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SECOND APPROXIMATION TO LAMINAR COMPRESSIBLE BOUNDARY LAYER ON FLAT PLATE IN SLIP FLOW

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SUMMARY

The first-order solution for the laminar compressible boundary-layer flow over a flat plate at constant wall temperature is given. The effect of slip at the wall as well as the interaction between the boundary-layer flow and the outer stream flow are taken into consideration. The solution is obtained explicitly in terms of the known zero order, or continuum, solution. No assumptions regarding the Prandtl number or viscosity-temperature law need be made. It is found that the first-order solution gives a decrease in heat transfer and, for supersonic flow, an increase in skin friction. For subsonic flow there is no first-order shear effect. The change in heat transfer is due to slip and the change in friction is due to the interaction of the zero- and first-order velocities at the outer edge of the boundary layer.

INTRODUCTION

With very high-altitude high-speed motion becoming of practical interest, the behavior of air flow in rarefied and semirarefied gases becomes of great importance. In this connection, as has been discussed by Tsien (reference 1), one can define four regimes of fluid flow. These may be termed, in order of increasing mean free path of the fluid molecule: continuum, slip, intermediate, and free molecule flows. Continuum flow, where the mean free path of the fluid is negligible compared with the boundary-layer thickness, has been exhaustively studied for some time, while the other regimes have not. These latter domains differ from the continuum flow both in the form of the equations of motion and in the boundary conditions. The change in the boundary conditions appears in the form of temperature and velocity discontinuities between a solid boundary and the fluid immediately adjacent to it.

The slip regime may be loosely defined to include flows such that the ratio of mean free path to boundary-layer thickness (which is shown in reference 1 to be proportional to the Mach number divided by the

square root of the Reynolds number) is between 0.01 and 1. It may be noted that this parameter is the Knudsen number based on boundary-layer thickness. In this regime the flow may be defined by the Burnett equations (references 2 and 3), which represent the second approximation to the Boltzman equation, the first approximation being the familiar Navier-Stokes and energy equations. The boundary conditions were first given to the corresponding degree of approximation in reference 4, where it is shown that, in spite of the fact that the Burnett equations are of higher order than the Navier-Stokes relations, the same number of boundary conditions are required.

Several authors have discussed special problems in slip flow (references 4 to 6, for example) such as the Couette flow between flat plates and concentric cylinders and an incompressible boundary-layer and stagnation-point flow. In references 7 to 10, slip flow over a flat plate is discussed from another point of view by analogy with the Rayleigh problem of an infinite plate suddenly set into steady motion in a viscous fluid. This approach is relatively simple, but is limited by questions of the validity of the analogy for slip flow.

In the present paper, the first-order solution of the compressible flat-plate boundary-layer problem is found for variable Prandtl number and arbitrary temperature-viscosity law. It is assumed only that the specific heats are constant. The solution is a perturbation on the known continuum solution. Although the Burnett equations are the applicable ones in the slip regime, these expressions are just the familiar continuum equations of motion with higher-order shear and heat-flux terms added. These added terms are at least of order $\varepsilon^2 \left(\varepsilon = \frac{M\sqrt{\gamma}}{\sqrt{Re}}\right)$. (All symbols are defined in appendix A.) The corresponding boundary conditions (reference 4) contain terms of first order in ε. Hence, to find the first-order effect of slip, the continuum equations of motion may be used. These equations are subjected to the boundary-layer assumptions and expanded in powers of ε to give two sets of equations, one for the zero-order and the other for the first-order quantities. The same procedure is followed with the boundary conditions. The zero-order system is then seen to be the usual continuum boundary-layer system. The first-order system is then solved explicitly in terms of the zero-order quantities. For the convenience of the reader who is primarily interested in the results of this investigation, the rather lengthy mathematical details of the analysis have been relegated to the appendices. The investigation was conducted at the NACA Lewis laboratory.

ANALYSIS

Differential Equations

If it is assumed that there exists a thin boundary layer close to the wall in the fluid, and appropriate nondimensional coordinates are introduced (appendix B), the equations of steady motion and the equation of state (herein assumed to be the general gas law) reduce to the familiar boundary-layer equations, which are correct to order ε (equations (B1) of appendix B). It may next be assumed that the velocities and thermodynamic properties of the fluid can be expanded in powers of the small, but nonzero, quantity ε. Thus, for example,

$$\left.\begin{aligned} u &= {}^{o}u + \varepsilon {}^{1}u + \ldots \\ T &= {}^{o}T + \varepsilon {}^{1}T + \ldots \end{aligned}\right\} \tag{1}$$

In order to find the differential equations governing these zero- and first-order quantities, equations (1) are substituted into the boundary-layer equations (B1) and the coefficients of the zero and first powers of ε are equated to zero. The resulting two sets of relations are equations (B2) and (B3) of appendix B. The first set contains only zero-order terms and is identical with the usual boundary-layer relations (equations (B1)), while the other set governs the first-order terms; it is the primary object of the present report to solve them subject to the appropriate boundary conditions.

Boundary Conditions

The boundary conditions at the plate have been discussed in detail by Schamberg in reference 4 (see also reference 6) and are given as equations (B4) of appendix B. If these are treated in the same manner as the differential equations, that is, if they are made nondimensional and expanded in powers of ε, equating the coefficients of the zero and first powers of ε to zero, two sets of boundary conditions (equations (B5) and (B6)) result. Finally, conditions far from the plate must be specified. At first glance, it might be expected that the conditions are simply that the properties approach the undisturbed stream values. To zero order, namely in the conventional boundary-layer theory, this is nearly true; but when higher-order approximations are desired, the mutual interaction of the boundary layer and outside stream must be considered. If, as is the case in this report, it is desired to use the known continuum boundary-layer solutions as the zero-order solution to the present problem, it is found that the zero-order solution is such that, as the outside of the boundary layer is approached, all the fluid properties approach the undisturbed stream values except the vertical velocity. This component, which is of magnitude $1/\sqrt{Re}$ and is

hence negligible to zero order, is a manifestation of the fact that the boundary layer causes a curved surface to be presented as the effective wall of the relatively inviscid outside stream flow. This outer flow, which to the present approximation satisfies the Prandtl-Glauert potential equation, can be solved subject to the known boundary condition on the vertical velocity at the edge of the boundary layer (see appendix B, equations (B8) and (B9)). The resulting axial velocity, and hence the pressure and temperature (the outside stream remains isoenergetic to first order) at the edge of the boundary layer, may then be found (equations (B10) and (B11)). These terms are of order ε for supersonic flow and of order ε^2, and hence zero to the present approximation, for subsonic flow. It then follows that the appropriate first-order boundary condition is simply that the first-order fluid properties in the boundary layer must reach these values (equations (B13) and (B14)) at the edge of the boundary layer. Actually, the first-order solution will be seen to behave in such a manner that these conditions yield insufficient information to make the solution unique. Hence a suitable added condition must be introduced. A sufficient one is that the momentum and displacement thicknesses be finite. It will be seen that this condition effectively requires that the various fluid properties approach their stream values exponentially.

Solution of Differential Equations

The zero-order system (equations (B2), (B5), and (B7)) is the usual boundary-layer system and has been treated exhaustively (references 11 to 13) under various assumptions regarding the Prandtl number and the viscosity-temperature relation, and its solution is partially described in appendix B. The first-order system (equations (B3), (B6), and (B13)) can be solved explicitly in terms of the known zero-order solution if similarity is again assumed in the usual form. The details of the rather involved analysis are given in appendix B following equations (B18). The resulting solution is, with use of equations (B19) and (B31),

$$
\left.
\begin{aligned}
{}^{1}u(x,y) &= \frac{1}{\sqrt{x}}\left[{}^{o}u_{\eta}\left({}^{o}\mu(T_w)a_1\sqrt{T_w} - \frac{K}{2}\int_0^{\eta}\frac{b-t}{{}^{o}\mu\,{}^{o}u_{\eta}}\,dt - \int_0^{\eta}\frac{\mu_T}{{}^{o}\mu}F(t)dt\right)\right] \\
{}^{1}T(x,y) &= \frac{1}{\sqrt{x}}\left[{}^{o}T_{\eta}\left({}^{o}\mu(T_w)a_1\sqrt{T_w} - \frac{K}{2}\int_0^{\eta}\frac{b-t}{{}^{o}\mu\,{}^{o}u_{\eta}}\,dt - \int_0^{\eta}\frac{\mu_T}{{}^{o}\mu}F(t)dt\right) + F(\eta)\right] \\
{}^{1}v(x,y) &= \frac{1}{2\sqrt{x}}\left[\eta\,{}^{1}u(x,y) - {}^{o}\xi\left(\frac{K\gamma M^2}{\sqrt{x}} + \frac{{}^{1}T(x,y)}{{}^{o}T(\eta)}\right)\right] \\
&\text{where} \\
&\eta = y/\sqrt{x} \\
F(\eta) &= \frac{{}^{o}\mu}{{}^{o}\lambda}\,e^{-\int_0^{\eta}\frac{{}^{o}\xi\,Pr\,dt}{2\,{}^{o}\lambda\,{}^{o}T}}\left\{{}^{o}\lambda(T_w)(c_1 - a_1)\sqrt{T_w}\,{}^{o}T_{\eta}(0) - \frac{K}{2}\int_0^{\eta}(t-b)\left[(\gamma-1)M^2Pr\,{}^{o}u + \frac{{}^{o}\lambda\,{}^{o}T_{\eta}}{{}^{o}\mu\,{}^{o}u_{\eta}}\right]\frac{e^{\int_0^{t}\frac{{}^{o}\xi\,Pr\,dr}{2\,{}^{o}\lambda\,{}^{o}T}}}{{}^{o}\mu}\,dt\right\}
\end{aligned}
\right\} \quad (2)
$$

Also, it should be noted that

$$
\left.
\begin{aligned}
K &= 0 \quad \text{for subonsic flow} \\
&\text{and for supersonic flow} \\
K &= -\frac{1}{M\sqrt{\gamma(M^2-1)}}\left[\sqrt{x}\;{}^{o}v(x,\infty)\right] \\
&= -\frac{1}{M\sqrt{\gamma(M^2-1)}}\sqrt{Re_x}\,\frac{\delta^*}{x^*}
\end{aligned}
\right\} \quad (3)
$$

where δ^* is the zero-order displacement thickness.

