' = \frac{1}{2\pi}\left\{16\,\gamma\, C_{D_f}\left(E^2 - k'^2 K^2\right)\left[1 + \left(\frac{\pi}{\gamma} - 1\right)k'\right]\right\}^{1/3} \tag{A4}$$

It remains to assign values to the constants γ and C_{D_f}.

Inspection of figure 5 shows that the bodies under consideration do not differ greatly in shape from prolate ellipsoids of revolution. Thus, for the present purposes, it will be sufficient to use the approximation

$$\gamma = \frac{5}{2}$$

corresponding to the surface area of a prolate ellipsoid the minor axis of which is small compared to its major axis. Equation (A4) for the optimum thickness ratio now becomes

$$t_o' = \frac{1}{2\pi}\left\{40\ C_{D_f}\ (E^2-k'^2K^2)\ [1 + (\tfrac{2}{5}\pi-1)\ k']\right\}^{1/3} \tag{45}$$

If the cylindrical section is allowed to vanish, k' vanishes, and the last equation becomes

$$t_o'\Big|_{L=0} = \frac{1}{2\pi}\sqrt[3]{40\ C_{D_f}} \tag{46}$$

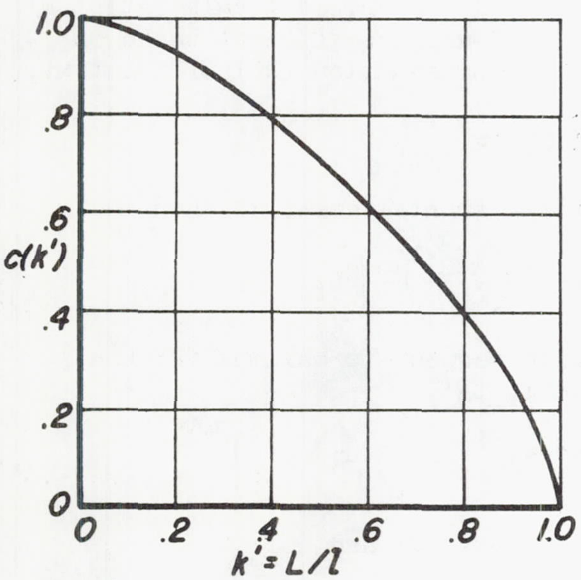
The optimum thickness ratio for bodies with a center section can therefore be expressed as

$$t_o' = c(k')\ t_o'\Big|_{L=0} \tag{47}$$

where

$$c\ (k') = \left\{(E^2-k'^2K^2)\left[1+\left(\frac{2}{5}\pi-1\right)k'\right]\right\}^{1/3}$$

A graph of $c(k')$ versus k' is shown in the sketch.



The remaining constant is the skin-friction drag coefficient C_{D_F}. For the purpose of illustration, an average value of 0.0025, corresponding to a turbulent boundary layer at a Mach number of about 1.7 and a Reynolds number of 13 million, was taken from the data of reference 9. The optimum thickness ratio for a body of revolution with no center section, and with prescribed length, considering both wave and friction drag, is then found to be (from equation (A6))

$$t_o'\Big|_{L=0} \cong \frac{1}{14}$$

By means of the sketch of the variation of $c(k')$, the results can be extended to bodies of revolution with a center section. Consider a body the center section of which makes up 10 percent of its total length. From the sketch and the above value of $t_o'\Big|_{L=0}$, it is seen that the optimum value of thickness ratio remains about 1/14, again showing that the effect of the added cylindrical portion upon the drag is small for small values of the length ratio k'. The optimum thickness ratio decreases to 1/19 for a body the center part of which is 50 percent of the total length.

