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BENDING OF THIN PLATES WITH COMPOUND CURVATURE

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SUMMARY

An analysis is presented for the deformation of a doubly curved thin plate under edge loads or surface loads for small deflections. This problem is approached from thin-shell theory so that the plate is to form part of a shell of revolution. The method developed is particularly useful for a plate whose radius of curvature in one direction is large compared with its length and width dimensions. The solution consists of an expansion about a parameter which depends on this fact.

An analytical solution is presented completely for a plate with an arbitrary meridian curve of small curvature and loaded by normal edge loads on one pair of opposite edges. Numerical calculations for the deflection and moment distribution are presented for a particular meridian curve. For the meridian curve chosen for the numerical example, part of the surface had a negative Gaussian curvature. Results show that the deflections and bending moment are largest at the part of the plate with negative Gaussian curvature.

The method is developed to the point that it may be applied readily to other problems of the deformations of doubly curved thin plates under edge or surface loads. The theory, however, is limited to small deflections of the plate or shell considered.

INTRODUCTION

This report is concerned with the behavior of a doubly curved thin plate under edge loads or surface loads. This problem is considered in the following way. The doubly curved plate is to form a part of a shell of revolution bounded by two meridians and two parallels. The meridian curve is assumed to have a radius of curvature much larger than the radius of curvature of a parallel. This situation is clearly presented in most airplane fuselage panels. The introduction of a small parameter dependent on this fact allows the equations for equilibrium of the shell of revolution to reduce to ones with constant coefficients. The solution of this sequence of problems then leads to a complete solution of the problem. It may be noted that the method so developed is equally valid, for analysis of the deformation of a shell of revolution with a meridian curve of small curvature and loaded in any manner whatsoever.

The thin-shell equations valid for small deformation of an arbitrary shell of revolution are first presented. These, in general, lead to a set of differential equations with variable coefficients. The perturbation method is then applied to reduce these equations to ones with constant coefficients. An analytic solution is presented completely for a plate with arbitrary meridian of small curvature loaded by normal edge loads on one pair of opposite edges and stress free on the other pair. In addition, all edges are free from shearing stresses. For this type of condition two infinite families of algebraic equations are obtained which are solved by retaining a finite number. Finally, numerical calculations for the deflection and moment distribution are given for a particular meridian curve.

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SYMBOLS

$A = Gh^3/12$	
a	constant defined by equation (10)
$B = Gh$	
$C = Eh/(1 - \nu^2)$	
$D = Eh^3/12(1 - \nu^2)$	
E	Young's modulus
$F^{(o)}, F^{(1)}, \ldots$	stress functions
$f(z)$	equation of meridian curve

G	shear modulus
G, H, K	known functions given from solution of each approximation
h	thickness of shell
l, θ_o	define length and width of plate
$M_\xi, M_\theta, M_{\xi\theta}$	moment resultants defined by equations (3)
$N_\xi, N_\theta, N_{\xi\theta}$	force resultants defined by equations (3)
p_ζ	surface load in direction normal to middle surface
R_θ, R_ξ	principal radii of curvature of middle surface of shell
u, v, w	displacements in θ, ξ, and ζ directions, respectively
x, y, z	rectangular Cartesian coordinates
α^2, r^2	fundamental magnitudes of first order of middle surface
$\alpha_n = n\pi/2$	
$\beta_m = m\pi/2$	
$\gamma_m = m\pi b/2$	
$\delta_n = n\pi l/2b$	
$\epsilon_\theta, \epsilon_\xi, \gamma_{\theta\xi}$	extensional strains and shear strain given by equations (8)
ζ	coordinate normal to middle surface
θ, ξ	curvilinear coordinates on middle surface of shell
$\kappa_\theta, \kappa_\xi, \tau$	change-of-curvature functions given by equations (8)
μ	small parameter defined by equation (10)
$\mu_m = m\pi/2l$	

ν	Poisson's ratio
$\nu_n = n\pi/2b$	
$\tau_{\xi\theta}$, $\tau_{\theta\zeta}$	shear stresses
$\tau_{\xi\xi}$, $\tau_{\theta\theta}$	normal stresses
ψ	function defined by equations (22)
Ω	"equivalent" potential function

BENDING OF SHELLS OF REVOLUTION

The differential equations governing the small deflection of shells of revolution are considered herein. In this section the equations pertinent to the later investigations are outlined.

Coordinate System on Shell

Consider a shell of revolution located as shown in figure 1; in addition to the x, y, and z Cartesian coordinate system a set of orthogonal curvilinear coordinates θ, ξ, and ζ is chosen on the middle surface of the shell such that the θ and ξ lines are lines of curvature of the middle surface and ζ is the distance normal to the middle surface. In addition, the θ and ξ lines are the parallels and meridians of the middle surface (for a shell of revolution). Note that ξ is a parameter along the meridian plane and θ is the polar angle measured from the xz-plane. The element of arc length will be:

$$ds^2 = \alpha^2\left(1 + \frac{\zeta}{R_\xi}\right)^2 d\xi^2 + r^2\left(1 + \frac{\zeta}{R_\theta}\right)^2 d\theta^2 + d\zeta^2 \tag{1}$$

where $\alpha^2 = \left(\frac{dz}{d\xi}\right)^2 + \left(\frac{dr}{d\xi}\right)^2$ and R_ξ and R_θ are the principal radii of curvature of the middle surface. It is noted that instead of ξ one can use z as the independent variable. Then the element of arc length in the ξ direction becomes

$$\alpha\, d\xi = \sqrt{1 + \left(\frac{dr}{dz}\right)^2}\, dz \tag{2}$$

One also notes that $r = r(z)$ is the equation of the meridian curve.

The sign convention as given, for example, by Sokolnikoff and Specht (reference 1) is used. Thus if τ_{ij} is the stress tensor, then the subscript i indicates direction of normal to plane under consideration and j, the direction of the component of stress. Tensile stresses and compressive stresses are positive and negative, respectively. Shear stresses on a particular plane are positive if the normal forces are positive on that plane and if they are acting in the direction of positive coordinate axes; otherwise, they are negative.

The stress resultants and moments, as defined later, have the following sign convention (see fig. 2). The resultant bending moments are positive if they cause positive stress on the positive side (ζ) of the middle surface. Resultant forces have the same sign convention as the stresses.

Assumptions in Analysis

The assumptions inherent in this analysis are the usual ones for thin-shell theory. They are amply presented in reference 2. Summarizing, they are:

(1) Material is isotropic and follows Hooke's law

(2) Thickness of shell is small compared with smallest radius of curvature of middle surface

(3) Displacements are small compared with thickness

(4) Straight lines normal to undeformed middle surface remain straight and normal to deformed middle surface

In addition, two other assumptions are made here. These are: First, the effect of transverse shear in the resultant force equations is small and can be neglected and, secondly, the effect of the displacements tangential to the middle surface of the shell in the changes of curvature is of higher order than that of the displacement normal to the surface. These assumptions are discussed more fully later in the text.

Differential Equations of Equilibrium

Force and moment resultants.- The stress resultants and moments are defined on the coordinate curves on the middle surface of the shell. They are given in units of force or moment per unit length of the middle surface. Thus the following definitions are made:

$$\left.\begin{aligned}
N_{\xi} &= \int \tau_{\xi\xi}\left(1 + \frac{\zeta}{R_{\theta}}\right) d\zeta \\
M_{\xi} &= \int \zeta\tau_{\xi\xi}\left(1 + \frac{\zeta}{R_{\theta}}\right) d\zeta \\
N_{\xi\theta} &= \int \tau_{\xi\theta}\left(1 + \frac{\zeta}{R_{\theta}}\right) d\zeta \\
M_{\xi\theta} &= \int \zeta\tau_{\xi\theta}\left(1 + \frac{\zeta}{R_{\theta}}\right) d\zeta \\
Q_{\xi} &= \int \tau_{\xi\xi}\left(1 + \frac{\zeta}{R_{\theta}}\right) d\zeta \\
M_{\theta} &= \int \zeta\tau_{\theta\theta}\left(1 + \frac{\zeta}{R_{\xi}}\right) d\zeta \\
N_{\theta} &= \int \tau_{\theta\theta}\left(1 + \frac{\zeta}{R_{\xi}}\right) d\zeta \\
M_{\theta\xi} &= \int \zeta\tau_{\theta\xi}\left(1 + \frac{\zeta}{R_{\xi}}\right) d\zeta \\
N_{\theta\xi} &= \int \tau_{\theta\xi}\left(1 + \frac{\zeta}{R_{\xi}}\right) d\zeta \\
Q_{\theta} &= \int \tau_{\theta\zeta}\left(1 + \frac{\zeta}{R_{\xi}}\right) d\zeta
\end{aligned}\right\} \qquad (3)$$

with the integrals evaluated between the limits $-h/2$ to $h/2$.

The differential equations of equilibrium for the force and moment resultants may be obtained from physical considerations (e.g., reference 2) or by integration across the thickness of the shell of the differential equations for the stresses (reference 3). Thus the differential equations for the stress resultants are:

$$\frac{\partial}{\partial\theta}\left(\alpha N_{\theta}\right) + \frac{\partial}{\partial\xi}\left(rN_{\xi\theta}\right) + \frac{\partial r}{\partial\xi} N_{\theta\xi} + \frac{\alpha r}{R_{\theta}} Q_{\theta} = 0 \tag{4a}$$

$$\frac{\partial}{\partial\xi}\left(rN_{\xi}\right) + \frac{\partial}{\partial\theta}\left(\alpha N_{\theta\xi}\right) - \frac{\partial r}{\partial\xi} N_{\theta} + \frac{\alpha r}{R_{\xi}} Q_{\xi} = 0 \tag{4b}$$

$$\frac{\partial}{\partial\xi}\left(rQ_{\xi}\right) - \frac{\alpha r}{R_{\theta}} N_{\theta} - \frac{\alpha r}{R_{\xi}} N_{\xi} + \alpha r p_{\zeta} + \frac{\partial}{\partial\theta}\left(\alpha Q_{\theta}\right) = 0 \tag{4c}$$

where

$$p_{\zeta} = \left[\left(1 + \frac{\zeta}{R_{\theta}}\right)\left(1 + \frac{\zeta}{R_{\xi}}\right)\tau_{\zeta\zeta}\right]_{-h/2}^{h/2}$$

In these equations it has been stipulated that there are no tangential forces applied to the middle surface of the shell and no body forces are presented. The equations for moment equilibrium are

$$\frac{\partial\left(\alpha M_{\theta}\right)}{\partial\theta} + \frac{\partial\left(rM_{\xi\theta}\right)}{\partial\xi} - r\alpha Q_{\theta} + \frac{\partial r}{\partial\xi} M_{\theta\xi} = 0 \tag{5a}$$

$$\frac{\partial\left(rM_{\xi}\right)}{\partial\xi} + \frac{\partial}{\partial\theta}\left(\alpha M_{\theta\xi}\right) - r\alpha Q_{\xi} - \frac{\partial r}{\partial\xi} M_{\theta} = 0 \tag{5b}$$

$$N_{\theta\xi} - N_{\xi\theta} = \frac{M_{\xi\theta}}{R_{\xi}} - \frac{M_{\theta\xi}}{R_{\theta}} \tag{5c}$$

Equation (5c) is not obtained from integration of the stress-equilibrium equation in the ζ direction. It is implied by the relation of equality of cross shears

$$\tau_{\theta\xi} = \tau_{\xi\theta} \tag{6}$$

that is, if equation (6) is rewritten in the form

$$\tau_{\theta\xi}\left(1 + \frac{\zeta}{R_\xi}\right) - \tau_{\xi\theta}\left(1 + \frac{\zeta}{R_\theta}\right) = -\frac{1}{R_\theta}\zeta\tau_{\theta\xi}\left(1 + \frac{\zeta}{R_\xi}\right) + \frac{1}{R_\xi}\zeta\tau_{\xi\theta}\left(1 + \frac{\zeta}{R_\theta}\right)$$

and integrated across the thickness of the shell, equation (5c) results.

