NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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TEMPERATURES, THERMAL STRESS, AND SHOCK IN HEAT-GENERATING PLATES OF CONSTANT CONDUCTIVITY AND OF CONDUCTIVITY THAT VARIES LINEARLY WITH TEMPERATURE

By S. V. Manson

Lewis Flight Propulsion Laboratory
Cleveland, Ohio

NACA

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TEMPERATURES, THERMAL STRESS, AND SHOCK IN HEAT-GENERATING PLATES OF CONSTANT CONDUCTIVITY AND OF CONDUCTIVITY THAT VARIES LINEARLY WITH TEMPERATURE

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SUMMARY

Working formulas are presented for the steady-state temperatures and thermal stress in heat-generating infinite plates of constant conductivity, and of conductivity that decreases linearly with temperature as the temperature increases, for the case in which the heat is generated uniformly throughout the plate thickness and both faces of the plate are equally cooled. In addition to exact formulas for the variable-conductivity plates, simpler approximate formulas are derived, and criteria for their applicability are indicated.

It is shown that, of all planes in the plate, the plate surfaces are always under the greatest tension, and the midplane is under the greatest compression.

A criterion is indicated for determining the surface cooling conditions under which the thermal shocks at the surface and midplane will be smaller than, equal to, or greater than the steady-state thermal stresses at those planes. The shocks exceed the steady-state stresses only when the initial surface-cooling conditions of the transient state are more severe than the surface cooling conditions of the steady state. The criterion is expressed for heat-generating plates by a simple relation between a steady-state heat-transfer parameter of the heat-generating plate and the Biot number of the transient state.

The dimensionless parameters governing the transient temperatures and thermal stresses in materials of linearly varying conductivity are derived by a similarity study of the conduction equation and boundary conditions of the transient state. A numerical technique for solving the transient-state equations is indicated in detail. The method is employed to obtain numerical values of the transient temperatures and stresses for a variety of parameter combinations, selected to test the thermal-shock criterion. The results of the calculations substantiate the shock criterion.

As a result of the study an insight is obtained into the steady and transient temperature and stress mechanisms in heat-generating plates of constant conductivity and of conductivity that varies linearly with temperature.

INTRODUCTION

Thermal stresses in the steady and transient states have acquired increasing interest for aircraft: in ductile-material applications, as in turbine disks and blades; and in brittle-material applications, as in nozzle diaphragms, combustion-chamber linings, high-temperature coatings, and recently in solids that generate heat internally. 2920

A substantial literature exists on the steady-state thermal stresses arising from temperature nonuniformity in solids of cylindrical and rectangular geometries. The greater part of the literature on transient-state stresses (shock), however, is concerned with suddenly heated or cooled bodies of initially uniform temperature. No systematic study of shock in bodies of initially nonuniform temperature nor any criterion for ready prediction of when the shock will exceed the steady-state stress was found.

An additional consideration which appears not to have been studied is the question whether thermal conductivity variations produce significant stress effects, in those applications for which the operating temperature range is large. Examples of solids of substantially varying conductivity are low-carbon steel, which halves its conductivity in the range from room temperature to 1450° F, and Inconel X which doubles its conductivity in the same temperature range (ref. 1). Cermets and other high-temperature materials may also exhibit substantial conductivity variations with temperature. If conductivity variations are considered, it is desirable to identify the parameters governing the temperature in both the steady and transient states, inasmuch as such an identification facilitates design and the choice of test conditions.

A specific problem of current interest is that of thermal stress and shock in solids that generate heat internally. If the power per unit volume of solid is high, and if the solid has low conductivity or unfavorable creep properties, the thermal stress or shock may be excessive. For heat-generating solids the effect of substantially decreasing conductivity with increasing temperature is also of considerable interest, inasmuch as such conductivity variation aggravates the steady stresses.

As part of its research in materials for aircraft, the NACA Lewis laboratory has investigated the steady and transient temperatures, and the steady and transient thermal stresses, in infinite plates which generate heat uniformly throughout their interior in the steady state and which, on suddenly ceasing to generate heat, are simultaneously immersed in an environment of constant temperature. The study has been made for cases in which the plate conductivity is constant and for cases in which the conductivity decreases linearly with temperature as the temperature increases.

The steady-state temperature distribution in a heat-generating plate of linearly varying conductivity was not found in the literature, and hence is derived herein; for comparison and completeness, the well-known steady-state temperature distribution in a heat-generating plate of constant conductivity is also presented. The transient temperature distribution in plates having an initially nonuniform temperature distribution and a thermal conductivity that varies continually with location and time was also not found in the literature. Such a plate cannot be readily analyzed by the existing procedures for composite plates, the conductivities of which are generally treated as uniform and as time-independent in a region of any one material; hence an extension of the well-known finite-difference procedures was employed. The transient temperatures in a constant-conductivity plate having an initially nonuniform temperature distribution are presented in an infinite-series solution in reference 2. For consistency with the calculations on variable-conductivity plates, and also to eliminate series-convergence considerations for the short time-periods of interest in the thermal-shock study, the finite-difference procedure was preferred over the infinite-series solution for the constant-conductivity plates.

In the study of heat-generating plates reported herein, formulas for the steady-state temperatures and stresses are presented. A criterion is indicated for determining whether the steady stress or thermal shock will be the greater under anticipated conditions of operation. The dimensionless groups governing the transient temperatures and thermal stresses in materials of linearly varying conductivity are derived. A numerical technique is indicated for solving the transient-state conduction equation for variable-conductivity materials, and typical curves of transient temperature and stress are presented.

ANALYSIS

General Stress Considerations

The relation for thermal stress in a plate of thickness $2L$ (fig. 1), in which the temperature varies only in the x-direction, and in which end effects are not considered, is given in reference 3, and in the notation of this report is

$$\sigma_y = \sigma_z = -\frac{\beta E T}{1-\nu} + \left(\frac{1}{1-\nu}\right)\frac{1}{2L}\int_{-L}^{L}\beta E T\,dx + \frac{3}{2(1-\nu)}\frac{x}{L^3}\int_{-L}^{L}\beta E T x\,dx \qquad (1)$$

(Symbols are defined in appendix A.) The geometric variable x may be changed to the dimensionless form x/L, and the temperature variable T may be changed to θ defined by the relation

$$\theta \equiv T - T_e \tag{2}$$

where T_e is a constant. If βE is satisfactorily constant over the temperature range of interest in an application, and if use is made of the facts that 2920

$$\frac{1}{2} \int_{-1}^{1} T_e \, d\left(\frac{x}{L}\right) = T_e$$

and

$$\int_{-1}^{1} T_e \left(\frac{x}{L}\right) d\left(\frac{x}{L}\right) = 0$$

equation (1) may be written

$$\sigma_y = \sigma_z = -\frac{\beta E \theta}{1-\nu} + \left(\frac{1}{1-\nu}\right)\left(\frac{1}{2}\right) \beta E \int_{-1}^{1} \theta \, d\left(\frac{x}{L}\right) + \frac{3}{2(1-\nu)} \left(\frac{x}{L}\right) \beta E \int_{-1}^{1} \theta \left(\frac{x}{L}\right) d\left(\frac{x}{L}\right) \tag{3}$$

From equation (2), the θ value at the plate surface is

$$\theta_s = T_s - T_e \tag{4}$$

and if the steady-state surface temperature is denoted by $T_{s,o}$,

$$\theta_{s,o} = T_{s,o} - T_e \tag{5}$$

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and $\theta_{s,o}$ is a constant.

If σ is employed to represent both σ_y and σ_z, which are equal to each other, and if σ^* is defined as

$$\sigma^* \equiv \frac{\sigma(1-\nu)}{\beta E \theta_{s,o}} \tag{6}$$

equation (3) may be rewritten in the dimensionless form

$$\sigma^* \equiv \sigma \frac{(1-\nu)}{\beta E \theta_{s,o}} = -\left(\frac{\theta}{\theta_{s,o}}\right) + \frac{1}{2} \int_{-1}^{1} \left(\frac{\theta}{\theta_{s,o}}\right) d\left(\frac{x}{L}\right) + \frac{3}{2}\left(\frac{x}{L}\right) \int_{-1}^{1} \left(\frac{\theta}{\theta_{s,o}}\right)\left(\frac{x}{L}\right) d\left(\frac{x}{L}\right) \tag{7}$$

Equation (7) involves no explicit time dependence; if $\theta/\theta_{s,o}$ is known throughout the plate the dimensionless stress is determined regardless of whether $\theta/\theta_{s,o}$ is a steady or transient value. The first term of the right member of equation (7) is the local value of $\theta/\theta_{s,o}$ at a plane chosen for consideration, and may vary as x/L varies; the second term is the average value of $\theta/\theta_{s,o}$ over the plate thickness, and for an existing temperature distribution is constant. The third term is a "moment of a temperature-moment" and may vary with x/L, but if the temperature distribution is symmetrical about the plate center line, $(x/L = 0)$, the term vanishes, and the local dimensionless stress is equal to the difference between the plate average and the local values of $\theta/\theta_{s,o}$. Thus in the case of a symmetrical temperature distribution the thermal stress arises simply from the tendency of the local plane to expand or contract to a length different from that corresponding to the plate average temperature.

The case of a temperature distribution symmetrical about $x/L = 0$ is the case under consideration in this report. As has been mentioned, equation (7) reduces, in this case, to

$$\sigma^* \equiv \frac{\sigma(1-\nu)}{\beta E \theta_{s,o}} = -\left(\frac{\theta}{\theta_{s,o}}\right)_{x/L} + \left(\frac{\theta}{\theta_{s,o}}\right)_{av} \tag{8}$$

Equation (8) indicates that there is no stress at the plane x/L where the temperature is the same as the plate average temperature. If a positive value of σ^* denotes tension, and a negative value of σ^* denotes compression, equation (8) indicates that planes at which the temperature exceeds the plate average temperature are in compression, and planes at which the temperature is less than the plate average temperature are in tension. Thus, for any plate temperature distribution symmetrical about the midplane the surface and midplane are under the greatest stresses of opposite sign. For heat-generating plates equally cooled at both surfaces, the surface is under the greatest tension, and the midplane under the greatest compression, of all planes in the plate.

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Steady-State Temperature and Stress in Heat-Generating Plates

The actual magnitudes of surface and midplane stresses of heat-exchanging plates can be determined if the distribution of $\theta/\theta_{s,o}$ is known explicitly. For heat-generating plates in steady state, the following temperature distributions are derived in appendix B:

(a) For a plate material of constant thermal conductivity,

$$\frac{\theta}{\theta_{s,o}} = 1 + \frac{q'''L^2}{2K\theta_{s,o}}\left[1 - \left(\frac{x}{L}\right)^2\right] \tag{9}$$

(b) For a plate material having a conductivity that varies linearly in the temperature range between environment and surface temperatures,

$$\frac{\theta}{\theta_{s,o}} = \frac{1 - \frac{K_{s,o}}{K_e}\sqrt{1 - 2\left(\frac{K_e}{K_{s,o}}\right)\left(\frac{K_e}{K_{s,o}} - 1\right)\left(\frac{q'''L^2}{2K_e\theta_{s,o}}\right)\left[1 - \left(\frac{x}{L}\right)^2\right]}}{\left(1 - \frac{K_{s,o}}{K_e}\right)} \tag{10}$$

(An approximate formula for $\theta/\theta_{s,o}$ is discussed in appendix B.)

<u>Steady-state stresses for heat-generating plate of constant conductivity.</u> - The average value of $\theta/\theta_{s,o}$ with equation (9) is

$$\left(\frac{\theta}{\theta_{s,o}}\right)_{av} \equiv \frac{1}{2}\int_{-1}^{1}\left(\frac{\theta}{\theta_{s,o}}\right)d\left(\frac{x}{L}\right) = \frac{1}{2}\int_{-1}^{1}\left\{1 + \frac{q'''L^2}{2K\theta_{s,o}}\left[1 - \left(\frac{x}{L}\right)^2\right]\right\}d\left(\frac{x}{L}\right)$$

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or

$$\left(\frac{\theta}{\theta_{s,o}}\right)_{av} = 1 + \frac{2}{3}\left(\frac{q''' L^2}{2K\theta_{s,o}}\right) \qquad (11)$$

On substituting equations (9) and (11) into equation (7) or (8),

$$\sigma^* \equiv \frac{\sigma(1-\nu)}{\beta E\theta_{s,o}} = \left[\left(\frac{x}{L}\right)^2 - \frac{1}{3}\right]\left(\frac{q''' L^2}{2K\theta_{s,o}}\right) \qquad (12)$$

Equation (12) indicates zero stress at $\frac{x}{L} = \frac{1}{\sqrt{3}}$, where the temperature is equal to the plate average temperature. At the plate surface, $\frac{x}{L} = \pm 1$, the stress is

$$\sigma^*_{s,o} \equiv \frac{\sigma_{x,o}(1-\nu)}{\beta E\theta_{s,o}} = \frac{2}{3}\left(\frac{q''' L^2}{2K\theta_{s,o}}\right) \qquad (13)$$

at the plate midplane, $x/L = 0$, the stress is

$$\sigma^*_{m,o} \equiv \frac{\sigma_{m,o}(1-\nu)}{\beta E\theta_{s,o}} = -\frac{1}{3}\left(\frac{q''' L^2}{2K\theta_{s,o}}\right) \qquad (14)$$

<u>Steady-state stress for heat-generating plate of linearly varying conductivity.</u> - The average value of $\theta/\theta_{s,o}$ with equation (10) is

$$\left(\frac{\theta}{\theta_{s,o}}\right)_{av} = \frac{1}{2}\int_{-1}^{1}\left\{\frac{1 - \frac{K_{s,o}}{K_e}\sqrt{1 - 2\left(\frac{K_e}{K_{s,o}}\right)\left(\frac{K_e}{K_{s,o}} - 1\right)\left(\frac{q''' L^2}{2K_e\theta_{s,o}}\right)\left[1 - \left(\frac{x}{L}\right)^2\right]}}{\left(1 - \frac{K_{s,o}}{K_e}\right)}\right\} d\left(\frac{x}{L}\right)$$

If A is defined as

$$A \equiv 2\left(\frac{K_e}{K_{s,o}}\right)\left(\frac{K_e}{K_{s,o}} - 1\right)\left(\frac{q''' L^2}{2K_e\theta_{s,o}}\right) \tag{15}$$

and if only positive values of A are considered (positive values of A correspond to plates of conductivity that decreases as temperature increases, as indicated in appendix B),

$$\left(\frac{\theta}{\theta_{s,o}}\right)_{av} = \frac{1 - \frac{1}{2}\left(\frac{K_{s,o}}{K_e}\right)\left\{1 + \frac{1 - A}{2\sqrt{A}} \ln\left(\frac{1 + \sqrt{A}}{1 - \sqrt{A}}\right)\right\}}{\left(1 - \frac{K_{s,o}}{K_e}\right)} \tag{16}$$

By use of equations (10) and (16) in equation (8), the stress at any x/L can be computed. At the surface, $x/L = \pm 1$, the stress is

$$\sigma^*_{s,o} \equiv \frac{\sigma_{s,o}(1-\nu)}{\beta E\theta_{s,o}} = \frac{\frac{K_{s,o}}{K_e}\left[1 - \frac{1 - A}{2\sqrt{A}} \ln\left(\frac{1 + \sqrt{A}}{1 - \sqrt{A}}\right)\right]}{2\left(1 - \frac{K_{s,o}}{K_e}\right)} \tag{17}$$

at the midplane, $x/L = 0$, the stress is

$$\sigma^*_{m,o} \equiv \frac{\sigma_{m,o}(1-\nu)}{\beta E\theta_{s,o}} = \frac{\frac{K_{s,o}}{K_e}\left[1 + \frac{1 - A}{2\sqrt{A}} \ln\left(\frac{1 + \sqrt{A}}{1 - \sqrt{A}}\right) - 2\sqrt{1 - A}\right]}{2\left(1 - \frac{K_{s,o}}{K_e}\right)} \tag{18}$$

(Approximate formulas for $\sigma^*_{s,o}$ and $\sigma^*_{m,o}$ are derived in appendix C.)

Transient Temperatures and Thermal Shock

The $\theta/\theta_{s,o}$ distribution (and hence the stress) in the transient state is governed by the familiar equation

$$\rho c \frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial x}\left(K \frac{\partial\theta}{\partial x}\right) \tag{19}$$

in the plate interior; and at the surface, by the standard relation

$$q'' \equiv -K_s \left(\frac{\partial\theta}{\partial x}\right)_s = h\theta_s \tag{20}$$

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Equation (19) contains no source term because it is assumed for the analysis herein that the heat-generating plate has suddenly ceased to generate heat at the instant that it is immersed in the transient cooling environment. It is noted that, whereas in the steady state the constant T_e is arbitrarily assignable, the use of θ_s in equation (20) indicates that in the transient state the value assigned to T_e must be the constant temperature of the environment into which the plate is immersed on ceasing to generate heat. In practice it is convenient to take T_e the same for both the steady and unsteady states, even though the steady-state coolant may not have the temperature T_e.

A calculation procedure for obtaining numerical values of $\theta/\theta_{s,o}$ from equations (19) and (20) is indicated in appendix D, and in appendix E a derivation is presented of the dimensionless groups governing $\theta/\theta_{s,o}$ in the transient state. In terms of the derived groups, values of $\theta/\theta_{s,o}$ computed for an arbitrarily prescribed initial temperature distribution may be presented in a form useful for similar systems. From a knowledge of the $\theta/\theta_{s,o}$ distribution the transient thermal stress (shock) is obtained by use of equation (8) of this section, as is indicated in some additional detail in the Calculation Procedure of appendix D.

CRITERION FOR DOMINANCE OF STEADY-STATE THERMAL STRESS OR TRANSIENT THERMAL SHOCK

In establishing a criterion for whether the steady-state stress or the thermal shock will be greater, the heat-transfer process in the steady state and in the early stages of the transient state may be examined in some detail, and inferences be drawn concerning the simultaneous stress variation. The ultimate guide is equation (8) of the Analysis section, which indicates that the stress is proportional to $(\theta_{av} - \theta_{x/L})$. Inasmuch as the greatest stresses occur at $x/L = 0$ and ± 1, that is, at the plate midplane and surfaces, the following considerations are directed at a determination of the manner in which the midplane and surface temperatures vary in relation to the plate average temperature.