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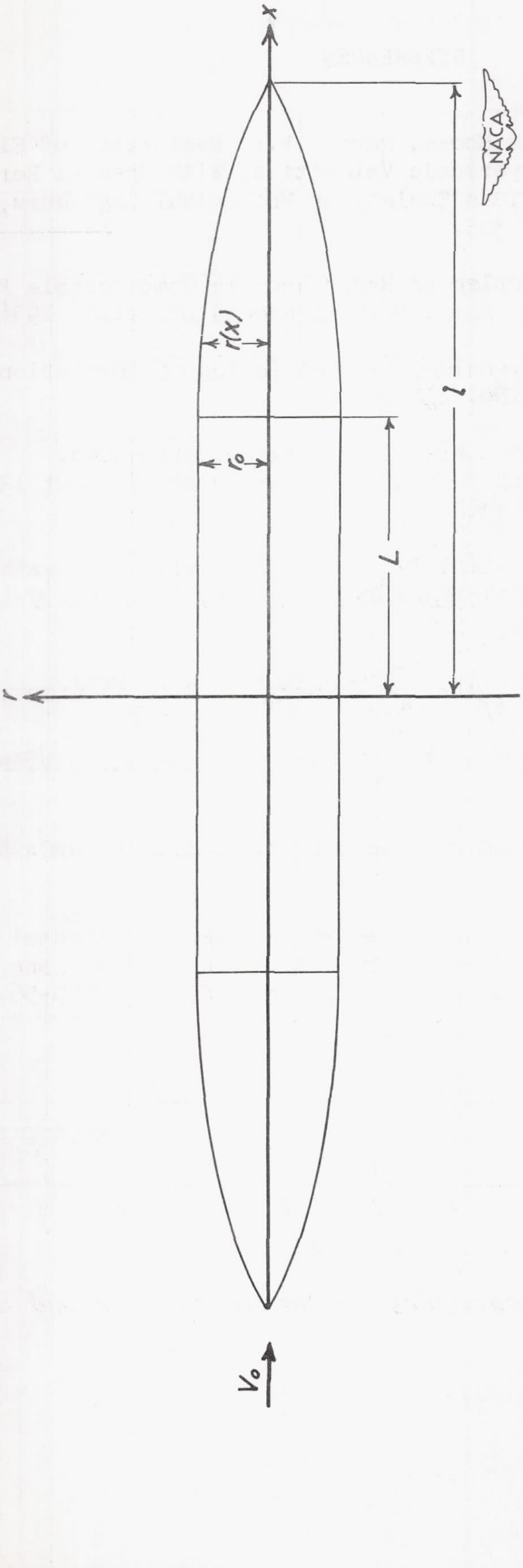

Figure 1.— Type of body considered and notation used.