Relations between force and moment resultants and strains of middle surface.- The relations between the stress resultants and the strains of the middle surface are obtained from stress-strain relations and definitions (3) utilizing the assumption that $\left(1 + \frac{\zeta}{R_\theta}\right) \approx 1$ and $\left(1 + \frac{\zeta}{R_\xi}\right) \approx 1$ in equations (3). (See references 3, 4, and 5.) These relations are:

$$\left.\begin{aligned}
N_\xi &= C\left(\epsilon_\xi + \nu\epsilon_\theta\right) \\
N_\theta &= C\left(\epsilon_\theta + \nu\epsilon_\xi\right) \\
N_{\xi\theta} &= N_{\theta\xi} = B\gamma_{\xi\theta} \\
M_\theta &= D\left(\kappa_\theta + \nu\kappa_\xi\right) \\
M_\xi &= D\left(\kappa_\xi + \nu\kappa_\theta\right) \\
M_{\xi\theta} &= M_{\theta\xi} = A\tau
\end{aligned}\right\} \tag{7}$$

where $A = \frac{Gh^3}{12}$, $B = hG$, $C = \frac{hE}{1 - \nu^2}$, and $D = \frac{Eh^3}{12\left(1 - \nu^2\right)}$.

The extensional strains and changes of curvature may be given in terms of the displacement of an arbitrary point in the middle surface of the shell. Let u, v, and w be the components of the displacement of a point on the middle surface where u is along the tangent to a parallel circle, v, along the tangent to the meridian curve, and w, along the normal to the surface. Thus the extensional strains and changes of curvature of the middle surface are

$$\left.\begin{aligned}
\epsilon_\theta &= \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r\alpha}\frac{\partial r}{\partial \xi} v + \frac{w}{R_\theta} \\
\epsilon_\xi &= \frac{1}{\alpha}\frac{\partial v}{\partial \xi} + \frac{w}{R_\xi} \\
\gamma_{\theta\xi} &= -\frac{u}{\alpha r}\frac{\partial r}{\partial \xi} + \frac{1}{\alpha}\frac{\partial u}{\partial \xi} + \frac{1}{r}\frac{\partial v}{\partial \theta} \\
\kappa_\theta &= \frac{1}{r}\left(\frac{1}{R_\theta}\frac{\partial u}{\partial \theta} - \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\right) + \frac{1}{\alpha r}\frac{\partial r}{\partial \xi}\left(\frac{v}{R_\xi} - \frac{1}{\alpha}\frac{\partial w}{\partial \xi}\right) \\
\kappa_\xi &= \frac{1}{\alpha}\frac{\partial}{\partial \xi}\left(\frac{v}{R_\xi} - \frac{1}{\alpha}\frac{\partial w}{\partial \xi}\right) \\
\tau &= -\frac{1}{\alpha r}\frac{\partial r}{\partial \xi}\left(\frac{u}{R_\theta} - \frac{1}{r}\frac{\partial w}{\partial \theta}\right) + \frac{1}{\alpha}\frac{\partial}{\partial \xi}\left(\frac{u}{R_\theta} - \frac{1}{r}\frac{\partial w}{\partial \theta}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{v}{R_\xi} - \frac{1}{\alpha}\frac{\partial w}{\partial \xi}\right)
\end{aligned}\right\} \tag{8}$$

In the latter part of this report the above set of equations will be applied to doubly curved plates which have been "cut" from a nearly circular cylindrical shell. Thus the principal radius of curvature R_ξ will be quite large. Moreover, the assumption $(h/R) \ll 1$ $\big(R$ is the smaller of R_ξ and $R_\theta\big)$ implies that in these problems u and v are second order small in κ_θ, κ_ξ, and τ. Finally, if terms of higher order than h^3 in the values of κ_θ, κ_ξ, and τ are rejected one obtains in place of the last three relations in equations (8) the following equations:

$$\left.\begin{aligned}
\kappa_\theta &= -\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{\alpha^2 r}\frac{\partial r}{\partial \xi}\frac{\partial w}{\partial \xi} \\
\kappa_\xi &= -\frac{1}{\alpha}\frac{\partial}{\partial \xi}\left(\frac{1}{\alpha}\frac{\partial w}{\partial \xi}\right) \\
\tau &= \frac{1}{\alpha r^2}\frac{\partial r}{\partial \xi}\frac{\partial w}{\partial \theta} - \frac{1}{\alpha}\frac{\partial}{\partial \xi}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{\alpha}\frac{\partial w}{\partial \xi}\right)
\end{aligned}\right\} \qquad (9)$$

One notes that in equation (5c) the terms $M_{\xi\theta}/R_\xi$ and $M_{\theta\xi}/R_\theta$ are of the order of h^3/R_ξ and h^3/R_θ and can be rejected, thus leading to

$$N_{\xi\theta} = N_{\theta\xi}$$

agreeing with equations (7). A further simplification can be made by rejecting the transverse shears Q_θ and Q_ξ in the force equations since they are of the order of h^3/R_θ and h^3/R_ξ, respectively, while the remainder of the terms are at most of the order of h/R_ξ and h/R_θ.

ANALYSIS OF CURVED PLATES

In the following sections the equations will be applied to the bending of doubly curved plates which form a part of a shell of revolution. The equations as given above are, in general, intractable to the problem at hand. However, in the application, for example, to curved panels of an airplane fuselage the radius of curvature of the meridian curve is many times the length and width of the panel. Hence a solution to the problem can be obtained by an expansion in terms of a parameter which will be small when the radius of curvature of the meridian curve is large. Then the solution will appear, for example, in w as

$$w = w^{(0)} + \mu w^{(1)} + \mu^2 w^{(2)} + \ldots$$

where μ is the parameter concerned.

To this purpose the equation of the meridian curve is represented by the relation

$$r = a + \mu f(z) \tag{10}$$

where a is a constant, μ is the small parameter dependent on the maximum deviation of the meridian curve from the straight line $r = a$, and $f(z)$ is the equation of the meridian curve.

It is more convenient in the following sections to use the coordinate z instead of ξ. Thus one replaces ξ by z in all equations above.

The radii of curvature R_ξ and R_θ are given by the relations

$$\left.\begin{aligned} \frac{1}{R_\theta} &= -\frac{1}{r\left[1 + (dr/dz)^2\right]^{1/2}} \\ \frac{1}{R_\xi} &= \frac{\left(d^2r/dz^2\right)}{\left[1 + (dr/dz)^2\right]^{3/2}} \end{aligned}\right\} \tag{11}$$

In accordance with the first paragraph of this section all dependent variables will be expanded in power series of the parameter μ. Then solutions of the different approximations are obtained. For small values of the parameter μ, probably two approximations are sufficient to lead to a complete solution of the problem.

The expansions required are:

$$\left.\begin{aligned} \epsilon_\theta &= \epsilon_\theta{}^{(o)} + \mu\epsilon_\theta{}^{(1)} + \mu^2\epsilon_\theta{}^{(2)} + \ldots \\ \kappa_\theta &= \kappa_\theta{}^{(o)} + \mu\kappa_\theta{}^{(1)} + \mu^2\kappa_\theta{}^{(2)} + \ldots \end{aligned}\right\} \tag{12}$$

$$u = u^{(o)} + \mu u^{(1)} + \mu^2 u^{(2)} + \ldots \tag{13}$$

$$N_\theta = N_\theta^{(o)} + \mu N_\theta^{(1)} + \mu^2 N_\theta^{(2)} + \ldots \tag{14}$$

$$M_\theta = M_\theta^{(o)} + \mu M_\theta^{(1)} + \mu^2 M_\theta^{(2)} + \ldots \tag{15}$$

and similar expansions are obtained for ϵ_z, $\epsilon_{\theta z}$, κ_z, τ, N_z, $N_{\theta z}$, M_z, and $M_{\theta z}$.

The expansions of the various terms have been summarized in appendixes A and B. The superscripts correspond to the order of the approximation.

Equations for First Approximation

The first approximation yields essentially the same equations as that of a circular cylinder, as would be expected. One can obtain two simultaneous equations for a stress function $F^{(o)}$ and the normal displacement $w^{(o)}$ from the pertinent equations in appendix B as follows: Let

$$\left.\begin{aligned} N_\theta^{(o)} &= \frac{\partial^2 F^{(o)}}{\partial z^2} \\ N_z^{(o)} &= \frac{1}{a^2}\frac{\partial^2 F^{(o)}}{\partial \theta^2} \\ N_{\theta z}^{(o)} &= -\frac{\partial^2 F^{(o)}}{a\,\partial\theta\,\partial z} \end{aligned}\right\} \tag{16}$$

where the superscript corresponds to the first approximation. Then it is seen that equations (B13) for force equilibrium are automatically satisfied. Moreover, a compatibility equation can be constructed from the strains (equation (B7)) and is

$$\frac{\partial^2 \epsilon_\theta^{(o)}}{\partial z^2} + \frac{1}{a^2}\frac{\partial^2 \epsilon_z^{(o)}}{\partial \theta^2} - \frac{1}{a}\frac{\partial^2 \gamma_{\theta z}^{(o)}}{\partial z\,\partial\theta} = -\frac{1}{a}\frac{\partial^2 w^{(o)}}{\partial z^2} \tag{17}$$

The resultant force and strain relations give

$$\left.\begin{aligned} \epsilon_z^{(o)} &= \frac{1}{hE}\left[N_z^{(o)} - \nu N_\theta^{(o)}\right] \\ \epsilon_\theta^{(o)} &= \frac{1}{hE}\left[N_\theta^{(o)} - \nu N_z^{(o)}\right] \\ \gamma_{z\theta}^{(o)} &= \frac{N_{z\theta}^{(o)}}{hG} \end{aligned}\right\} \tag{18}$$

The substitution of equations (18) and (16) into equation (17) leads to

$$\nabla^4 F^{(o)} = -\frac{hE}{a}\frac{\partial^2 w^{(o)}}{\partial z^2} \tag{19}$$

where

$$\nabla^4 = \frac{\partial^4}{\partial z^4} + 2\frac{\partial^4}{\partial s^2 \partial z^2} + \frac{\partial^4}{\partial s^4}$$

$$s = a\theta$$

The moment-equilibrium equation together with its relation to the changes in curvature (equations (B4, (B5), and (B10)) gives the second equation for $w^{(o)}$:

$$\nabla^4 w^{(o)} = \frac{1}{aD}\frac{\partial^2 F^{(o)}}{\partial z^2} + \frac{p_\zeta}{D} \tag{20}$$

Equations (19) and (20) are a pair of simultaneous equations for $w^{(o)}$ and $F^{(o)}$ subjected to appropriate boundary conditions. The effect of curvature in the θ direction is evident from the terms multiplied by $1/a$. Thus one obtains the flat-plate equations if $(1/a) \longrightarrow 0$, that is, coupling of the two equations disappears. It is noted that these equations are exact for bending of a circular cylindrical shell with small deflections.