For convenience in discussion the half-plate may be assumed divided into N thin segments of equal width δ. If 1 square foot of frontal area is considered, and the plate is assumed to generate q''' Btu per second per cubic foot uniformly throughout its interior in the steady state, then each segment of width δ generates $q''' \delta$ Btu per second. The steady-state temperature profile (i.e., local slope), is such that each segment receives all the heat generated by segments more interior than itself and transmits not only the heat received but also the additional amount $q''' \delta$ that it generates; hence the profile steepens outward. The segment bounded by the plate surface receives $(N-1)\ \delta q'''$, generates $\delta q'''$, and transmits $N\delta q'''$ to the sink. If h_{st} represents the steady-state heat-transfer coefficient,

$$h_{st}\ (T_s - T_c) \equiv h_{st.}(\theta_s - \theta_c) = N\delta q''' = q'''\ L \qquad (21)$$

Associated with the steady-state temperature profile and heat flow is a thermal-stress distribution, everywhere proportional to the difference between the average temperature and the local temperature, $\left[\left(\frac{\theta}{\theta_{s,o}}\right)_{av} - \left(\frac{\theta}{\theta_{s,o}}\right)\right]_o$, where the subscript o denotes steady state.

At the instant that the plate ceases to generate heat and is immersed into the coolant of temperature T_e, the temperature profile in the plate interior is that of the steady state and the heat flow throughout the plate and into the plate surface is the same as in the steady state. Accordingly, in a short time interval $\Delta\tau$ of the transient state each element of thickness δ (except possibly the element next to the plate surface) delivers outward all the heat received from segments more interior than itself, and in addition gives up $\delta q''' \Delta\tau$ heat units from its own volume. Inasmuch as the heat source has ceased to generate, the amount of heat $\delta q''' \Delta\tau$ given up by each segment is derived from the enthalpy of the segment itself; hence the temperature of every interior segment falls by $q''' \Delta\tau/\rho c$ degrees. The level of the interior temperature is thus decreased, but the temperature profile everywhere except near the surface is exactly as before. Accordingly, in the next time interval $\Delta\tau$ each interior segment, delivering heat at the rate determined by the local temperature profile (slope), again gives up $\delta q''' \Delta\tau$ heat units. It is seen that in a finite period of the transient state, before the plate interior can be affected by the cooling conditions at the surface, the interior temperature level must fall uniformly with time, at the rate $q'''/\rho c$ at every point; and the interior temperature profile must remain for a period exactly the same as in the steady state, regardless of the nature of the surface cooling conditions. It is accordingly seen that the only influence the midplane region can exert on the plate average temperature early in the transient state is to promote a decrease of the average temperature at the rate $q'''/\rho c$.

The major factor capable of modifying the rate at which the average temperature changes is the surface temperature, which is at all times responsive to the external cooling conditions. Hence the stress-shock criterion must emerge from a study of the temperature variations in the plate as determined by the three possible kinds of cooling condition at the plate surface, as follows:

If h is the constant heat-transfer coefficient in the transient state, $T_{s,o}$ the initial temperature of the plate surface, and T_e the constant temperature of the transient-state environment, the heat flow from plate to environment at the instant of immersion is

$$h(T_{s,o} - T_e) \equiv h\theta_{s,o} \tag{22}$$

and three possibilities exist:

$$\begin{aligned} &(a) \quad h\theta_{s,o} = h_{st} \, (\theta_{s,o} - \theta_c) \\ &(b) \quad h\theta_{s,o} < h_{st} \, (\theta_{s,o} - \theta_c) \\ &(c) \quad h\theta_{s,o} > h_{st} \, (\theta_{s,o} - \theta_c) \end{aligned} \tag{23}$$

Case (a). - In case (a) of equations (23) the plate surface is subject to initial transient-state cooling conditions equal to those of the steady state. Then the environment removes heat at exactly the rate of its delivery from plate midplane to surface, with no heat retention or extraordinary heat flow occurring anywhere in the plate. Then not only all interior temperatures, but the surface temperature as well, initially fall at the rate $q'''/\rho c$, and for a period the temperature profile throughout the plate (except for second-order changes at the surface) will remain the same as in the steady state. During this period of constant temperature profile the average plate temperature of course also falls at the rate $q'''/\rho c$; hence for this transient-state period the thermal stresses everywhere in the plate must, to first-order accuracy, be constant at the values of the steady state. Ultimately the surface temperature decreases enough that $h\theta_s$ is perceptibly smaller than $h\theta_{s,o}$. At this time

$$(N - 1)\ \delta q''' < h\theta_s < h\theta_{s,o} = N\delta q'''$$

and

$$-K \left[\frac{\partial \theta}{\partial (x/L)}\right]_s < -K \left[\frac{\partial \theta}{\partial (x/L)}\right]_{s,o} \tag{24}$$

The decrease of the temperature slope at the plate surface indicated by inequality (24) arises because the segment next to the surface can no longer deliver heat to the environment at the rate prevailing in the steady state and must retain a fraction of the enthalpy it formerly delivered from its own volume. Thus the temperature of the surface segment can no longer fall at the rate $q'''/\rho c$ by which the temperatures of nearby points are falling. Inasmuch as the average plate temperature reflects the change at the surface, the average temperature now also falls at a rate slower than $q'''/\rho c$, although not as slowly as the temperature of the surface segment itself. At this time both the midplane and surface temperatures approach more closely than formerly to the average temperature, and the thermal stresses at the two planes must decrease to below the steady-state values.

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It is thus seen that if the cooling conditions at the surface at the beginning of the transient state are equal to those of the steady state, the stresses everywhere in the plate must for a period remain effectively constant at the steady-state values; and that the first changes in surface and midplane stresses are decreases to values below those of the steady state.

In case (a) it is noteworthy, as for the surface segment just considered, that every segment of the formerly-heat-generating plate is at some time subject to an undiminishing heat inflow at one face and a diminishing sink strength at the other face, and that the segment adjusts the rate of delivery of its own enthalpy to satisfy the heat-flow conditions at both faces. In this manner the plate temperature profile flattens inwardly until the midplane is affected by the cooling conditions at the surface. Throughout the period of initial profile flattening the stresses at the surface and midplane decrease.

As the surface temperature decreases, the ability of the environment to remove heat also decreases; and inasmuch as the steadily decreasing sink strength of the environment always requires time to affect the average and midplane temperatures, the temperatures of the interior will always drop more rapidly than the surface temperature. Thus in case (a) the surface and midplane stresses will decrease steadily to zero.

<u>Case (b).</u> - In case (b) the plate initially delivers heat to the surface at a rate greater than the environment can remove; hence the temperature of one or more segments near the surface rises immediately on

immersion, while simultaneously the interior temperatures fall at the rate $q'''/\rho c$. The average temperature must fall at a rate slightly lower than $q'''/\rho c$; hence the midplane and surface temperatures both come closer to the average than in the steady state, and both stresses must immediately decrease. For the same reason as in case (a), that the average and midplane temperatures each feel and respond to the changing sink strength of the environment later than the surface does, the temperature profile must flatten steadily inward, and the midplane and surface stresses must decrease steadily to zero.

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Case (c). - In case (c) of relation (23) the environment removes more heat than the plate interior initially delivers to the surface. The excess heat derives from segments near the surface, and the surface temperature must fall at a rate exceeding $q'''/\rho c$. Hence the average plate temperature will also fall more rapidly than $q'''/\rho c$, but less rapidly than the surface temperature. The midplane temperature, as previously discussed, falls initially at $q'''/\rho c$. Hence both the surface and midplane stresses must immediately rise to values greater than those of the steady state.

Inasmuch as the effect of the surface cooling conditions must penetrate into a substantial portion of the interior before the average temperature can begin to fall at the same rate as the surface temperature, the surface stress will continue to rise for a period. Similarly, the rate of decrease of midplane temperature lags behind that of the average temperature, and the midplane stress will also continue to rise. Ultimately the diminishing sink strength of the environment as surface temperature decreases, coupled with increasing heat flow from the plate interior, will lead to a leveling off and subsequent steady decrease of the surface stress. At a later time the midplane stress will reach its peak and subsequently subside.

Thus, it is seen that for formerly-heat-generating plates it is only in case (c), $h\theta_{st} > h_{st}(\theta_{s,o} - \theta_c)$, that the surface temperature initially falls more rapidly than the interior temperatures; and only in this case do the shocks at the surface and midplane exceed the steady-state stresses. (It is of interest that even in case (c) there are intermediate segments for which the stress initially decreases, because the stress becomes zero whereever the local temperature equals the average; in case (c) the location of the average temperature moves closer to the surface on plate immersion.)

Linearly varying conductivity. - The foregoing discussion has been for plates of constant conductivity. The effect of linear increase in conductivity as temperature decreases is now briefly considered. In general, the local temperature profile may be expected to dominate over the conductivity variation in determining the heat flow. Hence the general behavior of variable-conductivity plates may be expected to be similar to that of constant-conductivity plates. If it is noted, however, that $\Delta K/K$ per degree decrease in temperature is greater in the interior than near the surface, it follows that in cases (a) and (b) the temperature profile will flatten more rapidly, and the midplane and surface stresses will decrease more rapidly, than if conductivity is constant.

In the case (c) the surface stress will always tend initially to increase to a value above that of the steady state. The initial stress variation at the midplane, however, depends on the degree to which $h\theta_{s,o}$ exceeds $h_{st}\ (\theta_{s,o} - \theta_c)$, because the favorable conductivity variation at the midplane may permit the temperature there to decrease at a rate equal to or exceeding that of the plate average temperature. Inasmuch as the surface stress always tends to increase when $h\theta_{s,o}$ exceeds $h_{st}\ (\theta_{s,o} - \theta_c)$, a calculation of the transient stresses will be necessary. For a fixed ratio $h\theta_{s,o}/h_{st}\ (\theta_{s,o} - \theta_c)$ greater than unity, the maximum surface and midplane shocks in the variable-conductivity plate will be a smaller multiple of the steady-state values than in the constant-conductivity plate; the favorable K variation in the plate interior reduces the rates at which both the surface and midplane temperatures depart from the plate average temperature.

<u>Shock criterion in terms of hL/K_e and $q'''L^2/2K_e\theta_{s,o}$.</u> - The condition of equation (23a),

$$h\theta_{s,o} = h_{st}\ (\theta_{s,o} - \theta_c) \tag{23a}$$

which is the condition separating dominance by steady-state thermal stress from dominance by transient thermal shock, may be expressed as a relation between the transient-state parameter hL/K_e and the steady-state parameter $q'''L^2/2K_e\theta_{s,o}$ of heat-generating plates, as follows: From equation (21)

$$h_{st}\ (\theta_{s,o} - \theta_c) = q'''L$$

Then when equation (23a) is satisfied,

$$h\theta_{s,o} = q'''L \tag{25}$$

On multiplying both sides of equation (25) by $L/K_e\theta_{s,o}$

$$\frac{hL}{K_e} = 2\left(\frac{q''' L^2}{2K_e\theta_{s,o}}\right) \tag{26}$$

Thus if equation (26) is satisfied the steady-state stress and maximum transient shock will be equal to each other. The shock will exceed the stress only if

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$$\frac{hL}{K_e} > 2\left(\frac{q''' L^2}{2K_e\theta_{s,o}}\right) \tag{27}$$

RESULTS AND DISCUSSION

Steady State

Temperatures. - Formulas for the steady-state temperature distributions in heat-generating plates of constant conductivity, and of conductivity that changes linearly with temperature, are derived in appendix B and have been indicated in the Analysis section (eqs. (9) and (10)). The temperature variable employed in the derived formulas is θ, defined as $T - T_e$, where T is the local plate temperature, and T_e is a constant reference temperature. (T_e is not necessarily the temperature of the steady-state cooling environment. The use of θ puts the steady-state temperatures into the same form as the one useful in describing the plate transient temperatures that would occur on sudden exposure of the plate to a cooling environment of temperature T_e.) In the derived formulas the temperatures are expressed in dimensionless form by use of the ratio $\theta/\theta_{s,o}$, a parameter appearing among the dimensionless groups shown in appendix D to govern the transient temperatures in plates of linearly varying conductivity. From the ratio $\theta/\theta_{s,o}$ the quantity $\left[(\theta/\theta_{s,o}) - 1\right]$ is readily obtained and, being equal to $(T - T_{s,o})/\theta_{s,o}$, gives a direct measure of the temperature difference between the surface and an interior point in the heat-generating plate.

The formulas derived in appendix B show that in heat-generating plates of constant conductivity the ratio $\theta/\theta_{s,o}$ depends on x/L and on $q''' L^2/2K\theta_{s,o}$; in heat-generating plates of linearly varying conductivity the ratio $\theta/\theta_{s,o}$ depends on x/L, $q''' L^2/2K_e\theta_{s,o}$, and $K_{s,o}/K_e$. Curves illustrative of the distribution of $\theta/\theta_{s,o}$ are presented in figure 2 in the form $\left[(\theta/\theta_{s,o}) - 1\right]$ against x/L. Parts (a) to (d) of figure 2 are for the four values 0.0025, 0.005, 0.010, and 0.020 of the parameter $q''' L^2/2K_e\theta_{s,o}$, and in each part of the figure plates of constant conductivity, and plates of $K_{s,o}/K_e$ values of 0.5 and 0.2, are represented. (It is noteworthy that low values of $K_{s,o}/K_e$ do not necessarily imply a large slope of conductivity with temperature. A moderate conductivity slope with temperature, combined either with a large temperature difference $\theta_{s,o}$, or with a low value of reference conductivity K_e, or both, can result in values of $K_{s,o}/K_e$ substantially lower than unity.) The curves of figure 2 reflect the relations derived in appendix B:

For plates of constant conductivity the quantity $\left[\left(\theta/\theta_{s,o}\right) - 1\right]$ is parabolic with x/L, and at any x/L increases in direct proportion to $q''' L^2/2K\theta_{s,o}$. For plates of linearly varying conductivity, and for appropriate combinations of $q''' L^2/2K_e\theta_{s,o}$ and $K_{s,o}/K_e$ as discussed in detail in appendix B, the quantity $\left[\left(\theta/\theta_{s,o}\right) - 1\right]$ is also essentially parabolic with x/L, and at any x/L increases very nearly directly with $q''' L^2/2K_e\theta_{s,o}$ and inversely with $K_{s,o}/K_e$; these characteristics are illustrated in figure 2(a) to (c) wherein all the plates are behaving very nearly as constant-conductivity plates, with conductivity equal to that at the temperature of the plate surface. Figure 2(d) illustrates, however, in the case $\frac{K_{s,o}}{K_e} = 0.2$, that for combinations of $q''' L^2/2K_e\theta_{s,o}$ and $K_{s,o}/K_e$ outside the suitable range, as discussed in appendix B, variable-conductivity plates differ to a significant degree from constant-conductivity plates. A measure of the accuracy with which the constant-conductivity relation approximates the variable-conductivity relation is discussed in appendix B, where it is also shown that if K decreases as temperature increases there is a limiting value beyond which the parameter $q''' L^2/2K_e\theta_{s,o}$ cannot increase.

Stresses. - Relations have been indicated in the Analysis section (eqs. (13), (14), (17), and (18)) and in appendix C for the steady-state stresses in heat-generating plates of constant conductivity, and of conductivity that decreases linearly with temperature as temperature increases. Note has also been made in the Analysis section that in heat-generating plates equally cooled on both surfaces the greatest tensile stress occurs at the plate surfaces and the greatest compressive stress occurs at the plate midplane. Values of dimensionless steady-state surface stress $\sigma^*_{s,o}$, and midplane stress $\sigma^*_{m,o}$, in heat-generating plates of constant and of linearly varying thermal conductivity are shown in figure 3 for values of $q''' L^2/2K_e\theta_{s,o}$ from 0 to 0.0145 and for values of $K_{s,o}/K_e$ from 0 to 0.15. Figure 3 reflects the stress formulas derived in the Analysis section and in appendix C:

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For plates of constant conductivity both $\sigma_{s,o}^{*}$ and $\sigma_{m,o}^{*}$ increase in direct proportion to $q''' L^2/2K\theta_{s,o}$. For plates of conductivity that decreases linearly with temperature the stresses are at first very nearly directly proportional to $q''' L^2/2K_e\theta_{s,o}$ and inversely proportional to $K_{s,o}/K_e$. For the combinations of $q''' L^2/2K_e\theta_{s,o}$ and $K_{s,o}/K_e$ for which these direct and inverse proportionalities occur the plate is behaving as one having a constant conductivity equal to that at the temperature of the plate surface. The range of $q''' L^2/2K_e\theta_{s,o}$ in which σ^* is linear with $q''' L^2/2K_e\theta_{s,o}$ depends on the value of $K_{s,o}/K_e$, and the range increases as $K_{s,o}/K_e$ increases. Thus for $K_{s,o}/K_e$ of 0.15, curvature away from a straight line begins to occur at $q''' L^2/2K_e\theta_{s,o}$ of 0.003, while for $K_{s,o}/K_e$ above 0.375 linearity exists over the range of $q''' L^2/2K_e\theta_{s,o}$ (0 to 0.0145) shown in figure 3. For each $K_{s,o}/K_e$ smaller than unity there is a value of $q''' L^2/2K_e\theta_{s,o}$ above which the variation of σ^* ceases to be linear; whenever such curvature occurs the inverse proportionality between σ^* and $K_{s,o}/K_e$ also ceases to be valid. (It is noteworthy that when σ^* is linear with $q''' L^2/2K_e\theta_{s,o}$, σ^* is directly proportional to plate thickness, provided that $h_{st}(\theta_{s,o} - \theta_c)$, the heat dissipation per unit surface area, is constant. The constancy of $h_{st}(\theta_{s,o} - \theta_c)$ implies that, as plate thickness increases, the internal heat-generation rate q''' is decreasing as $1/L$.)