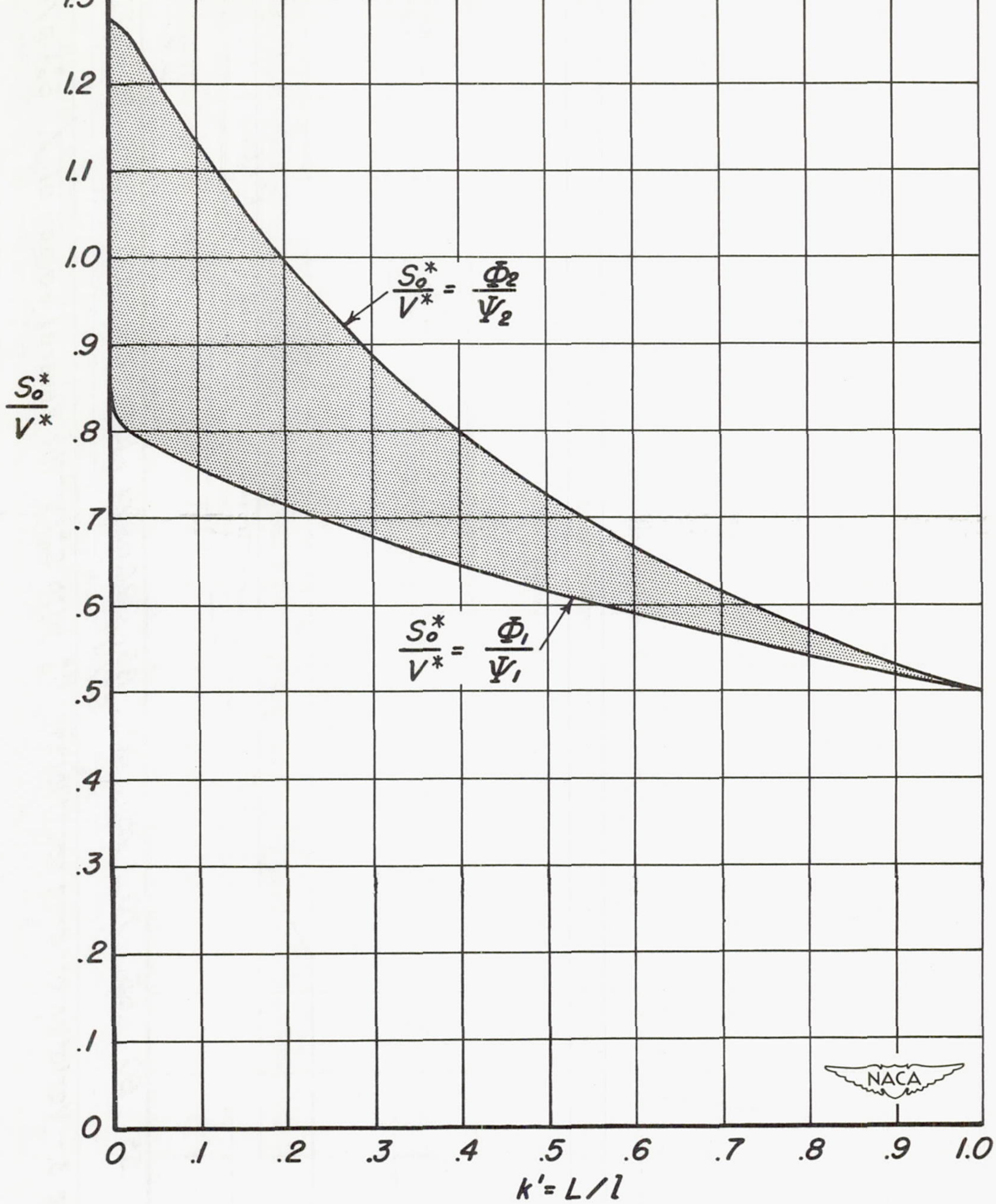

Figure 2.– Region of admissible values for the parameters k' and S_o^*/V^*.