Equations (19) and (20) can be combined into one equation for one complex function (cf. Reissner, reference 5). If equation (20) is multiplied by a constant k and added to equation (19) one obtains first

$$\nabla^4\left[F^{(o)} + kw^{(o)}\right] = \frac{\partial^2}{\partial z^2}\left[-\frac{hE}{a}w^{(o)} + \frac{kF^{(o)}}{aD}\right] + \frac{kp_\zeta}{D}$$

then

$$\nabla^4\left[F^{(o)} + i\sqrt{hED}w^{(o)}\right] = \frac{i\sqrt{hED}}{aD}\frac{\partial^2}{\partial z^2}\left[F^{(o)} + i\sqrt{hED}w^{(o)}\right] + \frac{p_\zeta}{D}k \qquad (21)$$

if $k = i\sqrt{hED}$ where $i = \sqrt{-1}$. Equation (21) is a differential equation for one complex function $\left[F^{(o)} + i\sqrt{hED}w^{(o)}\right]$. Furthermore, if one desires, equations (19) and (20) may be reduced to one eighth-order differential for a function ψ such that

$$\left.\begin{aligned} F^{(o)} &= -\frac{\partial^2\psi}{\partial z^2} \\ w^{(o)} &= \frac{a}{Eh}\nabla^4\psi \end{aligned}\right\} \qquad (22)$$

However, for practical calculation $F^{(o)}$ and $w^{(o)}$ may be represented by double Fourier series and equations (19) and (20) are used directly.

Equations for Second Approximation

The second approximation involves now the known functions of the first approximation on the right side of each equation. Let

$$\left.\begin{aligned} N_\theta{}^{(1)} &= \frac{\partial^2 F^{(1)}}{\partial z^2} + \Omega(\theta,z) \\ N_\xi{}^{(1)} &= \frac{1}{a^2}\frac{\partial^2 F^{(1)}}{\partial \theta^2} + \Omega(\theta,z) \\ N_{\theta\xi}{}^{(1)} &= -\frac{1}{a}\frac{\partial^2 F^{(1)}}{\partial\theta\,\partial z} \end{aligned}\right\} \qquad (23)$$

where the superscript corresponds to this approximation. With equations (23) force-equilibrium equations (B13) are satisfied if

$$\left.\begin{aligned}\frac{\partial \Omega}{\partial \theta} &= -\left[2\,\frac{df}{dz}\,N_{\theta z}{}^{(o)} + fN_z{}^{(o)}\right]\\ \frac{\partial \Omega}{\partial z} &= -\,\frac{1}{a}\left[\frac{df}{dz}\,N_z{}^{(o)} + f\,\frac{\partial N_z{}^{(o)}}{\partial z} - \frac{df}{dz}\,N_\theta{}^{(o)}\right]\end{aligned}\right\}\tag{24}$$

The relevant compatibility equation for this approximation is

$$\frac{\partial^2 \epsilon_\theta{}^{(1)}}{\partial z^2} + \frac{1}{a^2}\,\frac{\partial^2 \epsilon_z{}^{(1)}}{\partial \theta^2} - \frac{1}{a}\,\frac{\partial^2 \gamma_{z\theta}{}^{(1)}}{\partial \theta\,\partial z} = -\frac{1}{a}\,\frac{\partial^2 w^{(1)}}{\partial z^2} + G^{(o)}(\theta,z)\tag{25}$$

where $G^{(o)}(\theta,z)$ is a function determined by the first approximation. With the aid of the resultant force-strain relations

$$\left.\begin{aligned}\epsilon_\theta{}^{(1)} &= \frac{1}{hE}\left[N_\theta{}^{(1)} - \nu N_z{}^{(1)}\right]\\ \epsilon_z{}^{(1)} &= \frac{1}{hE}\left[N_z{}^{(1)} - \nu N_\theta{}^{(1)}\right]\\ \gamma_{z\theta}{}^{(1)} &= \frac{1}{hG}\,N_{z\theta}{}^{(1)}\end{aligned}\right\}\tag{26}$$

and equations (23), compatibility equation (17) gives

$$\nabla^4 F^{(1)} = -\,\frac{hE}{a}\,\frac{\partial^2 w^{(1)}}{\partial z^2} + H^{(o)}(\theta,z)\tag{27}$$

where $H^{(o)}(\theta,z)$ is a known function from the first approximation. The moment-equilibrium equation with equation (23) and the relations between resultant moment and change of curvature lead to

$$\nabla^4 w^{(1)} = \frac{1}{Da}\frac{\partial^2 F^{(1)}}{\partial z^2} + K^{(o)}(\theta,z) \tag{28}$$

where $K^{(o)}(\theta,z)$ is also a known function. The solution of equations (27) and (28) would give the second approximation. The effect of double curvature comes in at this stage. The method of solution follows the ones suggested previously. It is evident that the form of the equations for higher approximations will be similar in structure. Thus for the (n + 1)th approximation one has

$$\left.\begin{aligned} \nabla^4 F^{(n)} &= -\frac{hE}{a}\frac{\partial^2 w^{(n)}}{\partial z^2} + H^{(n-1)}(\theta,z) \\ \nabla^4 w^{(n)} &= \frac{1}{Da}\frac{\partial^2 F^{(n)}}{\partial z^2} + K^{(n-1)}(\theta,z) \end{aligned}\right\} \tag{29}$$

where $H^{(n-1)}$ and $K^{(n-1)}$ contain all of the known (n - 1)th functions $w^{(n-1)}$ and $F^{(n-1)}$. The functions $H^{(o)}(\theta,z)$, $G^{(o)}(\theta,z)$, and $K^{(o)}(\theta,z)$ are not given explicitly here but may be formulated readily from the expansions given in appendixes A and B.

Solution of Specific Problem

The developed differential equations will now be applied to a particular problem. The problem at hand is that of bending of a curved plate with two meridian curves and two parallel circles as boundaries. This plate is loaded along the edges z = Constant in compression (i.e., the load is along the direction of the z-axis). The boundaries of the plate are given by $z = \pm l$ and $\theta = \pm\theta_o$. (See fig. 3.)

Boundary conditions.- Four boundary conditions are needed on each edge, and with four edges there are sixteen conditions for w and F together.

The plate is supported at all edges and hinged at the edges so that bending moments are zero there. Thus,

$$\left.\begin{aligned} &\text{at } z = \pm l: \quad w = 0, \quad M_z = 0 \\ &\text{at } \theta = \pm\theta_o: \quad w = 0, \quad M_\theta = 0 \end{aligned}\right\} \tag{30a}$$

Furthermore, for applied load at edges $z = \text{Constant}$ it is required that at $z = \pm l$

$$\int_{-\theta_o}^{\theta_o} N_z a \, d\theta = \text{Applied load} \tag{30b}$$

(In most cases $N_z = \text{Constant}$ and the condition becomes equal to a constant.) In addition, it is required that the shear resultant be zero at all edges, that is,

$$\left.\begin{aligned} &\text{at} \quad z = \pm l: \quad N_{z\theta} = 0 \\ &\text{at} \quad \theta = \pm\theta_o: \quad N_{z\theta} = 0 \end{aligned}\right\} \tag{30c}$$

There are two more conditions needed and these are supplied at the edges $\theta = \pm\theta_o$. They may be formulated in either of two ways. If the plate has stiffeners attached at the edges which have infinite bending rigidity in the θ direction then the boundary condition at $\theta = \pm\theta_o$ is

$$u = 0 \tag{30d}$$

and if the stiffeners have zero bending rigidity in the θ direction the condition at $\theta = \pm\theta_o$ is

$$N_\theta = 0 \tag{30e}$$

Of course, the actual condition would be between these two. It may be noted that condition (26c) may be modified to have zero tangential displacements at the edges $\left(\text{then } N_{z\theta} \neq 0 \text{ at edges}\right)$. With the help of expansions (12), (13), (14), and (15) the above boundary conditions imply the following conditions for the different approximations:

$$
\left.\begin{array}{l}
\text{At } z = \pm l\text{:} \\
\qquad w = 0 \text{ gives } w^{(n)} = 0 \text{ for } n = 0, 1, 2, \ldots \\
\qquad M_z = 0 \text{ gives } M_z^{(n)} = 0 \text{ for } n = 0, 1, 2, \ldots \\
\text{At } \theta = \pm\theta_o\text{:} \\
\qquad w = 0 \text{ gives } w^{(n)} = 0 \\
\qquad M_\theta = 0 \text{ gives } M_\theta^{(n)} = 0 \text{ for } n = 0, 1, 2, \ldots \\
\text{At } z = \pm l\text{:} \\
\qquad N_z = \text{Constant gives } N_z^{(o)} = \text{Constant} \\
\qquad N_z^{(n)} = 0 \text{ for } n = 1, 2, \ldots \\
\text{At } \theta = \pm\theta_o\text{:} \\
\qquad \text{If } N_\theta = 0 \text{ then } N_\theta^{(n)} = 0 \text{ for } n = 0, 1, 2, \ldots \\
\qquad \text{If } u = 0 \text{ then } u^{(n)} = 0 \text{ for } n = 0, 1, 2, \ldots \\
\text{At } z = \pm l, \ \theta = \pm\theta_o\text{:} \\
\qquad N_{z\theta} = 0 \text{ gives } N_{z\theta}^{(n)} = 0 \text{ for } n = 0, 1, 2, \ldots
\end{array}\right\} \quad (31)
$$

Solution to first approximation.- For further application, consideration is given to stress-free edges at $\theta = \pm\theta_o$ (i.e., equation (30e)). A possible solution to equations (23) and (27) is that

$$w^{(o)} \equiv 0 \qquad (32)$$

and

$$F^{(o)} = Bs^2 \tag{33}$$

Equations (32) and (33) satisfy all boundary conditions given above and furthermore differential equations (19) and (20) are satisfied $\left(\text{with } p_\zeta \equiv 0\right)$. The constant B is determined by the load at the edges $z = \pm l$. If the loads at these edges are:

$$\left.\begin{aligned} N_z &= k = \text{Constant} \\ \text{then} \qquad B &= k/2 \end{aligned}\right\} \tag{34}$$

by equation (16). The displacement components $u^{(o)}$ and $v^{(o)}$ are given by

$$\left.\begin{aligned} u^{(o)} &= -\frac{\nu k}{hE} s \\ v^{(o)} &= \frac{1}{hE} kz \end{aligned}\right\} \tag{35}$$

<u>Solution to second approximation</u>.- The functions Ω, $H^{(o)}$, and $K^{(o)}$ must first be determined from the first approximation. Thus, one has

$$\left.\begin{aligned} \frac{\partial\Omega}{\partial\theta} &= 0 \\ \frac{\partial\Omega}{\partial z} &= -\frac{1}{a}\frac{df}{dz} k \end{aligned}\right\} \tag{36a}$$

or

$$\Omega = -\frac{kf}{a} \tag{36b}$$

$$H^{(o)}(\theta,z) = \frac{(2 - \nu)k}{a} \frac{d^2f}{dz^2} + \frac{d^3f}{dz^3} \frac{kz}{a} \tag{36c}$$

$$K^{(o)}(\theta,z) = -\frac{d^2f}{dz^2} \frac{k}{D} \tag{36d}$$

Differential equations (27) and (28) become now

$$\nabla^4 w^{(1)} = \frac{1}{Da} \frac{\partial^2 F^{(1)}}{\partial z^2} - \frac{k}{D} \frac{d^2f}{dz^2} \tag{37}$$

and

$$\nabla^4 F^{(1)} = -\frac{hE}{a} \frac{\partial^2 w^{(1)}}{\partial z^2} + \frac{(2 - \nu)k}{a} \frac{d^2f}{dz^2} + \frac{d^3f}{dz^3} \frac{kz}{a} \tag{38}$$

