NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2872

THE EFFECT OF INITIAL CURVATURE ON THE STRENGTH OF AN INELASTIC COLUMN

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SUMMARY

The reduction in column strength due to initial curvature is determined theoretically for a pin-ended idealized inelastic H-section column. Equations relating load and lateral deflection are obtained which permit a systematic variation in the parameters representing the stress-strain properties, column proportions, and initial curvature of the column. The results, presented graphically, show the effect of various combinations of these parameters on column strength.

INTRODUCTION

For many years the reduced-modulus load (ref. 1) was considered to be the maximum load that could be supported by a straight inelastic column. In 1947, however, an analysis by Shanley (ref. 2) indicated that the maximum load of a straight inelastic column is always less than the reduced-modulus load but greater than the tangent-modulus load; thus, the maximum load was located between limits. Subsequently, several investigators extended Shanley's work to determine more definitely the maximum load for inelastic columns (see, for example, refs. 3 to 7).

In reference 7 analytical results were obtained to show that the maximum load that an initially perfect plastic column could support was indeed included between the tangent-modulus and reduced-modulus loads and that its magnitude depended on the shape of the stress-strain curve. Real columns, however, are not straight but have some initial crookedness. The present study was made to determine how significant is the effect of initial out-of-straightness and whether or not reasonable amounts of it could account for the fact that maximum loads obtained experimentally tend to scatter about the tangent-modulus load.

In the present paper, therefore, the maximum loads for initially curved pin-ended idealized H-section columns of different proportions and materials are determined theoretically. The analysis is similar to that of Duberg and Wilder (ref. 7) in which the maximum load was determined for an initially straight idealized H-section column, the material of which could be represented by the Ramberg-Osgood stress-strain relationship. Comparisons are made between the maximum loads for initially curved columns and the maximum loads for corresponding straight columns.

SYMBOLS

A	cross-sectional area of column
b	column thickness (see fig. 1)
d	total lateral deflection of column at midheight after loading
d_o	initial lateral deflection of column at midheight before loading
e	dimensionless initial lateral deflection of column at midheight before loading, $2d_o/b$
E	Young's modulus
E_T	slope of stress-strain curve at stress corresponding to tangent-modulus load
f	dimensionless total lateral deflection of column at midheight after loading, $2d/b$
L	column length
n	Ramberg-Osgood stress-strain-curve shape parameter (see fig. 2)
P	column load, $\bar{\sigma}A$
P_1	load, $\sigma_1 A$
P_T	tangent-modulus load, $\sigma_T A$
x	longitudinal distance measured from end of column
y	total lateral deflection of column after loading
y_o	initial lateral deflection of column before loading
ϵ_1	elastic strain corresponding to stress σ_1
ϵ_E	elastic strain corresponding to Euler stress, $\pi^2\rho^2/L^2$
ϵ_L	compressive strain in left (concave) flange at midheight of column

ϵ_R	compressive strain in right (convex) flange at midheight of column
ρ	radius of gyration, $b/2$
σ_1	0.7E secant yield stress
σ_E	average cross-sectional stress corresponding to Euler load
σ_L	compressive stress in left (concave) flange at midheight of column
σ_R	compressive stress in right (convex) flange at midheight of column
σ_T	average cross-sectional stress corresponding to tangent-modulus load of column
$\bar{\sigma}$	average cross-sectional stress of column

Subscript:

max	maximum value

ANALYSIS

The idealized column under consideration (fig. 1) consists of two thin flanges of equal area separated by a web of infinite shear stiffness and of negligible area. The initial curvature of the column is in the plane of the web and is of the form

$$y_o = d_o \sin \frac{\pi x}{L} \tag{1}$$

The application of an axial load to the hinged ends of the column produces an additional lateral deflection which is assumed to be sinusoidal; hence, the total lateral deflection is

$$y = d \sin \frac{\pi x}{L} \tag{2}$$

In reference 7 the use of a single sine term to represent the deflected shape was shown to be sufficiently accurate for computing the maximum strength of an initially straight column and is, therefore, assumed to be sufficiently reliable for computing the maximum strength of a column with small initial curvature.

The amplitude of the lateral deflection d can be related to the average stress associated with the axial load by imposing the conditions of equilibrium, geometrical continuity, and compatibility of stress and strain at the midheight of the column. Equilibrium of vertical forces and moments requires, respectively, that at the midheight of the column

$$\sigma_L + \sigma_R = 2\bar{\sigma} \tag{3}$$

and

$$\sigma_L - \sigma_R = 4\bar{\sigma}\frac{d}{b} \tag{4}$$

which may be solved to give

$$\sigma_L = \bar{\sigma}(1 + f) \tag{5}$$

and

$$\sigma_R = \bar{\sigma}(1 - f) \tag{6}$$

In accordance with the assumption of infinite transverse shear stiffness, the midheight strains and deflections are related in the following manner:

$$\frac{\epsilon_L - \epsilon_R}{b} = \left[-\frac{d^2}{dx^2}(y - y_o)\right]_{x=\frac{L}{2}}$$

or

$$\frac{\epsilon_L - \epsilon_R}{b} = \frac{\pi^2}{L^2}(d - d_o) \tag{7}$$

By using the Ramberg-Osgood formula (ref. 8), presented graphically in figure 2, the stress-strain relations for σ_L/σ_1 and σ_R/σ_1 increasing become

$$\frac{\epsilon_L}{\epsilon_1} = \frac{\sigma_L}{\sigma_1} + \frac{3}{7}\left(\frac{\sigma_L}{\sigma_1}\right)^n \tag{8}$$

and

$$\frac{\epsilon_R}{\epsilon_1} = \frac{\sigma_R}{\sigma_1} + \frac{3}{7}\left(\frac{\sigma_R}{\sigma_1}\right)^n \tag{9}$$

However, when σ_R/σ_1 reaches a maximum value and begins to decrease, stress reversal has occurred and the assumption of elastic unloading requires that the following linear relation replace equation (9):

$$\frac{\epsilon_R}{\epsilon_1} = \frac{\sigma_R}{\sigma_1} + \frac{3}{7}\left[\left(\frac{\sigma_R}{\sigma_1}\right)_{max}\right]^n \tag{10}$$