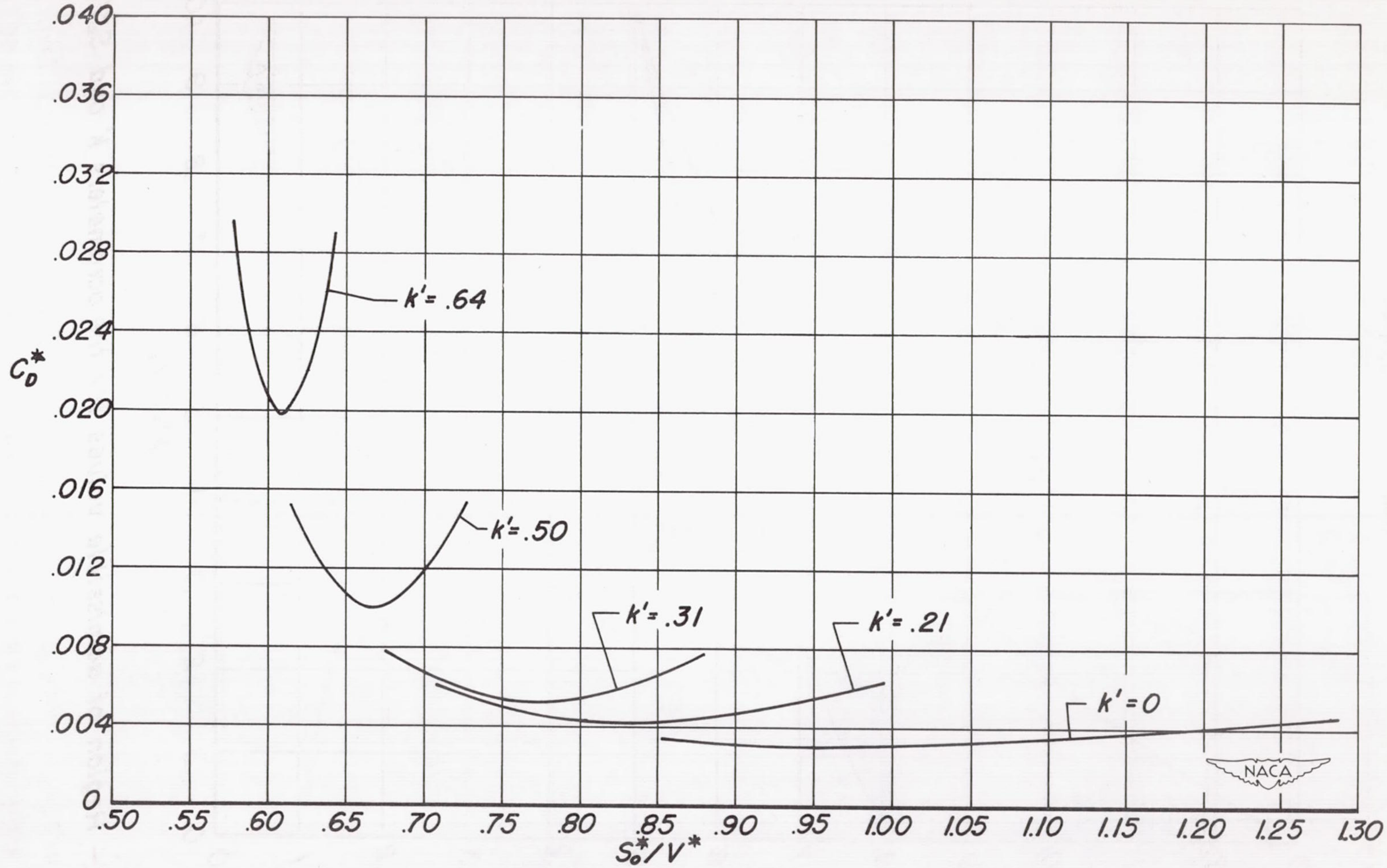

Figure 3.— Variation of drag coefficient C_D^* with S_o^*/V^* for several values of k'. $S_o^* = \pi/100$.