From these expressions, the local skin friction and local heat transfer can be found. The result is, to first order (appendix C),

$$c_f\sqrt{Re_x} = {}^{o}(c_f\sqrt{Re_x}) + \varepsilon_x{}^{1}(c_f\sqrt{Re_x}) = \left[2^{o}\mu^{o}u_\eta + \varepsilon_x(2M\sqrt{\gamma(M^2-1)}\ K^2\right]_{\eta=0}$$

$$\frac{Nu_x}{\sqrt{Re_x}} = {}^{o}\left(\frac{Nu_x}{\sqrt{Re_x}}\right) + \varepsilon_x{}^{1}\left(\frac{Nu_x}{\sqrt{Re_x}}\right) \tag{4}$$

$$= \frac{1}{T_a-T_w}\left\{{}^{o}\lambda^{o}T_\eta + \varepsilon_x\left[-\ a_1(\gamma-1)M^2Pr({}^{o}\mu^{o}u_\eta)^2\ \sqrt{T_w}\right]\right\}_{\eta=0}$$

where, again, equation (3) holds.

DISCUSSION OF RESULTS

In order to see where the various terms in equations (2) and (4) arise, it may be observed that, if there were no slip, a_1 and c_1 would be equal to zero (see the wall boundary conditions, equations (B4), (B5), and (B6)), while if there were no interaction between the boundary layer and the stream outside it, K would be zero. Thus, for example, for subsonic flow with no slip the entire first-order solution vanishes.

With reference to equations (4), it may be observed that, for subsonic flow, the second approximation (first-order solution) contributes nothing to the skin friction, while for supersonic flow this approximation has a contribution. This skin-friction effect is not due to the slip at the wall at all but is rather simply due to the interaction between the boundary layer and the free stream. It therefore represents the effect of a slightly decelerating flow, generated in this case by the boundary layer itself, on the boundary layer. On the other hand, the change in heat transfer due to the first-order solution is entirely a slip effect and there is no basic difference between the subsonic and supersonic cases.

Some comparison of these results with those of other authors may be made. For incompressible flow, the present solution for ${}^{1}u(x,y)$ is the same as that found in reference 5, and both predict zero first-order shear (for subsonic flow). The Rayleigh problem (references 7 to 10) also yields solutions predicting zero first-order shear. The heat-transfer solution found in reference 10 is incorrect because the frictional dissipation which was neglected is the source of the thermal energy involved in the first-order heat transfer. The solution given in reference 8 for heat transfer is also incorrect because, in the notation of that report, as Te (adiabatic equilibrium temperature) is a function of θ (time), the solution for T_2, from which the heat tranfer was found, is in error. If, however, the analysis of reference 8 were

carried through without error, the same result for first-order heat transfer as that found in this report would be obtained (appendix D).

If it were desired to find only the second approximation to the laminar compressible boundary layer for constant wall temperature, assuming no slip (see, for comparison, reference 14, where this problem is considered for the case of incompressible flow), the solutions already obtained would apply except that the factors a_1 and c_1 appearing in the solutions (equations (2) and (4)) introduced by the slip conditions would in this case be set equal to zero. In this connection, it may be observed that, when, at the start of the analysis, the Navier-Stokes equations were expanded in powers of ε, only even powers of ε appeared. Also the wall boundary conditions involve no odd powers of ε, and in fact are, except for the zero-order temperature, homogeneous. A superficial observation of the conditions to be applied at the outside of the boundary layer would perhaps lead one to expect that these also were, when expanded, independent of odd powers of ε (and would in fact be homogeneous except for the zero-order terms) and that therefore it would be proper to solve the problem by expanding in powers of ε^2. That this is not the case, at least for supersonic flow, has been shown here and the reason lies entirely in the conditions which arise in matching the boundary layer with the outside, essentially inviscid, stream.

NUMERICAL RESULTS

The first-order skin friction and heat transfer may be readily calculated provided the zero-order skin friction, heat transfer, and displacement thickness are known. The data of reference 13 have been used as the known zero-order solution because they are the best available continuum solution of the flat-plate laminar boundary-layer problem for high-speed flow. The notation of reference 13 is related to the present terminology by the following relations:

Present report	Reference 13
$-2KM\sqrt{\gamma(M^2-1)}$	$\frac{\delta^*}{x}\sqrt{Re_\delta}$
T_∞^*	T_δ
T_w^*	T_w
T_a^*	T_i
$^{\circ}(c_f\sqrt{Re_x})$	$(c_f\sqrt{Re})_\delta$
M	M
$^{\circ}\left(\frac{Nu_x}{\sqrt[3]{Pr}\ \sqrt{Re_x}}\right)$	$\left(\frac{Nu_x}{\sqrt[3]{Pr}\ \sqrt{Re}}\right)_\delta$

The results of the calculation are presented in table I, which is to be used with equations (4), together with, for comparison, the zero-order solutions as given in reference 13. For computational purposes, a_1 and c_1 were taken as 1.253 and 2.870, respectively, corresponding to values of the Maxwell reflection coefficient of 1.0 and of the accommodation coefficient of 0.9 (reference 6). Computations are shown for an ambient stream temperature T_∞^* of 400° R, which is fairly typical of high-altitude flight. The quantity ε_x is, in free air, for a standard atmosphere (reference 15), given by

$$\varepsilon_x \equiv \frac{M\sqrt{\gamma}}{\sqrt{Re_x}} = \sqrt{\frac{M}{x^*}}\ \alpha(H) \tag{5}$$

where H is the altitude. The variation of $\alpha(H)$ with altitude is shown in figure 1. Thus, for example, at an altitude of 200,000 feet, if the Mach number is 10, the plate temperature is 1200° R, and $x^* = 1$ foot, the first-order solution shows a 10-percent increase in the shear and the same amount of decrease in the heat transfer. At 300,000 feet and the same Mach number, temperature, and length, the first-order solution indicates 60-percent increase in shear and a similar decrease in heat transfer. Of course, in this latter case, ε_x is nearly one half and is certainly too large for the theory presented herein to give more than a qualitative answer.

CONCLUDING REMARKS

The first-order boundary-layer solution for a flat plate at constant temperature has been calculated for high-speed motion in a semirarefied gas. There are actually two effects involved. One is the effect of slip itself, as reflected in the boundary conditions at the wall, and the other is a result of the interaction between the boundary layer and the stream outside it. These two effects are independent of each other and the effects in the latter case are not the same in the subsonic and supersonic regimes. The slip causes a decrease in the heat transfer at the wall but has no effect on the shear to first order, while the second effect causes, for supersonic flow, an increase in the shear and does not affect the heat transfer. For subsonic flow, the second effect causes no change in either the shear or heat transfer to first order.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, August 6, 1952

APPENDIX A

SYMBOLS

The following symbols are used in this report:

A, A_1, A_2, A_3	constants of integration
a_1	$\sqrt{\frac{\pi}{2}}\frac{2-\bar{a}}{\bar{a}}$, where $\bar{a}$ is Maxwell's reflection coefficient (reference 6)
b	$2\sqrt{x}\lim_{\eta\to\infty} {}^{o}v(x,\eta)$
b_1, b_2, b_3	constants
c_f	local skin-friction coefficient
c_p	specific heat at constant pressure
c_1	$\left(\frac{2}{\gamma+1}\right)\left(\frac{1}{Pr}\right)\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(\frac{2-\bar{c}}{\bar{c}}\right)$, where $\bar{c}$ is the accommodation coefficient (reference 6)
F, f	functions defined by equations (B26) and (B31)
G	function defined by equation (D4)
H	altitude, ft
K	constant, $\sqrt{\frac{\gamma M^2}{M^2-1}}\lim_{y\to\infty}\sqrt{x}\,{}^{o}v(x,y)$
L	characteristic length, say, length of plate
l	l^*/L
l^*	quantity defined by equation (D5); approximately the molecular mean free path
M	Mach number of undisturbed stream
m	m^*/L

m^*	quantity defined by equation (D5)
Nu_x	local Nusselt number
P	nondimensional pressure, P^*/P^*_∞
P^*	pressure
Pr	Prandtl number in undisturbed stream
R	gas constant
Re	Reynolds number based on characteristic length and undisturbed stream values, $\frac{\rho^*_\infty u^*_\infty L}{\mu^*_\infty}$
Re_x	local Reynolds number along plate, $\frac{\rho^*_\infty u^*_\infty x^*_1}{\mu^*}$
r,s,t	dummy variables
S	function defined by equation (D4)
T	nondimensional temperature, T^*/T^*_∞
T_a	T^*_a/T^*_∞
T_w	T^*_w/T^*_∞
T^*	temperature
T^*_a	continuum adiabatic wall temperature
T^*_w	specified wall temperature
u,v	nondimensional cartesian velocities; $u = \frac{u^*}{u^*_\infty}$, $v = \frac{v^*}{u^*_\infty}\sqrt{Re}$
u^*,v^*	cartesian velocities, u^* being parallel to the plate and v^* normal to the plate
x,y	nondimensional cartesian coordinates; $x = x^*/L$, $y = \frac{y^*}{L}\sqrt{Re}$
x^*,y^*	cartesian coordinates, x^* being parallel to the plate and y^* normal to the plate

α	function of altitude defined by equation (5) and reference 15, $ft^{1/2}$
β	function defined by equations (B19) and (B31)
γ	ratio of specific heats
δ^*	displacement thickness
ε	slip parameter, $\frac{M\sqrt{\gamma}}{\sqrt{Re}}$
ε_x	$M\sqrt{\gamma}/\sqrt{Re_x}$
ζ^*	time
η	$y/\sqrt{x}$
θ	momentum thickness
λ	nondimensional heat conductivity, $\lambda^*/\lambda_\infty^*$
λ_T	$\left(\frac{d\lambda(T)}{dT}\right)_{T=^oT}$
λ^*	heat conductivity
μ	nondimensional viscosity, μ^*/μ_∞^*
μ_T	$\left(\frac{d\mu(T)}{dT}\right)_{T=^oT}$
μ_{TT}	$\left(\frac{d^2\mu(T)}{dT^2}\right)_{T=^oT}$
μ^*	viscosity
ξ	function defined by equations (B16), (B19), and (B31)
ρ	nondimensional density, ρ^*/ρ_∞^*
ρ^*	density
σ	function defined by equations (B19) and (B31)
τ	shear

Φ,φ velocity potentials

Subscripts:

∞ undisturbed stream

s stream conditions outside boundary layer

x,y,ζ^*,η partial derivatives with respect to respective coordinate except for Re_x, ε_x, and Nu_x, which are defined elsewhere

Superscript:

$*$ physical quantity

Presuperscript:

0,1 zero and first approximations, respectively

APPENDIX B

ANALYSIS

The Differential Equations and Boundary Conditions

On application of the boundary-layer assumptions and introduction of nondimensional coordinates, the equations of steady motion and the general gas law reduce to these familiar boundary-layer equations, correct to order ε,

$$\left.\begin{aligned}
&(\rho u)_x + (\rho v)_y = 0 \\
&\frac{1}{\gamma M^2} P_x + \rho(uu_x + vu_y) - (\mu u_y)_y = 0 \\
&P_y = 0 \\
&\frac{\gamma-1}{\gamma} uP_x - \rho(uT_x + vT_y) + \frac{1}{Pr}(\lambda T_y) + (\gamma-1)M^2 \mu u_y^{\,2} = 0 \\
&P = \rho T
\end{aligned}\right\} \quad \text{(B1)}$$

In the transformations involved the coordinate and velocity normal to the wall are multiplied by $1/\sqrt{Re}$. Thus the physical and velocity fields are stretched in one direction, namely, across the boundary layer. This is, indeed, just the usual boundary-layer procedure. If equations (1) are substituted into equations (B1), the resulting expressions are expanded, and the coefficients of the zero and first powers of ε equated to zero, two sets of equations result. These are

$$\left.\begin{aligned}
&({}^o\rho\,{}^o u)_x + ({}^o\rho\,{}^o v)_y = 0 \\
&\frac{1}{\gamma M^2}\,{}^oP_x + {}^o\rho({}^o u\,{}^o u_x + {}^o v\,{}^o u_y) - ({}^o\mu\,{}^o u_y)_y = 0 \\
&{}^oP_y = 0 \\
&\frac{\gamma-1}{\gamma}\,{}^o u\,{}^oP_x - {}^o\rho({}^o u\,{}^oT_x + {}^o v\,{}^oT_y) + \frac{1}{Pr}({}^o\lambda\,{}^oT_y)_y + (\gamma-1)M^2\,{}^o\mu\,{}^o u_y^{\,2} = 0 \\
&{}^oP = {}^o\rho\,{}^oT
\end{aligned}\right\} \quad \text{(B2)}$$

$$\left.\begin{aligned}
&({}^{o}\rho\,{}^{1}u+{}^{1}\rho\,{}^{o}u)_x + ({}^{o}\rho\,{}^{1}v+{}^{1}\rho\,{}^{o}v)_y = 0 \\
&\frac{1}{\gamma M^2}\,{}^{1}P_x + {}^{o}\rho({}^{o}u\,{}^{1}u_x+{}^{1}u\,{}^{o}u_x+{}^{o}v\,{}^{1}u_y+{}^{1}v\,{}^{o}u_y) + {}^{1}\rho({}^{o}u\,{}^{o}u_x+{}^{o}v\,{}^{o}u_y) - ({}^{o}\mu\,{}^{1}u_y+{}^{1}\mu\,{}^{o}u_y)_y = 0 \\
&{}^{1}P_y = 0 \\
&\frac{\gamma-1}{\gamma}\,({}^{1}u\,{}^{o}P_x+{}^{o}u\,{}^{1}P_x) - {}^{o}\rho({}^{1}u\,{}^{o}T_x+{}^{o}u\,{}^{1}T_x+{}^{1}v\,{}^{o}T_y+{}^{o}v\,{}^{1}T_y) - {}^{1}\rho({}^{o}u\,{}^{o}T_x+{}^{o}v\,{}^{o}T_y) + \\
&\frac{1}{Pr}\,({}^{o}\lambda\,{}^{1}T_y+{}^{1}\lambda\,{}^{o}T_y)_y + (\gamma-1)M^2({}^{1}\mu\,{}^{o}u_y{}^{2}+2\,{}^{o}\mu\,{}^{o}u_y\,{}^{1}u_y) = 0 \\
&{}^{1}P = {}^{o}\rho\,{}^{1}T + {}^{1}\rho\,{}^{o}T
\end{aligned}\right\} \tag{B3}$$

The boundary conditions at the plate are (references 4 and 6), to first order,

$$\left.\begin{aligned}
v^*(x^*,0) &= 0 \\
u^*(x^*,0) &= \left[\frac{\mu^*}{P^*}\left(a_1\sqrt{RT^*}\,\frac{\partial u^*}{\partial y^*} + \frac{3R}{4}\,\frac{\partial T^*}{\partial x^*}\right)\right] \\
T^*(x^*,0) &= T_w + \left[\frac{\mu^*}{P^*}\left(c_1\sqrt{RT^*}\,\frac{\partial T^*}{\partial y^*} - \frac{1}{2}\,u^*\,\frac{\partial T^*}{\partial x^*}\right)\right]
\end{aligned}\right\} \tag{B4}$$

where the terms in brackets are evaluated in terms of conditions in the fluid adjacent to the wall.

If these are written in terms of the dimensionless quantities, there follows, correct to order ε,

$$v(x,0) = 0$$

$$u(x,0) = \varepsilon\left(\frac{\mu}{P}\,a_1\sqrt{T_w}\,u_y\right)$$

$$T(x,0) = T_w + \varepsilon\left(\frac{\mu}{P}\,c_1\sqrt{T_w}\,T_y\right)$$

These relations may next be expanded in powers of ε to give the two sets of boundary conditions at the wall. These are

$$\left.\begin{aligned} {}^{o}v(x,0) &= 0 \\ {}^{o}u(x,0) &= 0 \\ {}^{o}T(x,0) &= T_w \end{aligned}\right\} \tag{B5}$$

$$\left.\begin{aligned} {}^{1}v(x,0) &= 0 \\ {}^{1}u(x,0) &= \frac{{}^{o}\mu}{{}^{o}P}\, a_1 \sqrt{T_w}\, {}^{o}u_y \\ {}^{1}T(x,0) &= \frac{{}^{o}\mu}{{}^{o}P}\, c_1 \sqrt{T_w}\, {}^{o}T_y \end{aligned}\right\} \tag{B6}$$

Equations (B5) are the zero-order or continuum boundary conditions, while equations (B6) govern the first-order quantities.