The elimination of ϵ_L and ϵ_R in equation (7) through the use of the stress-strain relations (8), (9), and (10), followed by the elimination of σ_L/σ_1 and σ_R/σ_1 by using equations (5) and (6), yields the following relations between the load parameter $\bar{\sigma}/\sigma_1$ and the lateral-deflection parameter f:

For σ_R/σ_1 increasing,

$$\frac{3}{14}\left[(1 + f)^n - (1 - f)^n\right]\left(\frac{\bar{\sigma}}{\sigma_1}\right)^n + f\,\frac{\bar{\sigma}}{\sigma_1} = \frac{\epsilon_E}{\epsilon_1}(f - e) \tag{11}$$

and, for σ_R/σ_1 decreasing,

$$\frac{3}{14}(1 + f)^n\left(\frac{\bar{\sigma}}{\sigma_1}\right)^n + f\,\frac{\bar{\sigma}}{\sigma_1} = \frac{3}{14}\left[\left(\frac{\sigma_R}{\sigma_1}\right)_{max}\right]^n + \frac{\epsilon_E}{\epsilon_1}(f - e) \tag{12}$$

An expression for the parameter ϵ_E/ϵ_1 in terms of σ_T/σ_1 may be derived as follows:

$$\frac{\epsilon_E}{\epsilon_1} = \frac{\sigma_E}{E}\,\frac{E}{\sigma_1} = \frac{\sigma_T}{E_T}\,\frac{E}{\sigma_1} = \frac{\sigma_T}{\sigma_1}\,\frac{E}{E_T} = \frac{\sigma_T}{\sigma_1}\left[1 + \frac{3n}{7}\left(\frac{\sigma_T}{\sigma_1}\right)^{n-1}\right]$$

or

$$\frac{\epsilon_E}{\epsilon_1} = \frac{\sigma_T}{\sigma_1} + \frac{3n}{7}\left(\frac{\sigma_T}{\sigma_1}\right)^n \tag{13}$$

Equations (11) and (12) may now be written in the following alternate forms:

For σ_R/σ_1 increasing,

$$\frac{3}{14}\left[(1 + f)^n - (1 - f)^n\right]\left(\frac{\bar{\sigma}}{\sigma_1}\right)^n + f\,\frac{\bar{\sigma}}{\sigma_1} = \left[\frac{\sigma_T}{\sigma_1} + \frac{3n}{7}\left(\frac{\sigma_T}{\sigma_1}\right)^n\right](f - e) \tag{14}$$

and, for σ_R/σ_1 decreasing,

$$\frac{3}{14}(1+f)^n\left(\frac{\bar{\sigma}}{\sigma_1}\right)^n + f\frac{\bar{\sigma}}{\sigma_1} = \frac{3}{14}\left[\left(\frac{\sigma_R}{\sigma_1}\right)_{max}\right]^n + \left[\frac{\sigma_T}{\sigma_1} + \frac{3n}{7}\left(\frac{\sigma_T}{\sigma_1}\right)^n\right](f - e) \qquad (15)$$

Equations (14) and (15) may be solved for values of $\bar{\sigma}/\sigma_1$ corresponding to assigned values of f when particular values are given to the initial-curvature parameter e, the stress-strain-curve shape parameter n, and the parameter σ_T/σ_1 which defines the column proportions for a given stress-strain curve. If σ_R/σ_1 is evaluated from equation (6) for each value of f assigned to equation (14), the point at which equation (15) must be used instead of equation (14) is readily apparent.

Solutions of equations (14) and (15) were obtained with the aid of SEAC, the National Bureau of Standards Eastern Automatic Computer, for several values of e ranging from 0.00001 to 0.05, n from 2 to 40, and σ_T/σ_1 from 0.1 to 1.0. The results for selected values of these parameters are described in the following section.

RESULTS

The effect of initial curvature on the strength of inelastic columns can be determined from solutions of the load-deflection relations derived in the preceding section. Solutions of the load-deflection equations for $n = 10$ with different combinations of the dimensionless tangent-modulus load P_T/P_1 and the dimensionless initial midheight deflection e are plotted in figure 3 to show the growth of the lateral deflection f as the load changes. The locus of the average stress of the column when the stress on the convex flange reaches a maximum value and begins to decrease and the locus of the maximum values of average stress are shown as dashed-line curves in each plot of this figure. Figure 3 confirms the expectation that, as the columns become more slender (P_T/P_1 decreasing), the maximum loads approach the Euler loads and occur at larger deflections for a given value of e.

For other values of n, corresponding load-deflection curves have the same general appearance but have different magnitudes of load and deflection parameters at stress reversal and at maximum load. For smaller values of n, the magnitude of the lateral-deflection parameter at stress reversal and maximum load is greater. The near straightness of the stress-reversal-load and maximum-load loci observed in figure 3 ($n = 10$) was also evident for the entire range of n considered.

The maximum loads for selected values of n from 2 to 40 are summarized in figure 4 in plots of P_{max}/P_1 against the Euler strain parameter $\frac{\pi^2}{(L/\rho)^2\epsilon_1}$. Included for comparison are the dimensionless tangent-modulus load P_T/P_1 against the Euler strain parameter and the dimensionless stress-strain curve σ/σ_1 against ϵ/ϵ_1. For a high value of n, say $n = 20$ or 40, figure 4(c) indicates that the difference between the maximum load for an initially straight column and the tangent-modulus load is relatively small. This fact was also observed in reference 7.

The reduction in strength caused by initial curvature is plotted as a percentage of the maximum load for the initially straight column in figure 5.

Figure 6 presents the ratio of the maximum load for the initially curved column to the tangent-modulus load for the corresponding straight column. The curves of this figure show that the maximum load for an initially straight column is always greater than the tangent-modulus load. The maximum load for an initially curved column is always less than the maximum load for the corresponding straight column and may even be less than the tangent-modulus load, depending upon the column proportions, the magnitude of the initial curvature, and the shape of the stress-strain curve.

Examination of the midheight flange stresses when stress reversal occurred in the convex flange showed that, for all the combinations of column proportions, materials, and initial curvatures considered, the stress in the concave flange approximately equaled the tangent-modulus stress for the initially straight column. This observation suggests a method for determining the theoretical critical load for an initially straight column from the experimental investigation of the corresponding initially curved column. The application of this experimental method to plates and built-up sections is beyond the scope of the present paper but is worthy of further investigation.