Solution of equations (37) and (38).- To satisfy the condition of simply supported edges assume a solution for $w^{(1)}$ in the form

$$w^{(1)} = \sum_{m=1,3...}^{\infty} \sum_{n=1,3...}^{\infty} w_{mn} \cos\left(\frac{m\pi}{2l} z\right) \cos\left(\frac{n\pi}{2b} s\right) \tag{39}$$

where $b = a\theta_o$. Insertion of equation (39) into equation (38) leads to

$$\nabla^4 F^{(1)} = \frac{hE}{a} \sum_{m=1,3...}^{\infty} \sum_{n=1,3...}^{\infty} \mu_m{}^2 w_{mn} \cos \mu_m z \cos \nu_n s +$$

$$\frac{(2 - \nu)k}{a} \frac{d^2f}{dz^2} + \frac{d^3f}{dz^3} \frac{kz}{a} \tag{40}$$

where $\mu_m = m\pi/2l$ and $\nu_n = n\pi/2b$ for abbreviation. The particular solution of equation (40) is

$$F_p^{(1)} = \sum_{m=1,3\ldots}^{\infty} \sum_{n=1,3\ldots}^{\infty} F_{mn} \cos \mu_m z \cos \nu_n z + J(z) \tag{41a}$$

where

$$\left.\begin{aligned} F_{mn} &= \frac{hE}{a} \frac{\gamma_m{}^2 b^2}{\left(\gamma_m{}^2 + \alpha_n{}^2\right)^2} w_{mn} \\ \frac{d^4 J(z)}{dz^4} &= \frac{(2 - \nu)k}{a} \frac{d^2 f}{dz^2} + \frac{d^3 f}{dz^3} \frac{kz}{a} \end{aligned}\right\} \tag{41b}$$

and

$$\gamma_m = \mu_m b$$

$$\alpha_n = \nu_n b$$

A complementary solution that can satisfy the boundary conditions is

$$F_c^{(1)} = \frac{kf}{2a} s^2 + \sum_{m=1,3\ldots}^{\infty} \left[\cos \mu_m z\left(A_m \cosh \mu_m s + C_m \mu_m s \sinh \mu_m s\right)\right] + \sum_{n=1,3\ldots}^{\infty} \left[\cos \nu_n s\left(B_n \cosh \nu_n z + D_n \nu_n z \sinh \nu_n z\right)\right] \tag{42}$$

To recapitulate, the boundary conditions which remain to be satisfied are, at $z = \pm l$,

$$N_z^{(1)} = \frac{\partial^2 F^{(1)}}{\partial s^2} - \frac{kf}{a} = 0 \tag{43a}$$

$$N_{z\theta}^{(1)} = -\frac{\partial^2 F^{(1)}}{\partial z \, \partial s} = 0 \tag{43b}$$

and, at $s = \pm b$,

$$N_\theta^{(1)} = \frac{\partial^2 F^{(1)}}{\partial z^2} = 0 \tag{43c}$$

$$N_{z\theta}^{(1)} = -\frac{\partial^2 F^{(1)}}{\partial z\ \partial s} = 0 \tag{43d}$$

Assertion of conditions (43a) and (43c) leads, respectively, to relations between the constants of

$$A_m = \frac{f_m l}{\beta_m^{\ 2} \cosh \gamma_m} - \gamma_m \tanh \gamma_m C_m \tag{44a}$$

$$B_n = -\delta_n \tanh \delta_n D_n \tag{44b}$$

where f_m is the coefficient of the Fourier cosine expansion of $\frac{d^2J(z)}{dz^2}$ given by

$$f_m = \int_{-l}^{l} \frac{d^2J(z)}{dz^2} \cos \mu_m x\ dz \tag{45}$$

and

$$\beta_m = \mu_m l$$

$$\delta_n = \nu_n l$$

The details of this calculation are given in appendix C. The remaining conditions, equations (43b) and (43d), give two infinite families of algebraic equations for the constants D_n and C_m. They are:

$$\sum_{n=1,3\ldots}^{\infty} \frac{4\alpha_n\beta_m\delta_n{}^2}{\left(\delta_n{}^2 + \beta_m{}^2\right)^2} \sin \alpha_n \sin \beta_m \cosh \delta_n D_n =$$

$$\frac{hE}{a} \gamma_m{}^3 b^2 \sum_{n=1,3\ldots}^{\infty} \frac{\alpha_n \sin \alpha_n}{\left(\gamma_m{}^2 + \alpha_n{}^2\right)^2} w_{mn} - \frac{b^2}{l} f_m \tanh \gamma_m -$$

$$C_m b \gamma_m \mu_m \left(\frac{\gamma_m}{\cosh \gamma_m} + \sinh \gamma_m\right) \quad \text{for} \quad m = 1,\ 3,\ 5\ .\ .\ . \tag{46}$$

and

$$\sum_{m=1,3\ldots}^{\infty} \frac{4\alpha_n{}^2\gamma_m{}^2\beta_m}{\left(\alpha_n{}^2 + \gamma_m{}^2\right)^2} \sin \alpha_m \sin \beta_m \cosh \gamma_m C_m =$$

$$- \sum_{m=1,3\ldots}^{\infty} \left[\frac{2f_m l \gamma_m{}^2 \sin \alpha_m \sin \beta_m}{\beta_m\left(\alpha_n{}^2 + \gamma_m{}^2\right)}\right] +$$

$$\frac{hEb^2}{a} \sum_{m=1,3\ldots}^{\infty} \left[\frac{\gamma_m{}^2\beta_m\alpha_n \sin \beta_m w_{mn}}{\left(\gamma_m{}^2 + \alpha_n{}^2\right)^2}\right] - \alpha_n D_n\left(\frac{\delta_n{}^2}{\cosh \delta_n} + \delta_n \sinh \delta_n\right)$$

$$\text{for} \quad n = 1,\ 3\ .\ .\ . \tag{47}$$

The details of the above calculation are also given in appendix C. At this point all boundary conditions are satisfied and it remains to satisfy differential equation (37). If the value of $F^{(1)}$ is inserted into equation (37), then this equation is satisfied if

$$w_{mn}\left[\left(\gamma_m{}^2 + \alpha_n{}^2\right) + \frac{hb^4E}{Da^2} \frac{\gamma_m{}^4}{\left(\gamma_m{}^2 + \alpha_n{}^2\right)^2}\right] = b^4K_{mn} - \frac{b^4}{Dl^2} \frac{\beta_m{}^2}{\alpha_n}\left[\frac{a_{mn}f_m l}{\beta_m{}^2 \cosh \gamma_m} -\right.$$

$$\left. C_m \frac{4\gamma_m{}^2\alpha_n{}^2}{\left(\alpha_n{}^2 + \gamma_m{}^2\right)^2} \cosh \gamma_m \sin \alpha_n + \frac{4b^4}{Dl^2} D_n \frac{\beta_m{}^3\delta_n{}^2}{\left(\delta_n{}^2 + \beta_m{}^2\right)^2} \cosh \delta_n \sin \beta_m\right]$$

$$\text{for any} \quad m \quad \text{and} \quad n \tag{48}$$

where K_{mn} is the coefficient of the Fourier double cosine expansion of the function

$$\frac{1}{D} \frac{d^2 J(z)}{dz^2} - \frac{d^2 f}{dz} \frac{k}{D}$$

and is given by

$$K_{mn} = \frac{1}{bl} \int_{-b}^{b} \int_{-l}^{l} \frac{1}{D} \left[\frac{d^2 J(z)}{dz} - \frac{d^2 f}{dz} k \right] \cos \mu_m z \cos \nu_n s \, ds \, dz \qquad (49)$$

The complete solution of the problem lies in the solution of the infinite families of algebraic equations for C_m and D_n and then w_{mn} from equations (44). An approximate numerical solution for C_m and D_n may be obtained as follows. It is assumed that coefficients with suffixes greater than some fixed number can be neglected. Then it is verified that increasing this number does not affect the coefficients with small suffixes.

Illustrative example.- As an example the following meridian curve is considered:

$$r = a + \mu a \left(1 - \frac{z^2}{l^2} \right)^2 \qquad (50)$$

It is easily verified that the slope dr/dz is zero at $z = \pm l$. Thus the load $N_z = k$ at $z = \pm l$ is normal to the edges. In addition, consider a plate such that $l = b$. The sign of the constant k determines compression or tension; that is, $k < 0$ indicates compression and $k > 0$, tension. The procedure in the numerical calculations is outlined in appendix D. The results of the calculations are summarized in table I and figure 4. It may be noted that all the calculations can be carried through with the sign of k arbitrary. Thus the sign of k which determines either compression or tension can be assigned at the last stage of the calculations.

DISCUSSION OF RESULTS

The numerical results for the example are given in figure 4. It is seen from this figure that maximum value of the deflection w

(normal to undeformed plate surface) is not at the center of the plate. This deflection appears to have a maximum value near the loaded edge. This is due probably to the choice of the form of the meridian curve which gives a negative Gaussian curvature for $z > \frac{1}{\sqrt{3}} l$ and a positive Gaussian curvature for $z < \frac{1}{\sqrt{3}} l$. The deflections are largest at the part of the plate with negative Gaussian curvature. The moment distribution curve shows the same trend. It is noted that the actual values of the deflections and moments are multiplied by the parameter μ. The order of magnitude of the parameter μ is about 0.01 to 0.02. Thus the order of magnitude of the normal deflection w is about 0.04 to 0.08 of the thickness of the shell. This indicates that the first two approximations of most problems (i.e., μ^0 and μ^1) will be sufficient to give a complete solution.

The convergence of the approximate solution of the infinite families of algebraic equations is quite rapid. These results are given in table I. A total of six coefficients was used; the dominant coefficients were the first two, $\overline{C}_1$ and $\overline{D}_1$.

CONCLUDING REMARKS

A method has been developed for the analysis of the deformation of doubly curved thin plates under edge and surface loads. Only small deflections (small compared with the thickness of the plate) are considered here. This method is particularly suited to the analysis of a plate with a large radius of curvature in one direction. This is clearly the situation existing in airplane fuselages.

For the problem of a doubly curved plate in edgewise compression two infinite families of algebraic equations were obtained in order to satisfy the boundary conditions that would exist in most airplane coverings. Results were obtained by replacing these infinite sets by finite ones (neglecting all coefficients beyond a certain suffix). Convergence of this approximate solution of the algebraic equations was quite rapid.

Since the order of magnitude of the parameter μ is small (in the hundredths) the first two approximations are usually sufficient to give the complete solution to similar problems. Moreover, the convergence of most expansion methods about a parameter is difficult to prove mathematically. However, in most practical applications the parameter μ will be small and only a few terms in the expansion will suffice for the solution. Of course, the solution so obtained will indicate the number of terms required in the expansion.

It should be noted again that the equations are valid only for deflections that are small compared with the thickness of the plate or shell considered. If, in the application of the method, large magnitudes of the deflection are obtained one must resort to a nonlinear theory.

A specific example is presented which can be immediately applied to the bending of thin plates with compound curvature for small deflections when loaded in edgewise compression or tension. The basic equations given are equally valid for a curved plate loaded by any other edge loads or by surface loads, but the form of the specific example as presented will change slightly. Moreover, any shell of revolution with a meridian of small curvature may be analyzed by this method.

The Pennsylvania State College
State College, Pa., August 28, 1951

APPENDIX A

EXPANSIONS OF TERMS IN POWERS OF PARAMETER μ

To facilitate computation the following expansions are particularly useful. All terms are expanded up to and including second powers in the parameter μ. With

$$r = a + \mu f(z) \tag{A1}$$

The following expansions can be immediately written:

$$\frac{1}{R_\theta} = -\frac{1}{a}\left[1 - \mu \frac{f}{a} + \mu^2\left(\frac{f^2}{a^2} - \frac{1}{2} {f_z}^2\right)\right]$$

$$\frac{1}{R_\xi} = \mu f_z$$

$$\alpha = 1 + \mu^2 \frac{{f_z}^2}{2}$$

$$\frac{\alpha r}{R_\xi} = \mu a f_{zz} + \mu^2 f f_{zz}$$

$$\frac{1}{r} = \frac{1}{a}\left(1 - \mu \frac{f}{a} + \mu^2 \frac{f^2}{a^2}\right)$$

$$\frac{1}{\alpha} = 1 - \mu^2 \frac{{f_z}^2}{2}$$

$$\frac{r_z}{r\alpha} = \mu \frac{f_z}{a} - \mu^2 \frac{f}{a^2} f_z$$

Differentiation with respect to z is indicated by the subscript z.