Transient State

Governing dimensionless parameters. - If a plate of arbitrary steady-state temperature distribution is suddenly immersed in a cooling environment of constant temperature T_e the subsequent temperature and thermal stress histories of the plate depend on the plate initial temperature distribution; on T_e; on the transient-state heat-transfer coefficient; on plate thickness; and on plate thermal diffusivity, which may vary with temperature. In order that a systematic study and presentation of transient temperatures and stresses may be made, a knowledge of the parameters that govern the transient state is necessary.

General transient-state parameters for variable-conductivity plates: In appendix E it is shown that if the conductivity of a plate material varies linearly with temperature in the range between plate and environment temperatures, the parameters governing the transient-temperature ratio $\theta/\theta_{s,o}$ at x/L (and hence the dimensionless stress σ^* at x/L), are hL/K_e, $\alpha_e\tau/L^2$, and $K_{s,o}/K_e$. The first two parameters are the Biot and Fourier numbers, respectively, with plate thermal conductivity evaluated at the constant temperature T_e of the cooling environment. The parameter $K_{s,o}/K_e$ is the ratio of plate conductivity at the initial surface temperature $T_{s,o}$ to the plate conductivity at temperature T_e. The indicated three parameters, then, are suitable for the systematic study and presentation of transient temperatures in plates of linearly varying conductivity. (For a fixed value of $K_{s,o}/K_e$ the Biot and Fourier numbers can equally well be based on $K_{s,o}$ instead of K_e; hence $hL/K_{s,o}$, $\alpha_{s,o}\tau/L^2$, and $K_{s,o}/K_e$ could equally well be regarded as the governing parameters.)

It is noteworthy that although the thermal conductivity of a plate 2920
material may vary substantially with temperature the constant conductivities K_e or $K_{s,o}$ in the Biot and Fourier numbers, together with the constant ratio $K_{s,o}/K_e$, are completely adequate to define the transient temperatures, for a prescribed initial temperature distribution. It is also noteworthy that plate materials having different conductivities at T_e, and different conductivity slopes with temperature, will have equal transient values of $\theta/\theta_{s,o}$ at x/L if the values of hL/K_e, $\alpha_e\tau/L^2$, and $K_{s,o}/K_e$ are equal in the two plate systems (provided, of course, that the initial distributions of $\theta/\theta_{s,o}$ with x/L are also equal. The requirement of similar steady-state temperature distributions signifies, for example, that plates having uniform initial $\theta/\theta_{s,o}$ can be compared only with other plates having uniform initial $\theta/\theta_{s,o}$; while plates having linear, or parabolic, initial $\theta/\theta_{s,o}$ distributions can only be compared, respectively, with other plates having linear, or parabolic, initial distributions).

Parameters for heat-generating plates: The derived steady-state formulas show that for heat-generating plates of constant conductivity the steady-state value of $\theta/\theta_{s,o}$ at x/L depends on $q''' L^2/2K\theta_{s,o}$, and for heat-generating plates of linearly varying conductivity the steady-state value of $\theta/\theta_{s,o}$ at x/L depends on $q''' L^2/2K_e\theta_{s,o}$ and $K_{s,o}/K_e$. Thus if a plate generates heat uniformly in the steady state, and then ceases to generate heat and is simultaneously immersed in a cooling environment of temperature T_e, the transient-temperature ratio $\theta/\theta_{s,o}$ at x/L is completely determined if a specification is made of $q''' L^2/2K_e\theta_{s,o}$, $K_{s,o}/K_e$, hL/K_e, and $\alpha_e\tau/L^2$. (For plates of constant conductivity, $q''' L^2/2K_e\theta_{s,o}$ is, of course, the same as $q''' L^2/2K\theta_{s,o}$, and $K_{s,o}/K_e$ equals unity.)

Criterion for Dominance of Thermal Stress or Shock and Typical Variations of Transient Temperature and Stress

In the Analysis section it was indicated that the relative magnitudes of steady-state stress and transient shock depend on the relative magnitudes of $h\theta_{s,o}$ and $h_{st}(\theta_{s,o} - \theta_c)$; or equivalently, for heat-generating plates, on the comparative magnitudes of hL/K_e and $q''' L^2/2K_e\theta_{s,o}$. The results of temperature and stress calculations for several values of the stated parameters are presented in figures 4 to 9; figures 4 to 6 are for plates of constant conductivity, and figures 7 to 9 are for plates of $K_{s,o}/K_e$ equal to 0.151. In all cases the plates are assumed to have generated heat uniformly throughout their interiors in the steady state, and are assumed to have ceased suddenly to generate heat and to have been immersed in a cooling environment of constant temperature T_e. All plates are assumed to be cooled equally on both faces, by constant heat-transfer coefficient h between plate and cooling environment.

Constant-conductivity plates, $hL/K = 2(q''' L^2/2K\theta_{s,o})$. - Figure 4 is for plates of constant conductivity, for a steady-state value of $q''' L^2/2K\theta_{s,o}$ equal to 0.01, and for a transient-state value of hL/K_e equal to 0.02. In figure 4(a), $\theta/\theta_{s,o}$ is shown against x/L for several values of $\alpha\tau/L^2$. For the calculation an interval of slightly less than 0.02 was employed for $\alpha\tau/L^2$, as satisfying the convergence criterion indicated in the Calculation Procedure of appendix D; most values of $\alpha\tau/L^2$ are omitted from the figure. In figure 4(b) the dimensionless surface stress σ_s^* and the dimensionless midplane stress σ_m^* are shown against $\alpha\tau/L^2$.

The temperatures and stresses in figure 4 illustrate the conclusions obtained in the Analysis for plates of constant conductivity, for $hL/K = 2\ (q''' L^2/2K\theta_{s,o})$: The temperature profile, and hence the stresses throughout the plate, remain for a period effectively constant at the steady-state values; and after the period of constancy the stresses at surface and midplane decrease.

Constant-conductivity plates, $hL/K < 2(q''' L^2/2K\theta_{s,o})$. - Figure 5 is for plates of constant conductivity, for a steady-state value of $q''' L^2/2K\theta_{s,o}$ equal to 0.01 and for a transient-state value of hL/K also equal to 0.01. Figure 5 illustrates the conclusions of the Analysis for the constant-conductivity case, $hL/K < 2(q''' L^2/2K\theta_{s,o})$: The temperatures near the surface initially rise, while the interior temperatures fall uniformly; hence the surface and midplane stresses immediately decrease to values below those of the steady state. The figure also shows that the temperature profile throughout the plate ultimately adjusts itself so that every segment transmits not only all the heat it receives, but also gives up very nearly the same amount of its own enthalpy as given by the other segments. When this adjustment occurs the temperature throughout the plate falls very nearly uniformly with time, as indicated by the curves for $\alpha\tau/L^2$ from 0.40 to 0.48. Inasmuch as this condition is like the one previously presented in figure 4 $(hL/K = 2q''' L^2/2K\theta_{s,o})$, the temperature and stress behaviors subsequent to $\alpha\tau/L^2$ of 0.48 will closely resemble the behavior of the previously presented case.

Constant-conductivity plates, $hL/K > 2(q''' L^2/2K\theta_{s,o})$. - Figure 6 is for plates of constant conductivity, for a steady-state value of $q''' L^2/2K\theta_{s,o}$ equal to 0.01, and for a transient-state value of hL/K equal to 0.03.

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Figure 6(a) shows that initially the temperatures near the surface fall more sharply than those of the interior, reflecting the fact that the cooling conditions at the surface are more severe than in the steady state. The temperature curve steepens progressively inward until, for the case shown, $\alpha\tau/L^2$ equals about 0.38, after which the shape of the temperature curve is such that each segment gives up very nearly the same enthalpy as every other segment; thereafter the temperature falls very nearly uniformly throughout the plate, until $\alpha\tau/L^2$ equals at least 0.937. The subsequent temperature history will closely resemble that for the case $hL/K = 2(q''' L^2/2K\theta_{s,o})$, illustrated in figure 4.

Figure 6(b) shows that at the instant of plate immersion σ_s^* and σ_m^* both increase, σ_s^* more rapidly than σ_m^*. At $\alpha\tau/L^2$ of about 0.4, for the case shown, both stresses have slowed their rate of increase almost to zero. The stress σ_m^* is effectively at its peak value, 0.0099, a 50 percent increase above the 0.0066 value of the steady state. At $\alpha\tau/L^2$ of 0.937, σ_s^* has passed its peak value and has started to decrease, while σ_m^* is still increasing very slowly prior to attaining its own peak, which will differ only slightly from the value at $\alpha\tau/L^2$ of 0.937. Thus the maximum transient value of σ_m^*, like σ_s^*, is about 50 percent higher than the steady-state value.

The temperature and stress variations shown in figure 6 illustrate the conclusion in the Analysis that when hL/K exceeds $2(q''' L^2/2K\theta_{s,o})$ the surface temperature will initially fall more sharply than the uniformly falling interior temperatures, and hence that the surface and mid-plane stresses will increase initially to values above those of the steady state before decreasing to zero.

Variable-conductivity plates, $hL/K_e = 2(q''' L^2/2K_e\theta_{s,o})$. - Figure 7 is for plates of conductivity that decreases linearly as temperature increases; the conductivity variation is represented by the ratio $K_{s,o}/K_e = 0.151$. The figure corresponds to a steady-state value of $q''' L^2/2K_e\theta_{s,o}$ equal to 0.01 and a transient-state value of hL/K equal to 0.02. In figure 7(a), $\theta/\theta_{s,o}$ is shown against x/L for several values of $\alpha_e\tau/L^2$. For the calculation an interval of 0.0756 was used for $\alpha_e\tau/L^2$, as required by the convergence criterion indicated in the Calculation Procedure of appendix D. The solid curves represent temperatures computed with the transient-state conductivity variation taken into account. The dashed curves represent temperatures computed when, with the actual steady-state temperature distribution of the variable-conductivity plate taken as the starting distribution, the conductivity variation in the transient state was neglected and the initial surface conductivity was employed for the entire transient-state calculation. Figure 7(b) shows dimensionless surface and midplane stresses, σ_s^* and σ_m^*, respectively, and the solid and dashed curves have the same significance as indicated for figure 7(a).

Figure 7(a) shows that the steady-state temperature distribution $(\alpha_e\tau/L^2 = 0)$ is much steeper than the steady-state distribution shown in figure 4(a) for constant-conductivity plates. The difference in steepness for equal values of $q''' L^2/2K_e\theta_{s,o}$ arises from the much lower plate surface and interior conductivities when $K_{s,o}/K_e$ equals 0.151 than when $K_{s,o}/K_e$ equals unity. Figure 7(a) shows that although the surface cooling conditions at the start of the transient state are equal to those of the steady state, the interior temperatures fall more rapidly than the temperatures near the surface. The difference in cooling rates of the interior and surface regions arises because although the conductivity increases everywhere in the plate as the temperature falls, the value of $\Delta K/K$ per degree temperature drop is greater in the interior than near the surface.

The dashed lines of figure 7(a) show that the computation based on surface conductivity, a value not much in error for the plate region near the surface, but noticeably higher than the interior conductivities, leads to an initial overestimation of the heat flow from the plate interior, and to a corresponding overestimation of the early rate of temperature drop in the plate interior. From the relation $h\theta_s = q''$ it is seen that when the net heat flow is overestimated θ_s must be higher than when q'' is accurately evaluated. Hence, for the time span under consideration $(\alpha_e\tau/L^2 \leq 1.2)$, the surface temperatures on the dashed curves are higher than those of the more accurate solid curves. The comparative long-term temperature histories, after the conductivity everywhere in the plate exceeds the initial surface value, would require further calculation to establish.

Figure 7(b) shows that as in the case of constant-conductivity plates when $hL/K = 2(q''' L^2/2K\theta_{s,o})$, the surface stress is for a short period effectively constant at the steady-state value, and then slowly decreases. (It may be noted that the steady-state stress is much higher than for the constant-conductivity plate at the same value of $q''' L^2/2K_e\theta_{s,o}$, because of the steep temperature variation in the case $K_{s,o}/K_e = 0.151$, as previously indicated. The steady-state stress is slightly lower than the value shown in figure 3 for $K_{s,o}/K_e = 0.150$ because of the slightly better conductivity represented by $K_{s,o}/K_e = 0.151$.) Unlike the case of constant-conductivity plates, the midplane stress decreases immediately upon plate immersion, reflecting the approach of midplane and average temperatures as the midplane conductivity and temperature drop increase. For values of $\alpha_e\tau/L^2$ greater than 1.2, both σ_s^* and σ_m^* will slowly but steadily decrease to zero. 2920

The dashed curves of figure 7(b) show that because of the overestimated flattening of the temperature curve exhibited in figure 7(a), the stresses at both surface and midplane are initially underestimated if computations are based on constant conductivity equal to that at the initial surface temperature. Thus for this case, at $\alpha_e\tau/L^2$ of 1.0, the surface stress is underestimated about 8 percent, and the midplane stress about 20 percent. Neither underestimation is very significant, however, if the steady-state stresses are tolerable, because as indicated in the Analysis, and as indicated also by figure 7(b), the transient stresses at surface and midplane do not exceed the steady-state values when hL/K_e equals $2(q''' L^2/2K\theta_{s,o})$. (A conservative estimate of the stresses would result if the less laborious computation with constant conductivity were based on conductivity equal to that at the initial midplane temperature.)

<u>Variable-conductivity plates, $hL/K_e < 2(q''' L^2/2K_e\theta_{s,o})$.</u> - Figure 8 is for plates of conductivity variation represented by $K_{s,o}/K_e = 0.151$. The figure corresponds to values of $q''' L^2/2K_e\theta_{s,o}$ and hL/K_e both equal to 0.01. As a whole, the transient temperatures (fig. 8(a)) and the transient stresses (fig. 8(b)) exhibit behaviors very similar to those in the analogous case of constant-conductivity plates indicated in figure 5. The modifications introduced by the conductivity variation are basically the same as those already discussed in figure 7. Also, as in figure 7, the use of constant conductivity equal to the initial surface value, instead of the variable local values, initially overestimates the cooling rate of the plate interior and underestimates the cooling rate of the surface region; hence underestimates the early surface and midplane stresses.

<u>Variable-conductivity plates, $hL/K_e > 2(q''' L^2/2K_e\theta_{s,o})$.</u> - Figure 9 is for plates having a linear conductivity variation represented by $K_{s,o}/K_e = 0.151$. The figure is for $q''' L^2/2K_e\theta_{s,o}$ equal to 0.01 and

for hL/K_e equal to 0.03; thus $hL/K_e > 2(q''' L^2/2K_e\theta_{s,o})$ and the initial transient-state surface cooling conditions are more severe than those of the steady state. For the temperature and stress calculations shown in figure 9, an interval of slightly less than 0.0747 was used for $\alpha_e\tau/L^2$. The solid and dashed curves have the same significance as indicated in figure 7.

2920 The solid lines of figure 9(a) show that in the early transient state ($\alpha_e\tau/L^2$ smaller than about 0.3) the midplane and surface regions cool more rapidly than the intermediate plate region. The midplane region cools relatively rapidly because $\Delta K/K$ per degree temperature decrease is relatively high. The surface region cools rapidly because the environment extracts heat at a rate greater than can be supplied by the plate interior, the net heat flow from which is primarily governed by the temperature profile and only secondarily by the conductivity variation. By the time $\alpha_e\tau/L^2$ is about 0.3, however, the plate temperature profile has adjusted itself so that each segment cools at very nearly the same rate as every other segment, a transient-state adjustment which was also noted in the cases of constant-conductivity plates herein considered. The dashed curves of figure 9(a) show, as in previous figures, that the initial surface conductivity overestimates the net heat flow from the plate, mainly from overestimating the cooling rate of the midplane region.

Figure 9(b) shows, as in the case of constant-conductivity plates, that when $hL/K > 2(q''' L^2/2K_e\theta_{s,o})$ the surface stress rises on immersion of the plate into the transient-state environment. The surface stress reaches a maximum and then decreases. It is noteworthy that the maximum percentage rise of surface stress for $K_{s,o}/K_e$ of 0.151 is about 20 percent of the steady-state value, compared with the 50-percent increase noted for constant-conductivity plates having the same $q'''L^2/2K_e\theta_{s,o}$ and hL/K_e (fig. 6). The favorable effect of an increasing conductivity on the transient-state stresses was indicated in the Analysis. As in previous figures, the constant-conductivity calculation, represented by the dashed curves of figure 9(b), underestimates the surface stress, but for the conditions of figure 9 the effect is less than 5 percent of the exact value.

The midplane stress is seen in figure 9(b) to undergo an initial decrease, because of the excess local-temperature decrease as compared with the average plate-temperature decrease. The midplane stress subsequently increases, however, to a value about 10 percent above that of the steady-state value, as compared with approximately 50-percent increase noted for σ_m^* in the constant-conductivity plates of figure 6. (At $\alpha_e\tau/L^2$ of about 1.0 in figure 10, the midplane stress is still increasing very slowly, but its achievement of a maximum and subsequent decrease are imminent.) The constant-conductivity calculation underestimates the midplane stress by about 18 percent of the exact value and changes the time at which the stress starts to rise after its initial decrease. As previously indicated, advantage may be taken of the smaller labor involved in the constant-conductivity calculation, while obtaining a conservatively high approximation of the stresses, by employing the conductivity corresponding to the initial temperature of the plate midplane.