CONCLUDING REMARKS

Previous studies have shown that, for an initially straight inelastic column, the amount by which the maximum load exceeds the tangent-modulus load is related to the shape of the stress-strain curve and the proportions of the column. The present analysis indicates that the maximum load for an initially curved inelastic column is always less than the maximum load for the corresponding straight column and may even be less than the tangent-modulus load, depending upon the column proportions, the magnitude of the initial curvature, and the shape of the stress-strain curve.

Langley Aeronautical Laboratory,
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Langley Field, Va., July 8, 1952.

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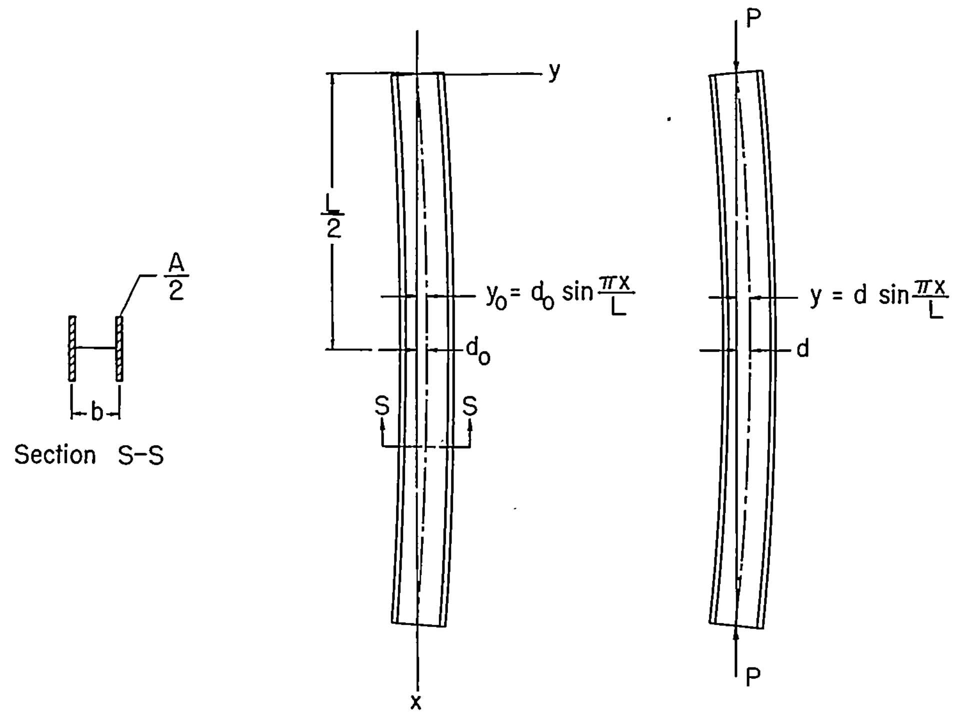


Figure 1.- Idealized H-section column.

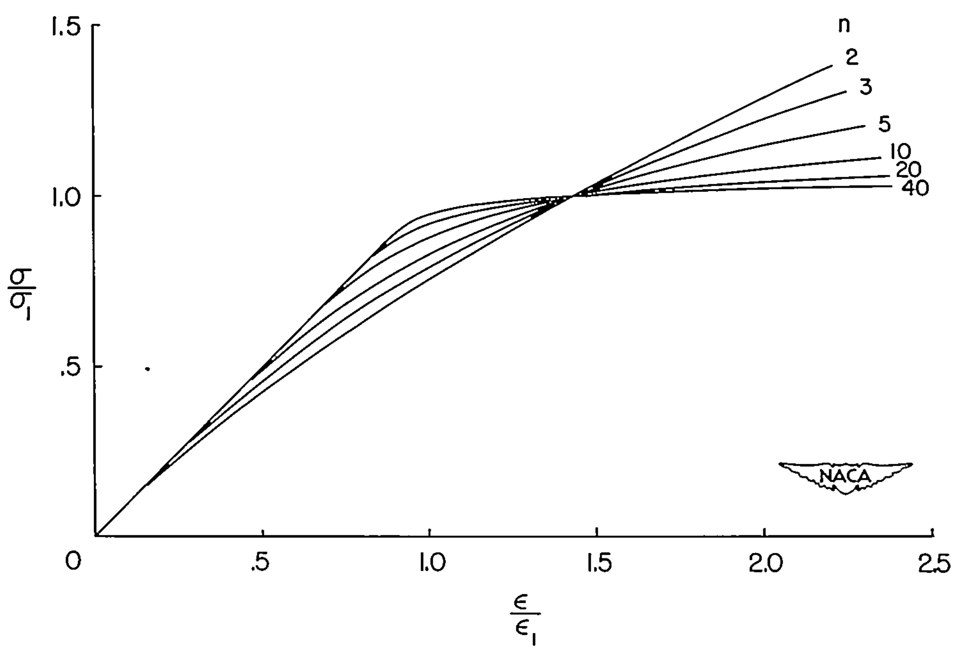


Figure 2.- Ramberg-Osgood dimensionless stress-strain curves.

$$\frac{\epsilon}{\epsilon_1} = \frac{\sigma}{\sigma_1} + \frac{3}{7}\left(\frac{\sigma}{\sigma_1}\right)^n.$$