APPENDIX B

EXPANSIONS OF EQUATIONS

Expansions of Equations for Strains

The strains have the following expansions up to and including the second powers of μ:

$$\left.\begin{aligned}
\epsilon_\theta{}^{(o)} &= \frac{1}{a}\frac{\partial u^{(o)}}{\partial\theta} - \frac{w^{(o)}}{a} \\
\epsilon_\theta{}^{(1)} &= \frac{1}{a}\frac{\partial u^{(1)}}{\partial\theta} - \frac{w^{(1)}}{a} - \frac{f}{a^2}\frac{\partial u^{(o)}}{\partial\theta} + \frac{f_z}{a}v^{(o)} + \frac{f}{a^2}w^{(o)} \\
\epsilon_\theta{}^{(2)} &= \frac{1}{a}\frac{\partial u^{(2)}}{\partial\theta} - \frac{w^{(2)}}{a} - \frac{f}{a^2}\frac{\partial u^{(1)}}{\partial\theta} + \frac{f_z}{a}v^{(1)} + \frac{f}{a^2}w^{(1)} + \\
&\quad \frac{f^2}{a^3}\frac{\partial u^{(o)}}{\partial\theta} - \frac{ff_z}{a^2}v^{(o)} - \left(\frac{f^2}{a^2} - \frac{1}{2}f_z{}^2\right)\frac{w^{(o)}}{a}
\end{aligned}\right\} \quad \text{(B1)}$$

$$\left.\begin{aligned}
\epsilon_z{}^{(o)} &= \frac{\partial v^{(o)}}{\partial z} \\
\epsilon_z{}^{(1)} &= \frac{\partial v^{(1)}}{\partial z} + f_{zz}w^{(o)} \\
\epsilon_z{}^{(2)} &= \frac{\partial v^{(2)}}{\partial z} + f_{zz}w^{(1)} - \frac{1}{2}f_z{}^2\frac{\partial v^{(o)}}{\partial z}
\end{aligned}\right\} \quad \text{(B2)}$$

$$\left.\begin{aligned}
\gamma_{z\theta}{}^{(0)} &= \frac{\partial u^{(0)}}{\partial z} + \frac{1}{a}\frac{\partial v^{(0)}}{\partial \theta} \\
\gamma_{z\theta}{}^{(1)} &= \frac{\partial u^{(1)}}{\partial z} + \frac{1}{a}\frac{\partial v^{(1)}}{\partial \theta} - \frac{f_z}{a}u^{(0)} - \frac{f}{a^2}\frac{\partial v^{(0)}}{\partial \theta} \\
\gamma_{z\theta}{}^{(2)} &= \frac{\partial u^{(2)}}{\partial z} + \frac{1}{a}\frac{\partial v^{(2)}}{\partial \theta} - \frac{f_z}{a}u^{(1)} - \frac{f}{a^2}\frac{\partial v^{(1)}}{\partial \theta} + {} \\
&\quad \frac{ff_z}{a^2}u^{(0)} - \frac{f_z{}^2}{2}\frac{\partial u^{(0)}}{\partial z} + \frac{f^2}{a^3}\frac{\partial v^{(0)}}{\partial \theta}
\end{aligned}\right\} \tag{B3}$$

$$\left.\begin{aligned}
\kappa_\theta{}^{(0)} &= -\frac{1}{a^2}\frac{\partial^2 w^{(0)}}{\partial \theta^2} \\
\kappa_\theta{}^{(1)} &= -\frac{1}{a^2}\frac{\partial^2 w^{(1)}}{\partial \theta^2} + \frac{2f}{a^3}\frac{\partial^2 w^{(0)}}{\partial \theta^2} - \frac{f_z}{a}\frac{\partial w^{(0)}}{\partial z} \\
\kappa_\theta{}^{(2)} &= -\frac{1}{a^2}\frac{\partial^2 w^{(2)}}{\partial \theta^2} + \frac{2f}{a^3}\frac{\partial^2 w^{(1)}}{\partial \theta^2} - \frac{f_z}{a}\frac{\partial w^{(1)}}{\partial z} - {} \\
&\quad \frac{1}{a^2}\left(2\frac{f^2}{a^2} + \frac{f}{a}\right)\frac{\partial^2 w^{(0)}}{\partial \theta^2}
\end{aligned}\right\} \tag{B4}$$

$$\left.\begin{aligned}
\kappa_z{}^{(0)} &= -\frac{\partial^2 w^{(0)}}{\partial z^2} \\
\kappa_z{}^{(1)} &= -\frac{\partial^2 w^{(1)}}{\partial z^2} \\
\kappa_z{}^{(2)} &= -\frac{\partial^2 w^{(2)}}{\partial z^2} + \frac{1}{2}f_z{}^2\frac{\partial^2 w^{(0)}}{\partial z^2} + f_z f_{zz}\frac{\partial w^{(0)}}{\partial z} + {} \\
&\quad \frac{f_z{}^2}{2}\frac{\partial^2 w^{(0)}}{\partial z^2}
\end{aligned}\right\} \tag{B5}$$

$$\left.\begin{aligned}
\tau^{(o)} &= -\frac{2}{a}\frac{\partial^2 w^{(o)}}{\partial\theta\,\partial z}\\
\tau^{(1)} &= -\frac{2}{a}\frac{\partial^2 w^{(1)}}{\partial z\,\partial\theta} + \frac{f_z}{a^2}\frac{\partial w^{(o)}}{\partial\theta} + \frac{f}{a^2}\frac{\partial^2 w^{(o)}}{\partial\theta\,\partial z} + \frac{f_z}{a^2}\frac{\partial w^{(o)}}{\partial\theta} + \frac{f}{a^2}\frac{\partial^2 w^{(o)}}{\partial z\,\partial\theta}\\
\tau^{(2)} &= -\frac{2}{a}\frac{\partial^2 w^{(2)}}{\partial z\,\partial\theta} + \frac{f}{a^2}\frac{\partial^2 w^{(1)}}{\partial z\,\partial\theta} + \frac{f_z}{a^2}\frac{\partial w^{(1)}}{\partial\theta} + \frac{f}{a^2}\frac{\partial^2 w^{(1)}}{\partial z\,\partial\theta} + \frac{f_z}{a^2}\frac{\partial w^{(1)}}{\partial\theta} -\\
&\quad \frac{2ff_z}{a^3}\frac{\partial w^{(o)}}{\partial\theta} + \frac{f_z{}^2}{a}\frac{\partial^2 w^{(o)}}{\partial\theta\,\partial z} - \frac{1}{a^3}\frac{\partial}{\partial z}\left[\frac{\partial w^{(o)}}{\partial\theta}f^2\right] - \frac{f}{a^3}\frac{\partial^2 w^{(o)}}{\partial z\,\partial\theta} +\\
&\quad -\frac{f_z{}^2}{2a}\frac{\partial^2 w^{(o)}}{\partial z\,\partial\theta}
\end{aligned}\right\}\qquad\text{(B6)}$$

Compatibility Equations

The following compatibility equations can be constructed from the values of the strains given above:

$$\frac{\partial^2\epsilon_\theta{}^{(o)}}{\partial z^2} + \frac{1}{a^2}\frac{\partial^2\epsilon_z{}^{(o)}}{\partial\theta^2} - \frac{1}{a}\frac{\partial^2\gamma_{z\theta}{}^{(o)}}{\partial z\,\partial\theta} = -\frac{1}{a}\frac{\partial^2 w^{(o)}}{\partial z^2}\qquad\text{(B7)}$$

$$\begin{aligned}
&\frac{\partial^2\epsilon_\theta(1)}{\partial z^2} + \frac{1}{a^2}\frac{\partial^2\epsilon_z(1)}{\partial\theta^2} - \frac{1}{a}\frac{\partial^2\gamma_{z\theta}(1)}{\partial z\,\partial\theta} = -\frac{1}{a}\frac{\partial^2 w^{(1)}}{\partial z^2} + \frac{f_{zz}}{a^2}\frac{\partial^2 w^{(o)}}{\partial\theta^2} +\\
&\frac{f_{zz}}{a^2}w^{(o)} + \frac{f_z}{a^2}\frac{\partial w^{(o)}}{\partial z} + \frac{f_z}{a^2}\frac{\partial w^{(o)}}{\partial z} + \frac{f}{a^2}\frac{\partial^2 w^{(o)}}{\partial z^2} -\\
&\frac{f_z}{a^2}\frac{\partial^2 u^{(o)}}{\partial\theta\,\partial z} - \frac{f}{a^2}\frac{\partial^3 u^{(o)}}{\partial\theta\,\partial z^2} + \frac{2f_{zz}}{a}\frac{\partial v^{(o)}}{\partial z} + \frac{f_z}{a}\frac{\partial^2 v^{(o)}}{\partial z^2} +\\
&\frac{f_{zzz}}{a}v^{(o)} + \frac{1}{a^3}f\frac{\partial^3 v^{(o)}}{\partial\theta^2\,\partial z} + \frac{f_z}{a^3}\frac{\partial^2 v^{(o)}}{\partial\theta^2}
\end{aligned}\qquad\text{(B8)}$$

The compatibility equations for the next and succeeding approximations are constructed similar to equations (B7) and (B8) obtaining in each case terms as $-\frac{1}{a}\frac{\partial^2 w^{(n-1)}}{\partial z^2}$ and so forth where n denotes the nth approximation.

Expansion of Equations for Moment Equilibrium

The last three equilibrium equations (equations (4c), (5a), and (5b)) can be combined into the single equation

$$\frac{\alpha}{r}\frac{\partial^2 M_\theta}{\partial \theta^2} + \frac{1}{r}\frac{\partial^2 (rM_{z\theta})}{\partial z\, \partial \theta} + \frac{1}{r}\frac{\partial r}{\partial z}\frac{\partial M_{z\theta}}{\partial \theta} + \frac{\partial}{\partial z}\left[\frac{1}{\alpha}\frac{\partial}{\partial z}\left(rM_z\right)\right] +$$

$$\frac{\partial^2 M_{z\theta}}{\partial \theta\, \partial z} - \frac{\partial}{\partial z}\left(\frac{M_\theta}{\alpha}\frac{\partial r}{\partial z}\right) - \frac{\alpha r}{R_\theta} N_\theta - \frac{\alpha r}{R_\xi} N_z + \alpha r p_\zeta = 0 \qquad \text{(B9)}$$

The expansions in powers of μ of equations (B6) lead to the following equations for each power of μ:

For μ^o:

$$\frac{1}{a}\frac{\partial^2 M_\theta{}^{(o)}}{\partial \theta^2} + \frac{\partial^2 M_{\theta z}{}^{(o)}}{\partial \theta\, \partial z} + a\frac{\partial^2 M_z{}^{(o)}}{\partial z^2} + \frac{\partial^2 M_{\theta z}{}^{(o)}}{\partial \theta\, \partial z} + N_\theta{}^{(o)} + p_\zeta a = 0 \qquad \text{(B10)}$$