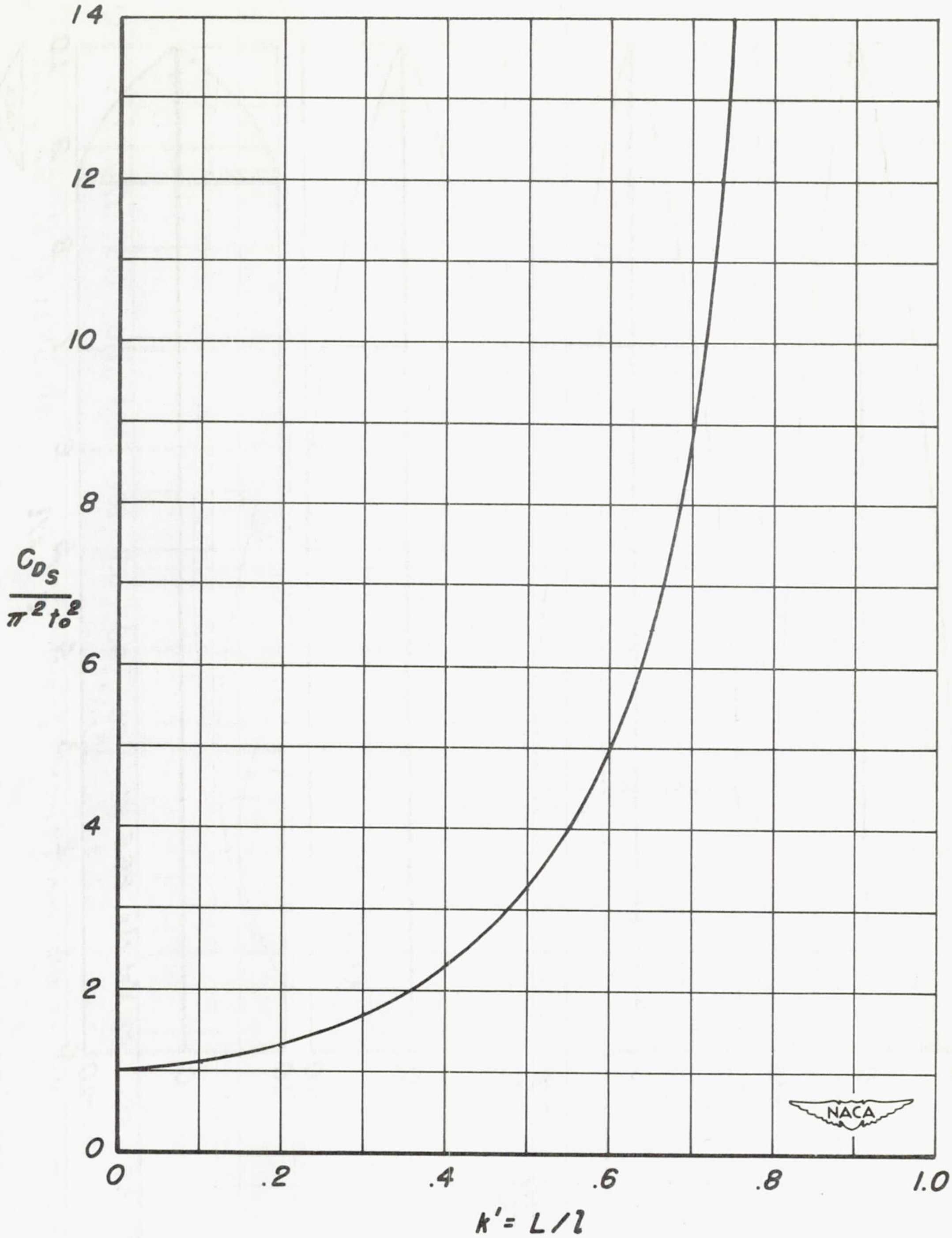

Figure 4.— Variation of drag coefficient based on frontal area with length of cylindrical section, given length-ratio and frontal-area parameter.