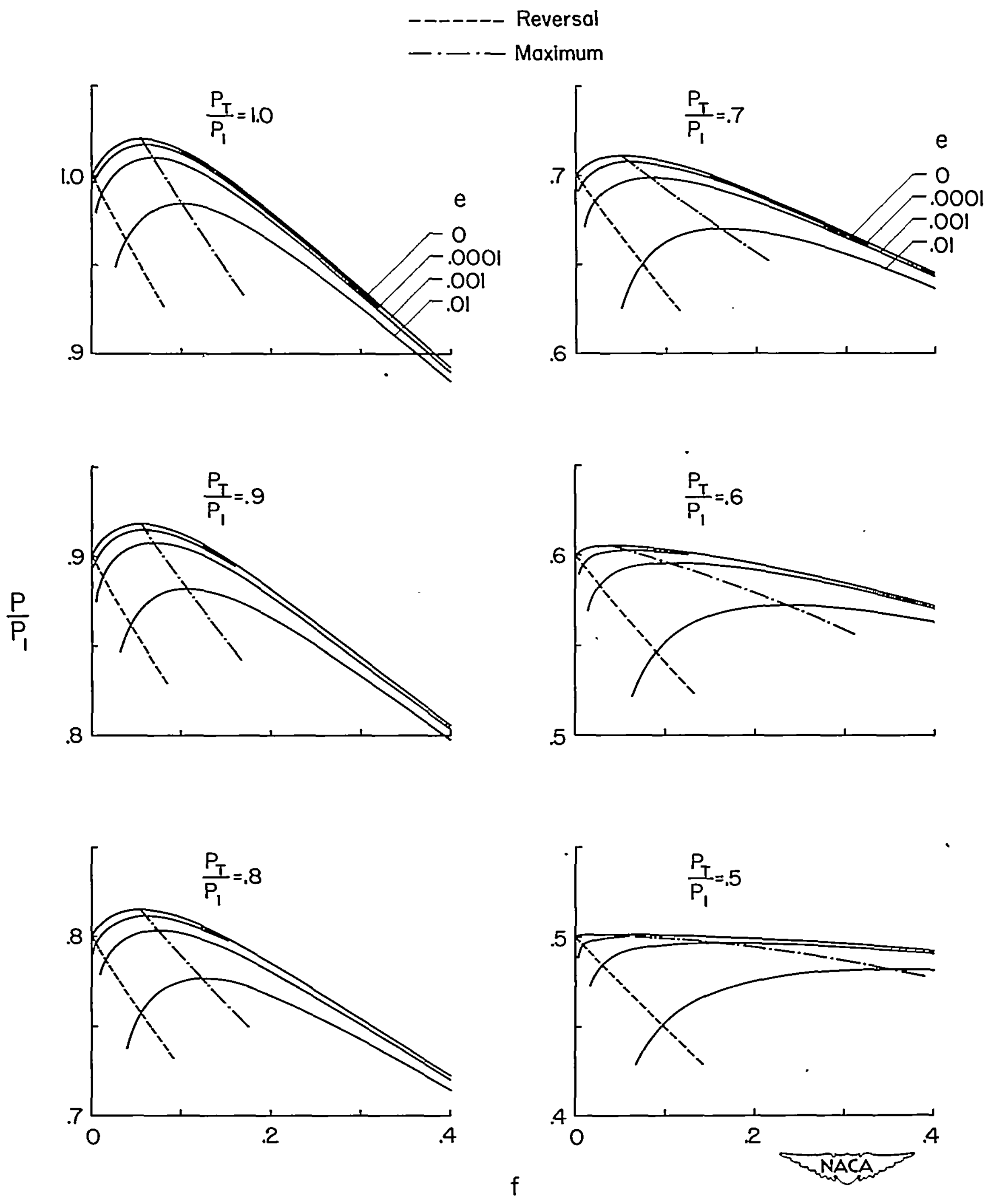
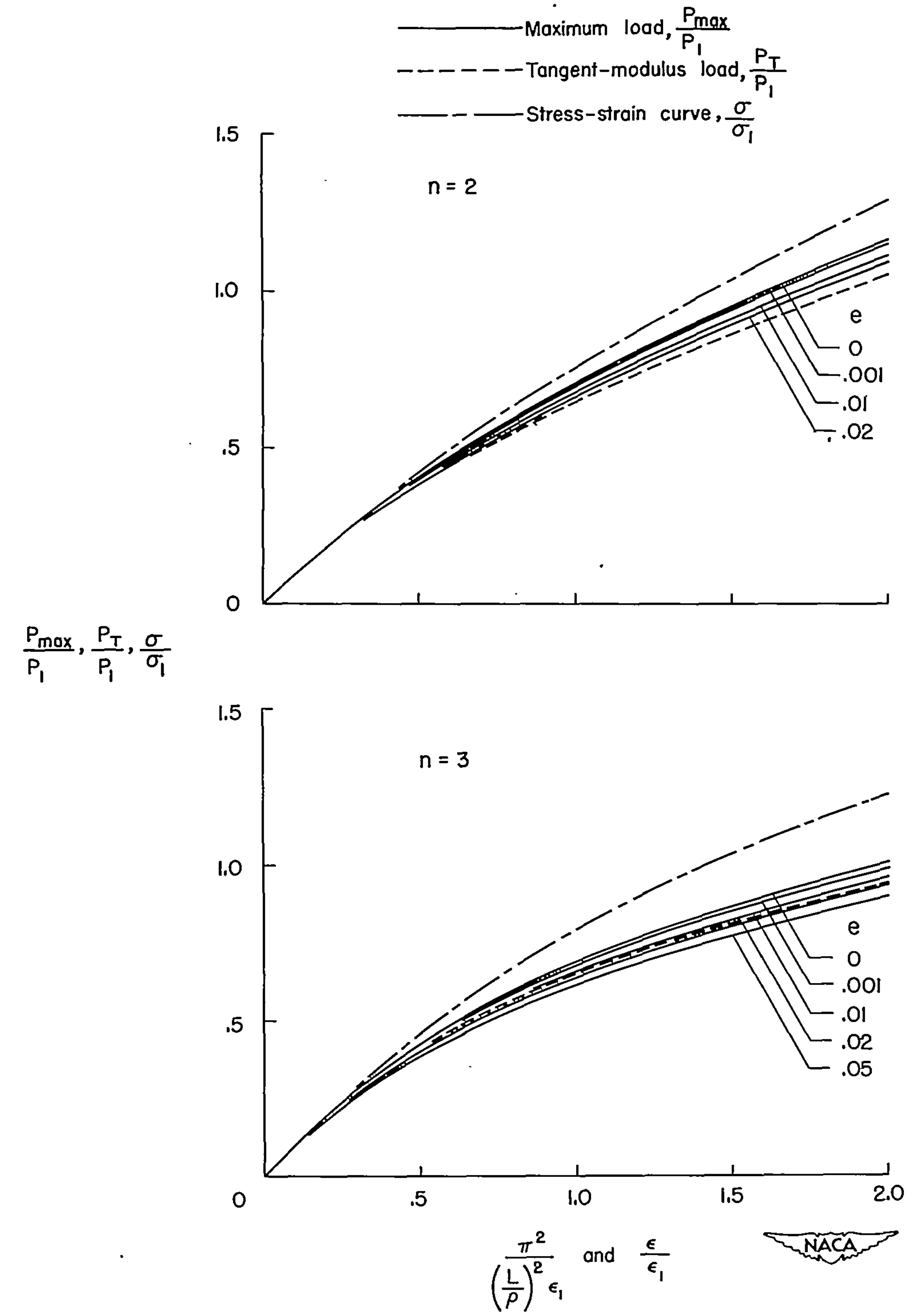