Typical variations. - It is seen from figures 4 to 9 for plates of constant conductivity, and of conductivity which decreases linearly as temperature increases, that the three possible variations of early transient-state temperature and stress are accounted for by exploration of the combinations (a) $hL/K_e = 2(q''' L^2/2K_e\theta_{s,o})$, (b) $hL/K_e < 2(q''' L^2/2K_e\theta_{s,o})$, and (c) $hL/K_e > 2(q''' L^2/2K_e\theta_{s,o})$. Hence it is seen that the comparison of hL/K_e and $2(q''' L^2/2K_e\theta_{s,o})$ is a suitable criterion for determining whether the steady stress or the transient shock will be the greater. It may also be seen that whereas values of $q''' L^2/2K_e\theta_{s,o}$ and hL/K_e other than those herein considered will lead to magnitudes of $\theta/\theta_{s,o}$ and σ^* different from those shown in the figures, the nature of the transient-state variations will be similar to those of either case (a), (b), or (c). That is, the curves of figures 4 to 9 are typical cooling and shock curves for formerly-heat-generating plates of constant conductivity, and of conductivity that decreases linearly as temperature increases. 2920

Conclusions. - From the discussion of figures 2 to 9 the following conclusions may be drawn:

1. The steady-state temperatures and stresses increase as $q''' L^2/2K_e\theta_{s,o}$ increases and as $K_{s,o}/K_e$ decreases.

2. In the early transient state the deep interior of the plate cools at a rate determined entirely by the steady-state local temperature profile and by the local conductivity variations as the early cooling proceeds; that is, the cooling rate at the midplane early in the transient state is independent of the cooling conditions at the surface. The surface region, on the other hand, cools at a rate determined simultaneously by the heat flow from the interior, the strength of the external sink, and the local conductivity variation. The parameters governing the transient temperature ratio $\theta/\theta_{s,o}$ at any point x/L are $q''' L^2/2K_e\theta_{s,o}$, $K_{s,o}/K_e$, hL/K_e, and $\alpha_e\tau/L^2$.

3. The severity of the thermal shock, as compared with the steady-state thermal stress, may be satisfactorily evaluated by a comparison between the initial surface cooling condition of the transient state

$h\theta_{s,o}$ and the steady-state surface cooling condition $h_{st}(\theta_{s,o} - \theta_c)$. For formerly-heat-generating plates as herein described, the stress-shock comparison may be made, equivalently, by a comparison of the Biot number hL/K_e and the steady-state quantity $2(q''' L^2/2K_e\theta_{s,o})$. The thermal shocks at both midplane and surface exceed the steady-state stress only when $h\theta_{s,o}$ exceeds $h_{st}(\theta_{s,o} - \theta_c)$, or equivalently, when hL/K_e exceeds $2(q''' L^2/2K_e\theta_{s,o})$.

4. If the conductivity of the plate material increases linearly as the temperature decreases, the maximum shocks of plate surface and midplane, when $hL/K_e > 2(q''' L^2/2K_e\theta_{s,o})$, are a smaller multiple of the steady-state values than when the conductivity of the plate material is constant.

5. Use of the initial surface conductivity for calculation of the transient temperatures and stresses initially overestimates the cooling rate and can substantially underestimate the early transient stresses. A conservative constant-conductivity calculation would be obtained by use of the steady-state conductivity of the plate midplane.

6. The conclusions derived from figures 4 to 9 for the transient state substantiate the indications obtained from the analysis presented herein.

RESULTS AND CONCLUSIONS

1. Working formulas are derived for the steady-state temperatures and thermal stresses in heat-generating infinite plates of constant conductivity, and of conductivity that decreases linearly with temperature as the temperature increases, for the case in which both faces of the plate are equally cooled. In addition to exact formulas for the variable-conductivity plates, simpler approximate formulas are derived, and criteria for their applicability are indicated.

2. It is shown that the plate surfaces are always under the greatest tension, and the midplane is under the greatest compression, of all planes in the plate.

3. A criterion is indicated for determining the surface cooling conditions under which the transient thermal shocks at the surface and midplane will be smaller than, equal to, or greater than the steady-state thermal stresses at those planes. The shocks exceed the steady-state stresses only when the initial surface-cooling conditions of the transient state are more severe than the surface-cooling conditions of the steady state. The criterion is expressed for heat-generating plates by a simple relation between a steady-state heat-transfer parameter of the heat-generating plate and the Biot number of the transient state.

4. The dimensionless parameters governing the transient temperatures and thermal stresses in materials of linearly varying conductivity are derived by a similarity study of the conduction equation and boundary conditions of the transient state. A numerical technique for solving the transient-state equations is indicated in detail. The method is employed to obtain numerical values of the transient temperatures and stresses for a variety of parameter combinations, selected to test the thermal shock criterion. The results of the calculations substantiate the shock criterion.

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5. As a result of the study, an insight is obtained into the steady and transient temperature and stress mechanisms in heat-generating plates of constant conductivity and of conductivity that varies linearly with temperature.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
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APPENDIX A

SYMBOLS

The following symbols are used in this report:

A	symbol for the quantity $2(K_e/K_{s,o})(K_e/K_{s,o} - 1)(q''' L^2/2K_e\theta_{s,o})$
c	specific heat of plate material, Btu/lb-^{o}F
E	modulus of elasticity, lb/sq in.
h	heat transfer coefficient in transient state, Btu/sec-ft^2-^{o}F
h_{st}	heat transfer coefficient in steady state, Btu/sec-ft^2-^{o}F
K	thermal conductivity of plate material, Btu/sec-ft-^{o}F
K_e	thermal conductivity of plate material at temperature T_e, Btu/sec-ft-^{o}F
K_s	thermal conductivity of plate material at surface temperature, Btu/sec-ft-^{o}F
L	half-thickness of plate, ft
m	coefficient in equation for thermal conductivity, $(K = mT + n)$
N	number of thin segments into which plate thickness is divided
n	constant in equation for thermal conductivity, $(K = mT + n)$
q''	heat flow per square foot of plate area normal to direction of heat flow, Btu/sec-ft^2
q'''	heat generation per unit volume, Btu/sec-ft^3
T	temperature, ^{o}F
x	coordinate, distance of plane from plate midplane, ft
α	thermal diffusivity of plate material, ft^2/sec
β	coefficient of linear thermal expansion, ft/ft-^{o}F
δ	thickness of segment in a plate that has been divided into N equal segments, ft

θ difference between local plate temperature T, and constant reference temperature T_e, $\theta \equiv T - T_e$, $^{\circ}F$

ν Poisson's ratio

ρ weight density of plate material, lb/ft^3

σ normal stress, lb/sq in.

σ^* dimensionless stress parameter, $\sigma^* \equiv \sigma(1-\nu)/\beta E \theta_{s,o}$ 2920

τ time, sec

Parameters:

hL/K Biot number

hL/K_e Biot number with plate conductivity evaluated at temperature T_e

$\alpha\tau/L^2$ Fourier number

$\alpha_e\tau/L^2$ Fourier number with plate conductivity evaluated at temperature T_e

Subscripts:

av average

c coolant of the steady state

e environment of the transient state

f fictive layer

i integral multiple of unit distance (Δx), $i = 1, 2, 3, \ldots$, and designates distance of local plane from prescribed reference plane

$i\pm1/2$ multiple of unit distance, (Δx)

j integral multiple of unit time-increment $(\Delta\tau)$, $j = 1, 2, 3, \ldots$

m midplane of plate

max maximum value taken on by variable

min	minimum value taken on by variable
o	steady state
s	surface of plate
y	direction
z	direction

APPENDIX B

TEMPERATURES

The temperatures in a plate are governed by the conservation of heat in the plate interior and at the surface. Heat conservation in the interior of a heat-generating plate of variable conductivity is expressed by the familiar equation

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$$\rho c\left(\frac{\partial T}{\partial \tau}\right) \doteq \frac{\partial}{\partial x}\left(\frac{K\, \partial T}{\partial x}\right) + q''' \qquad \text{(B1)}$$

If T_e is an arbitrarily assignable constant, and θ is defined as

$$\theta \equiv T - T_e \qquad \text{(B2)}$$

equation (B1) may be rewritten

$$\rho c\left(\frac{\partial \theta}{\partial \tau}\right) = \frac{\partial}{\partial x}\left(\frac{K\, \partial \theta}{\partial x}\right) + q''' \qquad \text{(B3)}$$

Heat conservation at a surface where the plate transfers heat to a coolant is expressed by the standard relation

$$q'' = -K_s\left(\frac{\partial \theta}{\partial x}\right)_s = h(\theta_s - \theta_c) \qquad \text{(B4)}$$

where

$$\theta_s \equiv T_s - T_e \qquad \text{(B5)}$$

and

$$\theta_c \equiv T_c - T_e \qquad \text{(B6)}$$

In the following discussion the plate is assumed to be equally cooled on both faces, in both the steady and transient states.

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Steady State

Heat-generating plate of constant conductivity. - In the steady state the left member of equation (B3) vanishes, and if the plate conductivity is constant, equation (B3) simplifies to

$$K \frac{d^2\theta}{dx^2} = -q''' \tag{B7}$$

with the condition $q''' > 0$ because the plate generates heat. Direct integration of equation (B7) and successive application of the conditions that at $x = 0$, $\frac{d\theta}{dx} = 0$, and at $x = \pm L$, $\theta = \theta_{s,o}$, lead to the familiar form

$$\theta = \theta_{s,o} + \frac{q''' L^2}{2K}\left[1 - \left(\frac{x}{L}\right)^2\right] \tag{B8}$$

or

$$\frac{\theta}{\theta_{s,o}} = 1 + \frac{q''' L^2}{2K\theta_{s,o}}\left[1 - \left(\frac{x}{L}\right)^2\right] \tag{B9}$$

The difference between local and surface temperatures is

$$\frac{\theta-\theta_{s,o}}{\theta_{s,o}} = \frac{q''' L^2}{2K\theta_{s,o}}\left[1 - \left(\frac{x}{L}\right)^2\right]$$

Heat-generating plates of variable conductivity. - If the plate conductivity varies linearly with temperature, the equation for conductivity may be written

$$K = mT + n \equiv m\theta + (mT_e + n) = m\theta + K_e \tag{B10}$$

Then equation (B3) becomes for the steady state

$$\frac{d}{dx}\left[(m\theta + K_e)\frac{d\theta}{dx}\right] = -q''' \tag{B11}$$

Direct integration yields

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$$(m\theta + K_e)\frac{d\theta}{dx} = -q'''\,x + C_1$$

Inasmuch as $\frac{d\theta}{dx} = 0$ at $x = 0$, $C_1 = 0$. A second integration yields

$$\frac{1}{2}\,m\theta^2 + K_e\theta = -\frac{q'''\,x^2}{2} + C_2$$

At the surfaces $x = \pm L$, $\theta = \theta_{s,o}$; therefore,

$$C_2 = \frac{1}{2}\,m\theta_{s,o}^2 + K_e\theta_{s,o} + \frac{q'''\,L^2}{2}$$

Thus

$$\frac{1}{2}\,m\theta^2 + K_e\theta - \left\{\frac{1}{2}\,m\theta_{s,o}^2 + K_e\theta_{s,o} + \frac{q'''\,L^2}{2}\left[1 - \left(\frac{x}{L}\right)^2\right]\right\} = 0 \tag{B12}$$

From equation (B10),

$$K_{s,o} = m\theta_{s,o} + K_e$$

It is now assumed that $K_e \neq K_{s,o}$, or $K_{s,o}/K_e \neq 1$, which is equivalent to assuming that the arbitrarily prescribable reference temperature T_e is taken different from the steady-state surface temperature $T_{s,o}$, and the quantity $\theta_{s,o} \equiv (T_{s,o} - T_e) \neq 0$. This assumption is made to permit solution for the ratio $\theta/\theta_{s,o}$, and is not necessary if $\theta_{s,o}$ is not desired to enter in the denominator of the solution. Then

$$m = \frac{K_{s,o} - K_e}{\theta_{s,o}} \tag{B13}$$

When equation (B13) is substituted into (B12) and solved for $\theta/\theta_{s,o}$ by the quadratic formula, the result is

$$\frac{\theta}{\theta_{s,o}} = \frac{1 - \frac{K_{s,o}}{K_e}\sqrt{1 - 2\left(\frac{K_e}{K_{s,o}}\right)\left(\frac{K_e}{K_{s,o}} - 1\right)\left(\frac{q''' L^2}{2K_e\theta_{s,o}}\right)\left[1 - \left(\frac{x}{L}\right)^2\right]}}{1 - \frac{K_{s,o}}{K_e}} \tag{B14}$$

Thus $\theta/\theta_{s,o}$ in the steady state is governed by x/L, $q''' L^2/2K_e\theta_{s,o}$, and $K_{s,o}/K_e$, and it is recalled that equation (B14) has meaning when $q''' \neq 0$, and $K_{s,o}/K_e \neq 1$. Equation (B14) is not inconvenient to use; nevertheless an approximate relation is instructive, and is discussed as follows:

Approximate formula for $\theta/\theta_{s,o}$ in variable-conductivity plates. - An approximate formula for $\theta/\theta_{s,o}$ may be obtained from equation (B14) as follows. The equation is of the form

$$\frac{\theta}{\theta_{s,o}} = \frac{1 - \frac{K_{s,o}}{K_e}\sqrt{1 - \phi}}{1 - \frac{K_{s,o}}{K_e}} \tag{B15}$$

where

$$\phi = 2\left(\frac{K_e}{K_{s,o}}\right)\left(\frac{K_e}{K_{s,o}} - 1\right)\left(\frac{q''' L^2}{2K_e\theta_{s,o}}\right)\left[1 - \left(\frac{x}{L}\right)^2\right] \tag{B16}$$

On restricting considerations to values of $\phi \geq 0$ (i.e., plates for which K decreases as temperature increases), on approximating $\sqrt{1 - \phi}$ by $(1 - \phi/2)$, and on substituting for ϕ in equation (B15),

$$\frac{\theta}{\theta_{s,o}} \simeq \frac{1 - \frac{K_{s,o}}{K_e}\left(1 - \frac{\phi}{2}\right)}{1 - \frac{K_{s,o}}{K_e}} \equiv \frac{1 - \frac{K_{s,o}}{K_e}\left\{1 - \left(\frac{K_e}{K_{s,o}}\right)\left(\frac{K_e}{K_{s,o}} - 1\right)\left(\frac{q''' L^2}{2K_e\theta_{s,o}}\right)\left[1 - \left(\frac{x}{L}\right)^2\right]\right\}}{1 - \frac{K_{s,o}}{K_e}} \tag{B17}$$

On carrying through the operations in the right member of equation (B17), 2920

$$\frac{\theta}{\theta_{s,o}} \simeq 1 + \frac{\left(\frac{q''' L^2}{2K_e\theta_{s,o}}\right)}{\left(\frac{K_{s,o}}{K_e}\right)}\left[1 - \left(\frac{x}{L}\right)^2\right] \tag{B18}$$

Alternately,

$$\frac{\theta}{\theta_{s,o}} \simeq 1 + \frac{q''' L^2}{2K_{s,o}\theta_{s,o}}\left[1 - \left(\frac{x}{L}\right)^2\right] \tag{B19}$$

Equation (B18) shows that for the ranges of $K_{s,o}/K_e$ and ϕ in which equation (B18) is a satisfactory approximation, $(\theta/\theta_{s,o}) - 1$ varies directly with $q''' L^2/2K_e\theta_{s,o}$ and inversely with $K_{s,o}/K_e$, and that the shape of $\theta/\theta_{s,o}$ with x/L is a parabola. Equation (B19), which is simply a reduced form of equation (B18), is of the same form as the equation for a plate of constant conductivity, the conductivity being that at the temperature of the plate surface. Thus equation (B19) contains no provision for conductivity variation in the plate interior, and if the conductivity decreases as temperature increases, equation (B19) will underestimate the plate interior temperatures (while if the conductivity increases with temperature the equation will overestimate the interior temperatures). Equation (B19) also fails to indicate that a limit exists for the feasible value of $q''' L^2/2K_e\theta_{s,o}$ if conductivity decreases as temperature increases, as is shown in the next section.

In order to consider the range of ϕ in which equation (B19) is a suitable approximation, equation (B15) may first be written

$$\frac{\theta}{\theta_{s,o}} = 1 + \frac{\left(\frac{K_{s,o}}{K_e}\right)\left(1 - \sqrt{1 - \phi}\right)}{1 - \frac{K_{s,o}}{K_e}} \tag{B20}$$

A comparison of equations (B18) and (B20) shows that the approximation resides in the quantity $\left[\frac{\theta}{\theta_{s,o}} - 1\right]$, which is equal to $\left(\frac{T-T_{s,o}}{\theta_{s,o}}\right)$ and measures the temperature rise in the plate interior. On replacing $\left[\frac{\theta}{\theta_{s,o}} - 1\right]$ by $\left(\frac{T-T_{s,o}}{\theta_{s,o}}\right)$, on denoting the values in equations (B20) and (B17) by the respective subscripts "exact" and "approximate," and on dividing equation (B20) by (B17), the result is

$$\frac{(T-T_{s,o})_{exact}}{(T-T_{s,o})_{approximate}} = \frac{2}{\phi}\left(1 - \sqrt{1 - \phi}\right) \tag{B21}$$

The function of ϕ in equation (B21) has the limit unity as ϕ becomes zero $(x/L \rightarrow \pm 1)$, and increases steadily to the value 2 as ϕ increases to its upper limit $(\phi = 1)$. Thus whereas equation (B19) necessarily gives the correct surface temperature, it underestimates the plate midplane temperature by the greatest amount, and the exact difference between midplane and surface temperatures may be twice as great as the value given by equation (B19) if ϕ is unity at the midplane. (From eq. (B16) it may be seen that ϕ increases in the direction of the plate interior.) If the error in $(T-T_{s,o})$ is defined by ε in the relation

$$\frac{(T-T_{s,o})_{exact}}{(T-T_{s,o})_{approximate}} \equiv (1 + \varepsilon) = \frac{2}{\phi}\left(1 - \sqrt{1 - \phi}\right) \tag{B22}$$

it is seen on solving for ϕ that the error ε occurs when

$$\phi = \frac{4\varepsilon}{(1 + \varepsilon)^2} \tag{B23}$$

Thus, for example, $(T-T_{s,o})_{exact} = 1.1\ (T-T_{s,o})_{approximate}$ when $\phi = 0.33$, and $(T-T_{s,o})_{exact} = 1.2\ (T-T_{s,o})_{approximate}$ when $\phi = 0.56$.