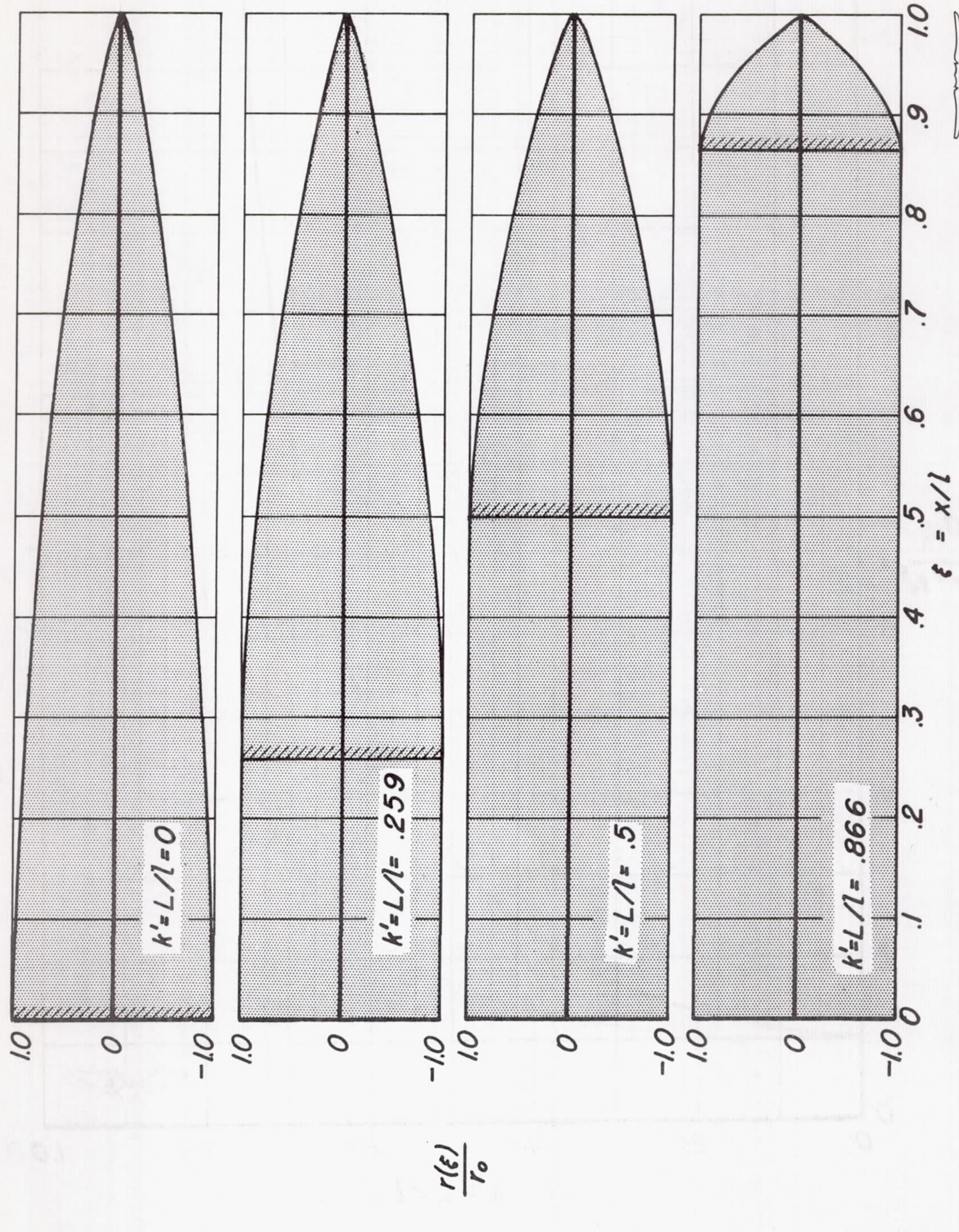

Figure 5.- Variation of body shape with k'; given length-ratio and frontal-area parameter.

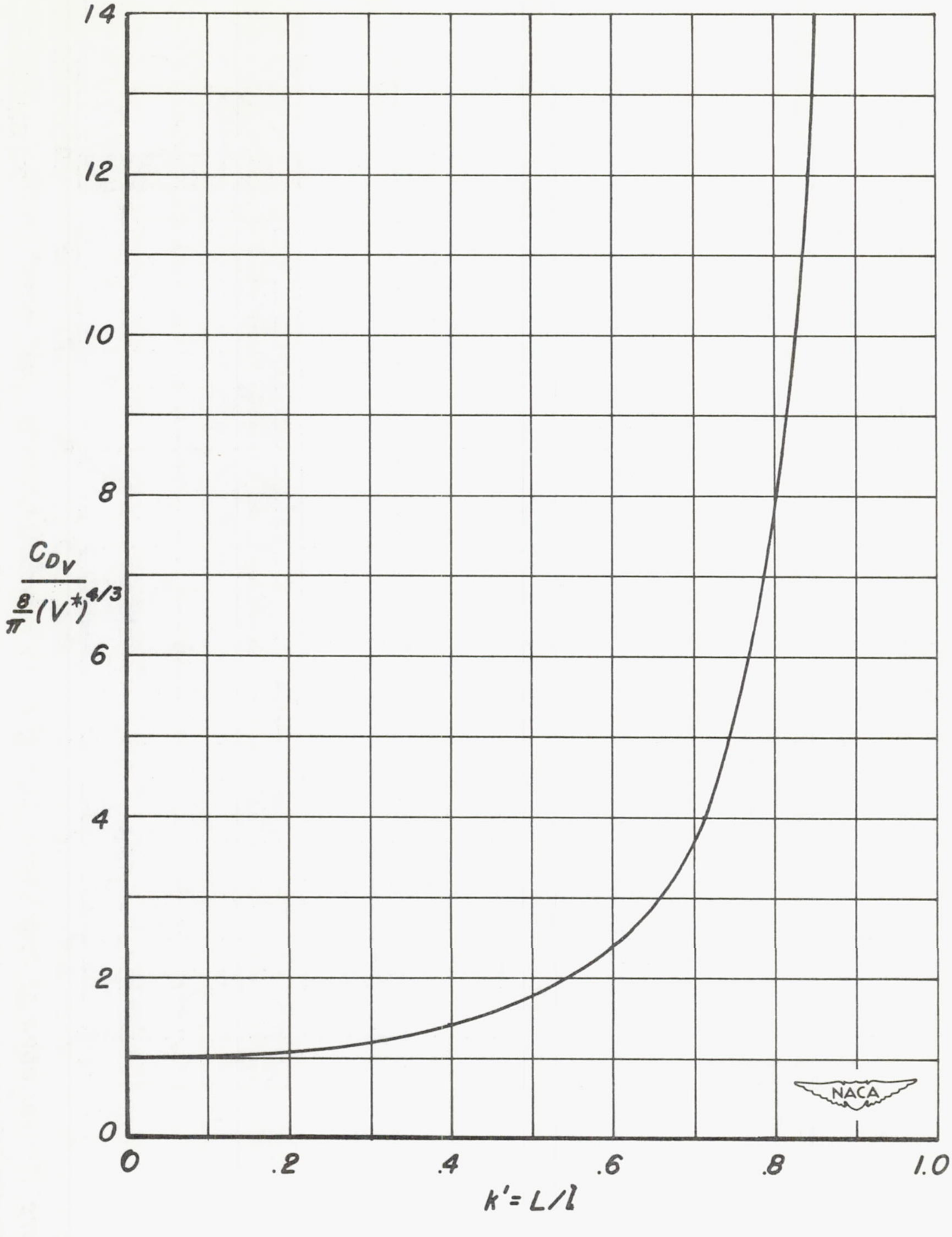

Figure 6.— Variation of drag coefficient based on volume$^{2/3}$ with length of cylindrical section, given length–ratio and volume parameter.