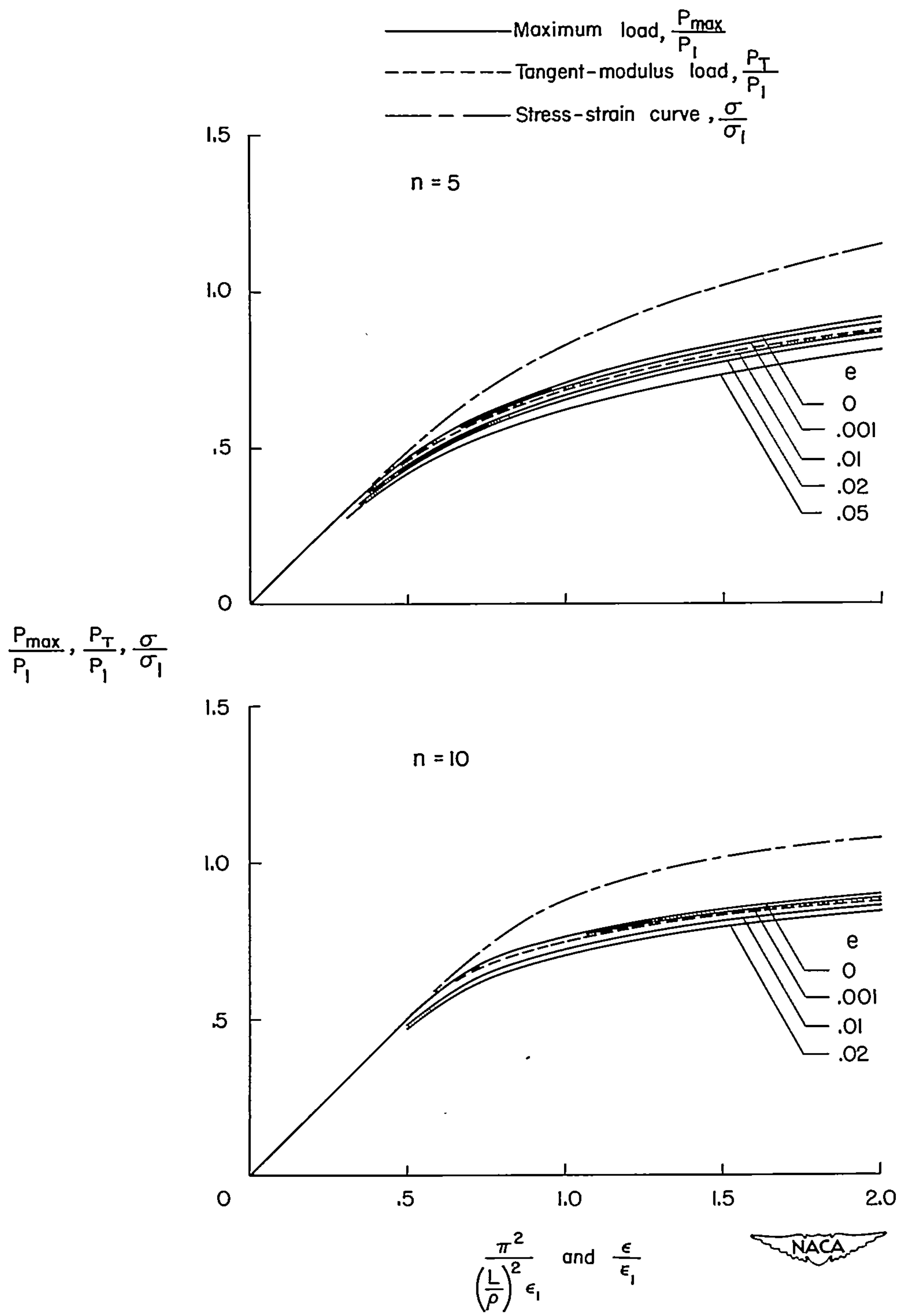