Limitation on $q''' L^2/2K_e\theta_{s,o}$ when conductivity decreases with temperature. - Inasmuch as $\theta/\theta_{s,o}$ is always real the expression under the radical in equation (B14) must never be less than zero. If the plate conductivity decreases as temperature increases a value of $K_{s,o}/K_e$ smaller than unity is associated with a positive value of $\theta_{s,o}$, and a value of $K_{s,o}/K_e$ greater than unity is associated with a negative value of $\theta_{s,o}$. (The possibility of a negative $\theta_{s,o}$ arises because T_e is arbitrarily assignable.) For either combination of $K_{s,o}/K_e$ and $\theta_{s,o}$, the quantity $2\left(\frac{K_e}{K_{s,o}}\right)\left(\frac{K_e}{K_{s,o}} - 1\right)\left(\frac{q''' L^2}{2K_e\theta_{s,o}}\right)$ is positive. On imposing the requirement that the radicand never be smaller than zero, the following limitation on the feasible value of $q''' L^2/2K_e\theta_{s,o}$ is obtained.

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$$\frac{q''' L^2}{2K_e|\theta_{s,o}|} < \frac{1}{2}\frac{\left(\frac{K_{s,o}}{K_e}\right)^2}{\left|1 - \frac{K_{s,o}}{K_e}\right|} \tag{B24}$$

where the symbol $|\ |$ indicates absolute value. Physically the limitation arises because if the assumed law of conductivity variation is valid the conductivity becomes "zero" at the plate midplane when the equality in expression (B24) is satisfied, and becomes "negative" when $q''' L^2/2K_e\theta_{s,o}$ exceeds the right member of relation (B24). (If the plate conductivity increases as temperature increases there is no limitation on $q''' L^2/2K_e\theta_{s,o}$; the sole requirement is that the plate surface temperature never acquire the value for which the conductivity is "zero.")

Relation between $q''' L^2/2K_e\theta_{s,o}$ and steady-state cooling conditions. - The fact that a plate generating q''' uniformly throughout its interior is maintained at a surface temperature $\theta_{s,o}$ in the steady state implies definite cooling conditions in this state. This fact may be used to obtain a relation for $q''' L^2/2K_e\theta_{s,o}$. From the heat balance

$$h_{st}\,(\theta_{s,o} - \theta_c) \doteq q''' L$$

Both sides of the equation may be multiplied by $L/2K_e\theta_{s,o}$; hence,

$$\frac{q''' \; L^2}{2K_e\theta_{s,o}} = \frac{1}{2}\left(\frac{h_{st} \; L}{K_e}\right)\left(\frac{\theta_{s,o} - \theta_c}{\theta_{s,o}}\right) \tag{B25}$$

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APPENDIX C

APPROXIMATE FORMULAS FOR STEADY-STATE STRESS IN HEAT-GENERATING PLATES OF LINEARLY VARYING CONDUCTIVITY

The formulas derived in the Analysis section for surface and mid-plane stresses in heat-generating plates of linearly varying conductivity are

$$\sigma^*_{s,o} = \frac{\frac{K_{s,o}}{K_e}\left[1 - \frac{1 - A}{2\sqrt{A}} \ln\left(\frac{1 + \sqrt{A}}{1 - \sqrt{A}}\right)\right]}{2\left(1 - \frac{K_{s,o}}{K_e}\right)} \tag{C1}$$

and

$$\sigma^*_{m,o} = -\frac{\frac{K_{s,o}}{K_e}\left[1 + \frac{1 - A}{2\sqrt{A}} \ln\left(\frac{1 + \sqrt{A}}{1 - \sqrt{A}}\right) - 2\sqrt{1 - A}\right]}{2\left(1 - \frac{K_{s,o}}{K_e}\right)} \tag{C2}$$

where

$$A \equiv 2\left(\frac{K_e}{K_{s,o}}\right)\left(\frac{K_e}{K_{s,o}} - 1\right)\left(\frac{q''' L^2}{2K_e \theta_{s,o}}\right) \tag{C3}$$

Approximate formulas useful for rapid estimate of the stresses may be obtained as follows: From reference 4 (p. 91), for $A < 1$,

$$\ln\left(\frac{1 + \sqrt{A}}{1 - \sqrt{A}}\right) = 2\left[A^{1/2} + \frac{A^{3/2}}{3} + \frac{A^{5/2}}{5} + \dots\right] \tag{C4}$$

If A is substantially smaller than unity (e.g., $A < 0.2$), terms beyond $\frac{A^{3/2}}{3}$ may be neglected; hence,

$$\ln\left(\frac{1+\sqrt{A}}{1-\sqrt{A}}\right) \simeq 2\left[A^{1/2} + \frac{A^{3/2}}{3}\right] \equiv 2\sqrt{A}\left(1 + \frac{A}{3}\right) \tag{C5}$$

Then

$$\frac{1-A}{2\sqrt{A}} \ln\left(\frac{1+\sqrt{A}}{1-\sqrt{A}}\right) \simeq (1-A)\left(1 + \frac{A}{3}\right) \equiv \left(1 - \frac{2A}{3} - \frac{A^2}{3}\right) \tag{C6}$$

On substituting equation (C6) into equation (C1),

$$\sigma^*_{s,o} \simeq \frac{\left(\frac{K_{s,o}}{K_e}\right)}{2\left(1 - \frac{K_{s,o}}{K_e}\right)} \left(\frac{2A}{3}\right)\left(1 + \frac{A}{2}\right)$$

Replacing $2A/3$ by its equivalent from equation (C3), and carrying through the algebraic operations, yields

$$\sigma^*_{s,o} \simeq \frac{2}{3}\left(\frac{q''' L^2/2K_e\theta_{s,o}}{K_{s,o}/K_e}\right)\left(1 + \frac{A}{2}\right) = \frac{2}{3}\left(\frac{q''' L^2}{2K_{s,o}\theta_{s,o}}\right)\left(1 + \frac{A}{2}\right) \tag{C7}$$

The midplane stress may be approximated as follows: On approximating $\sqrt{1-A}$ by $\left(1 - \frac{A}{2}\right)$, and on using equation (C6),

$$1 + \frac{1-A}{2\sqrt{A}} \ln\left(\frac{1+\sqrt{A}}{1-\sqrt{A}}\right) - 2\sqrt{1-A} \simeq 1 + \left(1 - \frac{2A}{3} - \frac{A^2}{3}\right) - 2\left(1 - \frac{A}{2}\right) = \frac{A}{3}(1-A)$$

Thus

$$\sigma^*_{m,o} \simeq -\frac{1}{3}\left(\frac{q''' L^2/2K_e\theta_{s,o}}{K_{s,o}/K_e}\right)(1-A) = -\frac{1}{3}\left(\frac{q''' L^2}{2K_{s,o}\theta_{s,o}}\right)(1-A) \tag{C8}$$

It is seen from the first forms of approximations (C7) and (C8) that if A is very small by comparison with unity, the surface and midplane stresses are directly proportional to $q''' L^2/2K_e\theta_{s,o}$ and inversely proportional to $K_{s,o}/K_e$; the second forms of equations (C7) and (C8) show that when A is very small by comparison with unity, the plate behaves as one having a conductivity that is constant at the value of the plate surface.

Comparison of equations (C7) and (C8) with formulas (13) and (14) of the Analysis for plates of constant conductivity shows that in a variable-conductivity plate the surface temperature is farther from the average, and the midplane temperature closer to the average, than in plates of constant conductivity. 2920

APPENDIX D

CALCULATION PROCEDURE

Temperature. - Steady-state temperatures were obtained by use of the equations derived in appendix B and given as equations (9) and (10) of the Analysis section. The transient temperatures were obtained as follows:

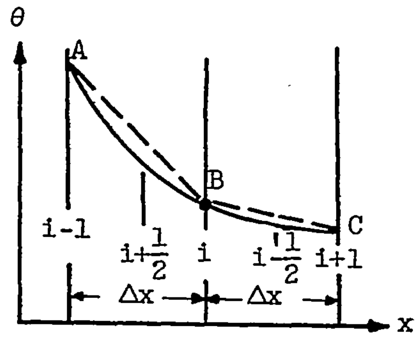
The plate transient temperatures are determined by the initial temperature distribution and by the conservation of heat. Heat conservation in the plate interior and at the plate surfaces is expressed alternately by the well-known equations

$$\rho c \frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left(K \frac{\partial \theta}{\partial x}\right) \tag{D1}$$

and

$$-K_s \left(\frac{\partial \theta}{\partial x}\right)_s = h\theta_s \tag{D2}$$

Inasmuch as only the early stages of the transient state were of interest a finite-difference solution, which for extended time spans might be less practical, was employed, as follows: In the accompanying sketch the designations $i-1$, i, $i+1$ refer to stations distant Δx from one another in the plate interior; $i-1/2$ and $i+1/2$ represent half-way stations. The curve ABC represents the local θ distribution after $j\Delta\tau$ time has elapsed. If station i is considered in the next time interval $\Delta\tau$



$$\left(\frac{\partial\theta}{\partial\tau}\right)_{i,j} = \lim_{\Delta\tau \to 0} \left[\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta\tau}\right] \sim \left[\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta\tau}\right] \tag{D3}$$

If stations $i-1/2$ and $i+1/2$ are considered at time $j\Delta\tau$ the slopes of the θ curve at these stations may be approximated by the slopes of the chords AB and BC; thus,

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$$(K_{i-\frac{1}{2},j})\left(\frac{\partial\theta}{\partial x}\right)_{i-\frac{1}{2},j} \sim (K_{i-\frac{1}{2},j})\left[\frac{\theta_{i,j} - \theta_{i-1,j}}{\Delta x}\right] \tag{D4}$$

$$(K_{i+\frac{1}{2},j})\left(\frac{\partial\theta}{\partial x}\right)_{i+\frac{1}{2},j} \sim (K_{i+\frac{1}{2},j})\left[\frac{\theta_{i+1,j} - \theta_{i,j}}{\Delta x}\right] \tag{D5}$$

Inasmuch as

$$\frac{\partial}{\partial x}\left[K\frac{\partial\theta}{\partial x}\right]_{i,j} = \lim_{\Delta x \to 0}\left[\frac{K_{i+\frac{1}{2},j}\left(\frac{\partial\theta}{\partial x}\right)_{i+\frac{1}{2},j} - K_{i-\frac{1}{2},j}\left(\frac{\partial\theta}{\partial x}\right)_{i-\frac{1}{2},j}}{\Delta x}\right]$$

the neglect of the limit process, and use of equations (D4) and (D5), leads to the approximation

$$\frac{\partial}{\partial x}\left[K\frac{\partial\theta}{\partial x}\right]_{i,j} \sim \frac{\left(K_{i+\frac{1}{2},j}\,\theta_{i+1,j}\right) - \left(K_{i-\frac{1}{2},j} + K_{i+\frac{1}{2},j}\right)\theta_{i,j} + \left(K_{i-\frac{1}{2},j}\,\theta_{i-1,j}\right)}{(\Delta x)^2} \tag{D6}$$

On substitution of equations (D3) and (D6) into equation (D1), and on solution for $\theta_{i,j+1}$, the obtained relation is

$$\theta_{i,j+1} = \theta_{i,j}\left\{1 - \frac{\Delta\tau}{(\Delta x)^2}\left[\frac{K_{i-\frac{1}{2},j} + K_{i+\frac{1}{2},j}}{\rho c}\right]\right\} + \frac{\Delta\tau}{(\Delta x)^2}\left[\frac{\left(K_{i+\frac{1}{2},j}\,\theta_{i+1,j}\right) + \left(K_{i-\frac{1}{2},j}\,\theta_{i-1,j}\right)}{\rho c}\right] \tag{D7}$$

If the θ distribution at time $j\Delta\tau$ is known, equation (D7) permits calculation of the θ distribution in the plate interior at time $(j+1)\,\Delta\tau$.

The temperatures prescribed by equation (D7) are constrained by equation (D2), to which the fictive-layer procedure was applied as recommended in reference 5. In this procedure the assumption is made that after the initial instant the temperatures at the plate surface and at the interior station distant $\Delta x/2$ from the surface lie on a straight line having the slope indicated by equation (D2). It is also assumed that this straight line may be extended a distance $\Delta x/2$ to the exterior of the plate, and that the so-introduced fictive layer of thickness $\Delta x/2$ may be treated as a segment of the plate. If θ_f is the θ value at the edge of the fictive layer (fig. 10) the algebraic equivalent of the assumption is that

$$\frac{(\theta_f - \theta_{\Delta x/2})_j}{\Delta x} = \left(\frac{\partial\theta}{\partial x}\right)_{s,j} = -\frac{h\theta_{s,j}}{K_{s,j}} = -\frac{h\left(\frac{\theta_f + \theta_{\Delta x/2}}{2}\right)_j}{K_{s,j}}$$

where $\theta_{\Delta x/2}$ is the value at station 1 of figure 10(a) or

$$\theta_{f,j} = \theta_{\Delta x/2,j}\left[\frac{1 - \frac{h\Delta x}{2K_{s,j}}}{1 + \frac{h\Delta x}{2K_{s,j}}}\right] \tag{D8}$$

Whereas this relation is employed for θ_f after the initial instant, the value of θ_f at zero time is assumed to be given by the relation

$$\frac{(\theta_f - \theta_s)_o}{\Delta x/2} = \left(\frac{\partial\theta}{\partial x}\right)_{s,o} = -\frac{h\theta_{s,o}}{K_{s,o}}$$

or

$$\theta_{f,o} = \theta_{s,o}\left(1 - \frac{h\Delta x}{2K_{s,o}}\right) \tag{D9}$$

A knowledge of θ_f permits use of equation (D7) to determine temperatures in the plate interior.

In applying equations (D7) through (D9) the increments $\Delta\tau$ and Δx were constant throughout the calculation at values determined by the convergence criterion indicated in reference 6.

$$\frac{\Delta\tau}{(\Delta x)^2} \leq \frac{1}{\left(\frac{K}{\rho c}\right)\left[2 + \frac{h\Delta x}{K}\right]} \tag{D10}$$

Inasmuch as $\Delta\tau$ and Δx may be chosen arbitrarily provided they satisfy equation (D10), the choice in this analysis was made according to the relation

$$\frac{\Delta\tau}{(\Delta x)^2} = \frac{1}{\left(\frac{K_{max}}{\rho c}\right)\left(2 + \frac{h\Delta x}{K_{min}}\right)}$$

wherein K_{max} and K_{min} were the respective maximum and minimum values of plate conductivity occurring during the total time span under consideration for thermal shock. This choice insured that equation (D10) would always be satisfied.

Dimensionless parameters for presentation of specific numerical results in a more generally useful form were derived by Nusselt's method of applying the similarity principle to heat transfer, as described in reference 7. The derivation of heat flow parameters is shown in appendix E.

Stress. - The stress was computed by use of equation (8) of the Analysis section. In the steady state, equations (13) and (14), and (17) and (18) of the Analysis were employed. In the transient state, the quantity $(\theta/\theta_{s,o})_{av}$ was obtained by planimeter integration under the transient curve of $(\theta/\theta_{s,o})$ over the plate thickness.

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APPENDIX E

PARAMETERS THAT GOVERN TRANSIENT TEMPERATURE IN PLATES OF LINEARLY VARYING CONDUCTIVITY

For the transient state it is assumed that the plate suddenly ceases to generate heat and is simultaneously immersed in a cooling environment of constant temperature T_e, to which the plate transfers heat by constant heat-transfer coefficient h. Equation (B3) becomes

$$\rho c \frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left(K \frac{\partial \theta}{\partial x}\right) \tag{E1}$$

and equation (B4) becomes

$$-K_s \left(\frac{\partial \theta}{\partial x}\right)_s = h\theta_s \tag{E2}$$

It may be noted that whereas in the steady state the temperature T_e was arbitrary, the use of θ in equation (E2) implies that for the transient state T_e must be equal to the temperature of the transient-state cooling environment. In practice it is convenient to use the same value of T_e for both the steady and transient states, because the results of the transient-state calculations depend on a knowledge of the steady-state temperature distribution, and it is convenient to have the original temperature distribution in the same terms as employed for the transient-state calculations. If the thermal conductivity of the plate is sufficiently linear in the temperature range of interest and can be written

$$K = mT + n \tag{E3}$$

it can be rewritten as

$$K = m(T - T_e) + (mT_e + n) = m\theta + K_e \tag{E3a}$$

Equations (E1) and (E2) become in terms of θ

$$\rho c \left(\frac{\partial\theta}{\partial\tau}\right) = \frac{\partial}{\partial x}\left[(m\theta + K_e)\frac{\partial\theta}{\partial x}\right]$$

$$= m\frac{\partial}{\partial x}\left(\theta\frac{\partial\theta}{\partial x}\right) + K_e\frac{\partial^2\theta}{\partial x^2} \qquad \text{(E1a)}$$

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and

$$h\theta_s = -m\theta_s\left(\frac{\partial\theta}{\partial x}\right)_s - K_e\left(\frac{\partial\theta}{\partial x}\right)_s \qquad \text{(E2a)}$$

The dimensionless groups which govern the transient temperatures may be determined by Nusselt's method of applying the similarity principle to heat transfer (ref. 7). Accordingly, plate systems 1 and 2 are now considered, and are required to be similar in geometry, time, and temperature. If x, τ, and θ represent local values, and if a, b, and f are constant independent of x, τ, and θ, the similarity requirement is expressible as

$$x_1 = ax_2 = \frac{L_1}{L_2}x_2$$

$$\tau_1 = b\tau_2 \qquad \text{(E4)}$$

$$\theta_1 = f\theta_2$$

The respective system constants, ρc, K_e, h, and m bear the constant ratios

$$(\rho c)_1 = g(\rho c)_2$$

$$K_{e,1} = i\ K_{e,2}$$

$$h_1 = j\ h_2 \qquad \text{(E4a)}$$

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$$m_1 = l\ m_2$$

It is noteworthy that $m_1 \neq m_2$ implies nonparallelism of K with θ in systems 1 and 2. The temperatures in plates 1 and 2 satisfy equation (E1a), i.e.,

$$(\rho c)_1 \frac{\partial \theta_1}{\partial \tau_1} = m_1 \frac{\partial}{\partial x_1}\left(\theta_1 \frac{\partial \theta_1}{\partial x_1}\right) + K_{e,1} \frac{\partial^2 \theta_1}{\partial {x_1}^2} \qquad \text{(E5)}$$

$$(\rho c)_2 \frac{\partial \theta_2}{\partial \tau_2} = m_2 \frac{\partial}{\partial x_2}\left(\theta_2 \frac{\partial \theta_2}{\partial x_2}\right) + K_{e,2} \frac{\partial^2 \theta_2}{\partial {x_2}^2} \qquad \text{(E6)}$$