Finally, to complete the definition of the problem, conditions far from the plate must be specified. Because it is desired that the zero-order problem be the usual continuum one, and because this introduces no inconsistency, the zero-order boundary conditions may be immediately specified. These are

$$ {}^{o}u(x,\infty) = {}^{o}P(x,\infty) = {}^{o}T(x,\infty) = 1 \tag{B7}$$

If it is now noted that the zero-order system (completely specified by equations (B2), (B5), and (B7)) is actually the desired familiar continuum one, it is then known that

$$ {}^{o}v(x,\infty) = \frac{b}{2}\sqrt{x} \tag{B8}$$

where b is a known constant. The outside stream satisfies the Prandtl-Glauert relation

$$(1-M^2)\,\varphi_{x^*x^*} + \varphi_{y^*y^*} = 0 \tag{B9}$$

where φ is the perturbation (from the undisturbed stream) velocity potential. The solution of equation (B9) subject to condition (B8) applied at the edge of the boundary layer will be found first for supersonic flow. The applicable general solution of equation (B9) is that φ is a function of $(x^* - \sqrt{M^2-1}\, y^*)$. Thus

$$u_s^* - u_\infty^* = -\frac{v_s^*}{\sqrt{M^2-1}} = \varphi_x^*(x^* - \sqrt{M^2-1}\ y^*)$$

or, in nondimensional coordinates,

$$u_s - 1 = \frac{-v_s}{\sqrt{M^2-1}\ \sqrt{Re}} = \Phi_x\left(x - \frac{\varepsilon\sqrt{M^2-1}}{\sqrt{\gamma}\ M}\right)$$

$$= \Phi_x(x)\ \left[1 + O(\varepsilon)\right]$$

Then, with use of equations (1), (B7), and (B8), at the edge of the boundary layer

$$^1u_s = -\frac{b}{2M\sqrt{\gamma(M^2-1)x}} \tag{B10}$$

Consideration will now be given to subsonic flow. A solution of equation (B9) corresponding to a linear source distribution of strength varying as $1/\sqrt{x^*}$ yields

$$u_s^* = -b_1\sqrt{2}\sqrt{\frac{\sqrt{x^{*2} - (M^2-1)y^{*2}} - x^*}{x^{*2} - (M^2-1)y^{*2}}} + u_\infty^*$$

$$\frac{v_s^*}{\sqrt{1-M^2}} = b_1\ \sqrt{2}\sqrt{\frac{\sqrt{x^{*2} - (M^2-1)y^{*2}} + x^*}{x^{*2} - (M^2-1)y^{*2}}}$$

or, in nondimensional coordinates,

$$u_s - 1 = -b_1\sqrt{2}\sqrt{\frac{\sqrt{x^2 - \frac{M^2-1}{\gamma M^2}\ \varepsilon^2 y^2} - x}{x^2 - \frac{M^2-1}{\gamma M^2}\ \varepsilon^2 y^2}} = \frac{b_1\varepsilon y}{x\sqrt{x}}\sqrt{\frac{1-M^2}{\gamma M^2}}\ \left[1 + O(\varepsilon)\right] \tag{B11}$$

$$\frac{\varepsilon v_x}{M\sqrt{\gamma(1-M^2)}} = b_1\sqrt{2}\sqrt{\frac{\sqrt{x^2 - \frac{M^2-1}{\gamma M^2}\varepsilon^2 y^2} + x}{x^2 - \frac{M^2-1}{\gamma M^2}\varepsilon^2 y^2}} = \frac{b_1}{\sqrt{x}}\left[1+O(\varepsilon)\right]$$

This last relation is of the same form as equation (B8) and hence these two expressions may be considered to define the desired flow outside the boundary layer for subsonic flow. Thus, by the preceding equations, b_1 is of order ε. Although b_1 is proportional to ε/M, which may apparently be large if M approaches zero, it is seen from the definition of ε that b_1 is proportional to $1/\sqrt{Re}$, which must be small in order that the boundary-layer assumption will hold in the first place. Hence, $u_s - 1$ is of order ε^2. Therefore, for subsonic flow,

$$^1u_s = 0 \tag{B12}$$

The pressure and temperature perturbations will now be described. For isentropic flow, which condition is satisfied to the present approximation by the outside stream,

$$T_s - 1 = -(\gamma-1)M^2(u_s-1)$$

$$P_s - 1 = -\gamma M^2(u_s-1)$$

Thus, for the first-order boundary conditions at the edge of the boundary layer,

$$\left.\begin{aligned} {}^1u(x,\infty) &= \frac{K}{\sqrt{x}} \\ {}^1T(x,\infty) &= -\frac{(\gamma-1)M^2K}{\sqrt{x}} \\ {}^1P(x,\infty) &= -\frac{\gamma M^2K}{\sqrt{x}} \end{aligned}\right\} \tag{B13}$$

where

$$K = 0 \quad \text{for subsonic flow}$$

$$K = \frac{-b}{2M\sqrt{\gamma(M^2-1)}} = \frac{-1}{M\sqrt{\gamma(M^2-1)}}\left[\sqrt{x}\ {}^{o}v(x,\infty)\right] \quad \text{for supersonic flow} \tag{B14}$$

In practical calculations, when the zero-order solution is known, K may be found in the following way:

$$K \equiv -\frac{1}{M\sqrt{\gamma(M^2-1)}}\left(\sqrt{x}\ \sqrt{Re}\ \frac{v_s^*}{u_\infty^*}\right) = -\frac{\sqrt{Re_x}}{M\sqrt{\gamma(M^2-1)}}\frac{d\delta^*}{dx}$$

where δ^* is the continuum displacement thickness and is known to vary as $\sqrt{x^*}$. Hence,

$$K \equiv -\frac{\sqrt{Re_x}}{2M\sqrt{\gamma(M^2-1)}}\frac{\delta^*}{x^*} \tag{B15}$$

Solution of Differential Equations

First, the zero-order system (equations (B2), (B5), and (B7)) may be considered. From the third of equations (B2) and the boundary condition equation (B7), ${}^{o}P(x,y) = 1$. Now, similarity may be assumed in the usual manner and there can be written, in the manner of Emmons and Brainerd (reference 11):

$$\left.\begin{aligned} {}^{o}u(x,y) &= {}^{o}u(\eta) \\ {}^{o}v(x,y) &= \frac{1}{2\sqrt{x}}\left[\eta\,{}^{o}u(\eta) - {}^{o}\xi(\eta)\right] \\ {}^{o}T(x,y) &= {}^{o}T(\eta) \end{aligned}\right\} \tag{B16}$$

where $\eta = y/\sqrt{x}$.

Then the system becomes the familiar one

$$\left.\begin{aligned}
\frac{{}^{o}u}{{}^{o}T} &= \left(\frac{{}^{o}\xi}{{}^{o}T}\right)_{\eta} \\
{}^{o}\xi\,{}^{o}u_{\eta} + 2\,{}^{o}T\left({}^{o}\mu\,{}^{o}u_{\eta}\right)_{\eta} &= 0 \\
\frac{{}^{o}\xi\,{}^{o}T_{\eta}}{{}^{o}T} + \frac{2}{Pr}\left({}^{o}\lambda\,{}^{o}T_{\eta}\right)_{\eta} + 2(\gamma-1)M^{2}\,{}^{o}\mu\,{}^{o}u_{\eta}{}^{2} &= 0
\end{aligned}\right\} \tag{B17}$$

$$\left.\begin{aligned}
{}^{o}u(0) &= {}^{o}\xi(0) = 0 \\
{}^{o}T(0) &= T_{w} \\
{}^{o}u(\infty) &= {}^{o}T(\infty) = 1
\end{aligned}\right\} \tag{B18}$$

This system or its equivalent has been solved by a number of investigators under various assumptions regarding the Prandtl number and the temperature-viscosity law (see, for example, references 11 to 13) and, hence, need not be considered further.

Next, the first-order system (equations (B3), (B6), and (B13)) may be considered. From the third of equations (B3) and the third of equations (B13) it follows immediately that

$${}^{1}P(x,y) = \frac{-K\gamma M^{2}}{\sqrt{x}}$$

Next, analogously with equations (B16), and following Kármán's suggestion (reference 16), it may be assumed that the first-order quantities have the form

$$\left.\begin{aligned}
{}^{1}u(x,y) &= \frac{1}{\sqrt{x}}\,{}^{1}\sigma(\eta) \\
{}^{1}v(x,y) &= \frac{1}{2x}\left[\eta\,{}^{1}\sigma(\eta) - {}^{1}\xi(\eta)\right] \\
{}^{1}T(x,y) &= \frac{1}{\sqrt{x}}\,{}^{1}\beta(\eta)
\end{aligned}\right\} \tag{B19}$$

and, hence, from the last of equations (B3),

$$^{1}\rho(x,y) = -\frac{1}{\sqrt{x}\ ^{o}T(\eta)}\left[K\gamma M^{2} + \frac{^{1}\beta(\eta)}{^{o}T(\eta)}\right]$$

Also, since the viscosity μ and heat conductivity λ are functions of temperature alone, $^{1}\mu$ and $^{1}\lambda$ may be found by expanding μ and λ in a Taylor series. Thus,

$$\mu(^{o}T+\varepsilon^{1}T) = \mu(^{o}T) + \varepsilon^{1}T\left(\frac{d\mu(T)}{dT}\right)_{T=^{o}T} = {}^{o}\mu + \varepsilon^{1}\mu$$