Figure 3.- Load–lateral-deflection curves for initially curved idealized H-section columns having various tangent-modulus loads. n = 10.



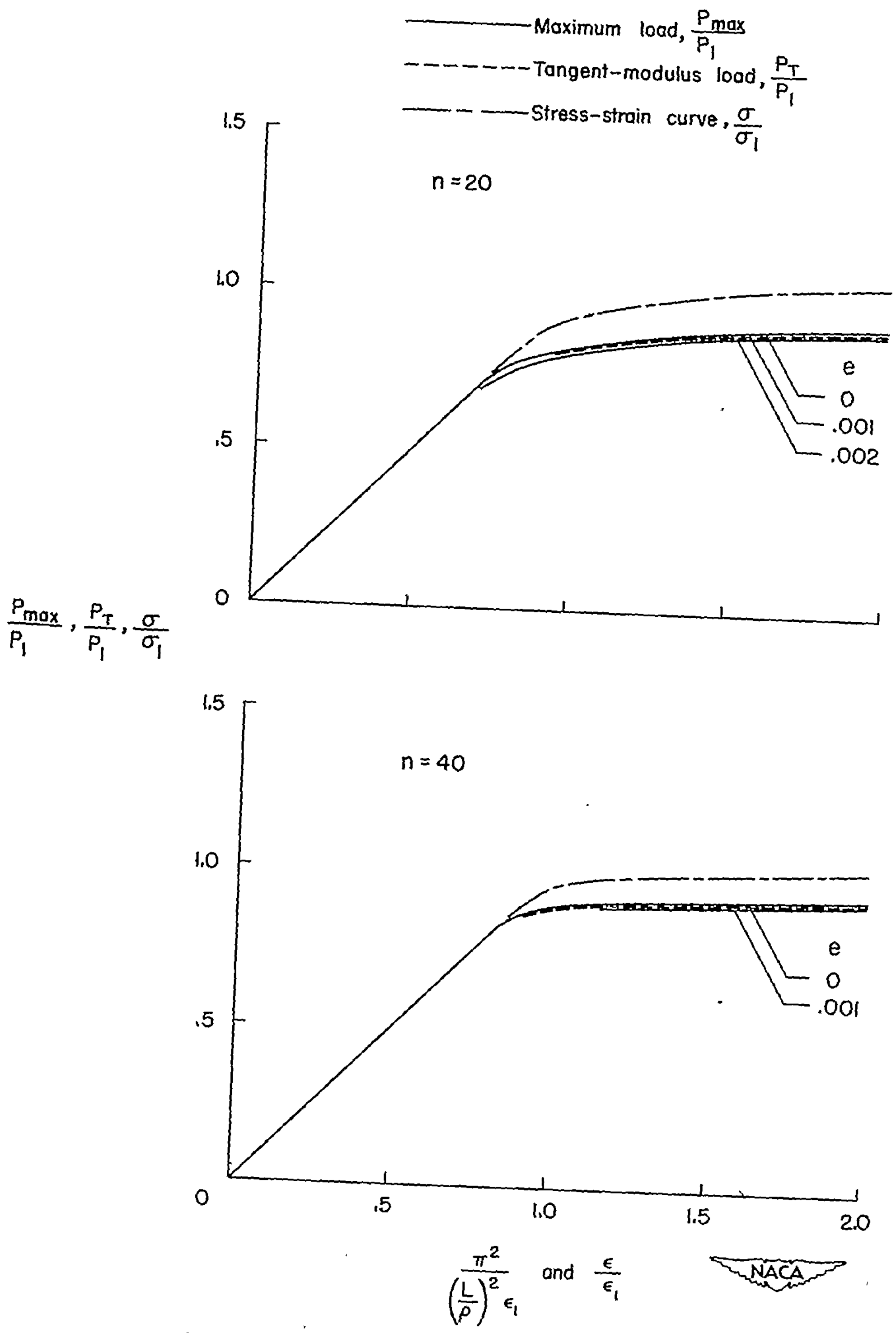
(a) n = 2 and 3.

Figure 4.- Maximum loads supported by idealized H-section columns with varying initial curvature.



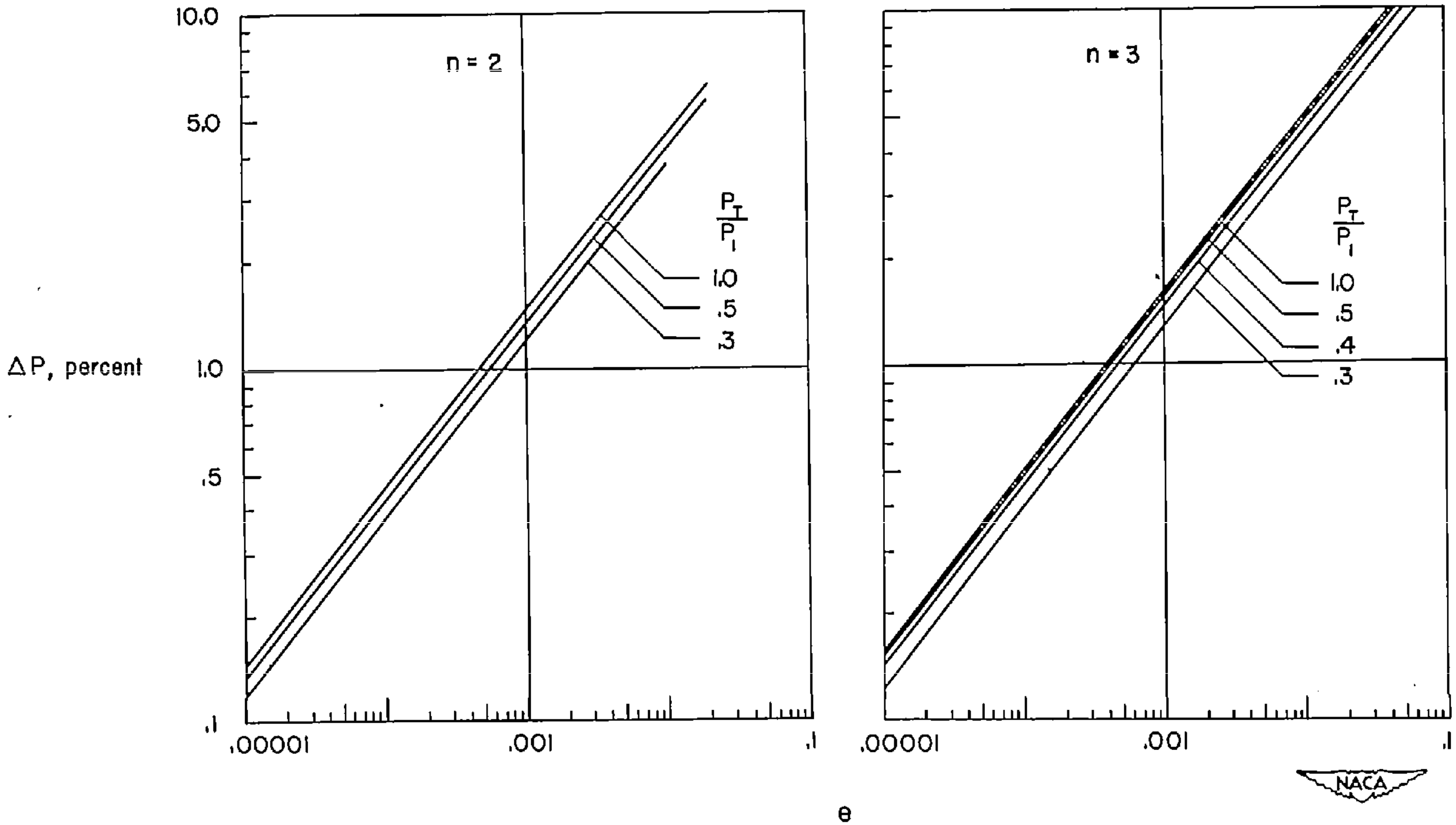
(b) n = 5 and 10.