For μ^1:

$$\frac{1}{a}\frac{\partial^2 M_\theta{}^{(1)}}{\partial \theta^2} + \frac{\partial^2 M_{\theta z}{}^{(1)}}{\partial \theta\, \partial z} + a\frac{\partial^2 M_z{}^{(1)}}{\partial z^2} + \frac{\partial^2 M_{\theta z}{}^{(1)}}{\partial \theta\, \partial z} + N_\theta{}^{(1)} + p_\zeta f -$$

$$\frac{f}{a^2}\frac{\partial^2 M_\theta{}^{(o)}}{\partial \theta^2} + \frac{2}{a} f_z \frac{\partial M_{z\theta}{}^{(o)}}{\partial \theta} + 2f_z\frac{\partial M_z{}^{(o)}}{\partial z} + f_{zz}M_z{}^{(o)} +$$

$$f\frac{\partial^2 M_z{}^{(o)}}{\partial z^2} - f_z\frac{\partial M_\theta{}^{(o)}}{\partial z} - f_{zz}M_\theta{}^{(o)} - af_{zz}N_z{}^{(o)} = 0 \qquad \text{(B11)}$$

For μ^2:

$$\frac{1}{a}\frac{\partial^2 M_\theta{}^{(2)}}{\partial\theta^2} + \frac{\partial^2 M_{\theta z}{}^{(2)}}{\partial\theta\,\partial z} + a\frac{\partial^2 M_z{}^{(2)}}{\partial z^2} + \frac{\partial^2 M_{\theta z}{}^{(2)}}{\partial\theta\,\partial z} + p_\zeta \frac{f_{zz}{}^2 a}{2} +$$

$$N_\theta{}^{(2)} - af_{zz}N_z{}^{(1)} - \frac{f}{a^2}\frac{\partial^2 M_\theta{}^{(1)}}{\partial\theta^2} + f\frac{\partial^2 M_z{}^{(1)}}{\partial z^2} + 2\frac{f_z}{a}\frac{\partial M_{\theta z}{}^{(1)}}{\partial\theta} +$$

$$2f_z\frac{\partial M_z{}^{(1)}}{\partial z} + f_{zz}M_z{}^{(1)} - f_z\frac{\partial M_\theta{}^{(1)}}{\partial z} - f_{zz}M_\theta{}^{(1)} - ff_{zz}N_z{}^{(o)} +$$

$$\left(\frac{f_z{}^2}{2} + \frac{f^2}{a^2}\right)\frac{\partial^2 M_\theta{}^{(o)}}{a\,\partial\theta^2} - \frac{a}{2}f_z{}^2\frac{\partial^2 M_z{}^{(o)}}{\partial z^2} - af_zf_{zz}\frac{\partial^2 M_{\theta z}{}^{(o)}}{\partial z^2} = 0 \qquad \text{(B12)}$$

Expansion of Equations for Force Equilibrium

The remaining force-equilibrium equations may be expanded in a similar way and then if each factor of μ^o, μ^1, and μ^2 is equated to zero the following equations result:

For μ^o:

$$\left.\begin{aligned} &\frac{\partial N_\theta{}^{(o)}}{a\,\partial\theta} + \frac{\partial N_{z\theta}{}^{(o)}}{\partial z} = 0 \\ &\frac{\partial N_z{}^{(o)}}{\partial z} + \frac{1}{a}\frac{\partial N_{z\theta}{}^{(o)}}{\partial\theta} = 0 \end{aligned}\right\} \qquad \text{(B13)}$$

For μ^1:

$$\left.\begin{aligned} &\frac{\partial N_\theta{}^{(1)}}{a\,\partial\theta} + \frac{\partial N_{z\theta}{}^{(1)}}{\partial z} + 2\frac{f_z}{a}N_{z\theta}{}^{(o)} + \frac{f}{a}\frac{\partial N_{z\theta}{}^{(o)}}{\partial z} = 0 \\ &\frac{\partial N_z{}^{(1)}}{\partial z} + \frac{1}{a}\frac{\partial N_{z\theta}{}^{(1)}}{\partial\theta} + \frac{f_z}{a}N_z{}^{(o)} + \frac{f}{a}\frac{\partial N_z{}^{(o)}}{\partial z} - \frac{f_z}{a}N_\theta{}^{(o)} = 0 \end{aligned}\right\} \qquad \text{(B14)}$$

For μ^2:

$$\left.\begin{aligned}
&\frac{\partial N_\theta^{(2)}}{a\ \partial\theta} + \frac{\partial N_{z\theta}^{(2)}}{\partial z} + 2\frac{f_z}{a}N_{z\theta}^{(1)} + \frac{f}{a}\frac{\partial N_{z\theta}^{(1)}}{\partial z} + \frac{f_z^{\,2}}{2a}\frac{\partial N_\theta^{(o)}}{\partial\theta} = 0\\
&\frac{\partial N_z^{(2)}}{\partial z} + \frac{1}{a}\frac{\partial N_{z\theta}^{(2)}}{\partial\theta} + \frac{f_z}{a}N_z^{(1)} + \frac{f}{a}\frac{\partial N_z^{(1)}}{\partial z} - \frac{f_z}{a}N_\theta^{(1)} +\\
&\frac{1}{2}\frac{f_z^{\,2}}{a}\frac{\partial N_{z\theta}^{(o)}}{\partial\theta} = 0
\end{aligned}\right\}\text{(B15)}$$

APPENDIX C

EVALUATION OF CONSTANTS OF INTEGRATION

The expression for the stress function F chosen to satisfy the differential equation for F and the boundary conditions is

$$F^{(1)} = \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} F_{mn} \cos \mu_m z \cos \nu_n s + J(z) + \frac{kf}{2a} s^2 + \sum_{m=1,3\ldots}^{\infty} \cos \mu_m z F_m(s) + \sum_{n=1,3\ldots}^{\infty} \cos \nu_n s F_n(z) \tag{C1}$$

where

$$F_m(s) = A_m \cosh \mu_m s + C_m \mu_m s \sinh \mu_m s$$

$$F_n(z) = B_n \cosh \nu_n z + D_n \nu_n z \sinh \nu_n z$$

and $\mu_m = m\pi/2l$, $\nu_n = n\pi/2b$, and $s = a\theta$ for abbreviation. Equation (C1) satisfies differential equation (21) if

$$\frac{d^4 J(z)}{dz^4} = (\nu - 2)\frac{k}{a} \frac{d^2 f}{dz^2} + \frac{kz}{2a} \frac{d^3 f}{dz^3} \tag{C2}$$

The boundary conditions which must be satisfied by F (equation C1) are, at $z = \pm l$:

$$N_z^{(1)} = \frac{\partial^2 F^{(1)}}{\partial s^2} - \frac{kf}{a} = 0 \tag{C3a}$$

$$N_{z\theta}^{(1)} = -\frac{\partial^2 F^{(1)}}{\partial z \, \partial s} = 0 \tag{C3b}$$

and, at $s = \pm b$:

$$N_{\theta}^{(1)} = \frac{\partial^2 F^{(1)}}{\partial z^2} = 0 \qquad \text{(C3c)}$$

$$N_{z\theta}^{(1)} = -\frac{\partial^2 F^{(1)}}{\partial z\,\partial s} = 0 \qquad \text{(C3d)}$$

Conditions (C3a) and (C3c) above lead to relations between the sets of constants A_m and C_m and B_n and D_n. Assertion of these conditions leads to

$$\sum_{m=1,3\ldots}^{\infty} \mu_m{}^2 F_m(b) \cos \mu_m z = \frac{d^2 J(z)}{dz^2} \qquad \text{(C4)}$$

and

$$\sum_{n=1,3\ldots}^{\infty} \nu_n{}^2 F_n(l) \cos \nu_n s = 0 \qquad \text{(C5)}$$

Equation (C5) immediately gives

$$B_n = -D_n \delta_n \tanh \delta_n \qquad \text{(C6)}$$

where $\delta_n = \nu_n l$.

Equation (C4) is a Fourier cosine expansion of the function $d^2J(z)/dz^2$ and its coefficients lead to

$$A_m = \frac{f_m}{l\mu_m{}^2 \cosh \mu_m b} - \frac{\mu_m b \sinh \mu_m b}{\cosh \mu_m b} C_m \qquad \text{(C7)}$$

where

$$f_m = \int_{-l}^{l} \frac{d^2J(z)}{dz^2} \cos \mu_m z \, dz \tag{C8}$$

To satisfy conditions (C3b) and (C3d) the following relations from equations (C1) and (C3) are obtained:

$$\sum_{m=1,3}^{\infty} \mu_m \sin \mu_m l \; F_m'(s) = \sum_{n=1,3\ldots}^{\infty} \left[\left(\sum_{m=1,3\ldots}^{\infty} F_{mn}\mu_m\nu_m \sin \mu_m l\right) - \nu_n F_n'(l)\right] \sin \nu_n s \tag{C9}$$

and

$$\sum_{n=1,3\ldots}^{\infty} \nu_n \sin \nu_n b \; F_n'(z) = \sum_{m=1,3\ldots}^{\infty} \left[\left(\sum_{n=1,3\ldots}^{\infty} F_{mn}\nu_n\mu_m \sin \nu_n b\right) - \mu_m F_m'(b)\right] \sin \mu_m z \tag{C10}$$

where the prime indicates differentiation with respect to either z or s. Equations (C9) and (C10) are Fourier sine expansions of functions $F_m'(s)$ and $F_n'(z)$, respectively. The coefficients of these series with the help of equations (C6) and (C7) reduce to the following sets of infinite algebraic equations:

$$\sum_{m=1,3\ldots}^{\infty}\left[\frac{4\beta_m\alpha_m{}^2\gamma_m{}^2}{\left(\alpha_n{}^2+\gamma_m{}^2\right)^2}\cosh\gamma_m\sin\alpha_n\sin\beta_m C_m\right]=$$

$$-\sum_{m=1,3\ldots}^{\infty}\frac{2f_m l\gamma_m{}^2\sin\alpha_n\sin\beta_m}{\beta_m\left(\alpha_n{}^2+\gamma_m{}^2\right)}+\sum_{m=1,3\ldots}^{\infty}F_{mn}\beta_m\alpha_n\sin\beta_m-$$

$$\alpha_n\left(\frac{\delta_n{}^2}{\cosh\delta_n}+\delta_n\sinh\delta_n\right)D_n \qquad \text{for} \quad n=1,3,5\ldots \qquad \text{(C11)}$$

and

$$\sum_{n=1,3\ldots}^{\infty}\left[\frac{4\alpha_n\delta_n{}^2\beta_m{}^2}{\left(\delta_n{}^2+\beta_m{}^2\right)^2}\cosh\delta_n\sin\beta_m\sin\alpha_n D_n\right]=$$

$$\sum_{n=1,3\ldots}^{\infty}F_{mn}\gamma_m\alpha_n\sin\alpha_n-\frac{b\gamma_m f_m\tanh\gamma_m}{\beta_m}-$$

$$b\gamma_m\mu_m\left(\frac{\gamma_m}{\cosh\gamma_m}+\sinh\gamma_m\right)C_m \qquad \text{for} \quad m=1,3,5\ldots \qquad \text{(C12)}$$

where $\beta_m = \mu_m l$, $\alpha_n = \nu_n b$, $\delta_n = \nu_n l$, and $\gamma_m = \mu_m b$. There exist also two relations given in the text whereby F_{mn} may be eliminated so that the two sets of equations contain only the unknowns C_m and D_n. These two infinite families of algebraic equations may then be solved approximately.