Therefore,

$$\left.\begin{aligned} ^{1}\mu(x,y) &= \frac{^{1}\beta(\eta)}{\sqrt{x}}\mu_{T} \\ \text{Similarly,}\qquad & \\ ^{1}\lambda(x,y) &= \frac{^{1}\beta(\eta)}{\sqrt{x}}\lambda_{T} \end{aligned}\right\} \tag{B20}$$

where

$$\mu_{T} \equiv \left(\frac{d\mu(T)}{dT}\right)_{T=^{o}T}$$

Now the expressions given by equations (B19) and (B20) may be put into the appropriate equations (B3), (B6), and (B13) to yield this system for the first-order terms.

$$\left.\begin{aligned}
&\left[\frac{{}^1\xi}{{}^oT} - \frac{{}^o\xi}{{}^oT}\left(K\gamma M^2 + \frac{{}^1\beta}{{}^oT}\right)\right]_\eta = 0 \\
&\frac{{}^ou\,{}^1\sigma}{{}^oT} + \frac{{}^o\xi\,{}^1\sigma_\eta}{{}^oT} + \frac{{}^ou_\eta}{{}^oT}\left[{}^1\xi - {}^o\xi\left(K\gamma M^2 + \frac{{}^1\beta}{{}^oT}\right)\right] + 2\left({}^1\beta\mu_T\,{}^ou_\eta + {}^o\mu\,{}^1\sigma_\eta\right)_\eta - K = 0 \\
&\frac{{}^ou\,{}^1\beta}{{}^oT} + \frac{{}^o\xi\,{}^1\beta_\eta}{{}^oT} + \frac{{}^oT_\eta}{{}^oT}\left[{}^1\xi - {}^o\xi\left(K\gamma M^2 + \frac{{}^1\beta}{{}^oT}\right)\right] + \frac{2}{Pr}\left({}^o\lambda\,{}^1\beta\right)_{\eta\eta} + (\gamma-1)\,{}^ouKM^2 + \\
&\qquad 2(\gamma-1)M^2\left({}^1\beta\mu_T\,{}^ou_\eta{}^2 + 2\,{}^o\mu\,{}^ou_\eta\,{}^1\sigma_\eta\right) = 0
\end{aligned}\right\} \tag{B21}$$

$$\left.\begin{aligned}
{}^1\xi(0) &= 0 \\
{}^1\sigma(0) &= {}^o\mu a_1\sqrt{T_w}\,{}^ou_\eta(0) \\
{}^1\beta(0) &= {}^o\mu c_1\sqrt{T_w}\,{}^oT_\eta(0) \\
{}^1\sigma(\infty) &= K \\
{}^1\beta(\infty) &= -(\gamma-1)M^2K
\end{aligned}\right\} \tag{B22}$$

Equations (B21) may be integrated explicitly in terms of the zero-order solution. From the first of equations (B21) and the boundary conditions on ${}^o\xi(\eta)$ and ${}^1\xi(\eta)$ (equations (B18) and (B22)), there follows immediately

$${}^1\xi(\eta) = {}^o\xi\left[K\gamma M^2 + \frac{{}^1\beta}{{}^oT}\right] \tag{B23}$$

With this result the last two equations of equations (B21) can be integrated once to give

$$\left.\begin{aligned}
&\frac{{}^o\xi}{{}^oT}\,{}^1\sigma + 2\left({}^1\beta\mu_T\,{}^ou_\eta + {}^o\mu\,{}^1\sigma_\eta\right) = 2A + K\eta \\
&\frac{{}^o\xi}{{}^oT}\,{}^1\beta + \frac{2}{Pr}\left({}^o\lambda\,{}^1\beta\right)_\eta + 2(\gamma-1)M^2\,{}^o\mu\,{}^ou_\eta\,{}^1\sigma = \frac{2A_2}{Pr} - 2(\gamma-1)M^2A\,{}^ou - (\gamma-1)M^2K\,{}^ou\eta
\end{aligned}\right\} \tag{B24}$$

It can readily be verified that a solution of the homogeneous equations corresponding to equations (B24) is

$$^{1}\sigma(\eta) = {}^{o}u_{\eta}(\eta)$$

$$^{1}\beta(\eta) = {}^{o}T_{\eta}(\eta)$$

and, hence, in general,

$$\left.\begin{aligned} ^{1}\sigma(\eta) &= {}^{o}u_{\eta}\left[A_1 + \int_0^{\eta} f(t)dt\right] \\ ^{1}\beta(\eta) &= {}^{o}T_{\eta}\left[A_1 + \int_0^{\eta} f(t)dt\right] + F(\eta) \end{aligned}\right\} \tag{B25}$$

If these expressions are put into equations (B24), the result is

$$({}^{o}\mu f + \mu_T F)\,{}^{o}u_{\eta} = A + \frac{K\eta}{2}$$

$$\left(\frac{{}^{o}\xi}{2\,{}^{o}T} + \frac{{}^{o}\lambda_{\eta}}{Pr}\right) F + \frac{{}^{o}\lambda}{Pr}\left({}^{o}T_{\eta} f + F_{\eta}\right) = \frac{A_2}{Pr} - (\gamma-1)M^2\,{}^{o}u\left(A + \frac{K\eta}{2}\right)$$

These equations may be solved to give

$$F(\eta) = \frac{{}^{o}\mu}{{}^{o}\lambda}\, e^{-\int_0^{\eta} \frac{{}^{o}\xi Pr\, dt}{2\,{}^{o}\lambda\,{}^{o}T}} \left\{A_3 + \int_0^{\eta}\left[A_2 - \left(A + \frac{K\eta}{2}\right)\left((\gamma-1)M^2 Pr\,{}^{o}u + \frac{{}^{o}\lambda\,{}^{o}T_{\eta}}{{}^{o}\mu\,{}^{o}u_{\eta}}\right)\right] \frac{e^{\int_0^{t} \frac{{}^{o}\xi Pr\, dr}{2\,{}^{o}\lambda\,{}^{o}T}}\, dt}{{}^{o}\mu}\right\} \tag{B26}$$

and

$$f(\eta) = \frac{A + K\eta/2}{{}^{o}\mu\,{}^{o}u_{\eta}} - \frac{\mu_T F(\eta)}{{}^{o}\mu}$$

It may be noted that, in the commonly assumed case of constant Prandtl number, ${}^{o}\mu = {}^{o}\lambda$ and hence

$$e^{-\int_0^\eta \frac{{}^{o}\xi \Pr\, dt}{2\,{}^{o}\lambda\,{}^{o}T}} = \frac{({}^{o}\mu\,{}^{o}u_\eta)^{\Pr}}{({}^{o}\mu\,{}^{o}u_\eta)^{\Pr}_{\eta=0}}$$

In order to apply the boundary conditions for large η, it is necessary to know the behavior of ${}^{1}\sigma(\eta)$ and ${}^{1}\beta(\eta)$ for large η. It is shown in appendix E (the result is derived for only ${}^{1}\sigma(\eta)$ while a similar analysis holds for ${}^{1}\beta(\eta)$) that for large η

$$\left.\begin{aligned}
{}^{1}\sigma(\eta) &= K + \left(A + \frac{bK}{2}\right) {}^{o}u_\eta \int_0^\eta \frac{dt}{{}^{o}\mu\,{}^{o}u_\eta} + \text{exponentially decaying terms} \\
{}^{1}\beta(\eta) &= -(\gamma-1)M^2K + \left[A_2 - (\gamma-1)M^2\Pr\left(A + \frac{bK}{2}\right)\right] ({}^{o}\mu\,{}^{o}u_\eta)^{\Pr} \int_0^\eta \frac{dt}{{}^{o}\mu({}^{o}\mu\,{}^{o}u_\eta)^{\Pr}} + \\
&\quad \text{exponentially decaying terms}
\end{aligned}\right\} \tag{B27}$$

and in the limit

$${}^{1}\sigma(\infty) = K$$

$${}^{1}\beta(\infty) = -(\gamma-1)M^2K$$

Thus for all values of the constants of integration, the last two boundary conditions of equations (B22) are satisfied. Now the displacement and momentum thicknesses, δ^* and θ, respectively, may be considered. These are, in nondimensional coordinates where the subscript s refers to stream values,

$$\delta^* = \int_0^\infty (\rho_s u_s - \rho u)dy = \sqrt{x}\int_0^\infty (\rho_s u_s - \rho u)d\eta$$

$$\theta = \int_0^\infty \rho u(u_s - u)dy = \sqrt{x}\int_0^\infty \rho u(u_s - u)d\eta$$

The contribution of the first-order slip term is

$$\left.\begin{aligned} \Delta\delta^* &= \sqrt{x}\int_0^\infty \left[{}^1\rho_s{}^0u_s + {}^0\rho_s{}^1u_s - ({}^1\rho{}^0u + {}^0\rho{}^1u)\right]d\eta \\ \Delta\theta &= \sqrt{x}\int_0^\infty \left[({}^1\rho{}^0u + {}^0\rho{}^1u)({}^0u_s - {}^0u) + {}^0\rho{}^0u({}^1u_s - {}^1u)\right]d\eta \end{aligned}\right\} \tag{B28}$$