When equations (E4) and (E4a) are employed in equation (E5)

$$\left(\frac{gf}{b}\right) (\rho c)_2 \frac{\partial \theta_2}{\partial \tau_2} = \left(\frac{lf^2}{a^2}\right) m_2 \frac{\partial}{\partial x_2}\left(\theta_2 \frac{\partial \theta_2}{\partial x_2}\right) + \left(\frac{if}{a^2}\right) K_{e,2} \frac{\partial^2 \theta_2}{\partial {x_2}^2} \qquad \text{(E5a)}$$

Equations (E5a) and (E6) are identical if

$$\left(\frac{gf}{b}\right)\left(\frac{a^2}{if}\right) = 1 \qquad \text{(E7)}$$

$$\left(\frac{lf^2}{a^2}\right)\left(\frac{a^2}{if}\right) = 1 \qquad \text{(E7a)}$$

Use of equations (E4) and (E4a) in equation (E7) gives

$$\frac{ga^2}{b_i} \equiv \frac{(\rho c)_1}{(\rho c)_2} \frac{{x_1}^2}{{x_2}^2} \frac{\tau_2}{\tau_1} \frac{K_{e,2}}{K_{e,1}} = 1$$

Inasmuch as $x_1 = \frac{L_1}{L_2} x_2$, and $\frac{K_e}{\rho c} \equiv \alpha_e$, 2920

$$\left(\frac{\alpha_e \tau}{L^2}\right)_1 = \left(\frac{\alpha_e \tau}{L^2}\right)_2 \tag{E8}$$

Thus, one dimensionless group is Fourier's modulus with the plate thermal diffusivity evaluated at the environment temperature. Equation (E8) shows that the times at which the temperatures of similarly located points are to be compared are in the ratio $\left(\frac{\alpha_e}{L^2}\right)_1 \Big/ \left(\frac{\alpha_e}{L^2}\right)_2 = \frac{\tau_2}{\tau_1}$. When equations (E4) and (E4a) are used in equation (E7a)

$$\frac{lf}{i} = \frac{m_1 \theta_1}{m_2 \theta_2} \frac{K_{e,2}}{K_{e,1}} = 1$$

or

$$\frac{\theta_1}{\theta_2} = \frac{K_{e,1}}{K_{e,2}} \frac{m_2}{m_1} \tag{E9}$$

Equation (E9) shows the actual ratio existing between temperature differences θ_1 and θ_2 of similarly located points when considered at the corresponding times designated by equation (E8). The explicit value of the θ ratio does not appear in the case of materials of constant thermal conductivity.

The dimensionless group that must have the same value in the two plate systems if equation (E9) is to be satisfied may be obtained by considering the surfaces of the two plates at zero time. If $\theta_{s,o}$ repsents initial temperature difference between surface and environment, equation (9) requires that at $\tau = 0$,

$$\left(\frac{m\theta_{s,o}}{K_e}\right)_1 = \left(\frac{m\theta_{s,o}}{K_e}\right)_2$$

But from equation (E3a), $m\theta_{s,o} = K_{s,o} - K_e$, where $K_{s,o}$ is the plate conductivity at the initial surface temperature. Hence,

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$$\left(\frac{K_{s,o} - K_e}{K_e}\right)_1 = \left(\frac{K_{s,o} - K_e}{K_e}\right)_2$$

or

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$$\left(\frac{K_{s,o}}{K_e}\right)_1 = \left(\frac{K_{s,o}}{K_e}\right)_2 \tag{E10}$$

Thus for two plates of linearly varying conductivity, temperature similarity can exist if the ratio of plate conductivity at the initial surface temperature to the plate conductivity at the environment temperature is the same in both systems.

For systems in which equation (E10) is satisfied equation (E4) is also valid and

$$\frac{\theta_1}{\theta_2} = f = \frac{K_{e,1}}{K_{e,2}} \frac{m_2}{m_1} = \frac{(\theta_{s,o})_1}{(\theta_{s,o})_2}$$

and

$$\left(\frac{\theta}{\theta_{s,o}}\right)_1 = \left(\frac{\theta}{\theta_{s,o}}\right)_2 \tag{E11}$$

Systems 1 and 2 both satisfy equation (E2a). Thus

$$(h\theta_s)_1 = -\left[m\theta_s\left(\frac{\partial\theta}{\partial x}\right)_s\right]_1 - \left[K_e\left(\frac{\partial\theta}{\partial x}\right)_s\right]_1 \tag{E12}$$

and

$$(h\theta_s)_2 = -\left[m\theta_s\left(\frac{\partial\theta}{\partial x}\right)_s\right]_2 - \left[K_e\left(\frac{\partial\theta}{\partial x}\right)_s\right]_2 \tag{E13}$$

On using equations (E4) and (E4a) in equation (E12),

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$$jf(h\theta_s)_2 = \left(\frac{lf^2}{a}\right)\left[-m\theta_s\left(\frac{\partial\theta}{\partial x}\right)_s\right]_2 + \left(\frac{if}{a}\right)\left[-K_e\left(\frac{\partial\theta}{\partial x}\right)_s\right]_2 \tag{E12a}$$

Equations (E12a) and (E13) are identical if

$$(jf)\ (a/if) = 1 \tag{E14}$$

and

$$\left(\frac{lf^2}{a}\right)\left(\frac{a}{if}\right) = 1 \tag{E15}$$

On using equations (E4) and (E4a) in (E14),

$$\frac{ja}{i} \equiv \frac{h_1}{h_2}\frac{L_1}{L_2}\frac{K_{e,2}}{K_{e,1}} = 1$$

or

$$\left(\frac{hL}{K_e}\right)_1 = \left(\frac{hL}{K_e}\right)_2 \tag{E16}$$

Biot's modulus with plate conductivity evaluated at environment temperature is another governing parameter. For temperature similarity of two plate systems the heat-transfer coefficients must be in the ratio $(L/K_e)_2/(L/K_e)_1$.

Equation (E15) yields again the requirement $(m\theta/K_e)_1 = (m\theta/K_e)_2$, which was obtained from equation (E7a).

Equations (E4), (8), (10), (11), and (16) show that results of numerical calculations for any type of initial temperature distribution can be presented in a form useful for similar systems having the same type of initial distribution by plots of $\theta/\theta_{s,o}$ against hL/K_e, with $\alpha_e\tau/L^2$ as parameter, for prescribed values of x/L and $K_{s,o}/K_e$. For general usefulness, dimensionless temperatures for the entire range of probable $K_{s,o}/K_e$ values, and for various values of hL/K_e and prescribed initial distribution, require calculation. (Numerical results for $K_{s,o}/K_e = 1$ and 0.15 are presented in this report for the initial temperature distribution occurring in a plate which in the steady state generates heat uniformly throughout its interior and is equally cooled on both faces.)

It is noteworthy that the dimensionless groups for plates of linear conductivity and constant volumetric specific heat, exchanging heat in the transient state with an environment of constant temperature T_e, all require the plate conductivity to be evaluated at the environment or surface temperature and include the dimensionless group $K_{s,o}/K_e$, the ratio of plate conductivities at initial surface and at environment temperatures. The ratio $K_{s,o}/K_e$ measures the rapidity with which the plate conductivity varies with temperature when $\theta_{s,o}$ and K_e are known, or indicates the ΔK experienced in going from T_e to T_s.

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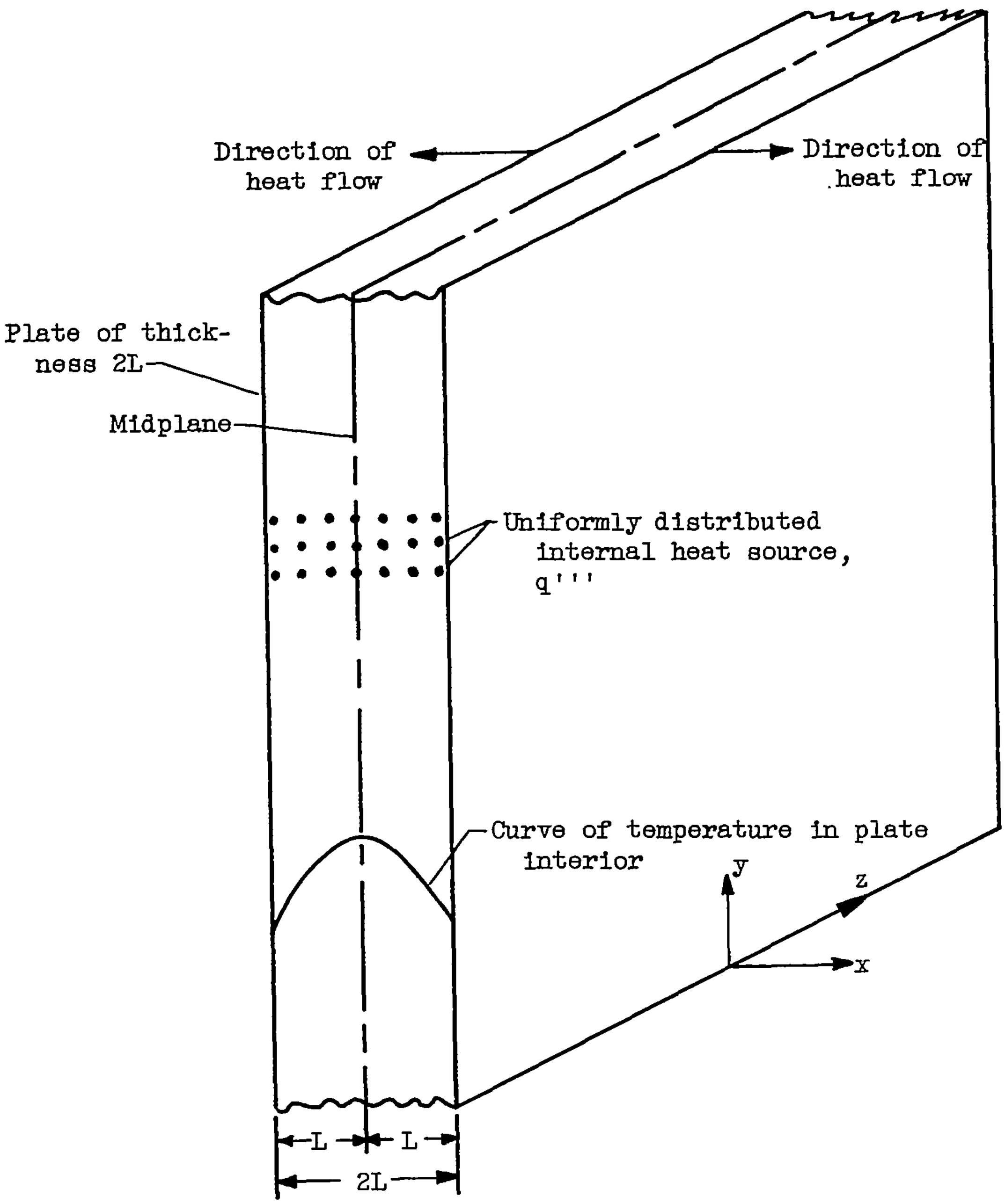
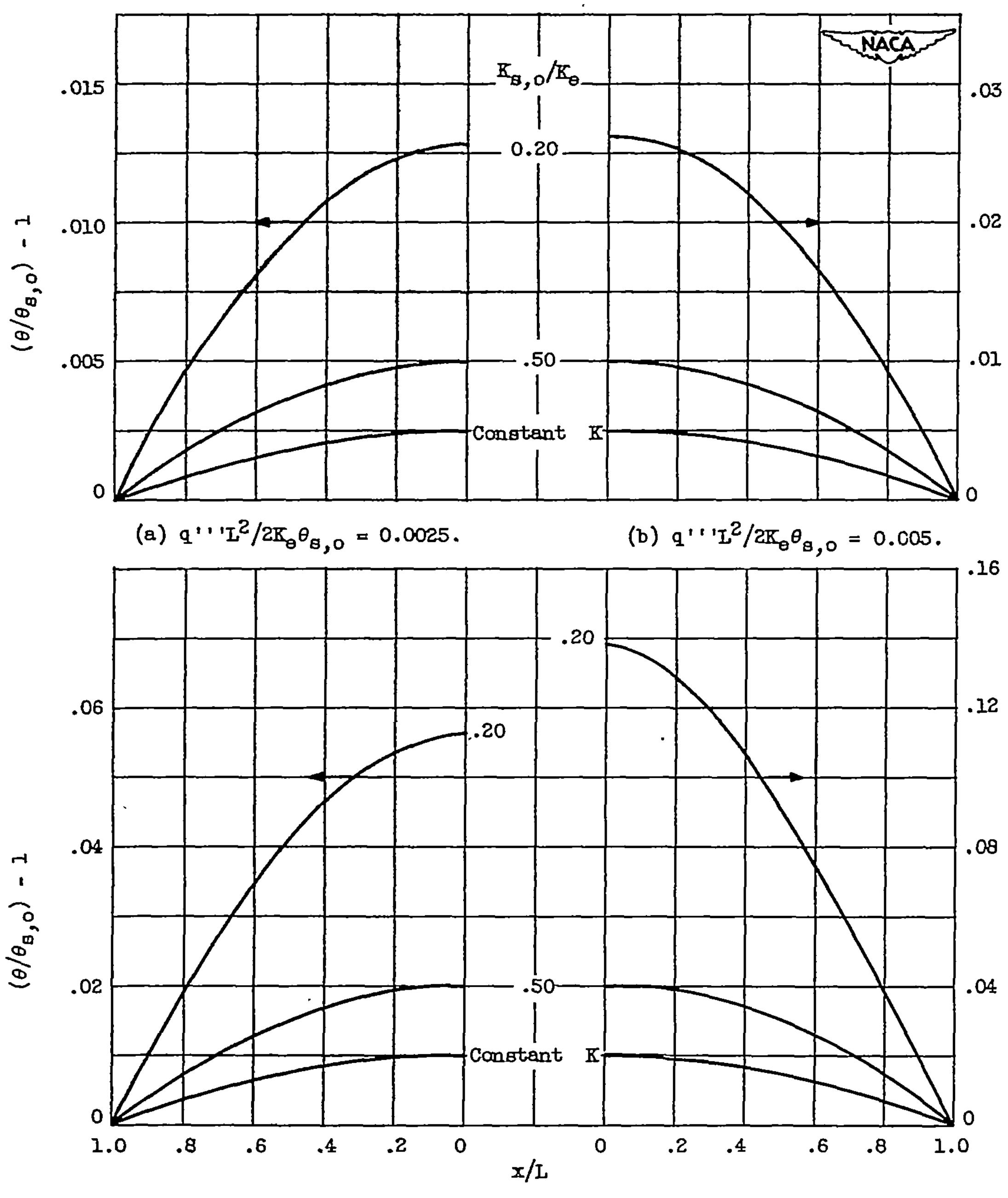


Figure 1. - System under consideration in steady state.



(a) $q'''L^2/2K_e\theta_{s,o} = 0.0025$.

(b) $q'''L^2/2K_e\theta_{s,o} = 0.005$.

(c) $q'''L^2/2K_e\theta_{s,o} = 0.01$.

(d) $q'''L^2/2K_e\theta_{s,o} = 0.02$.

Figure 2. - Distribution of $\theta/\theta_{s,o}$ for several values of $K_{s,o}/K_e$ and $q'''L^2/2K_e\theta_{s,o}$.