Figure 4.- Continued.



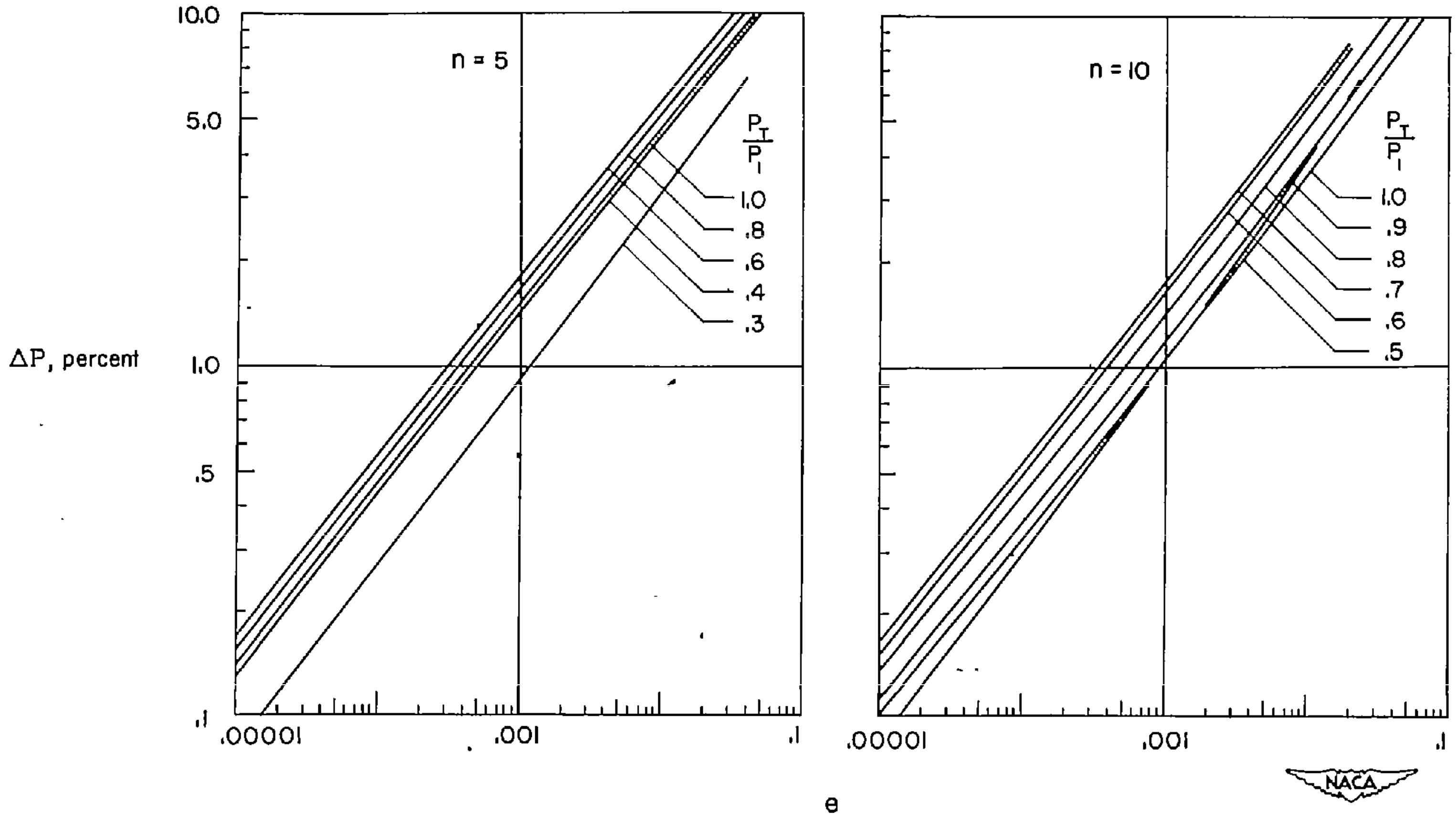
(c) n = 20 and 40.

Figure 4.- Concluded.



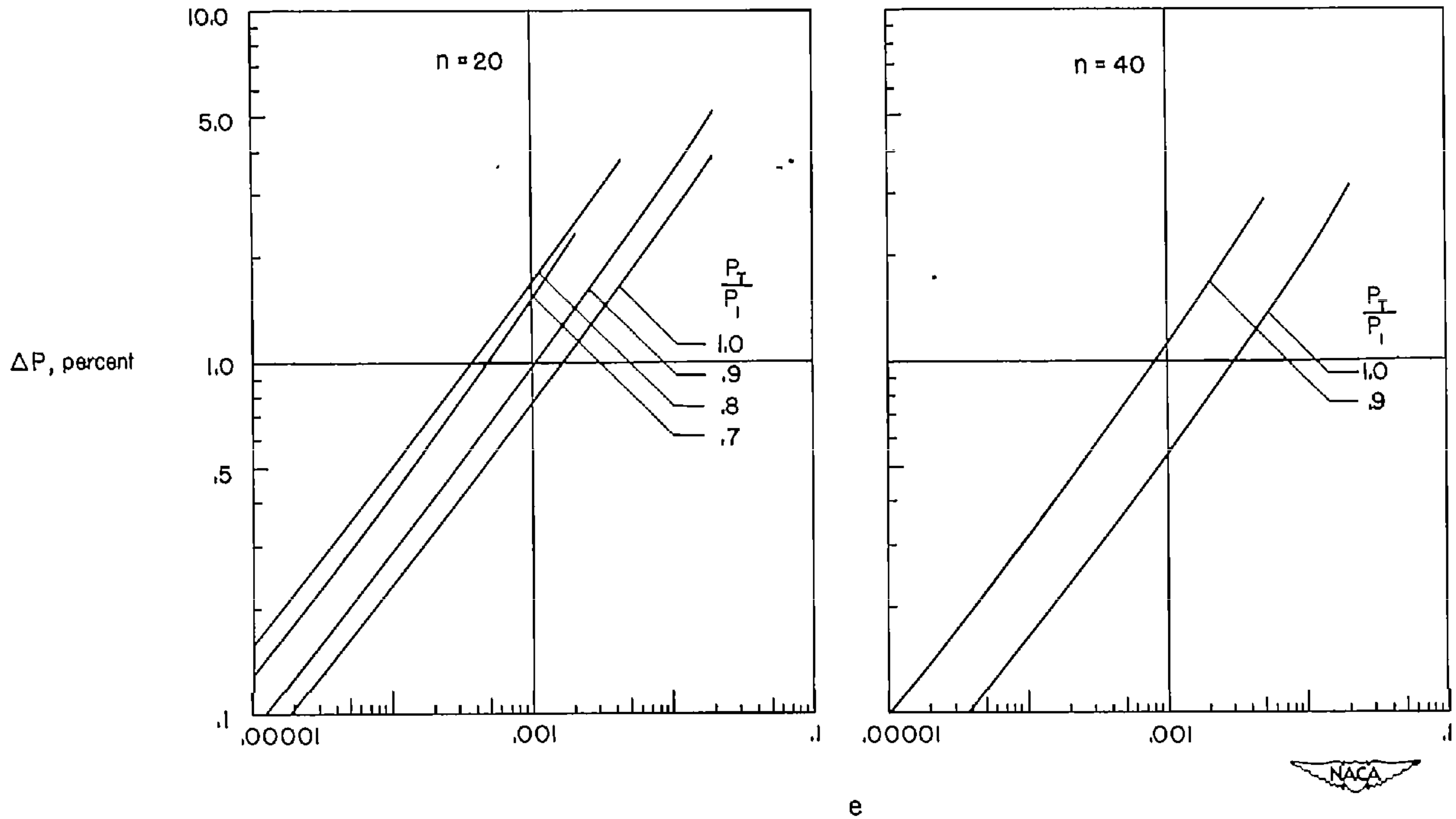
(a) n = 2 and 3.