In evaluating these integrals, equations (B27) can be used because all that is desired is to find the conditions for them to be finite. The exponentially vanishing terms can contribute only finite amounts to the integrals. It then follows (see appendix F) that in order for $\Delta\delta^*$ and $\Delta\delta$ to remain finite,

$$\left.\begin{aligned} A_2 &= 0 \\ A &= -\frac{bK}{2} \end{aligned}\right\} \tag{B29}$$

Finally, the conditions at the wall (equations (B22)) must be applied. From these there follows

$$\left.\begin{aligned} A_1 &= {}^0\mu(T_w)a_1\sqrt{T_w} \\ A_3 &= {}^0\lambda(T_w)(c_1 - a_1)\sqrt{T_w}\,{}^0T_\eta(0) \end{aligned}\right\} \tag{B30}$$

If equations (B26), (B29), and (B30) are put into equation (B25), there is obtained, finally,

$$
\left.
\begin{aligned}
{}^{1}\sigma(\eta) &= {}^{o}u_{\eta}\left[{}^{o}\mu(T_w)a_1\sqrt{T_w} - \frac{K}{2}\int_0^{\eta}\frac{b-t}{{}^{o}\mu\,{}^{o}u_{\eta}}\,dt - \int_0^{\eta}\frac{\mu_T}{{}^{o}\mu}F(t)dt\right] \\
{}^{1}\beta(\eta) &= {}^{o}T_{\eta}\left[{}^{o}\mu(T_w)a_1\sqrt{T_w} - \frac{K}{2}\int_0^{\eta}\frac{b-t}{{}^{o}\mu\,{}^{o}u_{\eta}}\,dt - \int_0^{\eta}\frac{\mu_T}{{}^{o}\mu}F(t)dt\right] + F(\eta) \\
\text{and}\quad {}^{1}\xi(\eta) &= {}^{o}\xi\left(K\gamma M^2 + \frac{{}^{1}\beta}{{}^{o}T}\right) \\
\text{where}\quad F(\eta) &= \frac{{}^{o}\mu}{{}^{o}\lambda}\,e^{-\int_0^{\eta}\frac{{}^{o}\xi Pr\,dt}{2\,{}^{o}\lambda\,{}^{o}T}}\left\{{}^{o}\lambda(T_w)(c_1-a_1)\sqrt{T_w}\,{}^{o}T_{\eta}(0) - \right. \\
&\qquad \left. \frac{K}{2}\int_0^{\eta}(\eta-b)\left[(\gamma-1)M^2Pr\,{}^{o}u + \frac{{}^{o}\lambda\,{}^{o}T_{\eta}}{{}^{o}\mu\,{}^{o}u_{\eta}}\right]\frac{e^{\int_0^{t}\frac{{}^{o}\xi Pr\,dr}{2\,{}^{o}\lambda\,{}^{o}T}}\,dt}{{}^{o}\mu}\right\}
\end{aligned}
\right\}\quad\text{(B31)}
$$

These four equations constitute the solution of the first-order boundary-layer problem for slip flow when the wall has a constant specified temperature T_w^*.

APPENDIX C

SKIN FRICTION AND HEAT TRANSFER

The local skin friction and local heat transfer at the wall are given by

$$\sqrt{Re_x}\, c_f = \sqrt{Re_x}\, \frac{\tau(x^*,0)}{\frac{1}{2}\rho_\infty^* u_\infty^{*2}} = 2(\mu u_\eta)_{\eta=0}$$

$$= 2\left[{}^o\mu\, {}^o u_\eta + \varepsilon_x(\mu_T{}^1\beta\, {}^o u_\eta + {}^o\mu\, {}^1\sigma_\eta)\right]_{\eta=0}$$

$$Nu_x/\sqrt{Re_x} = \left[\frac{x^*\lambda^*}{\lambda_\infty^*(T_a^*-T_w^*)}\left(\frac{\partial T^*}{\partial y^*}\right)\right]_{y=0}\left(\frac{1}{\sqrt{Re_x}}\right) = \frac{1}{T_a-T_w}\,(\lambda T_\eta)_{\eta=0}$$

$$= \frac{1}{T_a-T_w}\left[{}^o\lambda\, {}^o T_\eta + \varepsilon_x(\lambda_T{}^1\beta\, {}^o T_\eta + {}^o\lambda\, {}^1\beta_\eta)\right]_{\eta=0}$$

These become, with use of equations (B24) and the boundary conditions,

$$c_f\sqrt{Re_x} = {}^o(c_f\sqrt{Re_x}) + \varepsilon_x{}^1(c_f\sqrt{Re_x}) = \left[2\, {}^o\mu\, {}^o u_\eta + \varepsilon_x(2M\sqrt{\gamma(M^2-1)}\,K^2)\right]_{\eta=0}$$

$$\frac{Nu_x}{\sqrt{Re_x}} = {}^o\left(\frac{Nu_x}{\sqrt{Re_x}}\right) + \varepsilon_x{}^1\left(\frac{Nu_x}{\sqrt{Re_x}}\right) = \frac{1}{T_a-T_w}\left\{{}^o\lambda\, {}^o T_\eta + \varepsilon_x\left[-a_1(\gamma-1)M^2 Pr({}^o\mu\, {}^o u_\eta)^2\sqrt{T_w}\right]\right\}_{\eta=0}$$

APPENDIX D

SOLUTION OF THE RAYLEIGH PROBLEM FOR FIRST-ORDER SLIP

The mathematical statement of the Rayleigh problem for slip in an incompressible fluid is (reference 8) (in the present statement of the problem it is assumed, as in reference 9, that the fluid rather than the plate is suddenly accelerated)

$$\left.\begin{aligned} u^*_{\zeta^*} - \frac{\mu^*}{\rho^*} u^*_{y^*y^*} &= 0 \\ u^*(\zeta^*,0) &= l^*(u^*_{y^*})_{y=0} \\ u^*(\zeta^*,\infty) &= u^*_\infty \\ u^*(0,y^*) &= u^*_\infty \end{aligned}\right\} \tag{D1}$$

$$\left.\begin{aligned} T^*_{\zeta^*} - \frac{\lambda^*}{\rho^* c_p} T^*_{y^*y^*} &= \frac{\mu^*}{\rho^* c_p} (u^*_{y^*})^2 \\ T^*(\zeta^*,0) &= T^*_w + m^*(T^*_{y^*})_{y=0} \\ T^*(\zeta^*,\infty) &= T^*_\infty \\ T^*(0,y^*) &= T^*_\infty \end{aligned}\right\} \tag{D2}$$

Equations (D1) can be solved exactly. If ζ^* is chosen such that $\zeta^* = x^*/0.346u^*_\infty$, in order that the zero-order shear match the Blasius solution, there results, in nondimensional coordinates,

$$u(x,y) = \operatorname{erf}\left[\frac{0.294y}{\sqrt{x}}\right] + e^{\frac{y}{l\sqrt{Re}} + \frac{x}{0.346 l^2 Re}} \operatorname{erfc}\left[\frac{0.294y}{\sqrt{x}} + \frac{\sqrt{x}}{0.588 l\sqrt{Re}}\right]$$

from which there can be obtained

$$c_f\sqrt{Re_x} = 0.664 + O(\epsilon^2)$$

which is the result obtained in references 8 and 9, and which agrees with equation (4) of this report for subsonic flow.

Next, equations (D2) can be solved exactly by standard methods. The solution is, in nondimensional coordinates,

$$T(x,y) = T_w - S(x,y) + \int_0^x S(t,0)G(x-t,y)\,dt \tag{D3}$$

where

$$\left.\begin{aligned}
S(x,y) &= T_w - 1 + \frac{(\gamma-1)M^2Pr^{\frac{1}{2}}}{2\sqrt{\pi(0.346)}}\int_0^x ds \int_{-\infty}^{\infty} \left[u_r(s,r)\right]^2 \frac{e^{-\frac{0.346Pr(r-y)^2}{4(x-s)}}}{\sqrt{x-s}}\,dr \\
&\text{and} \\
G(x,y) &= \frac{0.959}{m\sqrt{RePrx}}\,e^{-\frac{0.346Pry^2}{4x}} - \\
&\frac{2.888}{m^2RePr}\,e^{\left(\frac{y}{m\sqrt{Re}} + \frac{x}{0.346m^2RePr}\right)}\,\operatorname{erfc}\left(\frac{0.294y\sqrt{Pr}}{\sqrt{x}} + \frac{\sqrt{x}}{0.588m\sqrt{RePr}}\right)
\end{aligned}\right\} \tag{D4}$$

From this, to order l ($l\sqrt{Re}$ is of the same order as ε),

$$\frac{Nu_x}{\sqrt{Re_x}} = \frac{1}{T_a-T_w}\left[(1-T_w+B)\ (0.332)\ \sqrt{Pr} - (0.332)^2\,\frac{l}{\sqrt{x}}\,(\gamma-1)M^2\sqrt{Pr}\ \sqrt{Re}\right]$$

However, if equations (D1) and (B4) are compared, there follows:

$$\left.\begin{aligned}
\frac{l^*}{L} &= l = \frac{\mu^*}{p^*}\,\frac{a_1\sqrt{RT^*}}{L} = \frac{\varepsilon a_1\sqrt{T_w}}{Re} \\
\frac{m^*}{L} &= m = \frac{\mu^*}{p^*}\,\frac{c_1\sqrt{RT^*}}{L} = \frac{\varepsilon c_1\sqrt{T_w}}{\sqrt{Re}}
\end{aligned}\right\} \tag{D5}$$

Then

$$\frac{Nu_x}{\sqrt{Re_x}} = \frac{1}{T_a - T_w}\left[(1-T_w+B)\ (0.332)\ \sqrt{Pr} - \varepsilon_x a_1(\gamma-1)M^2 Pr(0.332)^2\ \sqrt{T_w}\right] \tag{D6}$$

where

$$B = \frac{2(\gamma-1)M^2}{\pi\sqrt{2/Pr-1}}\cos^{-1}\sqrt{\frac{Pr}{2}}$$

The zero-order portion of equation (D6) has been essentially found by Emmons (reference 17), who has pointed out that, except for $Pr = 1$, this solution is not quite the same as that found from the usual boundary-layer theory. The first-order term in equation (D6), however, exactly matches the corresponding term of equation (4) of the present report.