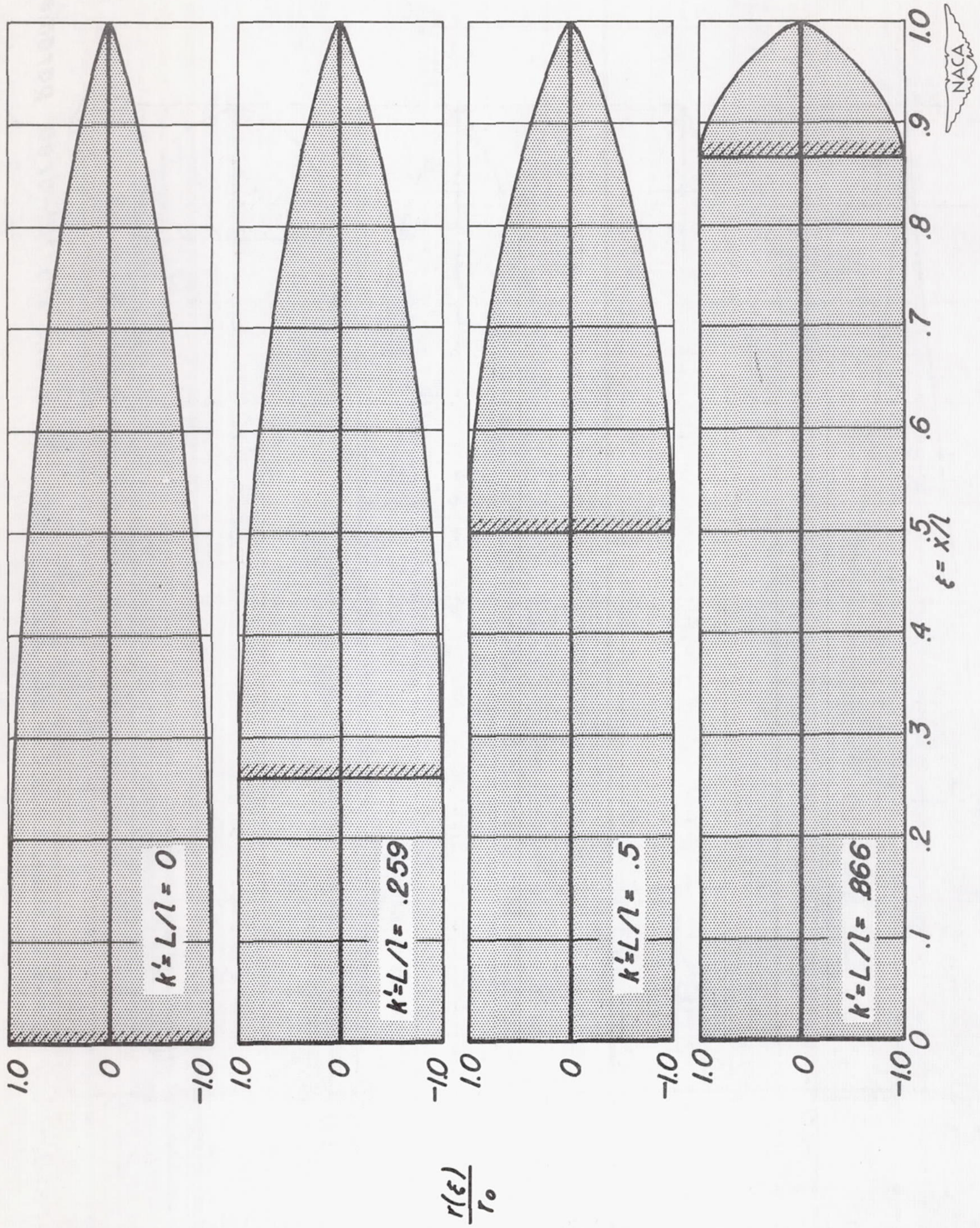

Figure 7.— Variation of body shape with k'; given volume parameter and length ratio.