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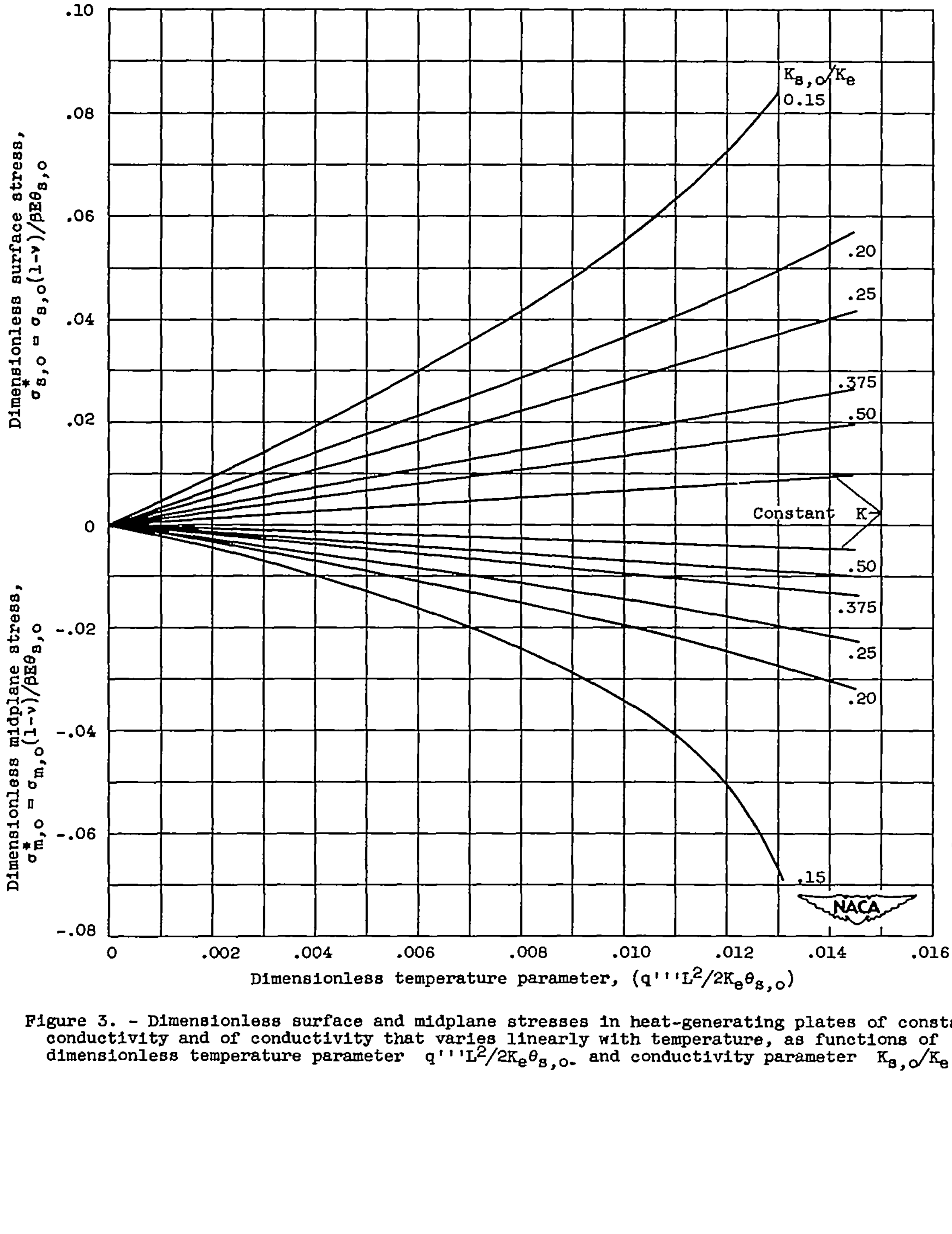
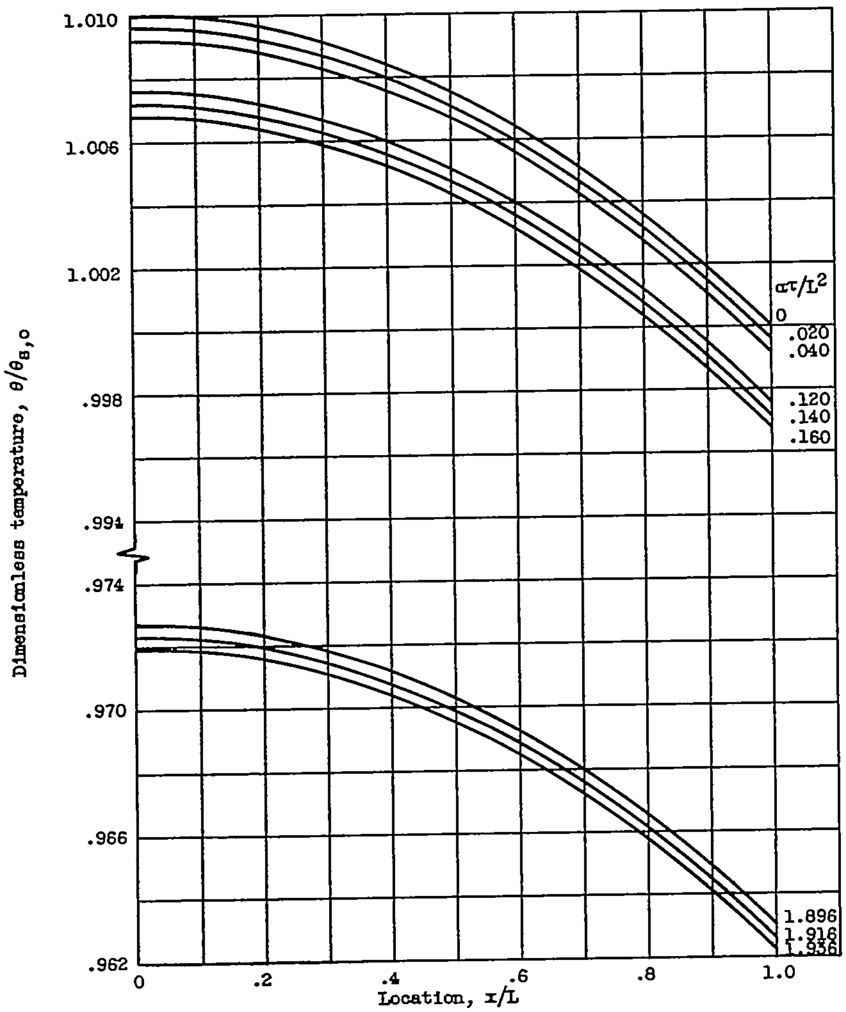
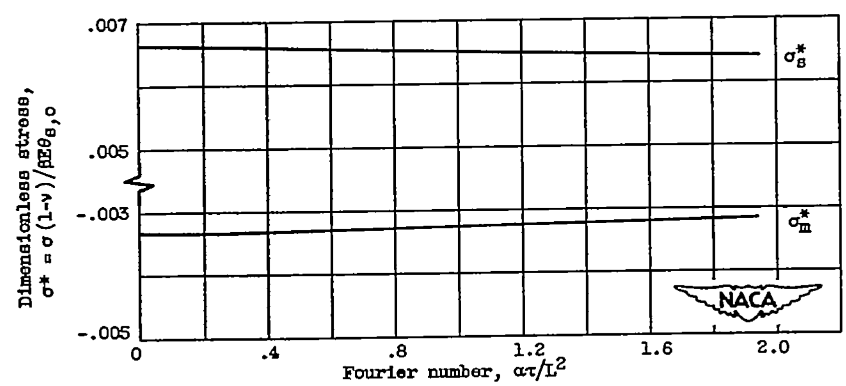


Figure 3. - Dimensionless surface and midplane stresses in heat-generating plates of constant conductivity and of conductivity that varies linearly with temperature, as functions of dimensionless temperature parameter $q'''L^2/2K_e\theta_{s,o}$ and conductivity parameter $K_{s,o}/K_e$.



(a) Temperature distribution.



(b) Surface and midplane stresses, σ_s^* and σ_m^*.

Figure 4. - Transient temperatures and stresses in formerly-heat-generating plates of constant conductivity. $q'''L^2/2K\theta_{s,o} = 0.01$; $\frac{hL}{K} = 2\left(\frac{q'''L^2}{2K\theta_{s,o}}\right) = 0.02$.

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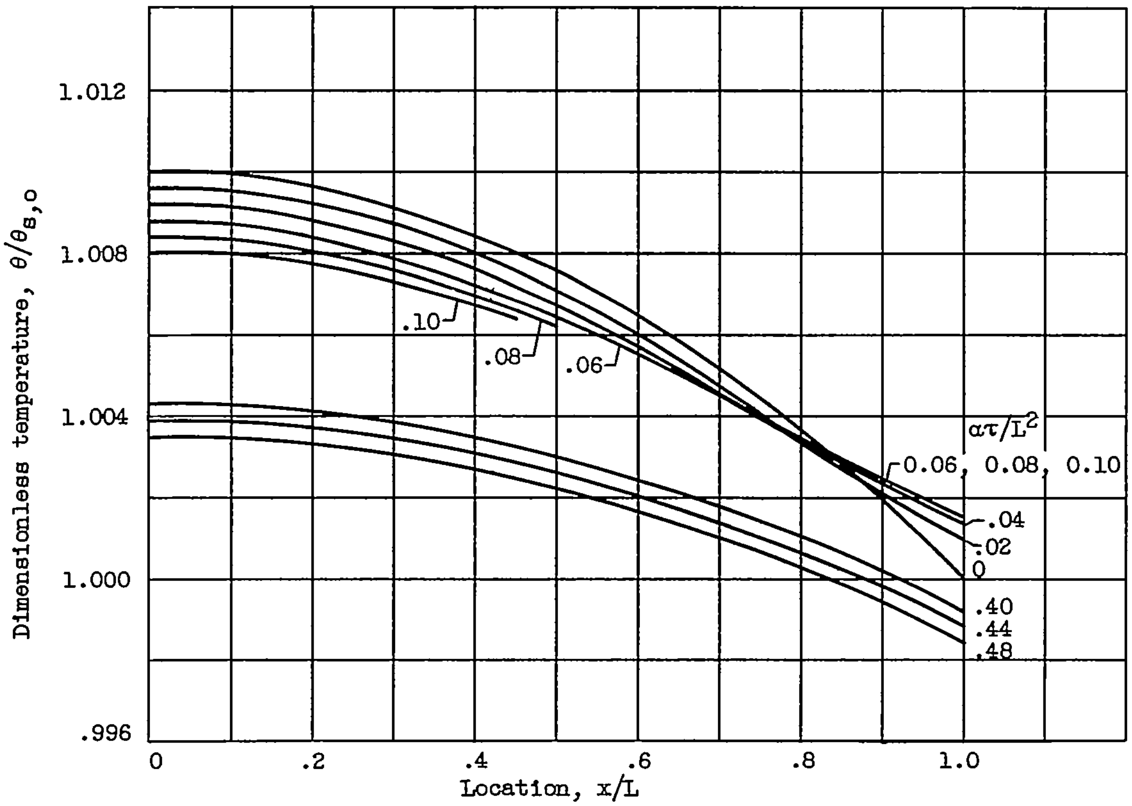
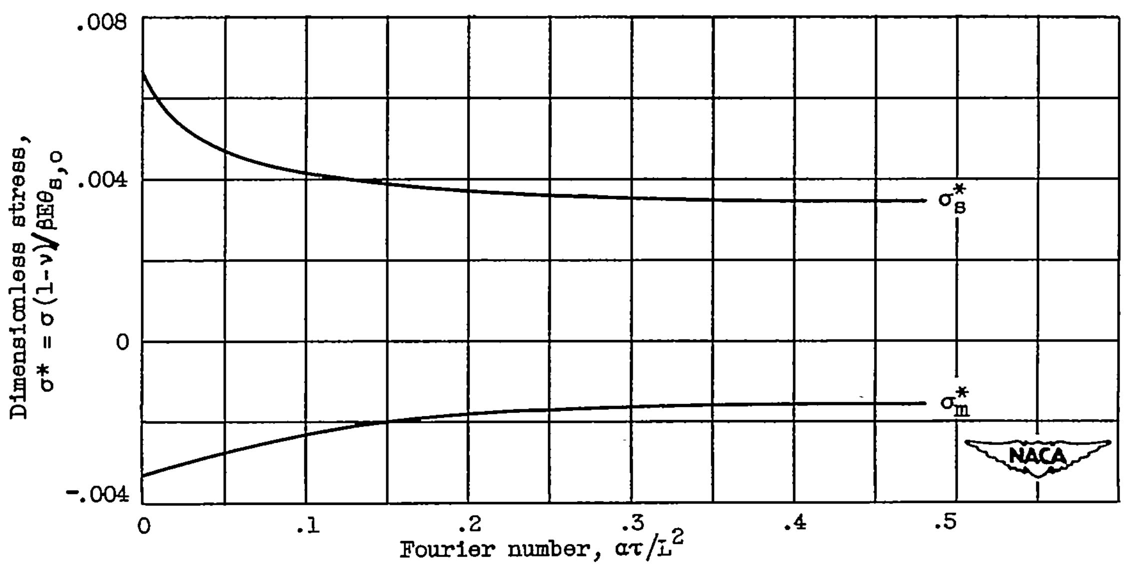
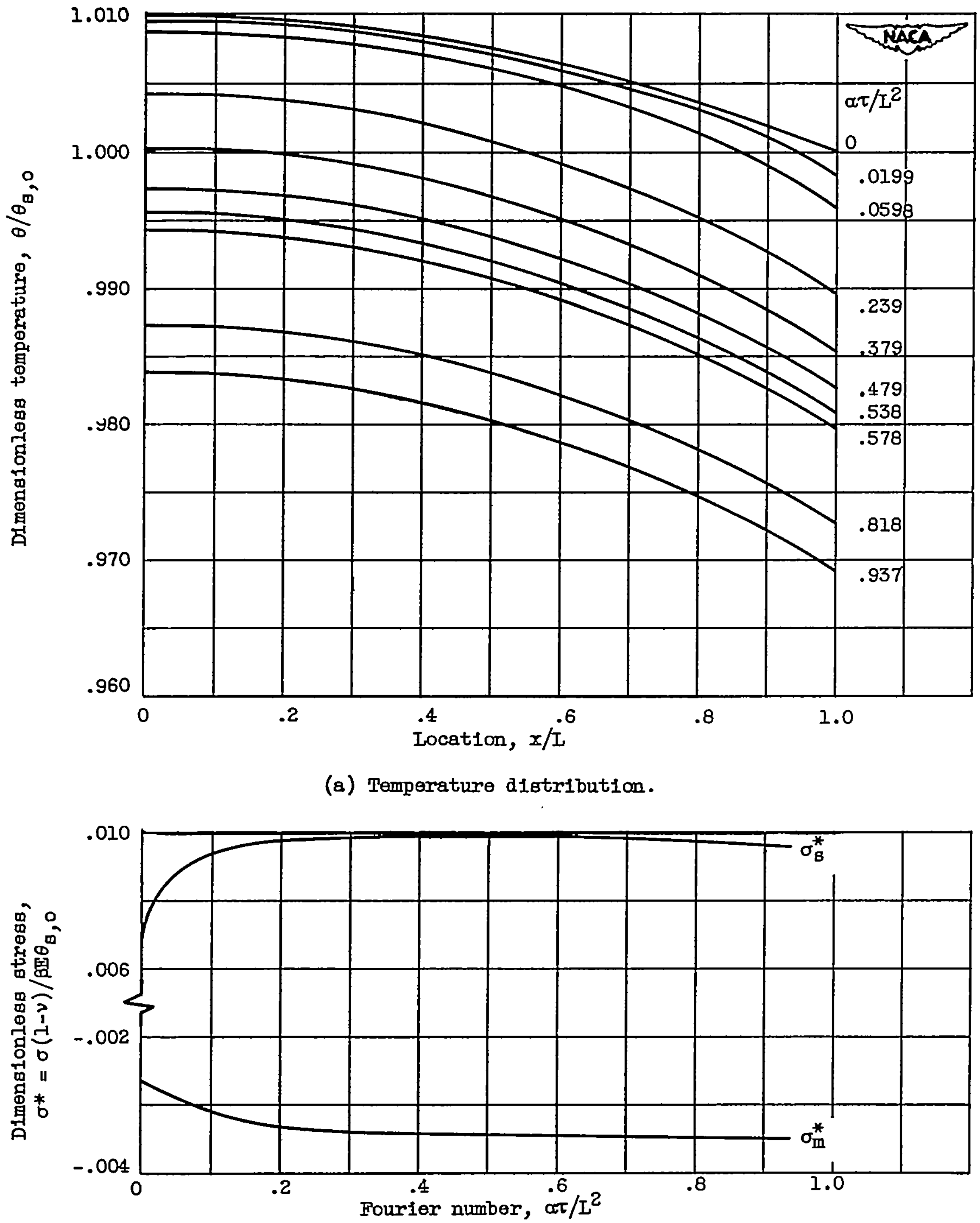


(a) Temperature distribution.



(b) Surface and midplane stresses.

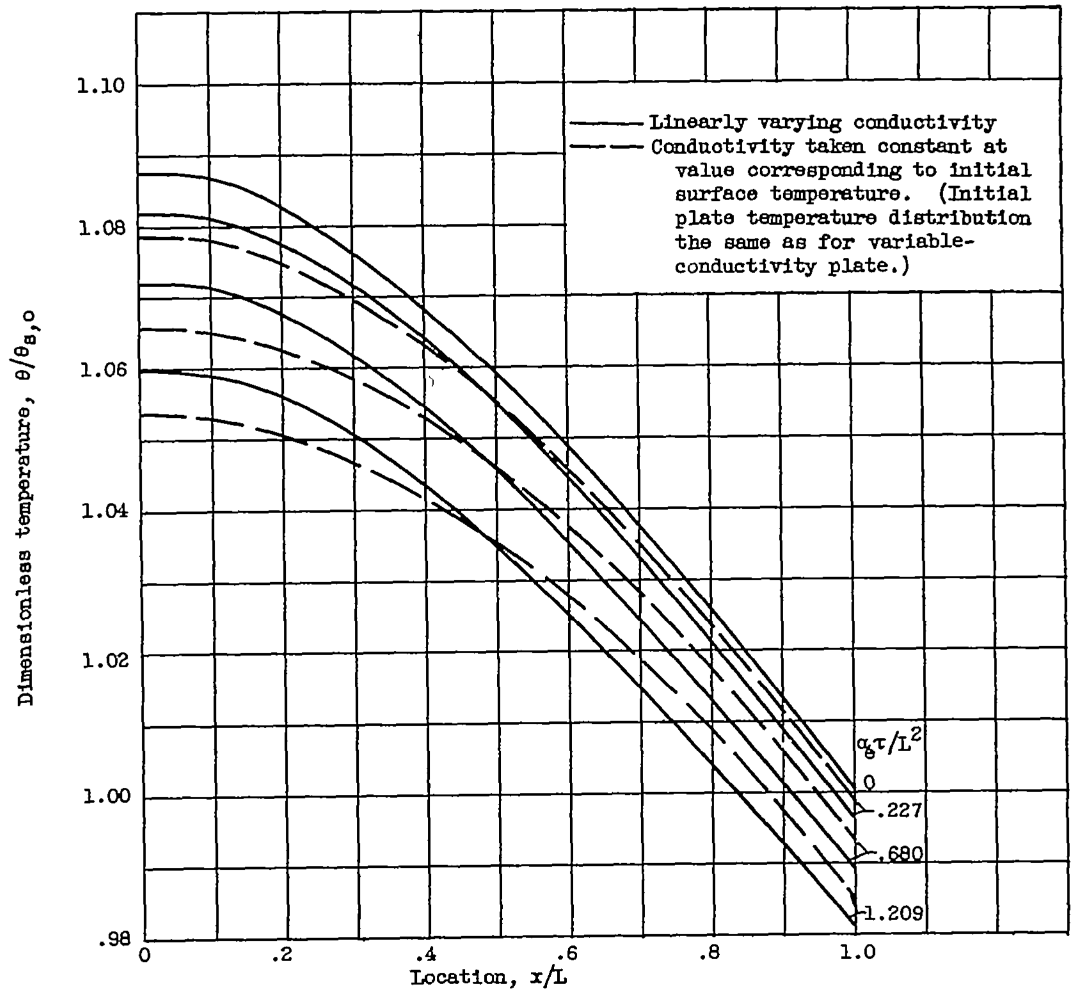
Figure 5. - Transient temperatures and stresses in formerly-heat-generating plates of constant conductivity. $q'''L^2/2K\theta_{s,o} = 0.01$, $hL/K_e = 0.01 < 2\left(\frac{q'''L^2}{2K\theta_{s,o}}\right)$.