APPENDIX E

BEHAVIOR OF ${}^{1}\sigma(\eta)$ FOR LARGE η

From equations (B25) and (B26),

$$
{}^{1}\sigma(\eta) = {}^{o}u_{\eta}\left(A_1 + \int_0^{\eta} \frac{\left(A + \frac{Kt}{2}\right)dt}{{}^{o}\mu\,{}^{o}u_{\eta}} - \int_0^{\eta} \frac{\mu_T}{{}^{o}\lambda} e^{-\int_0^{t} \frac{{}^{o}\xi Pr\,dr}{2{}^{o}\lambda\,{}^{o}T}} \left\{A_3 + \int_0^{t}\left[A_2 - \left(A + \frac{Kr}{2}\right)\left((\gamma - 1)M^2 Pr\,{}^{o}u_{Pr} + \frac{{}^{o}\lambda\,{}^{o}T_{\eta}}{{}^{o}\mu\,{}^{o}u_{\eta}}\right)\right] \frac{e^{\int_0^{r} \frac{{}^{o}\xi Pr\,ds}{2{}^{o}\lambda\,{}^{o}T}}\,dr}{{}^{o}\mu}\right\} ds\right) \tag{E1}
$$

First, it is well-known that the behavior of the zero-order solution is such that, as $\eta \to \infty$,

$$
\begin{aligned}
&{}^{o}u(\eta) \to 1 \\
&{}^{o}u(\eta) - 1 \to 0 \quad \text{as} \quad e^{-b_1\eta^2} \quad (b_1 > 0) \\
&{}^{o}u_{\eta} \to 0 \quad \text{as} \quad e^{-b_1\eta^2} \\
&{}^{o}T(\eta) \to 1 \\
&{}^{o}T_{\eta}(\eta) \to 0 \quad \text{as} \quad ({}^{o}u_{\eta})^{Pr} \\
&{}^{o}\xi(\eta) \to \eta - b
\end{aligned} \tag{E2}
$$

Most of the terms in equation (E1) will be shown to vanish exponentially. For the terms multiplied by A_1 and A_3, this follows immediately by the use of equation (E2). Some of the other terms will be seen to vanish exponentially if equation (E1) is multiplied by e^{η} and the product is shown to vanish. The terms in the last integral may be considered. Although $e^{\eta}\,{}^{o}u_{\eta}(\eta) \to 0$, the integral is infinite and hence the product is an indeterminate form, and it is appropriate to apply l'Hospital's rule. Thus,

$$I_1 = \lim_{\eta\to\infty}\left(e^{\eta}\,{}^{o}\mu\,{}^{o}u_{\eta}\int_0^{\eta}\left\{\frac{\mu_T}{{}^{o}\lambda}\,e^{-\int_0^t \frac{{}^{o}\xi Pr\,ds}{2{}^{o}\lambda\,{}^{o}T}}\int_0^t\left[A_2 - \left(A + \frac{Kr}{2}\right)\left((\gamma - 1)M^2Pr\,{}^{o}uPr + \frac{{}^{o}\lambda\,{}^{o}T_{\eta}}{{}^{o}\mu\,{}^{o}u_{\eta}}\right)\frac{e^{\int_0^r \frac{{}^{o}\xi Pr\,ds}{2{}^{o}\lambda\,{}^{o}T}}}{{}^{o}\mu}\right]dr\right\}dt\right)$$

$$= \lim_{\eta\to\infty}\left[\left(\frac{e^{\eta}\,{}^{o}\mu\,{}^{o}u_{\eta}}{-1 + \frac{{}^{o}\xi}{2{}^{o}\mu\,{}^{o}T}}\right)\frac{\mu_T}{{}^{o}\lambda}\,e^{-\int_0^{\eta}\frac{{}^{o}\xi Pr\,dt}{2{}^{o}\lambda\,{}^{o}T}}\int_0^{\eta}\left[A_2 - \left(A + \frac{Kt}{2}\right)\left((\gamma - 1)M^2Pr\,{}^{o}uPr + \frac{{}^{o}\lambda\,{}^{o}T_{\eta}}{{}^{o}\mu\,{}^{o}u_{\eta}}\right)\frac{e^{\int_0^t \frac{{}^{o}\xi Pr\,dr}{2{}^{o}\lambda\,{}^{o}T}}}{{}^{o}\mu}\right]dt\right]$$

Again, the integral approaches infinity and its coefficient is zero. Thus

$$I_1 = \lim_{\eta\to\infty}\left\{\frac{e^{\eta}\,{}^{o}u_{\eta}\,\frac{\mu_T}{{}^{o}\lambda}\left[A_2 - \left(A + \frac{K\eta}{2}\right)\left((\gamma-1)M^2Pr\,{}^{o}u + \frac{{}^{o}\lambda\,{}^{o}T_{\eta}}{{}^{o}\mu\,{}^{o}u_{\eta}}\right)\right]}{\frac{{}^{o}u}{2{}^{o}\mu\,{}^{o}T} + \left(\frac{{}^{o}\xi}{2{}^{o}\mu\,{}^{o}T} - 1\right)^2 - {}^{o}T_{\eta}\left[\frac{{}^{o}\xi\,\mu_T}{2{}^{o}\mu^2\,{}^{o}T} + \left(\frac{\mu_{TT}}{\mu_T} - \frac{\lambda_T}{{}^{o}\lambda}\right)\left(\frac{{}^{o}\xi}{2{}^{o}\mu\,{}^{o}T} - 1\right)\right]}\right\}$$

By the use of equations (E2), it is seen that the denominator approaches $\frac{1}{2} + \left(\frac{\eta-b}{2} - 1\right)^2$ plus smaller terms. The numerator contains only terms of the form $e^{\eta}\,{}^{o}u_{\eta}$, $e^{\eta}\,{}^{o}T_{\eta}$, $\eta e^{\eta}\,{}^{o}u_{\eta}$, and $\eta e^{\eta}\,{}^{o}T_{\eta}$, all of which vanish. Hence, finally, $I_1 = 0$ and therefore the last term in equation (E1) vanishes exponentially. Next, the term involving K in the first integral is considered. This can be written

$$I_2 = {}^{o}u_{\eta}\,\frac{K}{2}\left[\int_0^{\eta}\frac{b}{{}^{o}\mu\,{}^{o}u_{\eta}}\,dt - 2\int_0^{\eta}\frac{\left({}^{o}\mu\,{}^{o}u_{\eta}\right)_{\eta}}{\left({}^{o}\mu\,{}^{o}u_{\eta}\right)^2}\,dt + 2\int_0^{\eta}\frac{\left({}^{o}\mu\,{}^{o}u_{\eta}\right)_{\eta}}{\left({}^{o}\mu\,{}^{o}u_{\eta}\right)^2}\left(1 - \frac{(t-b)}{{}^{o}\xi}\,{}^{o}T\right)dt\right] \tag{E3}$$

The last integral can be shown to vanish exponentially.

$$I_3 = \lim_{\eta\to\infty}\left[\left({}^{o}\mu\,{}^{o}u_\eta\right)e^{\eta}\int_0^{\eta}\frac{\left({}^{o}\mu\,{}^{o}u_\eta\right)\eta}{\left({}^{o}\mu\,{}^{o}u_\eta\right)^2}\left(1 - \frac{{}^{o}T(t-b)}{{}^{o}\xi}\right)dt\right]$$

$$= \lim_{\eta\to\infty}\left[\frac{{}^{o}\mu\,{}^{o}u_\eta\ e^{\eta}}{{}^{o}\xi/2\,{}^{o}\mu\,{}^{o}T - 1}\left(\frac{-{}^{o}\xi/2\,{}^{o}\mu\,{}^{o}T}{{}^{o}\mu\,{}^{o}u_\eta}\right)\left(1 - \frac{{}^{o}T(\eta-b)}{{}^{o}\xi}\right)\right]$$