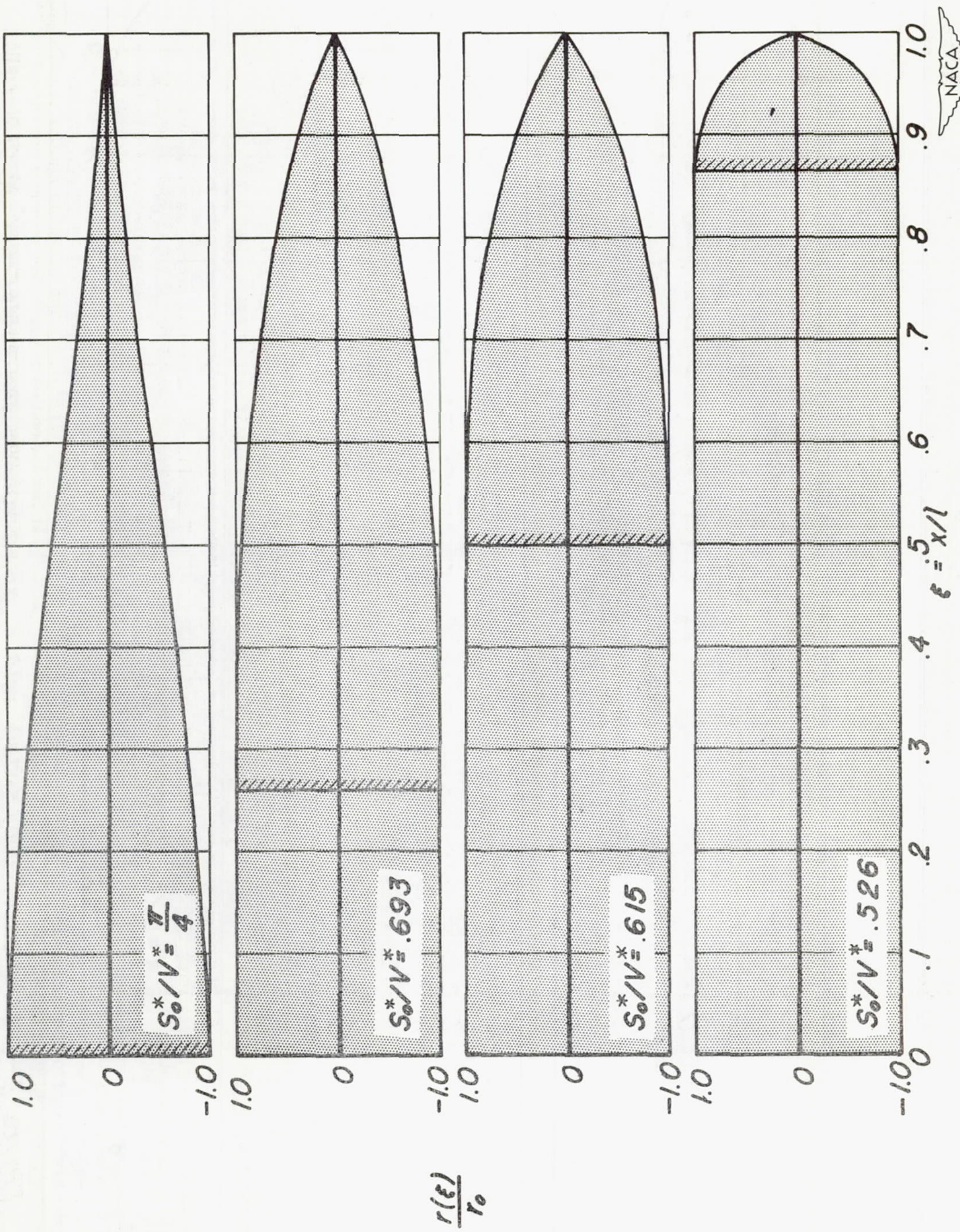

Figure 8.— Variation of body shape with S_o^*/V^*; given volume and frontal-area parameters.