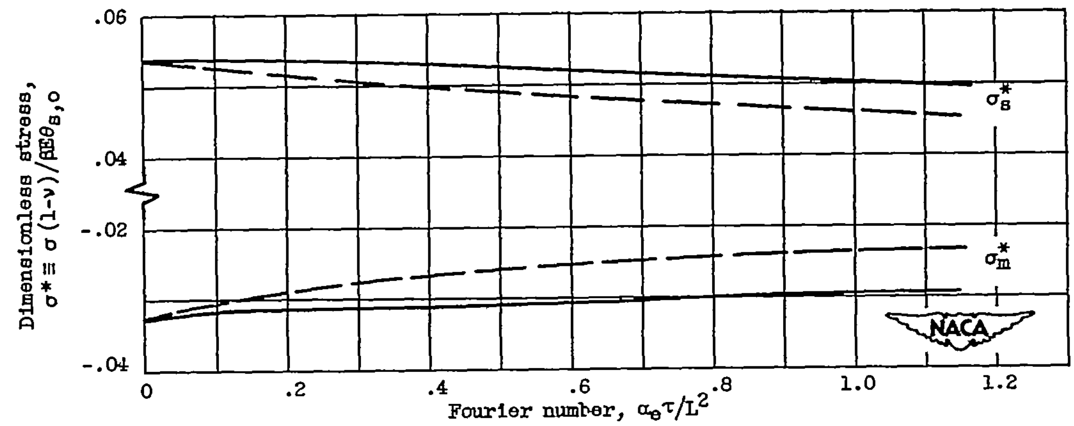
(a) Temperature distribution.

(b) Surface and midplane stresses.

Figure 6. - Transient temperatures and stresses in formerly-heat-generating plates of constant conductivity. $q'''L^2/2K\theta_{s,o} = 0.01$, $hL/K = 0.03 > 2\left(\frac{q'''L^2}{2K\theta_{s,o}}\right)$.

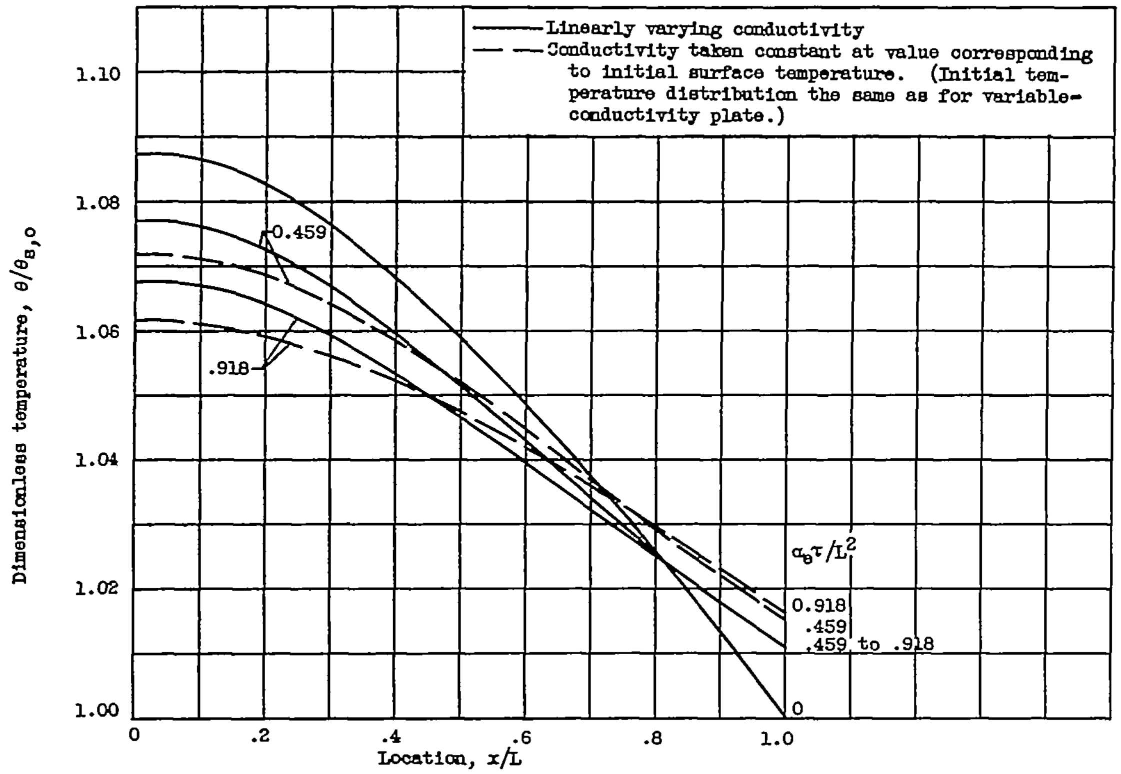


(a) Temperature distribution.

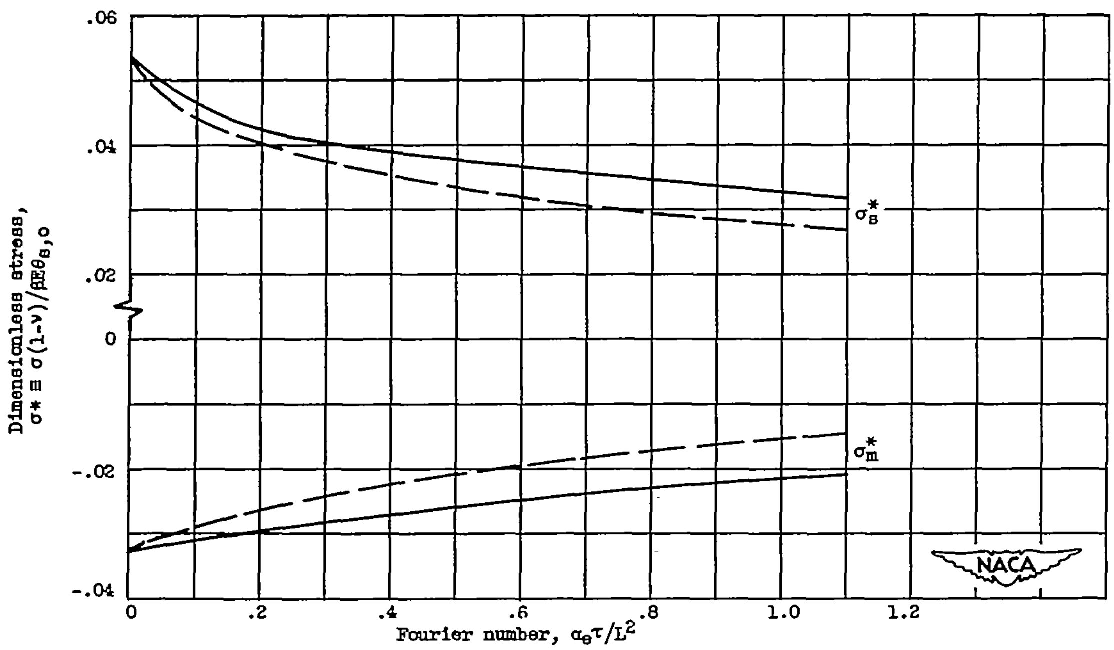


(b) Surface and midplane stresses, σ_s^* and σ_m^*.

Figure 7. - Transient temperatures and stresses in formerly-heat-generating plates of variable conductivity. $K_{s,o}/K_e = 0.151$, $q'''L^2/K_e\theta_{s,o} = 0.01$, $hL/K_e = 2\left(\frac{q'''L^2}{2K_e\theta_{s,o}}\right) = 0.02$.



(a) Temperature distribution.



(b) Surface and midplane stresses, σ^*_s and σ^*_m.

Figure 8. - Transient temperatures and stresses in formerly-heat-generating plates of variable conductivity. $K_{s,o}/K_e = 0.151$, $q'''L^2/2K_e\theta_{s,o} = 0.01$, $hL/K_e = 0.01 < 2\left(\frac{q'''L^2}{2K_e\theta_{s,o}}\right)$.

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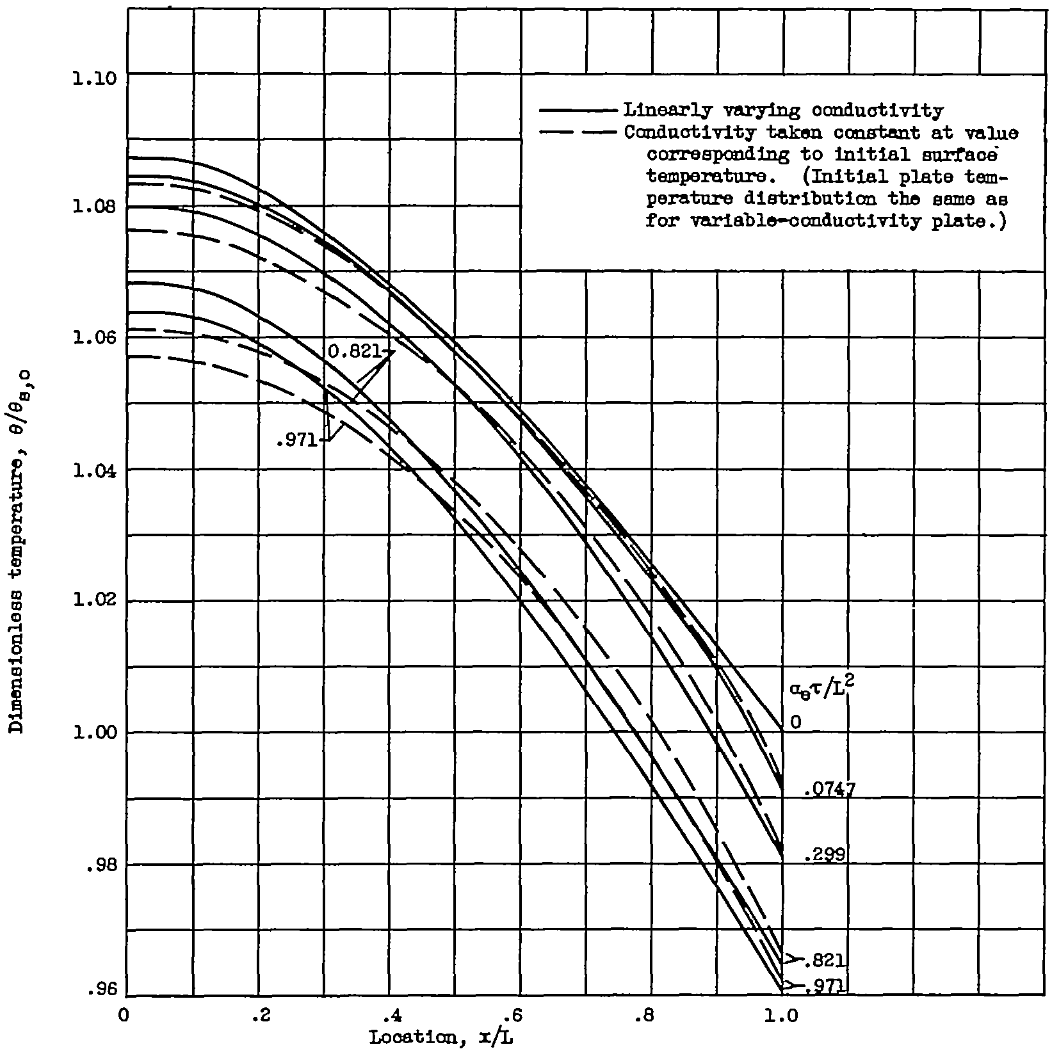
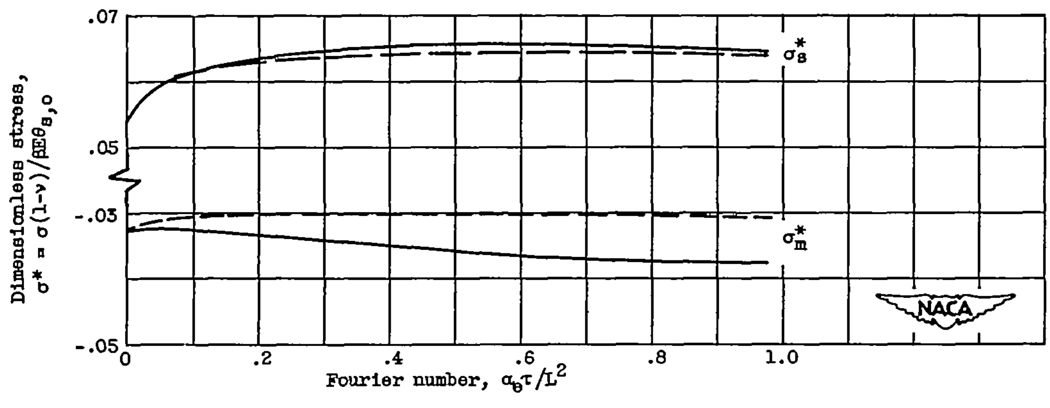
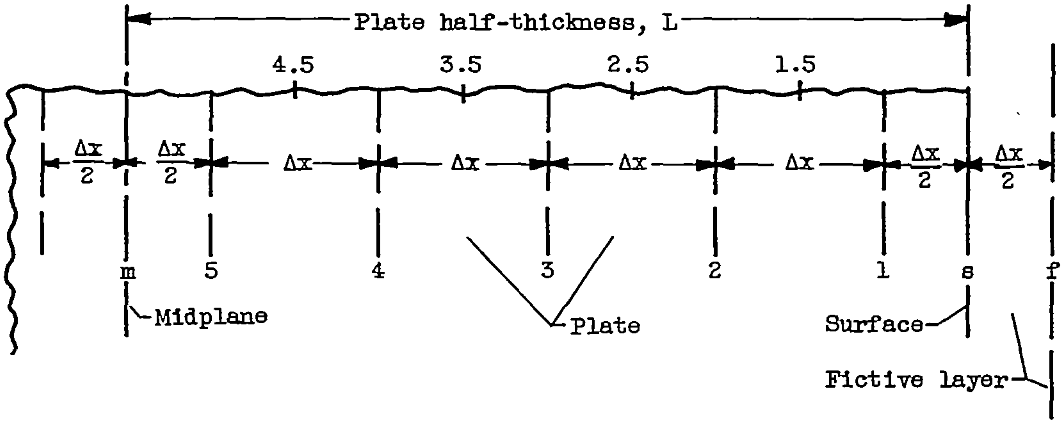


(a) Temperature distribution.

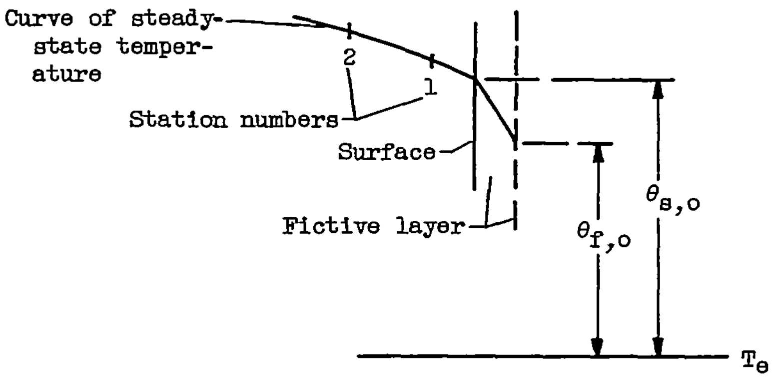


(b) Surface and midplane stresses, σ_s^* and σ_m^*.

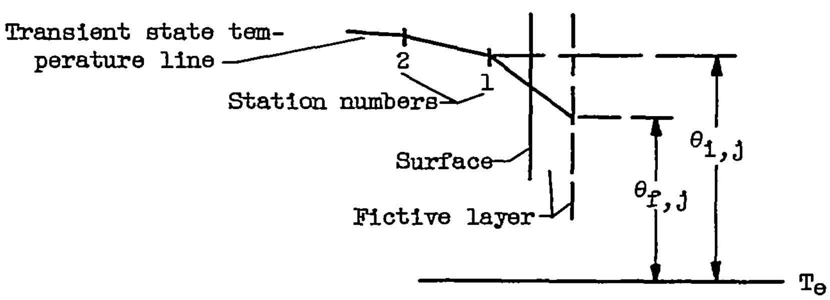
Figure 9. - Transient temperatures and stresses in formerly-heat-generating plates of variable conductivity. $K_{s,o}/K_e = 0.151$, $q'''L^2/2K_e\theta_{s,o} = 0.01$, $hL/K_e = 0.03 > 2\left(\frac{q'''L^2}{2K_e\theta_{s,o}}\right)$.



(a) Plate divisions and fictive layer.



(b) Assumed fictive-layer temperature $\theta_{s,o}$ at start of transient state, $\theta_{f,o} = \theta_{s,o}\left[1 - \frac{h\Delta x}{2K_{s,o}}\right]$.



(c) Assumed fictive layer temperature $\theta_{f,j}$ at time $j\Delta\tau$ of transient state, $\theta_{f,j} = \theta_{1,j}\left[1 - \frac{h\Delta x}{2K_{s,j}}\right] \Big/ \left[1 + \frac{h\Delta x}{2K_{s,j}}\right]$.

Figure 10. - Details of finite-difference procedure for obtaining temperatures in transient state.

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