But, by equation (E2),

$$\lim_{\eta\to\infty}\left(\frac{{}^{o}\xi/2\,{}^{o}\mu\,{}^{o}T}{{}^{o}\xi/2\,{}^{o}\mu\,{}^{o}T - 1}\ \frac{\eta\,{}^{o}T}{{}^{o}\xi}\right)\to 1$$

Therefore,

$$I_3 = \lim_{\eta\to\infty}\left[\frac{e^{\eta}}{\eta}\left(\frac{{}^{o}\xi}{{}^{o}T} - (\eta-b)\right)\right]$$

This is again an indeterminate form. Therefore,

$$I_3 = \lim_{\eta\to\infty}\left[\frac{e^{\eta}}{1-\eta}\left(\frac{{}^{o}u}{{}^{o}T} - 1\right)\right]\to 0$$

Then, when these results are used and the indicated integration in equation (E3) is carried out, it follows that, for large values of η,

$${}^{1}\sigma(\eta) = K + \left(A + \frac{bK}{2}\right){}^{o}u_\eta\int_0^{\eta}\frac{dt}{{}^{o}\mu\,{}^{o}u_\eta} + \text{exponentially vanishing terms}$$

Finally, it can be shown that as η approaches infinity, the term in this expression involving the integral vanishes as $1/\eta$. Hence,

$${}^{1}\sigma(\infty) = K$$

APPENDIX F

BOUNDARY CONDITIONS AT INFINITY

Consideration will now be given to the expression for $\Delta\theta$ in equation (B28). From the form of the solution, the term $({}^{1}\rho\,{}^{o}u+{}^{o}\rho\,{}^{1}u)$ is bounded in absolute value by some constant, b_2. Then, with the knowledge that ${}^{o}u_s = 1$ and that ${}^{o}u - {}^{o}u_s$ is always nonnegative,

$$\left|\int_0^\infty ({}^{o}\rho\,{}^{1}u+{}^{1}\rho\,{}^{o}u)({}^{o}u_s-{}^{o}u)d\eta\right| < b_2\int_0^\infty (1-{}^{o}u)d\eta$$

which is finite. The remaining term in $\Delta\theta$ is, if equations (B19) and (B27) are used,

$$I_4 = \sqrt{x}\left(A + \frac{bK}{2}\right)\int_0^\infty\left(\frac{{}^{o}u}{{}^{o}T}\,{}^{o}u_\eta\int_0^t \frac{ds}{{}^{o}\mu\,{}^{o}u_\eta}\right)dt \tag{F1}$$

But

$$\lim_{\eta\to\infty}\left(\eta\,\frac{{}^{o}u}{{}^{o}T}\,{}^{o}u_\eta\int_0^\eta \frac{ds}{{}^{o}\mu\,{}^{o}u_\eta}\right) = 2$$

and is always such that, for some b_3,

$$\eta\,\frac{{}^{o}u}{{}^{o}T}\,{}^{o}u_\eta\int_0^\eta \frac{ds}{{}^{o}\mu\,{}^{o}u_\eta} > b_3 > 0 \quad \text{if} \quad \eta > 1$$

Hence,

$$I_4 > \sqrt{x}\left(A + \frac{bK}{2}\right)b_3\int_1^\infty \frac{dt}{t} = \sqrt{x}\left(A + \frac{bK}{2}\right)b_3\lim_{\eta\to\infty}(\ln\eta)$$

Therefore, $\Delta\theta$ is only bounded if

$$A = -\frac{bK}{2} \tag{F2}$$

Now, if equation (F2) is used and equations (B27) are put into the expression for $\Delta\delta^*$ (equation (B28)), the result is

$$\Delta\delta = \int_0^\infty \left[- KM^2 \left(1 - \frac{{}^o u}{{}^o T}\right) + K \left(\frac{{}^o T - 1}{{}^o T}\right)\right]\left(1 + (\gamma-1)M^2 \frac{{}^o u}{{}^o T}\right) d\eta +$$

$$A_2 \int_0^\infty \left[\frac{{}^o u({}^o\mu\, {}^o u_\eta)^{Pr}}{{}^o T^2} \int_0^\eta \frac{dt}{{}^o\mu({}^o\mu\, {}^o u_\eta)^{Pr}}\right] d\eta$$

The first integral is finite because its integrand approaches zero exponentially. If the same procedure is observed as in the case of the expression in equation (F1), it follows that the second integral is infinite and, therefore, A_2 must be zero in order that $\Delta\delta^*$ be finite.

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TABLE I - VARIATION OF SKIN FRICTION AND HEAT TRANSFER WITH WALL TEMPERATURE AND MACH NUMBER FOR NOMINAL STREAM STATIC TEMPERATURE OF 400° R

T_w (°R)	M	${}^{o}(c_f\sqrt{Re_x})$	${}^{1}(c_f\sqrt{Re_x})$	${}^{o}(Nu/\sqrt{Re_x})$	${}^{1}(Nu/\sqrt{Re_x})$
1182	4.55	0.591	0.958	0.311	0.881
	6.31	.572	.738	.319	.525
	9.70	.538	.684	.325	.417
	14.83	.481	.752	.313	.321
2246	6.68	0.531	1.460	0.312	1.356
	10.20	.503	1.085	.319	.640
	15.50	.461	0.967	.305	.450

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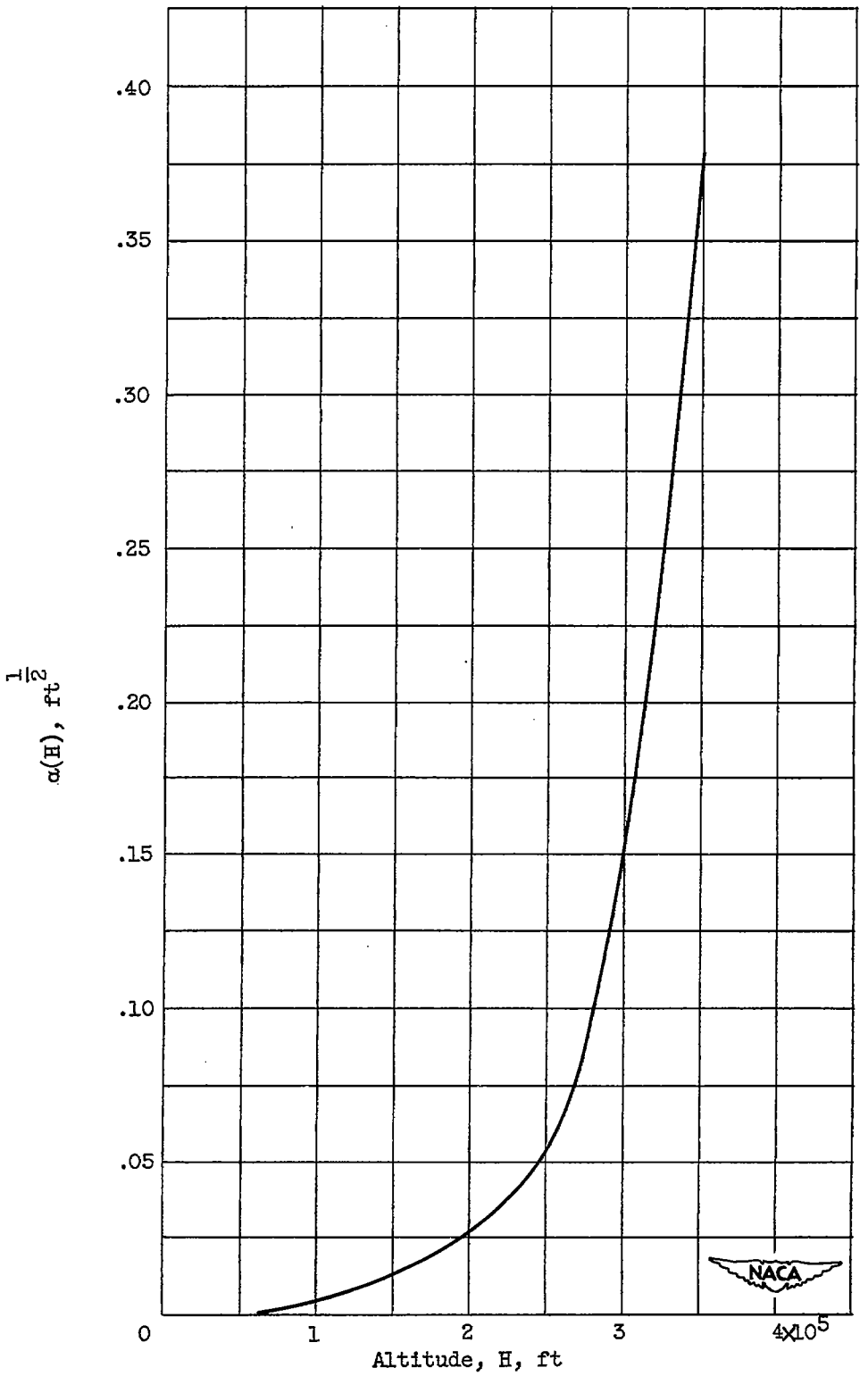

Figure 1. - Variation with altitude of $\alpha(H)$ for use in equation (5).