APPENDIX D

PROCEDURE IN NUMERICAL CALCULATIONS

The procedure in the numerical calculation follows here. First, equations (42), (43), and (44) are made dimensionless by using the quantities

$$\left.\begin{aligned} \bar{C}_m &= \frac{C_m}{kl^2} & \qquad \bar{D}_n &= \frac{D_n}{kl^2} \\ \bar{w}_{mn} &= \frac{w_{mn}}{h} & \qquad \bar{K}_{mn} &= \frac{K_{mn}aD}{k} \\ & & \bar{f}_m = \frac{f_m}{kl} & \end{aligned}\right\} \qquad \text{(D1)}$$

Then equations (42), (43), and (44) become

$$\bar{w}_{mn}R_{mn} = \bar{K}_{mn} + S_{mn} + \bar{C}_m T_{mn} + \bar{D}_n V_{mn} \qquad \text{for any } m,n \qquad \text{(D2)}$$

$$\sum_{m=1,3\ldots}^{\infty} \bar{C}_m\left(W_{mn} - \frac{Y_{mn}T_{mn}}{R_{mn}}\right) = \sum_{m=1,3\ldots}^{\infty} X_{mn} + \sum_{m=1,3\ldots}^{\infty} Y_{mn}\left(\frac{\bar{K}_{mn}}{R_{mn}} + \right.$$

$$\left.\frac{S_{mn}}{R_{mn}}\right) + \bar{D}_n\left(Z_n + \sum_{m=1,3\ldots}^{\infty} Y_{mn}\frac{V_{mn}}{R_{mn}}\right) \qquad \text{for } n = 1,3,5\ldots \qquad \text{(D3)}$$

$$\sum_{n=1,3\ldots}^{\infty} \bar{D}_n\left(I_{mn} - \frac{V_{mn}J_{mn}}{R_{mn}}\right) = \sum_{n=1,3\ldots}^{\infty} J_{mn}\left(\frac{\bar{K}_{mn}}{R_{mn}} + \frac{S_{mn}}{R_{mn}}\right) +$$

$$L_m + \bar{C}_m\left(M_m + \sum_{n=1,3\ldots}^{\infty} \frac{T_{mn}}{R_{mn}} J_{mn}\right) \qquad \text{(D4)}$$

These functions are tabulated at the end of this section. The physical parameters which appear can be combined into three groups:

$$\frac{kb^4}{haD} \qquad \frac{h^2Eb^2}{kl^2a} \qquad \frac{hEb^4}{Da^2} \tag{D5}$$

As a numerical example, assume that

$$\left.\begin{aligned} \frac{b}{a} &= \frac{l}{a} = 0.3 \\ \frac{b}{h} &= 600 \\ \frac{kb^2}{D} &= 10 \end{aligned}\right\} \tag{D6}$$

These values correspond, for example, to a fuselage panel with these dimensions:

$$\left.\begin{aligned} h &= 0.03 \text{ in.} \\ a &= 60 \text{ in.} \\ b &= l = 18 \text{ in.} \end{aligned}\right\} \tag{D7}$$

The procedure in the numerical example is as follows:

(1) The particular solution $J(z)$ is given by

$$J(z) = \frac{k}{l^4}\frac{z^6}{30} - (2 - \nu)k\left(\frac{z^2}{2} - \frac{z^4}{6l^2} + \frac{z^6}{30l^4}\right)$$

(2) The Fourier coefficients f_m are:

$$f_m = -2k(2 - \nu)l \sin \beta_m\left[-\frac{1}{(2 - \nu)}\frac{1}{\beta_m} + \frac{4(2\nu - 1)}{(2 - \nu)}\frac{1}{\beta_m^3} + \frac{1 - \nu}{2 - \nu}\frac{24}{\beta_m^5}\right]$$

(3) The Fourier coefficients K_{mn} are:

$$K_{mn} = \frac{1}{Dbl}\int_{-b}^{b}\int_{-l}^{l}\left[\frac{d^2J(z)}{dz^2} - \frac{d^2f}{dz^2}k\right]\cos\mu_m z\cos\nu_n s\,ds\,dz$$

$$= \frac{2k}{Da}\,\frac{\sin\alpha_n\sin\beta_m}{\alpha_n\beta_m}\left[\left(5 - 2\nu - \frac{20a^2}{l^2}\right) + \frac{1}{\beta_m^2}\left(-4\nu - 4 + \frac{48a^2}{l^2}\right) + \frac{\nu - 1}{\beta_m^4}\right]$$

(4) The functions as defined by equations (D2), (D3), and (D4) are tabulated and summed where required. Values of these functions are given in table II.

(5) The two families of algebraic equations are then set up for a finite number of unknown coefficients D_n and C_m neglecting all coefficients beyond a certain suffix. (In the numerical example a total of six coefficients was used.) These equations are then solved by Crout's method. The results are given in table I.

The functions used in the numerical example are defined as follows:

$$R_{mn} = \frac{\left(\gamma_m^2 + \alpha_n^2\right)^2 + \dfrac{hEb^4}{Da^2}\,\dfrac{\gamma_m^4}{\left(\gamma_m^2 + \alpha_n^2\right)^2}}{kb^4/haD}$$

$$\bar{K}_{mn} = K_{mn}aD/k$$

$$S_{mn} = -\frac{2\bar{f}_m\alpha_n\sin\alpha_n}{\alpha_n^2 + \gamma_m^2}$$

$$T_{mn} = \frac{4\gamma_m{}^2\alpha_n\beta_m \cosh \gamma_m \sin \alpha_n}{\left(\alpha_n{}^2 + \gamma_m{}^2\right)^2}$$

$$V_{mn} = \frac{4\beta_m{}^3\delta_n{}^2}{\left(\delta_n{}^2 + \beta_m{}^2\right)^2} \cosh \delta_n \sin \beta_m$$

$$W_{mn} = \frac{4\gamma_m{}^3\alpha_n{}^2}{\left(\alpha_n{}^2 + \gamma_m{}^2\right)^2} \sin \beta_m \cosh \gamma_m \sin \alpha_n$$

$$X_{mn} = -\frac{2\bar{f}_m\gamma_m \sin \alpha_n \sin \beta_m}{\alpha_n{}^2 + \gamma_m{}^2}$$

$$Y_{mn} = \frac{h^2Eb^2}{kl^2a} \frac{\gamma_m{}^3\alpha_n}{\left(\gamma_m{}^2 + \alpha_n{}^2\right)^2} \sin \beta_m$$

$$Z_n = -\left(\frac{\delta_n{}^3}{\cosh \delta_n} + \delta_n{}^2 \sinh \delta_n\right)$$

$$I_{mn} = \frac{4\beta_m{}^2\delta_n{}^3}{\left(\delta_n{}^2 + \beta_m{}^2\right)^2} \sin \alpha_n \sin \beta_m \cosh \delta_n$$

$$J_{mn} = \frac{h^2Eb^2}{kl^2a} \frac{\gamma_m{}^3\alpha_n \sin \alpha_n}{\left(\gamma_m{}^2 + \alpha_n{}^2\right)^2}$$

$$L_m = -\overline{f}_m \tanh \gamma_m$$

$$M_m = -\gamma_m{}^2\left(\frac{\gamma_m}{\cosh \gamma_m} + \sinh \gamma_m\right)$$

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TABLE I

RESULTS OF NUMERICAL EXAMPLE

(a) Deflection coefficients and values of moments at center line

[Upper sign for compression and lower sign for tension when two signs are present]

z/l	$M_\theta^{(1)}b^2/Dh$	$w^{(1)}/h$	s/b	$M_z^{(1)}b^2/Dh$
0	∓2.4191	±3.00997	0	±25.432
.2	±10.374	±2.6122	.2	∓17.892
.5	±30.767	±3.4933	.5	±3.6157
.8	±71.944	±3.85909	.8	±114.240
1.0	0	0	1.0	0

(b) Values of unknowns for different approximations of algebraic equations

	Number of unknowns		
	2	4	6
$\bar{D}_1$	-60.384	-52.232	-51.464
$\bar{D}_3$	--------	-.00265	-.005040
$\bar{D}_5$	--------	----------	.000100
$\bar{C}_1$	-1.6976	-1.70914	-1.7547
$\bar{C}_3$	--------	.049589	.049369
$\bar{C}_5$	--------	----------	-.000300
$\left[w^{(1)}/h\right]_{center}$		2.9934	3.00997

TABLE II

VALUES OF COEFFICIENTS OF ALGEBRAIC EQUATIONS

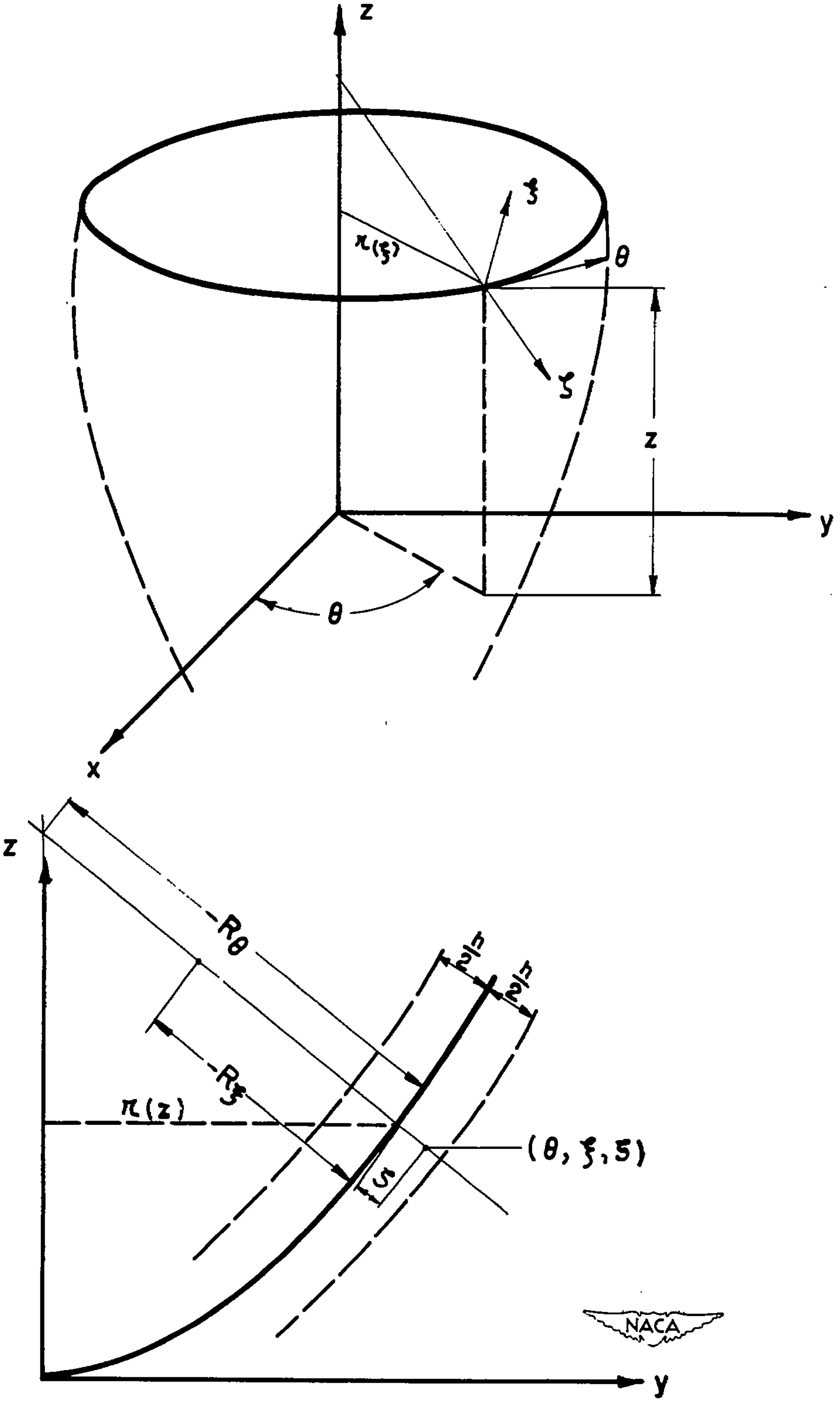
$W_{mn} - Y_{mn}\frac{T_{mn}}{R_{mn}}$			
m n	1	3	5
1	1.43331	-74.4314	1306.93
3	-1.16204	244.1042	-7550.46
5	.57467	-197.423	9894.78

$I_{mn} - \frac{V_{mn}J_{mn}}{R_{mn}}$			
m n	1	3	5
1	0.0013700	-0.00351	0.00368
3	-13.8618	5.19863	-6.13128
5	1327.299	2520.25	1484.91

n	$\sum_{m} Y_{mn}\left(\frac{\overline{K}_{mn} + S_{mn}}{R_{mn}}\right)$	$\sum_{m} Y_{mn}\frac{V_{mn}}{R_{mn}}$	$\sum_{m} X_{mn}$
1	-36.7113	6.64842	0.56935
3	11.5704	290.41846	.05352
5	-5.36824	7588.30616	-.09273

m	$\sum_{n} J_{mn}\left(\frac{\overline{K}_{mn} + S_{mn}}{R_{mn}}\right)$	$\sum_{n} J_{mn}\frac{T_{mn}}{R_{mn}}$
1	-7.99246	3.32264
3	55.24575	118.67377
5	-25.21255	3174.274

m	L_m	M_m
1	1.29746	-7.21918
3	.44046	-1237.68
5	-.26009	-79439.15



AT θ = CONSTANT PLANE

Figure 1.- Coordinate system.