Figure 5.- Percent reduction in maximum load due to initial curvature for idealized H-section columns. $\Delta P = \frac{\left(P_{max}\right)_{e=0} - \left(P_{max}\right)_{e\neq0}}{\left(P_{max}\right)_{e=0}} \times 100.$



(b) n = 5 and 10.

Figure 5.- Continued.



(c) n = 20 and 40.

Figure 5.- Concluded.

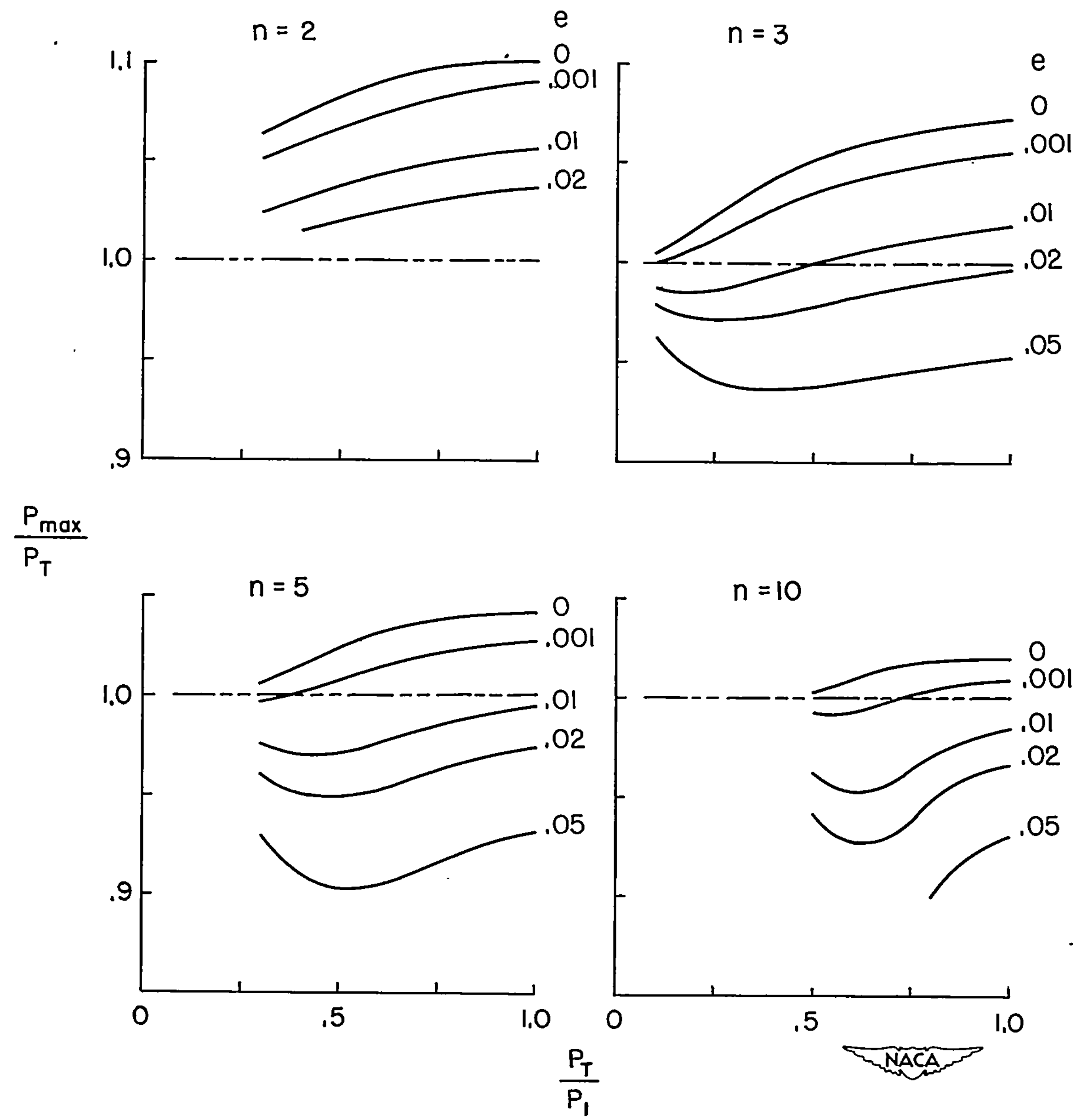


Figure 6.- Comparison between tangent-modulus load and maximum load for idealized H-section columns having initial curvature.