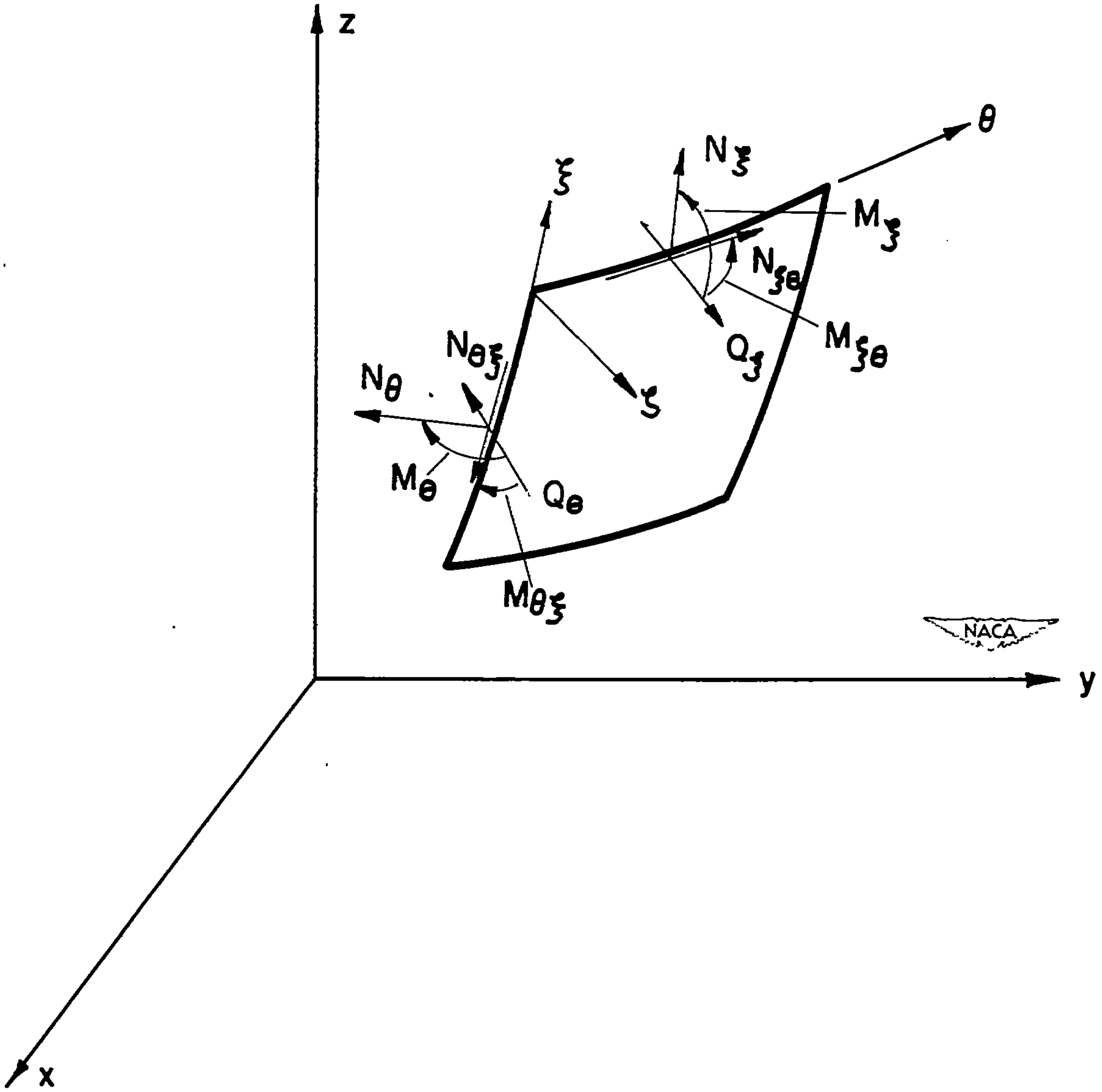


Figure 2.- Sign convention of resultant forces and moments.

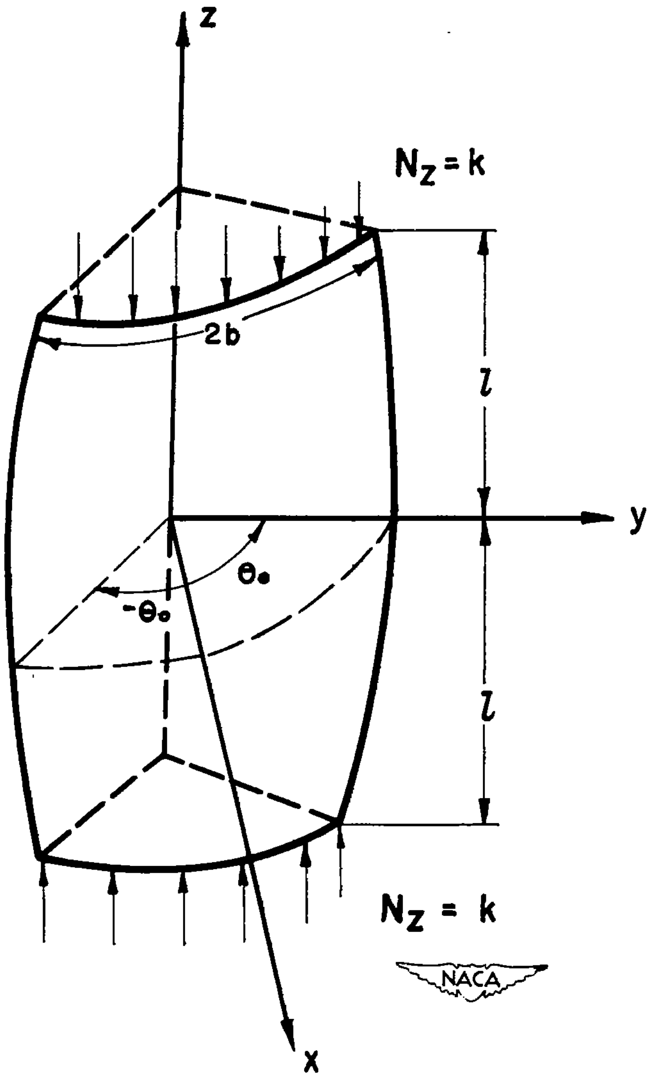


Figure 3.- Doubly curved plate loaded in compression.

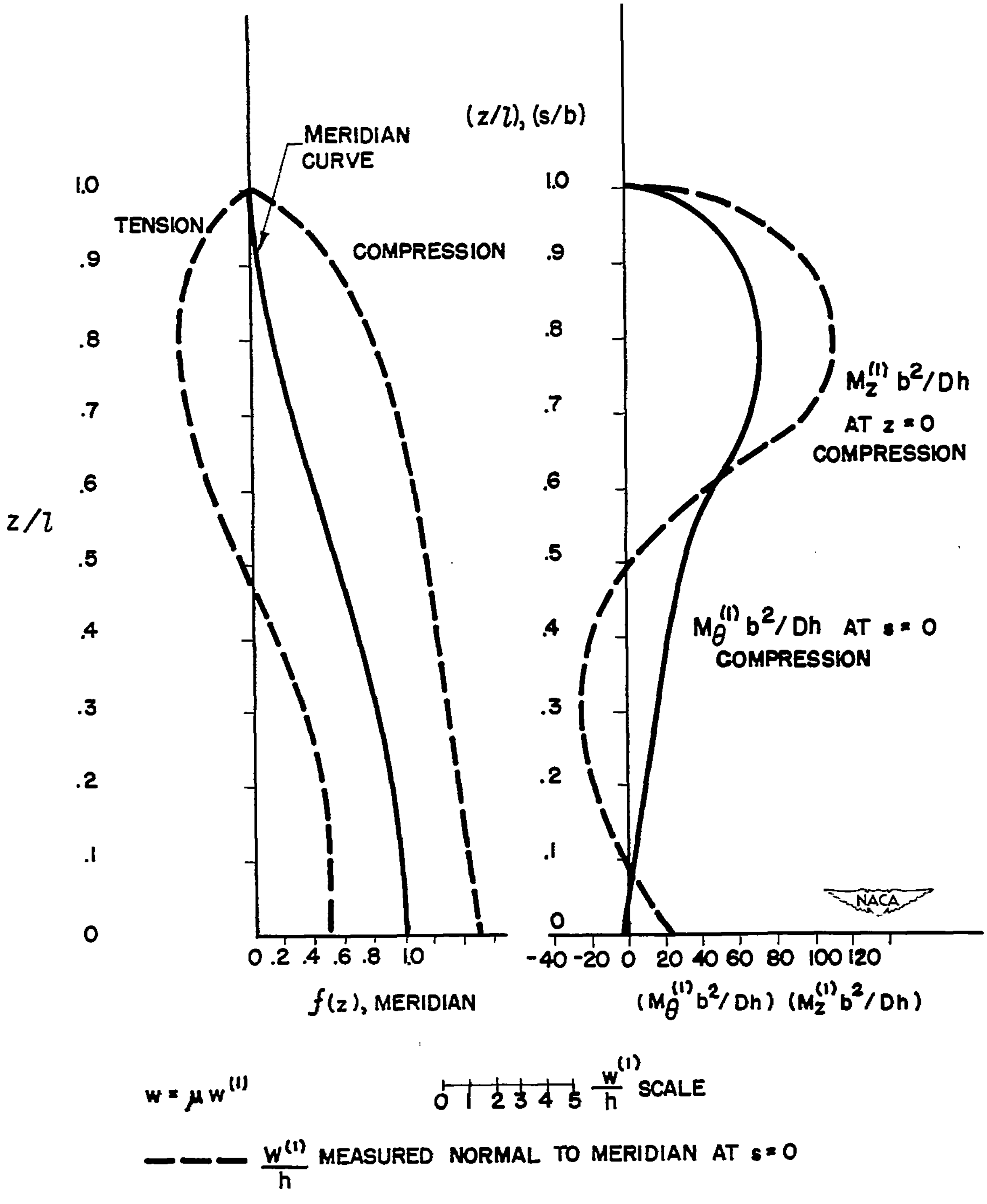


Figure 4.- Deflection normal to shell surface and moment distribution at center lines.

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